Tikhonov Regularisation for (Large) Inverse Problems

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joint work with C.J. Budd (Bath) and N.K. Nichols (Reading)

Ill-posed Problems

Given an operator A we wish to solve

$$Af = g.$$

It is well-posed if

- solution exits
- solution is unique
- is stable (A^{-1} continuous)

but ..

In finite dimensions existence and uniqueness can be imposed, but

- discrete problem of underlying ill-posed problem becomes ill-conditioned
- singular values of A decay to zero

An Illustrative Example

Fredholm first kind integral equation in 1D

$$g(x) = \int_0^1 k(x - x')f(x')dx' =: (Af)(x), \quad 0 < x < 1$$

- f light source intensity as a function of x
- g image intensity
- k kernel representing blurring effects, e.g. $k(x) = C \exp\left(-\frac{x^2}{2\gamma^2}\right)$, C, γ are positive parameters.

Discretisation

- use a piecewise smooth source $\mathbf{f}$
- determine $\mathbf{A}$ using standard numerical quadrature;

$$(\mathbf{A})_{ij} = hC \exp\left(-\frac{((i-j)h)^2}{2\gamma^2}\right), \quad 1 \le i, j \le n, \quad h = \frac{1}{n}$$

$\gamma = 0.05$, $C = \frac{1}{\gamma\sqrt{2\pi}}$.

Inverse Problem

Given the kernel k, and the blurred image g, determine the source f from $g = Af$, solve the discrete linear system

$$\mathbf{g} = \mathbf{A}\mathbf{f}.$$

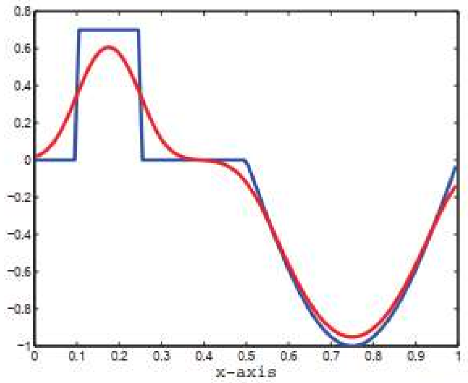

Figure: True solution and blurred image $\mathbf{g}$

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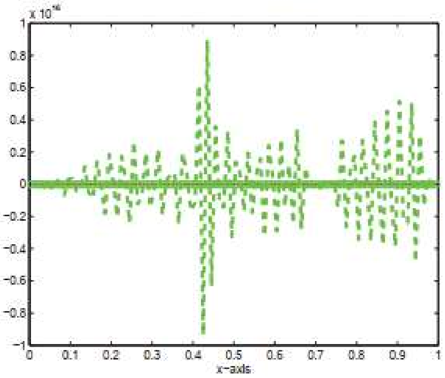


Figure: Naive Solution

Inverse Problem

Problem: data $\mathbf{g}$ are observed and contain noise and $\mathbf{A}$ is ill-conditioned:

$$\mathbf{g}_{\text{exact}} + \mathbf{e} = \mathbf{A}\mathbf{f},$$

$\mathbf{e}$ is unknown white noise.

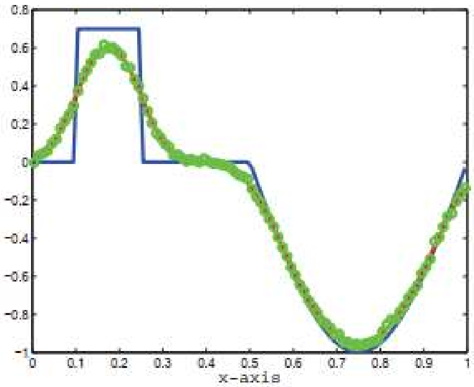


Figure: True solution and discrete noisy data

Inverse Problem

Problem: data $\mathbf{g}$ are observed and contain noise and $\mathbf{A}$ is ill-conditioned:

$$\mathbf{g}_{\text{exact}} + \mathbf{e} = \mathbf{A}\mathbf{f},$$

$\mathbf{e}$ is unknown white noise.

Singular Value Decomposition

Let

$$\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^T = \sum_{i=1}^{r} \sigma_i \mathbf{u}_i \mathbf{v}_i^T$$

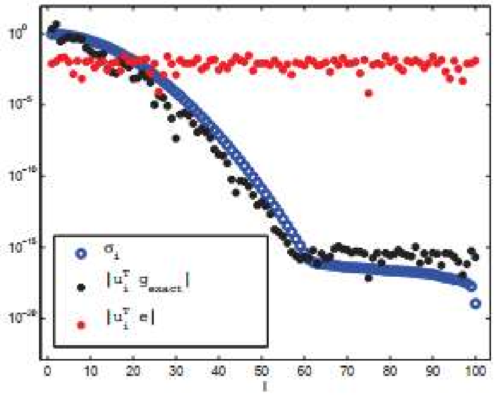
where

- $\mathbf{\Sigma} = \text{diag}(\sigma_1, \ldots, \sigma_r)$ and $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_r > 0$
- $\mathbf{U}^T\mathbf{U} = \mathbf{I}$ and $\mathbf{V}^T\mathbf{V} = \mathbf{I}$

Inverse Problem - Regularisation needed

Least squares solution (with and without noise)

$$
\begin{aligned}
\mathbf{f}_{\text{exact}} &= \mathbf{A}^{\dagger}\mathbf{g}_{\text{exact}} = \sum_{i=1}^{r} \frac{\mathbf{u}_i^T \mathbf{g}_{\text{exact}}}{\sigma_i}\mathbf{v}_i \\
\mathbf{f} &= \mathbf{A}^{\dagger}\mathbf{g} = \mathbf{A}^{\dagger}(\mathbf{g}_{\text{exact}} + \mathbf{e}) = \sum_{i=1}^{r} \frac{\mathbf{u}_i^T \mathbf{g}_{\text{exact}}}{\sigma_i}\mathbf{v}_i + \sum_{i=1}^{r} \frac{\mathbf{u}_i^T \mathbf{e}}{\sigma_i}\mathbf{v}_i \\
&= \mathbf{f}_{\text{exact}} + \sum_{i=1}^{r} \frac{\mathbf{u}_i^T \mathbf{e}}{\sigma_i}\mathbf{v}_i
\end{aligned}
$$



Tikhonov Regularisation

Regularised solution of the form

$$
\mathbf{f}_{\alpha} = \sum_{i=1}^{r} \frac{\sigma_i^2}{\sigma_i^2 + \alpha^2} \frac{\mathbf{u}_i^T \mathbf{g}}{\sigma_i}\mathbf{v}_i
$$

α regularisation parameter.
Solution $\mathbf{f}_{\alpha}$ to the minimisation problem

$$
\min_{\mathbf{f}} \left\{ \|\mathbf{g} - \mathbf{A}\mathbf{f}\|_2^2 + \alpha^2 \|\mathbf{f}\|_2^2 \right\}.
$$

Least squares solution $\mathbf{f}_{\alpha}$ to the linear system

$$
\begin{bmatrix} \mathbf{A} \\ \alpha\mathbf{I} \end{bmatrix} \mathbf{f} = \begin{bmatrix} \mathbf{g} \\ \mathbf{0} \end{bmatrix}.
$$

Normal equations

$$
(\mathbf{A}^T\mathbf{A} + \alpha^2\mathbf{I})\mathbf{f}_{\alpha} = \mathbf{A}^T\mathbf{g}.
$$

Tikhonov Regularisation

Regularisation parameter α

Regularised solution of the form

$$\mathbf{f}_\alpha = \sum_{i=1}^{r} \frac{\sigma_i^2}{\sigma_i^2 + \alpha^2} \frac{\mathbf{u}_i^T \mathbf{g}}{\sigma_i} \mathbf{v}_i$$

Filter factor $\dfrac{\sigma_i^2}{\sigma_i^2 + \alpha^2}$ as diagonal entries of the filter matrix $\mathbf{\Psi}$

Regularisation and perturbation error

$$\begin{aligned}
\mathbf{f}_\alpha &= \mathbf{V\Psi U}^T \mathbf{g}, \quad \mathbf{g} = \mathbf{g}_{\text{exact}} + \mathbf{e} \\
&= \mathbf{V\Psi\Sigma}^{-1}\mathbf{U}^T \mathbf{g}_{\text{exact}} + \mathbf{V\Psi\Sigma}^{-1}\mathbf{U}^T \mathbf{e} \\
&= \mathbf{V\Psi\Sigma}^{-1}\mathbf{U}^T \mathbf{U\Sigma V}^T \mathbf{f}_{\text{exact}} + \mathbf{V\Psi\Sigma}^{-1}\mathbf{U}^T \mathbf{e} \\
&= \mathbf{V\Psi V}^T \mathbf{f}_{\text{exact}} + \mathbf{V\Psi\Sigma}^{-1}\mathbf{U}^T \mathbf{e} \\
\mathbf{f}_{\text{exact}} - \mathbf{f}_\alpha &= (\mathbf{I} - \mathbf{V\Psi V}^T)\mathbf{f}_{\text{exact}} - \mathbf{V\Psi\Sigma}^{-1}\mathbf{U}^T \mathbf{e}
\end{aligned}$$

Regularisation error Perturbation error

Tikhonov Regularisation

Regularisation and perturbation error

$$\mathbf{f}_{\text{exact}} - \mathbf{f}_\alpha = (\mathbf{I} - \mathbf{V\Psi V}^T)\mathbf{f}_{\text{exact}} - \mathbf{V\Psi\Sigma}^{-1}\mathbf{U}^T \mathbf{e}$$

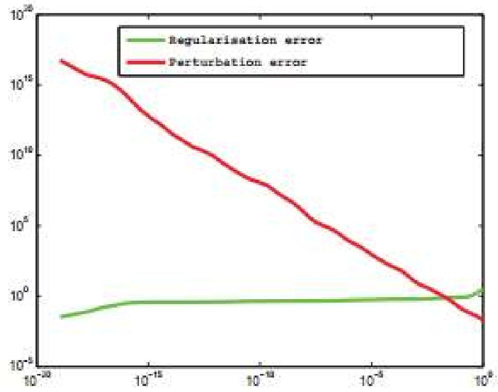


Figure: Regularisation and perturbation error

Tikhonov Regularisation

Illustrative example

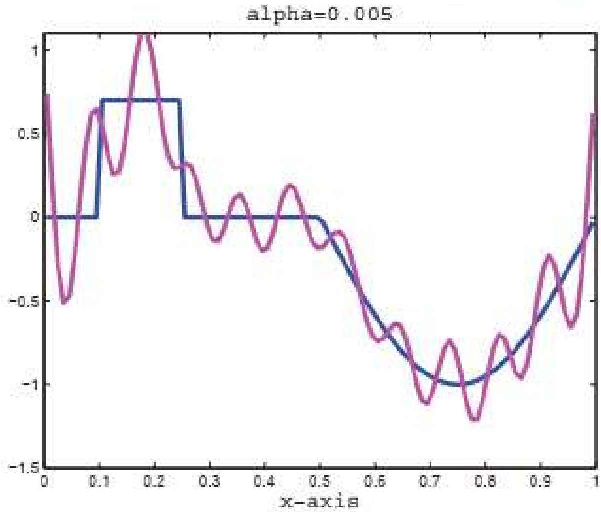


Figure: α too small

Illustrative example

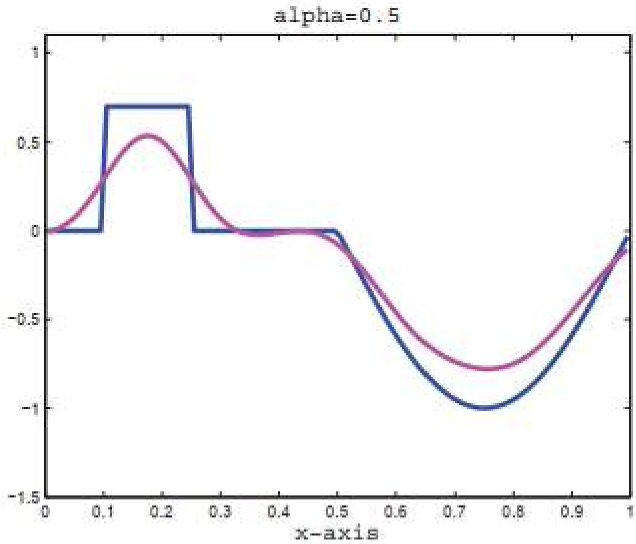


Figure: α too large

Illustrative example

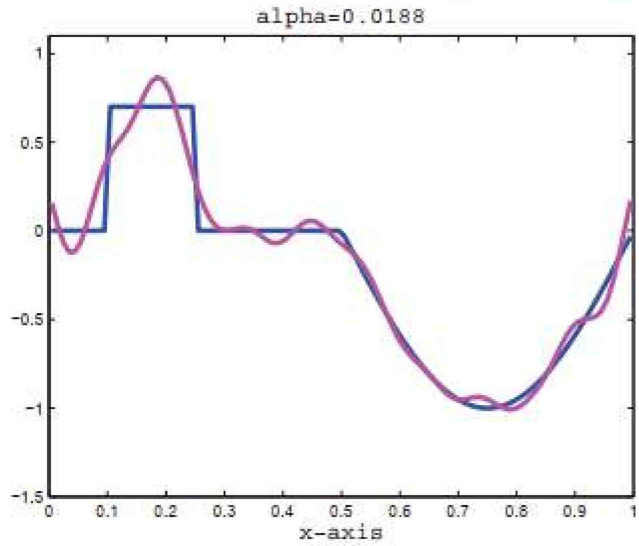


Figure: Good Value for α