Statistical Estimation for Physics

Bayesian Statistical Inference*

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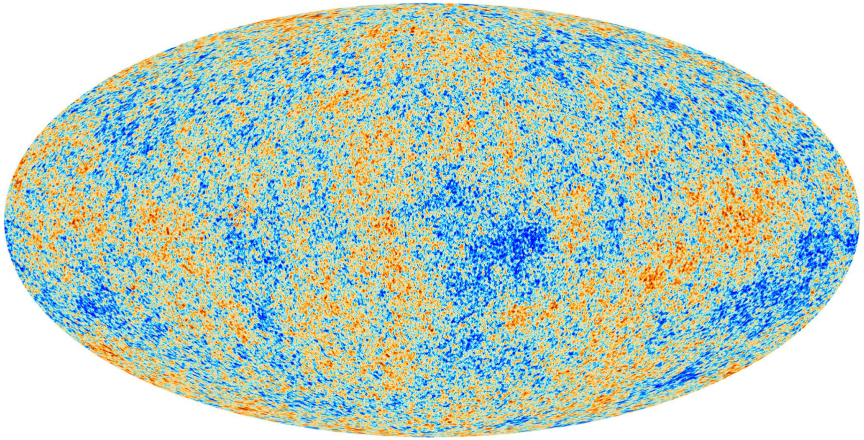
Senior Research Associate
(*DAMTP*)

*[Link]() to GitHub repository

The Cosmic Microwave Background (CMB) radiation

Full-sky map of the CMB temperature fluctuations



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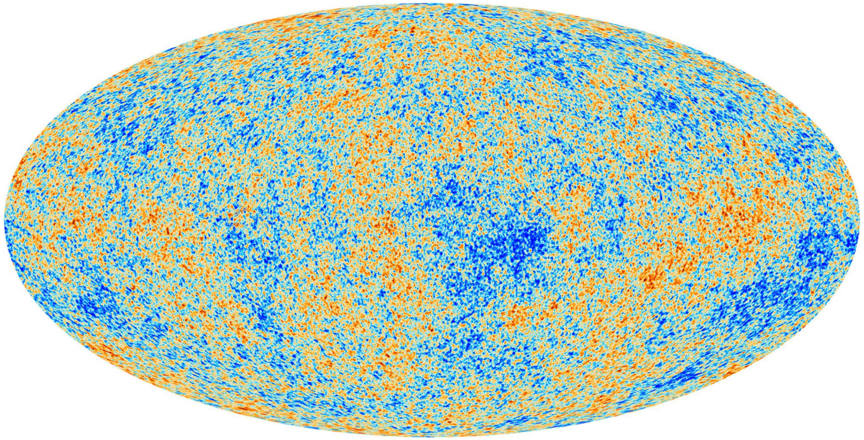

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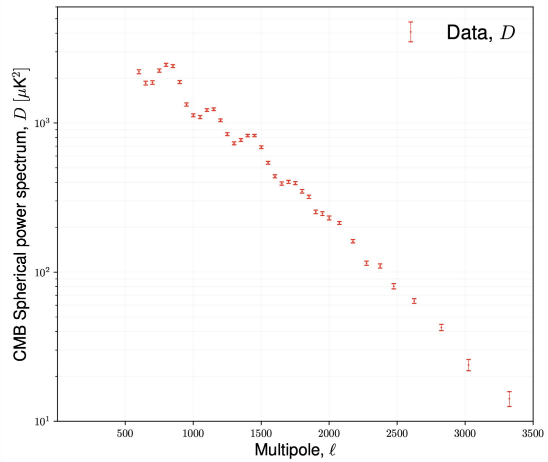
"Data compression"



large scales

small scales

A two-parameter model $D^{\mathrm{th}} = f(\mathrm{age}, M_\nu)$

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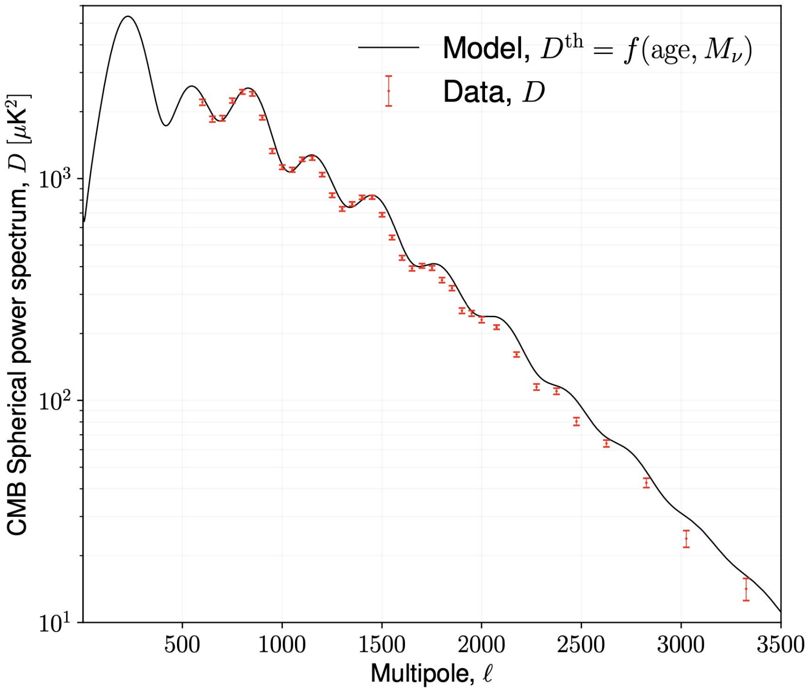

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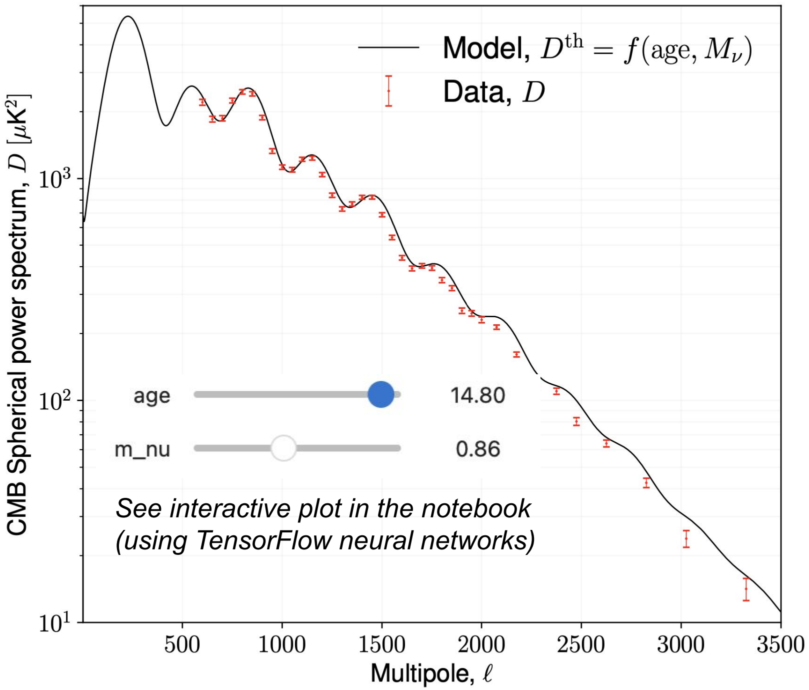

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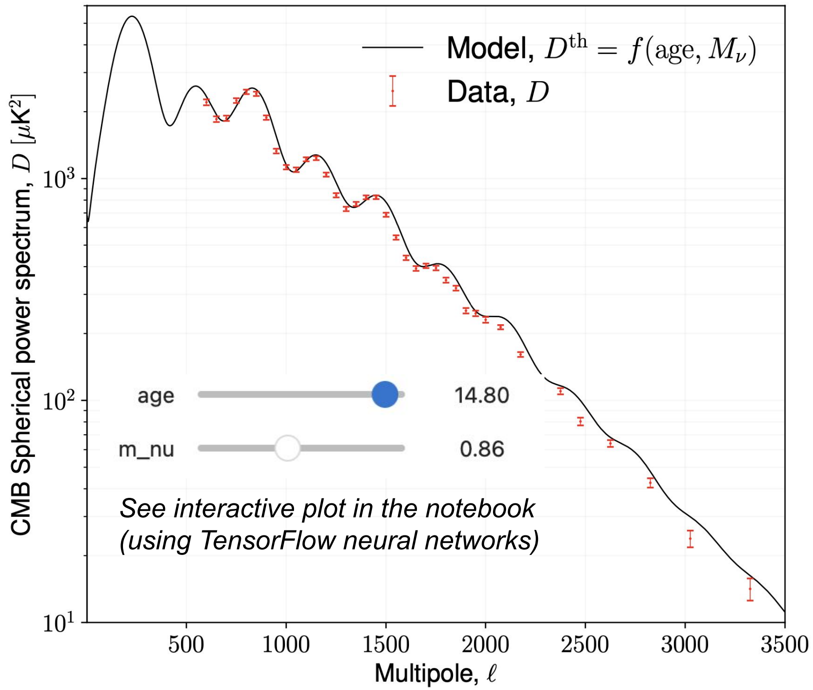
How old is the universe?



Bayes' theorem for Bayesian inference

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$D^{\mathrm{th}} = f(\theta)$: model

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- Evidence: normalizing constant $\longrightarrow$ $\boxed{P(\theta|D) \propto P(D|\theta)P(\theta)}$

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θ : parameters

The likelihood function

$$P(\theta|D) \propto P(D|\theta) P(\theta)$$

- *Definition:* probability of data for a given θ, a function of θ, $\mathcal{L}(\theta) = P(D|\theta)$
- Summarises experimental data
- Main focus of classical/frequentist analysis

The CMB likelihood

- Gaussian approximation $P(D|\theta) = \frac{1}{\sqrt{2\pi|C|}} \exp\{-\frac{1}{2}(D^{\mathrm{th}}(\theta) - D)^T C^{-1} (D^{\mathrm{th}}(\theta) - D)\}$

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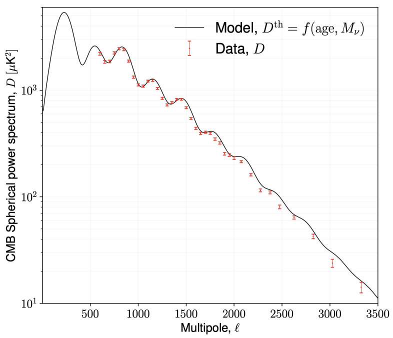

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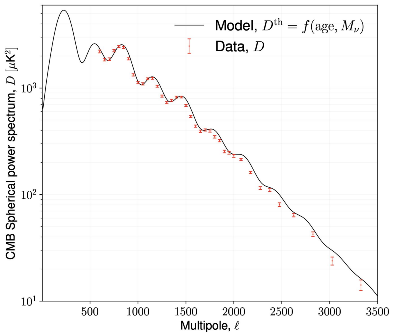
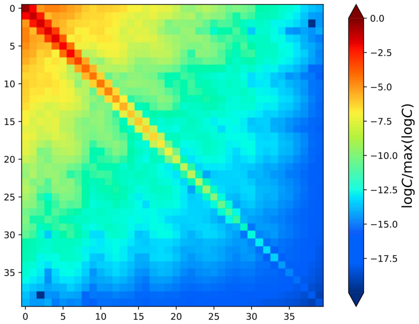


Covariance matrix (part of experimental data)



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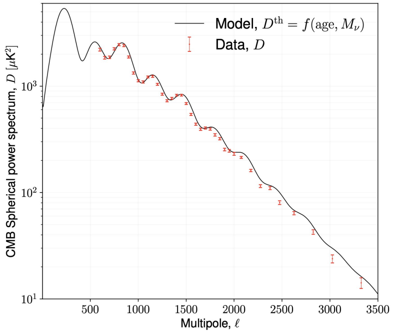

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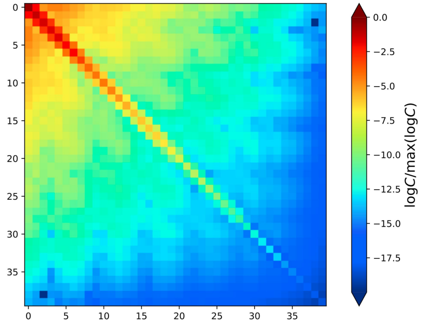

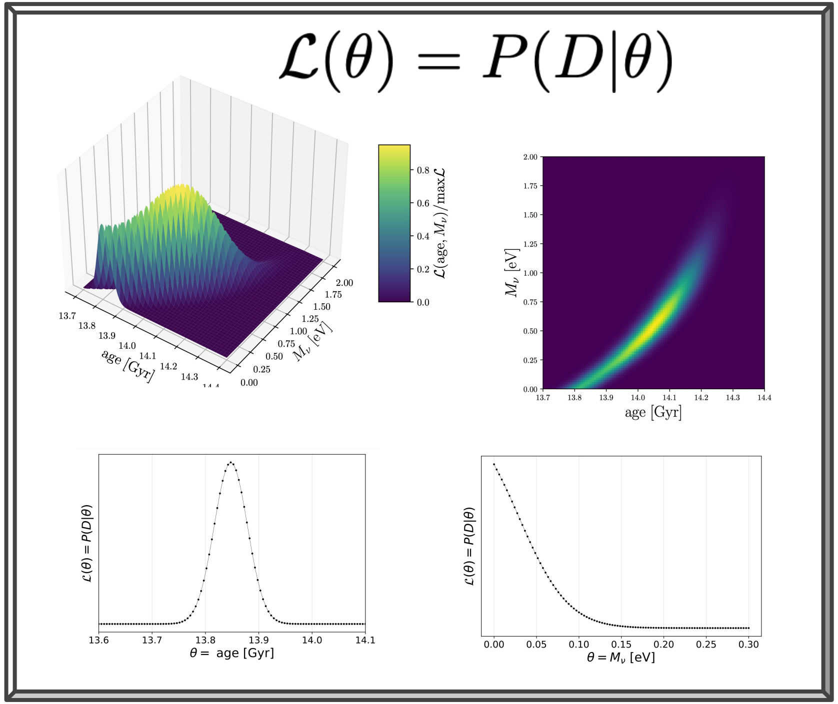

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- **Summarises prior knowledge** on the model
- Prior choice
  - Guiding principles
  - Educated intelligence

The posterior

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- *Definition:* multivariate/joint probability distribution for the model parameters (conditioned on D and model, given prior)

- Goal of Bayesian inference: solution to the problem

- ***Degree of belief***

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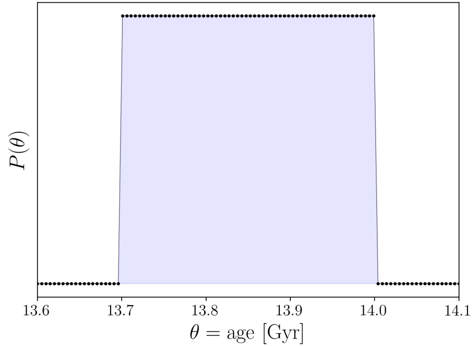
- *Definition:* multivariate/joint probability distribution for the model parameters (conditioned on D and model, given prior)
- Goal of Bayesian inference: solution to the problem
- ***Degree of belief***
- Does not exist in the classical/frequentist approach

Evaluating the posterior, a 1-parameter example

- Pick a **uniform prior**
- Idea: explore parameter space (sampling), compute prior and multiply by likelihood

$$P(\mathrm{age}|D) \propto P(D|\mathrm{age})P(\mathrm{age})$$



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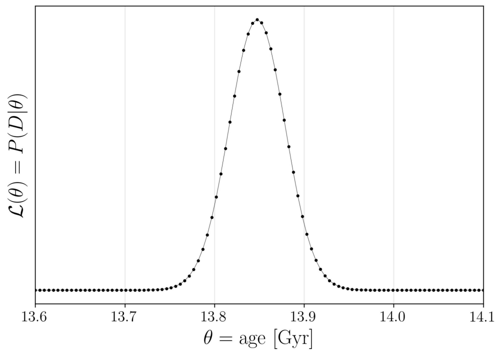
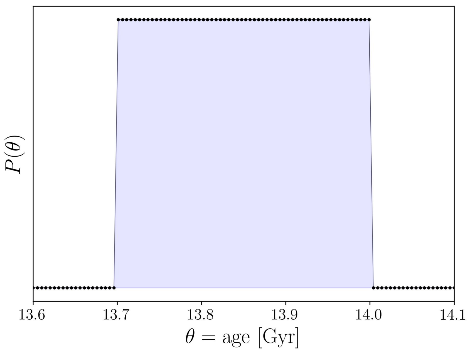


X



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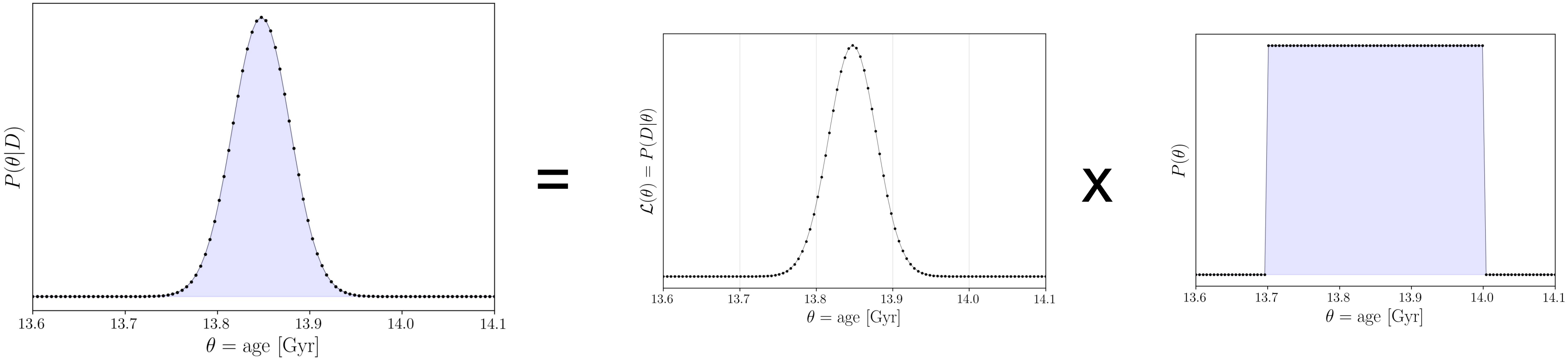

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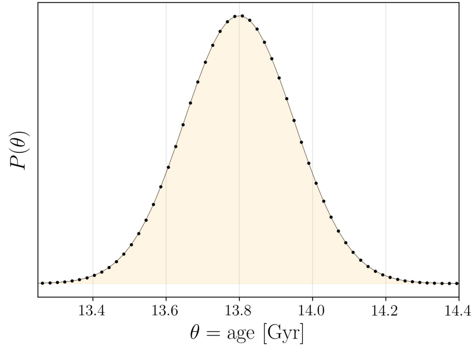

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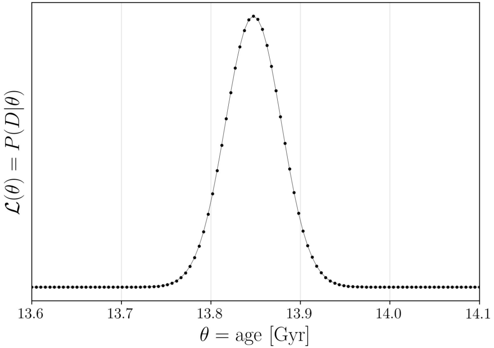

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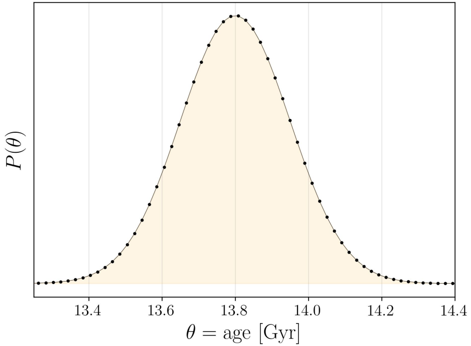

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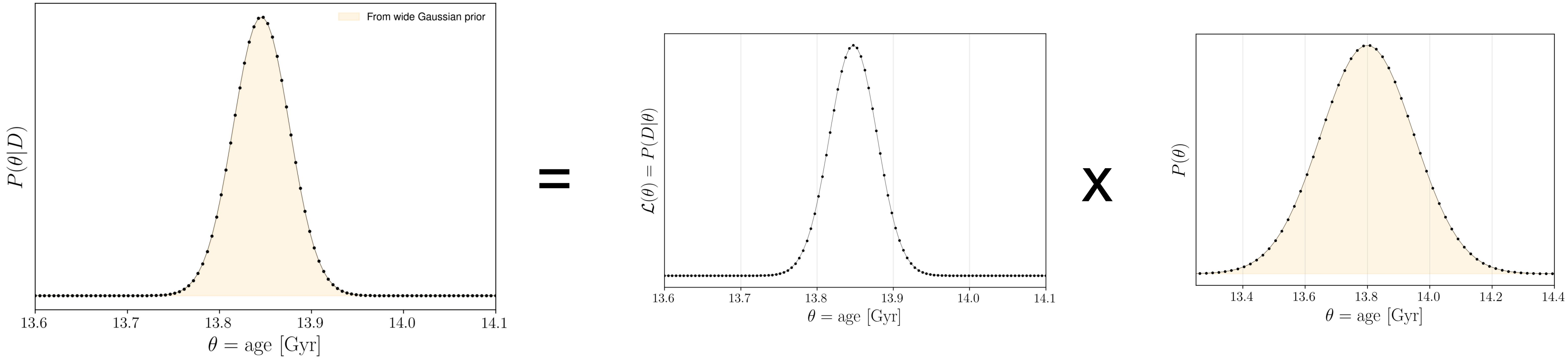

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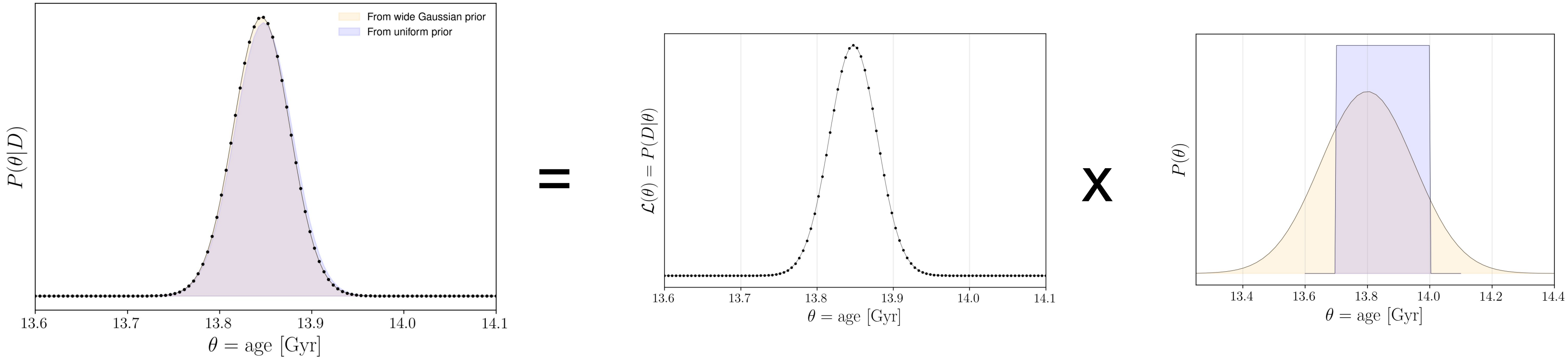

Two very different priors lead to the same posterior!

Bayesian credible intervals

- Set a X% Confidence Level (CL), e.g., X=68 or 95
- Find narrowest interval $t_1 < \theta < t_2$ such that:

$$X = 100 \int_{t_1}^{t_2} P(\theta|D) d\theta$$

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$$X = 100 \int_{t_1}^{t_2} P(\theta|D)d\theta$$

- Bayesian interpretation: **degree of belief** that the true value of theta lies within this interval (given D and model)

- Different from frequentist view!

Obtaining the 68% and 95%CL credible intervals

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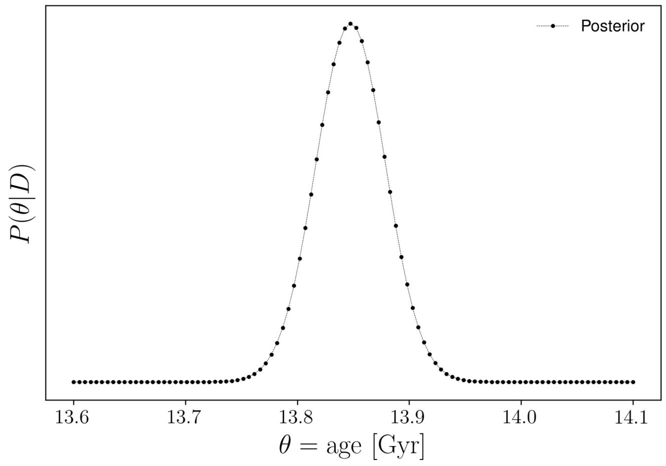

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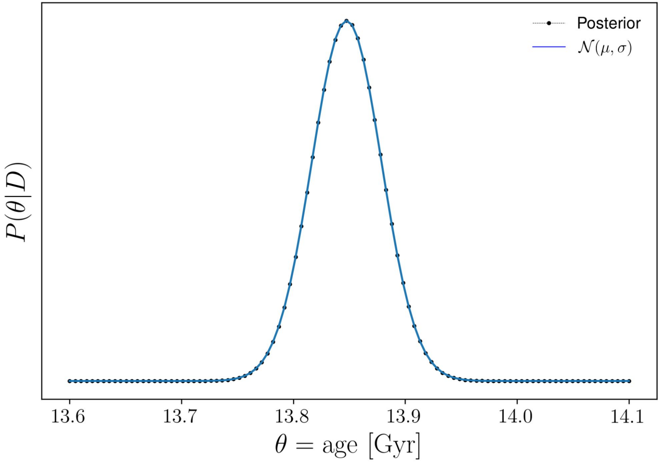

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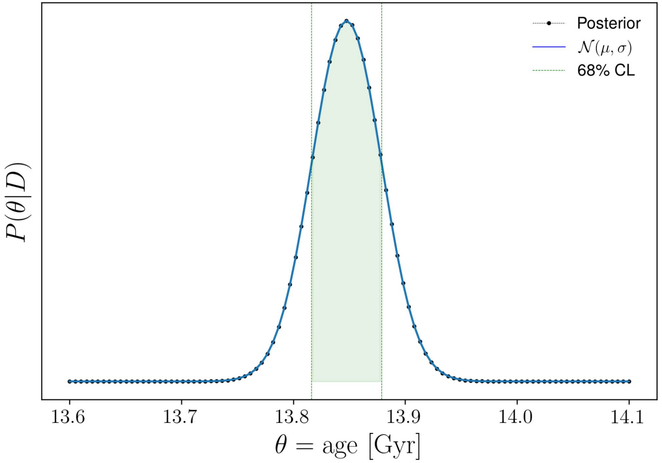

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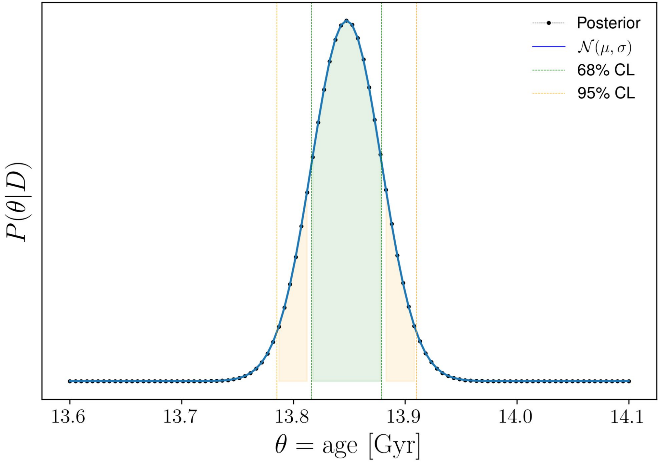

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- Uniform priors
  - $0.06 \text{ eV} < M_\nu < \text{few eV}$
  - $t_{\text{sun}} < \text{age} < \text{few}\, 10\text{Gyr}$

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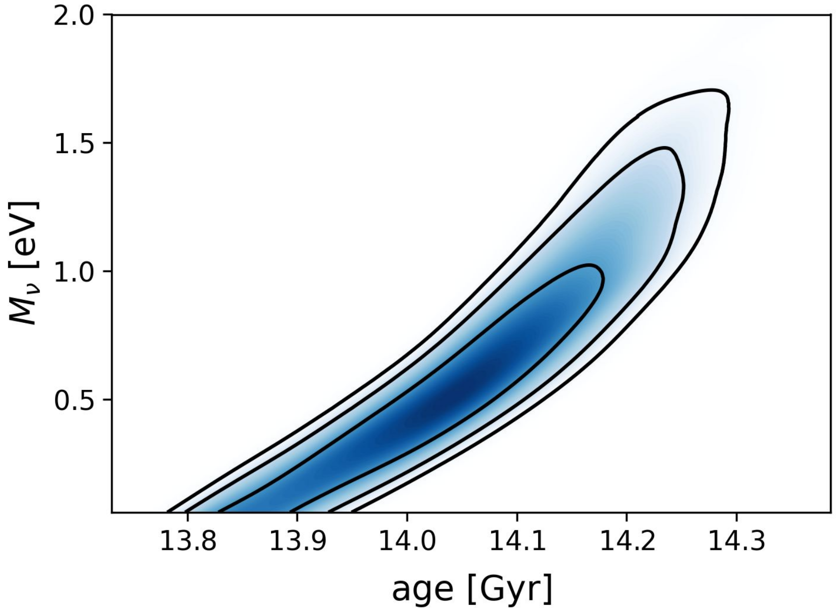
➔ 2D **posterior probability distribution**

➔ Obtained with MCMC* and KDE*
(see notebook)



*MCMC: Monte Carlo Markov Chains, KDE: Kernel Density Estimation

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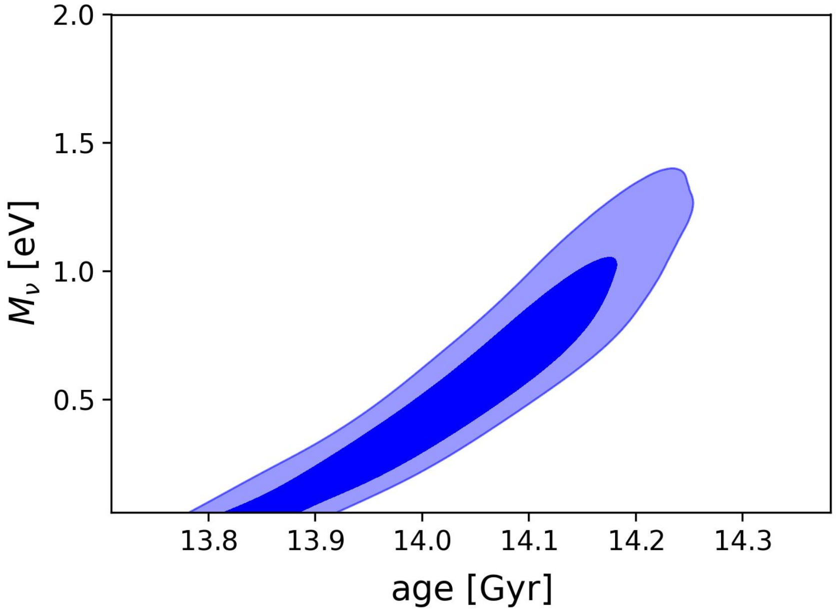
- ***Bayesian credible regions*** (68% and 95%CL)



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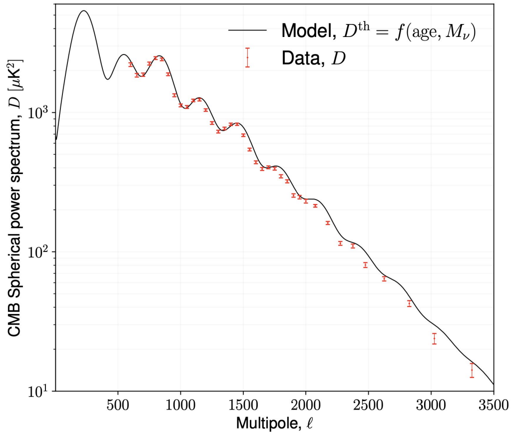
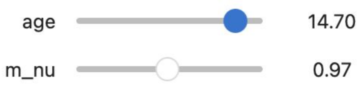
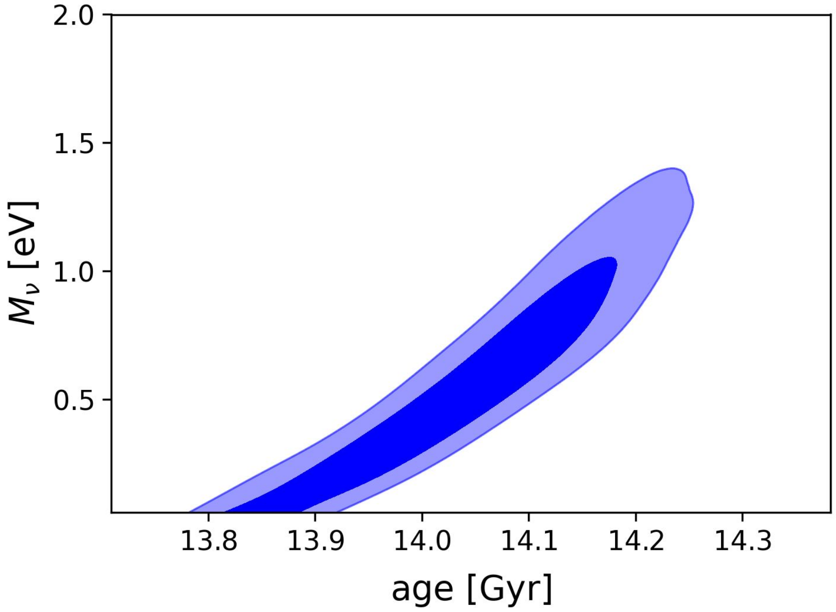


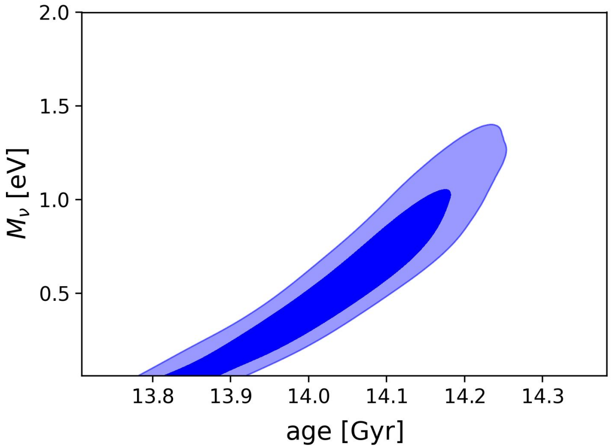




Eliminating a “*nuisance*” parameter

- Give-up hope of knowing neutrino mass precisely

- How old is the universe?



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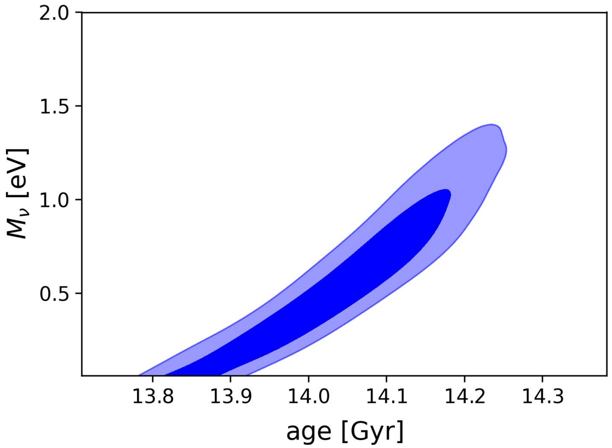

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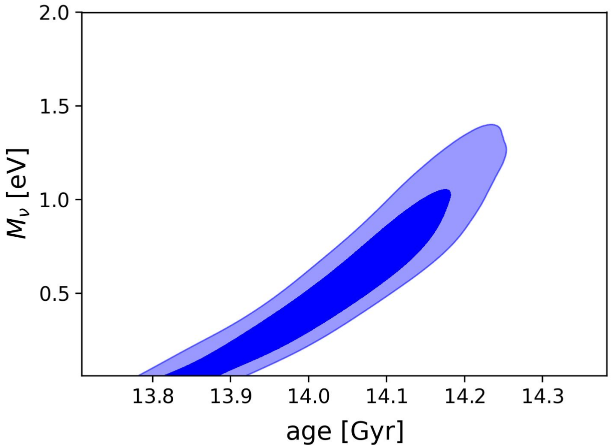

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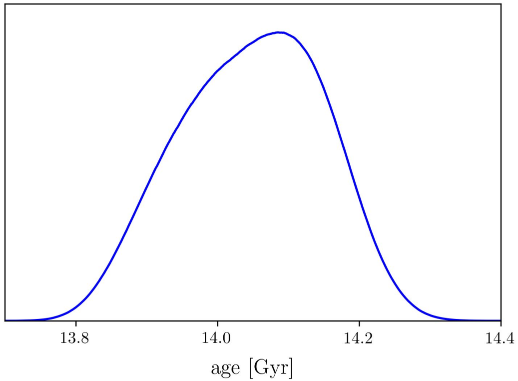
1D marginalised posterior PDF



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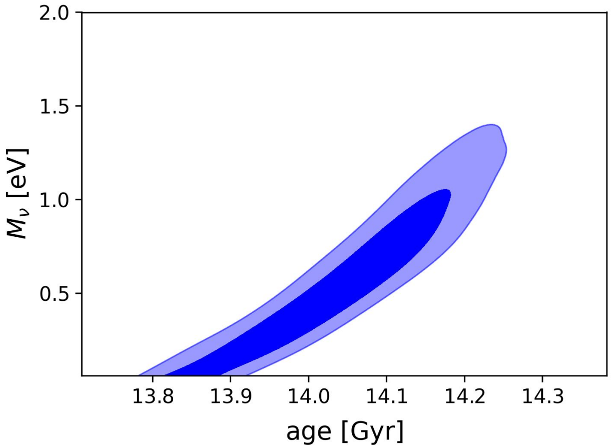

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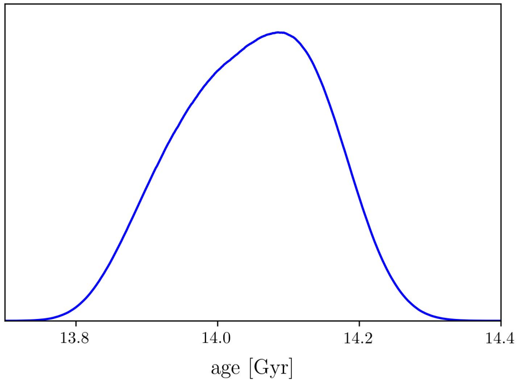
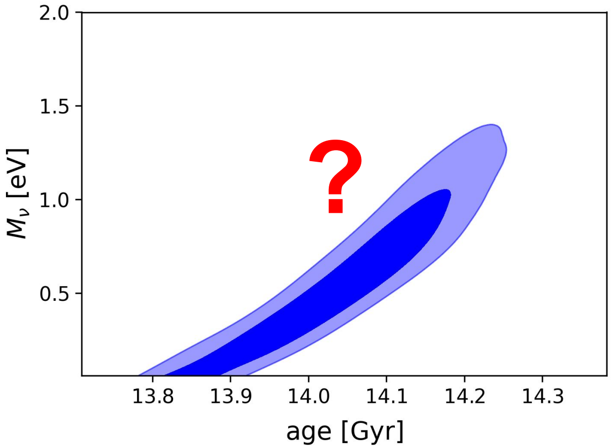


$$\boxed{\text{age} = 14.05 \pm 0.1\ \text{Gyr}(68\%\text{CL})}$$

Combining experiments

- A new neutrino mass measurements is available $P(M_\nu) \sim \mathcal{N}(\mu = 0.15, \sigma = 0.05)$



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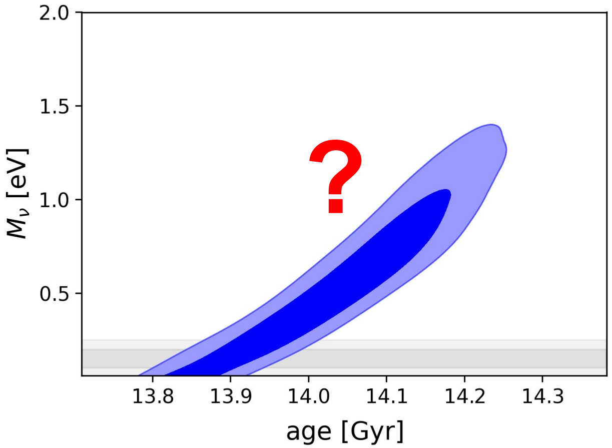

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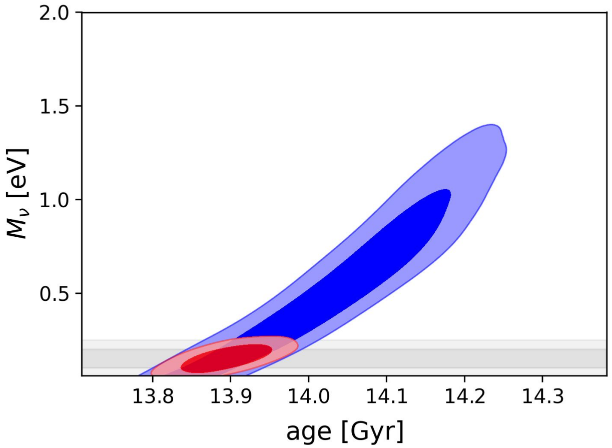

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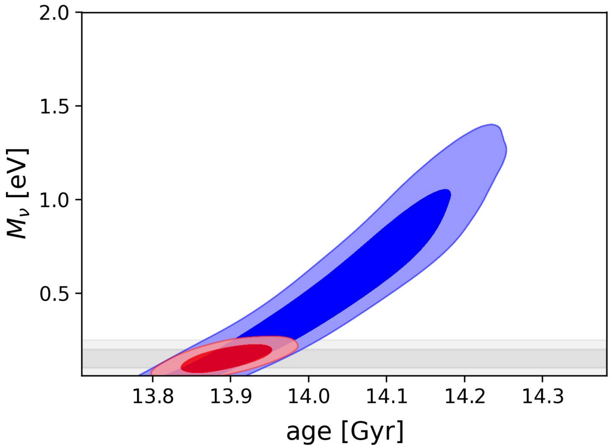

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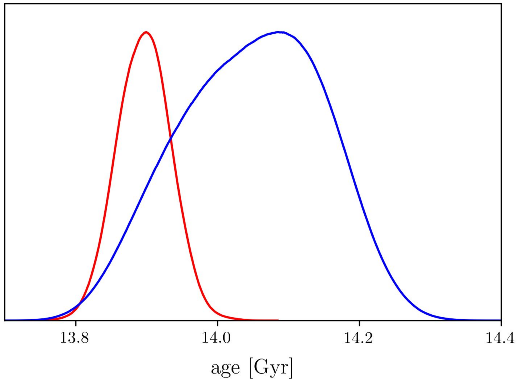
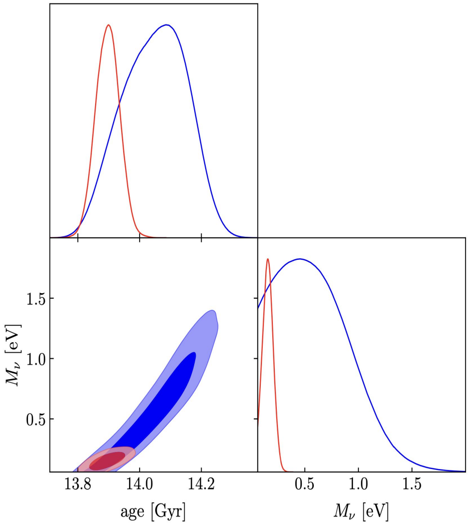


$\text{age} = 14.90 \pm 0.04\ \text{Gyr}(68\%\text{CL})$

Bayesian inference summary and key points

"What you know about θ after the data arrive is what you knew before [P(θ)], and what the data told you [P(D|θ)]." (MacKay)

- Interpretation of Bayes theorem $P(\theta|D) = \dfrac{P(D|\theta)P(\theta)}{P(D)}$
- Effects of prior choices
- Calculating the posterior probability distribution
- Bayesian credible intervals
- Marginalisation over nuisance parameters
- Adding new information/Combining experiments
- Approach directly applicable to large number of free parameters and priors



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