Bayesian Interpolation with Deep Linear Networks

Boris Hanin (Princeton ORFE) and Alexander Zlokapa (MIT Physics)

PNAS 2023 (arXiv:2212.14457)

Motivating Questions

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- $\underline{P<\infty}$: Deviation from linear model depends on

$$\frac{1}{N_1} + \cdots + \frac{1}{N_L} \simeq \frac{L}{N}$$

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Q3. How do models make predictions on test data?

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Effective depth of posterior: $$P\left(\frac{1}{N_1} + \cdots + \frac{1}{N_L}\right) \simeq \frac{PL}{N}$$

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T3. Deep networks with universal priors learn same posteriors as shallow networks with optimal data-dependent priors

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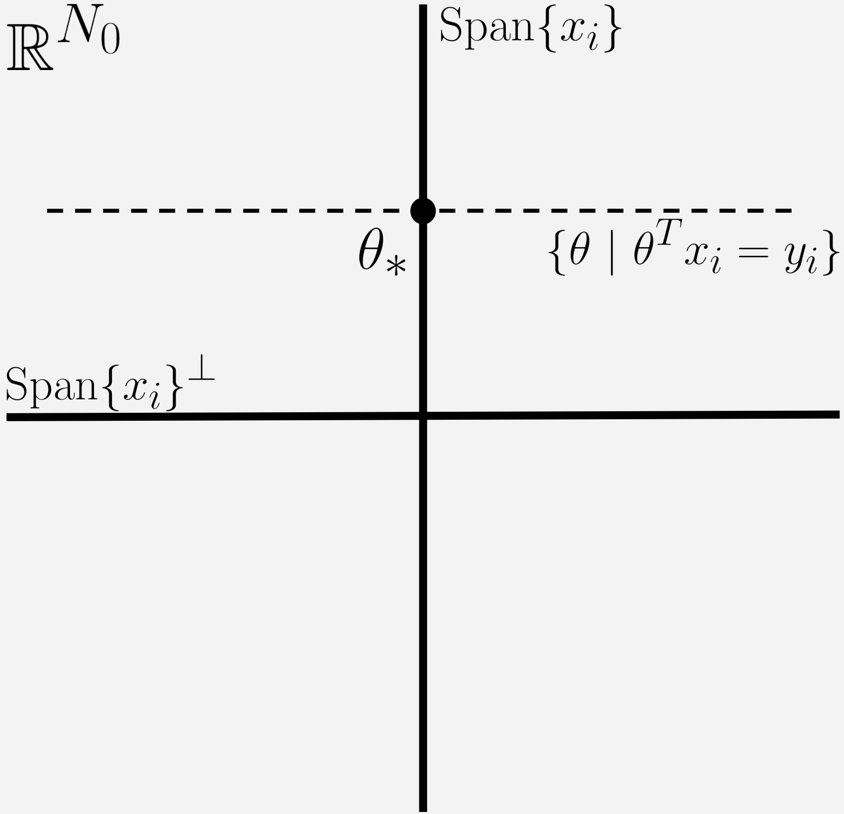

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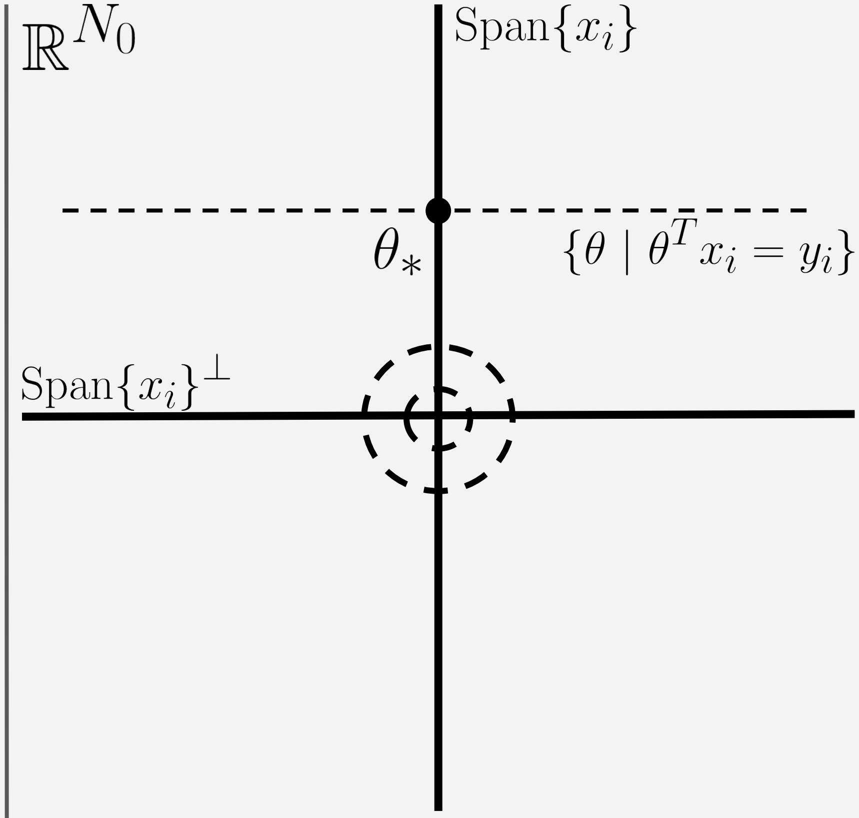

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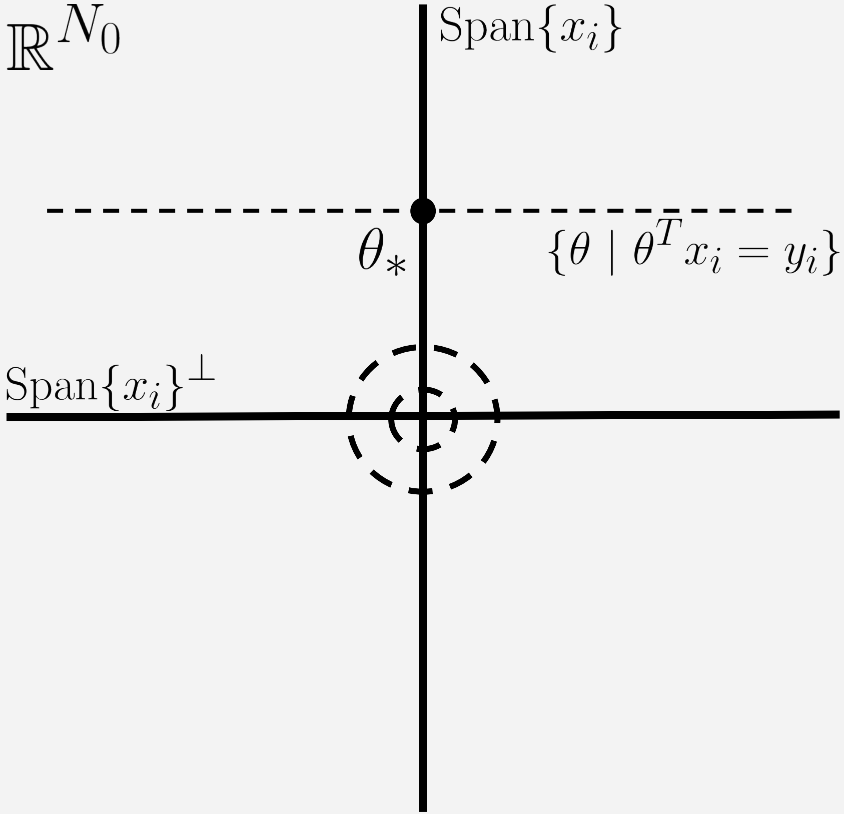

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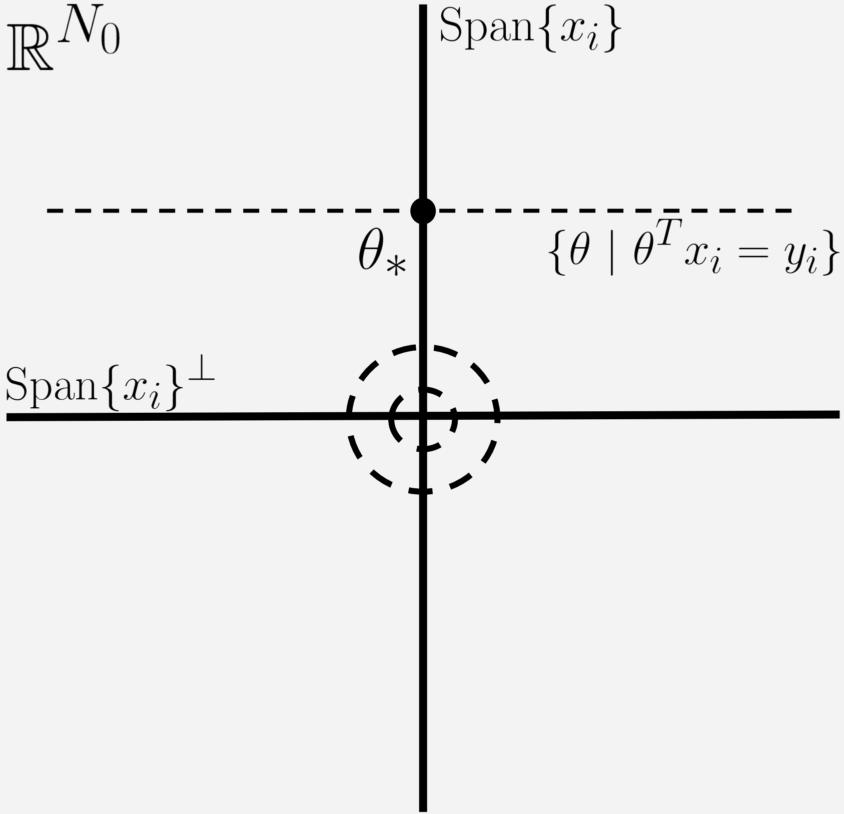

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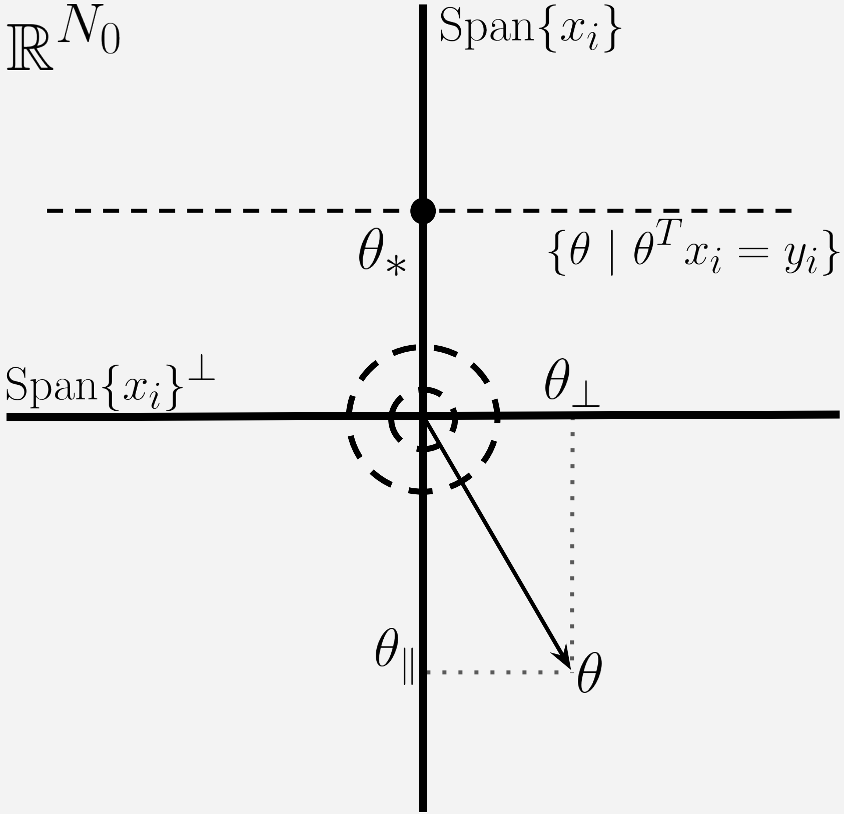

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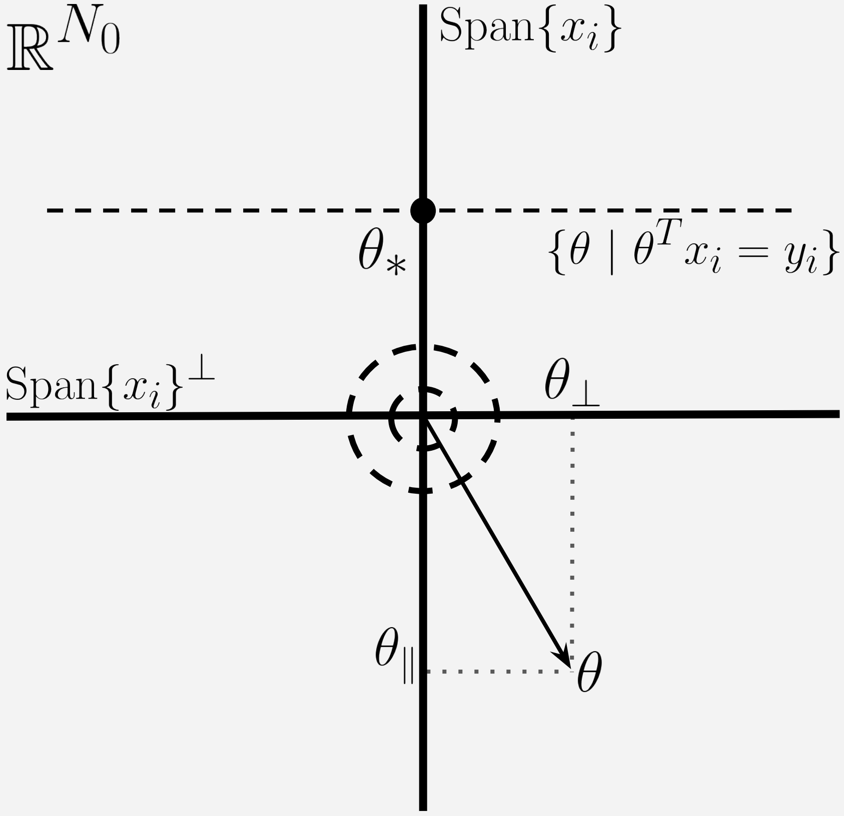
$$\Rightarrow \theta = \theta_* + ||\theta_\perp|| \cdot \text{unif}$$



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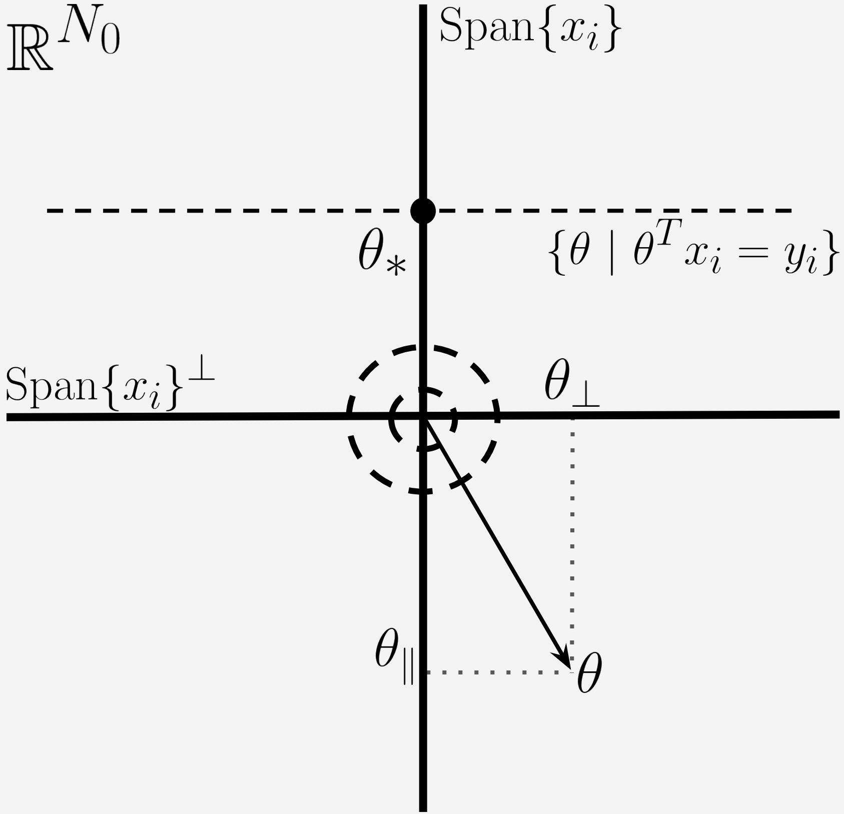
3 $||\theta_\perp||$ is a learnable data-dependent scale for predictions in new directions



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Evidence and Posterior Via G-Functions

Thm. Write $Z(t) = \mathbb{E}_{\mathrm{post}}\left[\exp\left\{-it\cdot\theta\right\}\right].$ Then,

$$Z(t) = \left(\frac{4\pi}{||\theta_*||^2}\right)^{\frac{P}{2}} \times e^{-i\langle\theta_*,\, t_{||}\rangle} \times \prod_{\ell=1}^{L}\Gamma\left(\frac{N_\ell}{2}\right)^{-1}$$

$$\times \sum_{k=0}^{\infty}\frac{(-||t^{\perp}||^2 M)^k}{k!}\, G_{L+1}\left(\frac{||\theta_*||^2}{4M}\,\middle|\,\frac{P}{2}, \frac{N_1}{2}+k, \ldots, \frac{N_L}{2}+k\right)$$

where $M = \prod_{\ell=0}^{L} 2\sigma^2/N_\ell$ and

$$G_\ell(z\,|\,b_1,\ldots,b_\ell) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} ds\; z^s \prod_{j=1}^{\ell}\Gamma\left(1+b_j-s\right)$$

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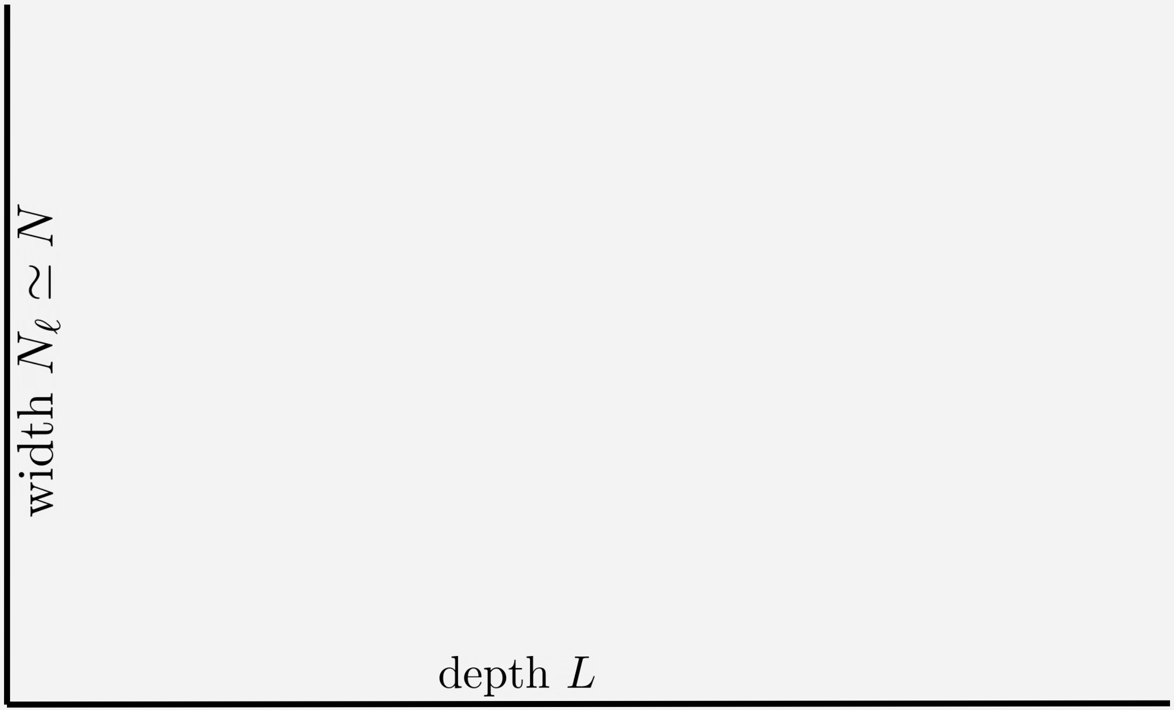

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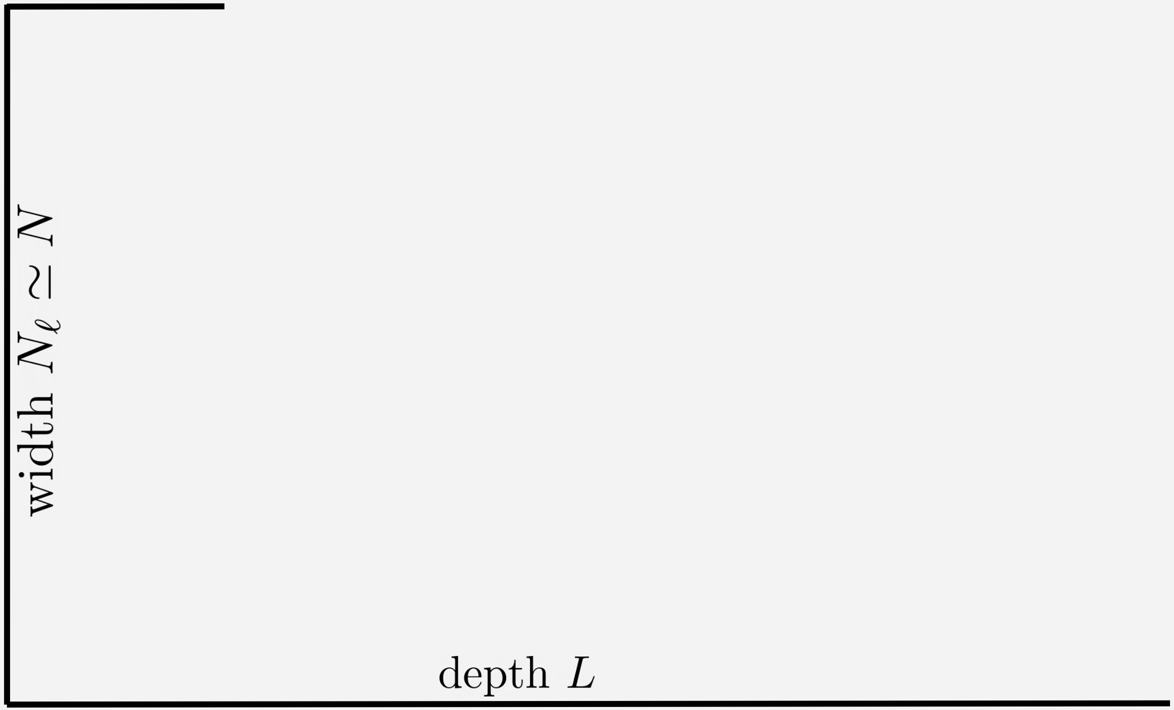

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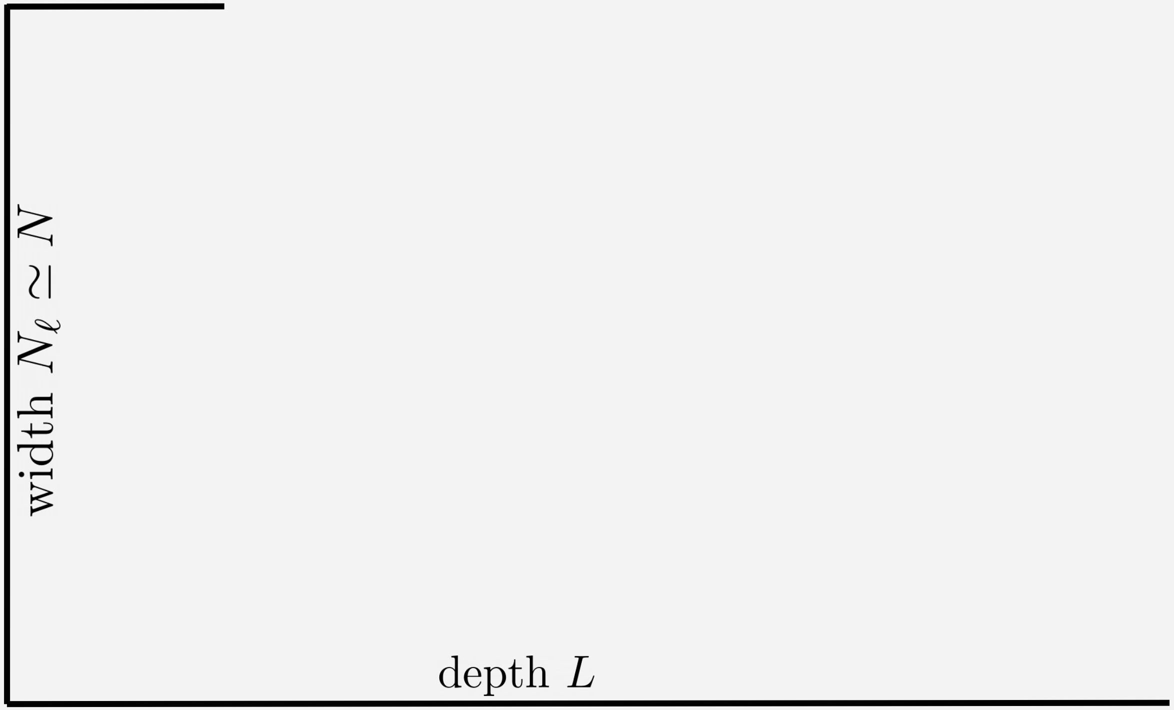

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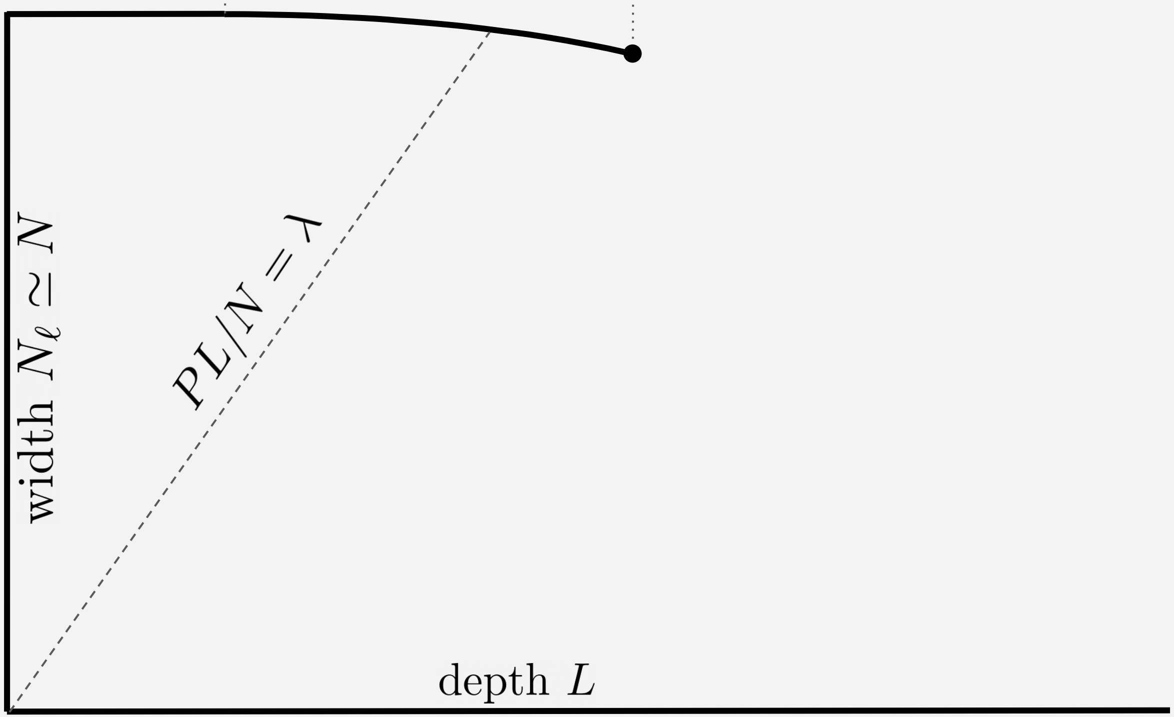

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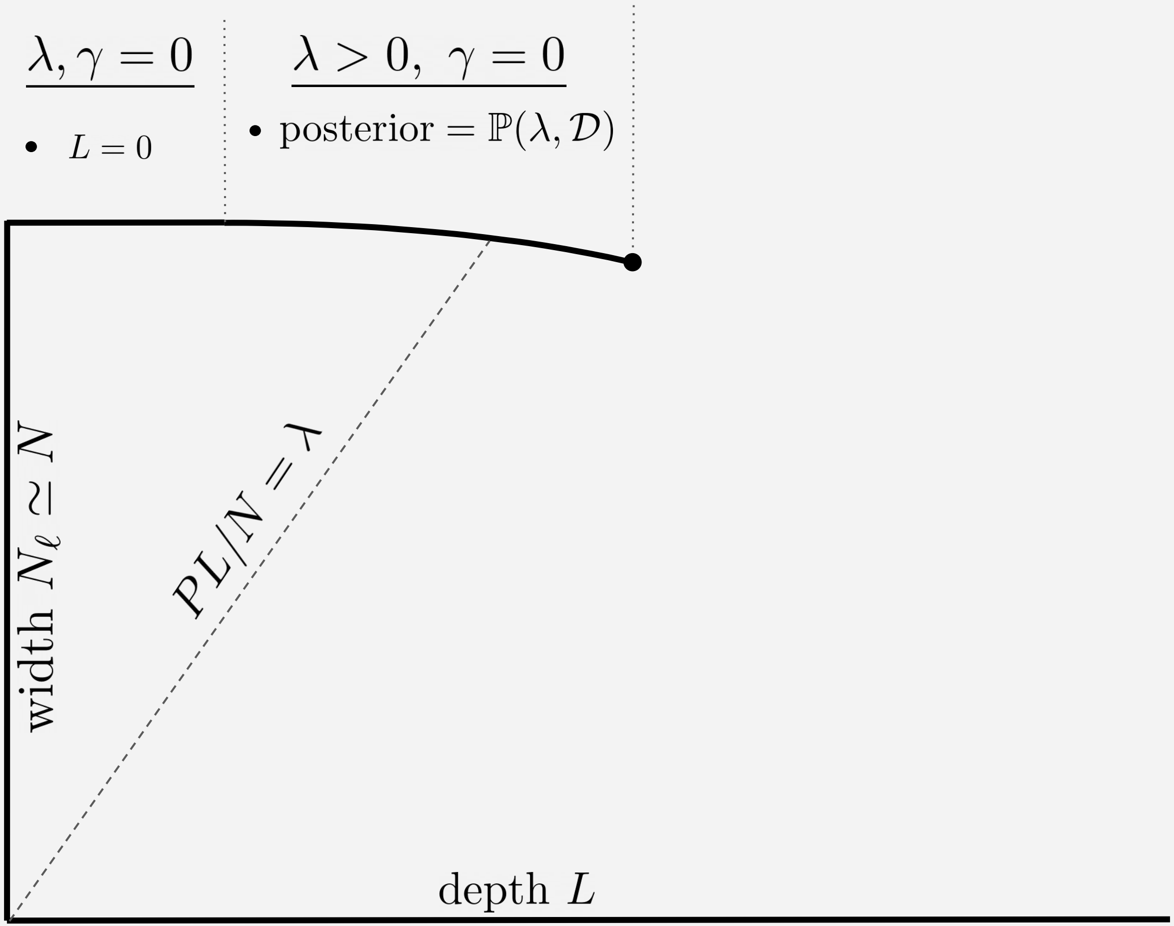

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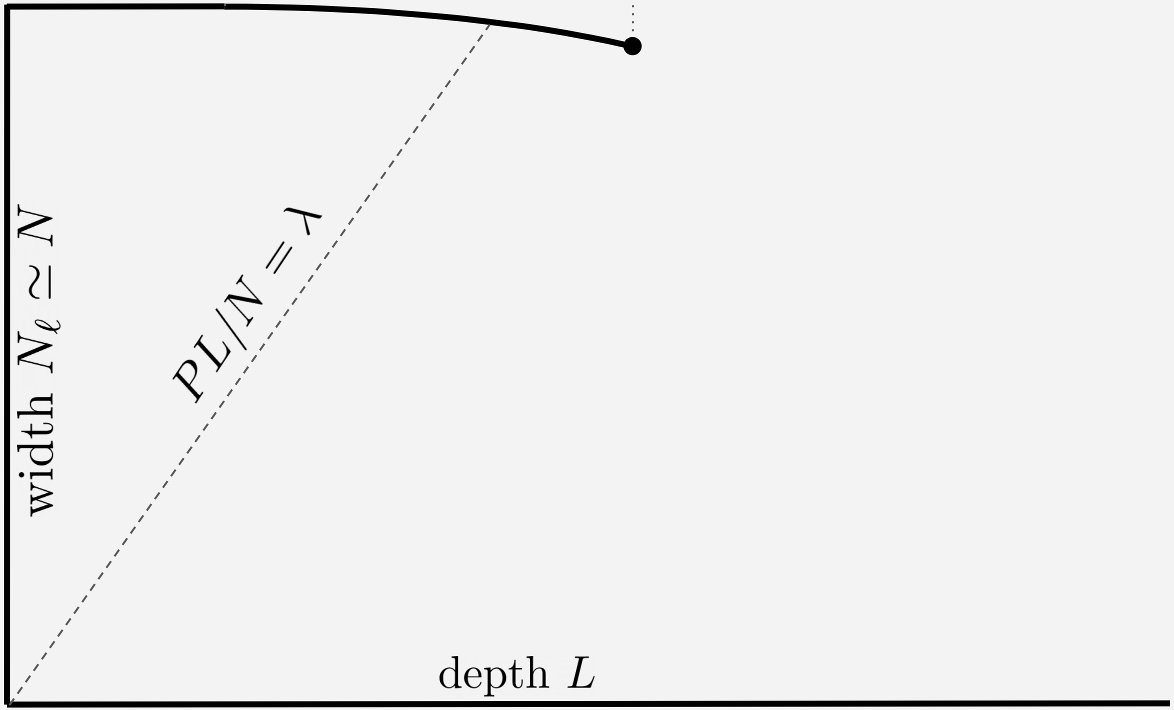

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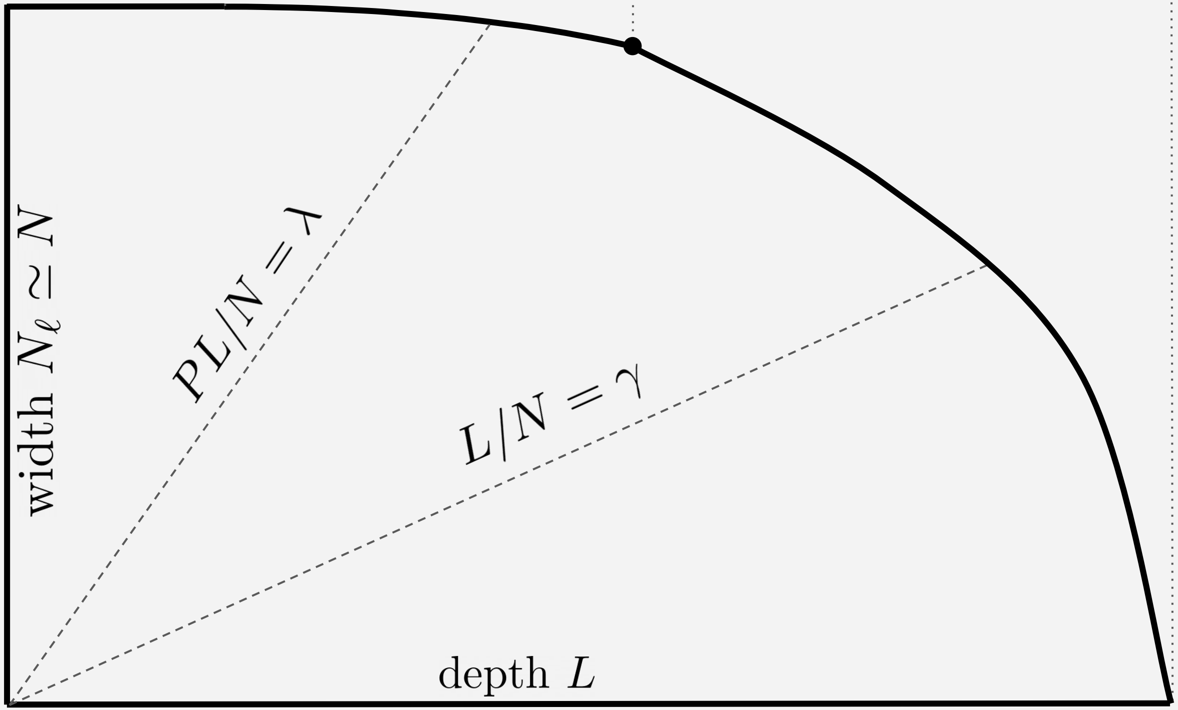

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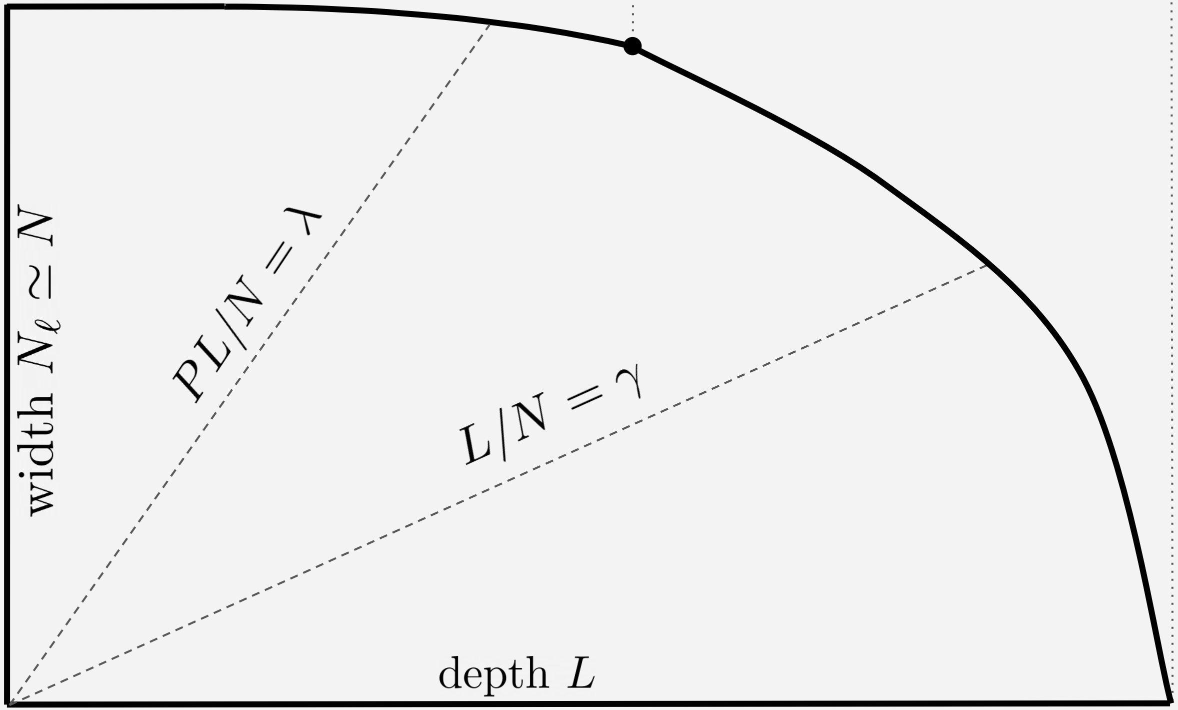

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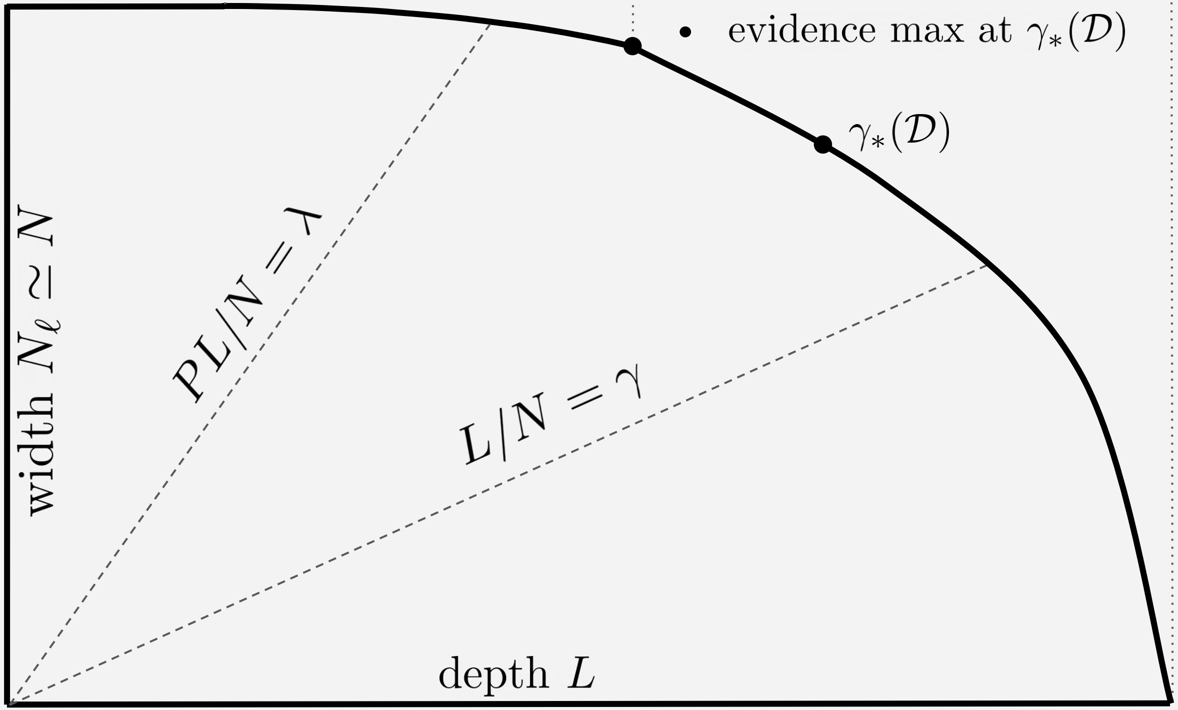
- evidence max at $\gamma_*(\mathcal{D})$



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Deep Linear Networks Learn Optimal Features

Thm. Define $\sigma_* := \mathrm{argmax}_\sigma \lim_{\substack{P, N_0 \to \infty \\ P/N_0 \to \alpha_0}} Z_\infty(\mathcal{D} \mid L, N_\ell, \sigma)$

Then,

$$\lim_{\substack{P, N_0 \to \infty \\ P/N_0 \to \alpha_0}} \mathbb{P}_{\text{post}}(||\theta_\perp|| \mid L, N_\ell, \sigma = \sigma_*) = \delta_{\kappa^*}$$

for $\kappa^* = \kappa^*(\alpha_0, ||\theta_*||)$. Moreover,

$$\lim_{\substack{P, N_0 \to \infty \\ P/N_0 \to \alpha_0 \\ PL/N \to \lambda}} \mathbb{P}_{\text{post}}(||\theta_\perp|| \mid L, N_\ell, \sigma = 1) = \delta_{\kappa(\lambda)}$$

where

$$\kappa(\lambda) \to \begin{cases} \kappa^*, & \lambda \to \infty \\ L = 0 \text{ post}, & \lambda \to 0 \end{cases}.$$