Emergence of Newtonian Dynamics from Metric Inertial Systems

A Structural Derivation of Second-Order Deterministic Evolution

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Abstract

We prove that within a specific axiomatic framework, second-order dynamics emerges as the unique minimal form compatible with three structural constraints: persistence, additivity, and compensation. Let $(\mathbf{S}, d)$ be a metric space with sufficient differential structure. We show that any first-order dynamical law $\mathrm{d}x/\mathrm{d}t = f(x, \mathcal{F})$ on configuration space $\mathbf{S}$ necessarily violates at least one constraint, establishing that velocity must be promoted to an independent state variable—yielding second-order dynamics on $\mathbf{S}$ (equivalently, first-order dynamics on the tangent bundle $T\mathbf{S}$). Classical Newtonian dynamics in Euclidean space is recovered as a special case. We distinguish carefully between the *form* of dynamical laws (which we claim is structurally constrained) and their *content* (specific potentials, coupling constants), which remains empirical.

Scope: Within the class of smooth deterministic dynamics on Riemannian state spaces satisfying Persistence, Additivity, and Compensation, any non-trivial dynamics must be of second order on configuration space. Systems outside this class—including stochastic, discrete, or strongly nonlinear systems—require separate treatment. Our result does not contradict the existence of first-order Hamiltonian formulations on phase space; rather, it explains *why* the phase-space doubling of variables is necessary.

1 Introduction

Newton established a method: define state, define change, identify causes, derive evolution. The resulting framework has proven extraordinarily successful. Yet a question remains: Is the second-order character of these laws contingent, or does it reflect structural constraints on coherent dynamical systems?

We argue for the latter, with important caveats. We show that under specific axioms, second-order dynamics is *necessary*—not as a physical discovery but as a mathematical consequence of the axioms. Our claim is conditional: *if* a system satisfies persistence, additivity, and compensation, *then* its evolution must be second-order. The axioms themselves abstract from physical intuition, and whether any given system satisfies them is an empirical question.

We distinguish throughout between:

- **Form**: The mathematical structure of dynamical laws (first-order vs. second-order)
- **Content**: Specific potentials, masses, coupling constants (empirically determined)

Our claim concerns form only. That gravity obeys an inverse-square law with coupling constant G is empirical; that coherent dynamical systems admit second-order description is (we argue) structural.

1.1 Relation to Existing Work

The derivation of equations of motion from structural principles has a long history. Maupertuis, Euler, Lagrange, and Hamilton showed that second-order dynamics follows from stationary action [8, 9]. The present approach differs in starting point: rather than action principles, we begin with constraints on how states evolve and interactions combine.

Our persistence postulate abstracts Newton's first law; additivity abstracts superposition; compensation abstracts the third law and momentum conservation. We do not claim these are purely logical necessities—they are abstractions of physical intuitions, elevated to axioms. The contribution is showing their joint implication: second-order dynamics.

Related frameworks include Abraham and Marsden's geometric mechanics [1], Barbour's shape dynamics [5], Friston's free energy principle [7], and Amari's information geometry [2]. Our approach is closest to the latter two in spirit, though technically distinct.

1.2 Structural vs. Dynamical Explanation

Our result contributes to ongoing discussions in philosophy of physics concerning *structural* versus *dynamical* explanations of physical law. Variational principles (Hamilton's principle, geodesic formulations) show that second-order equations follow from extremizing action; this is a structural derivation from a different starting point. The Hamilton–Jacobi formulation rewrites dynamics as a first-order PDE for the action function, and geodesic equations can be recast as Hamiltonian flows on cotangent bundles—all first-order formulations that encode the same physics.

Our contribution is orthogonal: we ask what structural constraints force *velocity to become an independent state variable*. The existence of first-order reformulations on phase space ($T\mathbf{S}$ or $T^*\mathbf{S}$) does not contradict our result; rather, it illustrates it. The "extra" variables in those formulations (momenta, velocities) are precisely what our axioms demand. Our theorem explains *why* the phase-space doubling is necessary: because Persistence, Additivity, and Compensation cannot be jointly satisfied on configuration space alone.

2 Formal Framework

2.1 State Space

Let $(\mathbf{S}, d)$ be a metric space. We impose additional structure:

Axiom 2.1 (Differential Structure)**.** $\mathbf{S}$ admits embedding in a smooth manifold M such that the metric d derives from a Riemannian metric g on M. That is, $d(x, y) = \inf\{L(\gamma) : \gamma \text{ a smooth path from } x \text{ to } y\}$, where $L(\gamma) = \int \sqrt{g(\dot{\gamma}, \dot{\gamma})}\, \mathrm{d}t$.

This axiom restricts our framework to metric spaces with geodesic structure and well-defined tangent spaces. Pure metric spaces without differential structure (e.g., discrete graphs with shortest-path distance) fall outside our scope.

Definition 2.2. A *state* is a point $x \in \mathbf{S}$. The *tangent space* at x, denoted $T_x\mathbf{S}$, is the space of velocity vectors at x.

2.2 Evolution and Trajectories

Definition 2.3. An *evolution* is a twice-differentiable curve $x : I \to \mathbf{S}$, where $I \subseteq \mathbb{R}$ is an interval.

Definition 2.4. The *velocity* at time t is $v(t) = \mathrm{d}x/\mathrm{d}t \in T_{x(t)}\mathbf{S}$. The *acceleration* is $a(t) = \nabla_v v$, the covariant derivative of velocity along the trajectory.

In a Riemannian manifold, acceleration includes connection terms: in local coordinates,

$$a^i = \frac{\mathrm{d}^2 x^i}{\mathrm{d}t^2} + \Gamma^i_{jk} \frac{\mathrm{d}x^j}{\mathrm{d}t} \frac{\mathrm{d}x^k}{\mathrm{d}t}, \tag{1}$$

where Γ^i_{jk} are Christoffel symbols. In flat Euclidean space, these vanish and we recover $a = \mathrm{d}^2x/\mathrm{d}t^2$.

Lemma 2.5 (Velocity-Independence in Second-Order Systems)**.** *In a second-order system $\nabla_v v = F/m$, two systems at the same position x but with different velocities $v_1 \neq v_2$ can experience the same force F and receive the same acceleration $a = F/m$. The response to interaction is independent of current velocity.*

Lemma 2.6 (Velocity-Dependence in First-Order Systems)**.** *In a first-order system $\mathrm{d}x/\mathrm{d}t = f(x, \ldots)$, velocity is* determined by *position (and any external parameters). It is impossible to have two systems at the same position with different velocities—the concept is undefined. Therefore, there can be no "velocity-independent response to interaction."*

The emergence of second-order dynamics is precisely the emergence of this independence: velocity becomes a free state variable, allowing interaction to act universally regardless of current motion.

2.3 Dynamical Laws

Definition 2.7. A *first-order law on configuration space* has the form $\mathrm{d}x/\mathrm{d}t = f(x, \mathcal{F})$ for some function f on $\mathbf{S}$, where $\mathcal{F}$ represents any external parameters or interactions.

Definition 2.8. A *second-order law on configuration space* has the form $\nabla_v v = g(x, v)$ for some function $g : T\mathbf{S} \to T\mathbf{S}$ (fiber-preserving).

Remark 2.9 (Phase-Space Reformulations)**.** In differential geometry and Hamiltonian mechanics, any second-order equation on configuration space $\mathbf{S}$ can be rewritten as a first-order system on the tangent bundle $T\mathbf{S}$ (or cotangent bundle $T^*\mathbf{S}$) by treating (x, v) as the state. For example, $m\nabla_v v = F$ becomes the first-order system:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = v, \quad \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{F}{m} - \Gamma(v, v). \tag{2}$$

This is the standard Hamiltonian or Lagrangian formulation on phase space.

Our theorem concerns a different question: We ask whether dynamics can be formulated with *position alone* as the state variable—that is, first-order on $\mathbf{S}$ itself, not on $T\mathbf{S}$. The emergence of second-order dynamics is precisely the emergence of *velocity as an independent state variable* required by the axioms. The "extra" variables in Hamiltonian formulations (v or p) are exactly what our axioms force into the ontology.

Throughout this paper, "first-order" means a law whose state space is $\mathbf{S}$ and whose evolution is $\mathrm{d}x/\mathrm{d}t = f(x, \mathcal{F})$; we do not count phase-space reformulations of second-order equations as "first-order" in this sense.

3 Structural Postulates

We state three postulates and then prove their joint incompatibility with first-order dynamics.

Axiom 3.1 (Persistence)**.** In the absence of interaction, geodesics are preserved: $\nabla_v v = 0$ when $F = 0$.

Interpretation: Unperturbed systems follow geodesics—paths of constant velocity in the Riemannian sense. In flat space, geodesics are straight lines at constant speed. Change of velocity requires a cause. This abstracts Newton's first law.

Axiom 3.2 (Additivity)**.** When multiple interactions act, their effects on acceleration combine linearly: if F_1 causes acceleration a_1 and F_2 causes a_2 (each acting alone), then $F_1 + F_2$ causes $a_1 + a_2$.

Formal statement: Let $L : \{\text{interaction fields}\} \to \{\text{acceleration fields}\}$ be the map from interaction to resulting acceleration. Then L is linear: $L(F_1 + F_2) = L(F_1) + L(F_2)$ and $L(\lambda F) = \lambda L(F)$ for $\lambda \in \mathbb{R}$.
Interpretation: Superposition holds. Combined causes produce combined effects. This abstracts the physical principle that forces add vectorially.

Axiom 3.3 (Compensation)**.** In a closed system, total momentum is conserved: $\sum_i m_i v_i =$ constant along trajectories.

Formal statement: For a system of n components with positions $x_1, \ldots, x_n$ and masses $m_1, \ldots, m_n$, define total momentum $\mathbf{P} = \sum_i m_i v_i$. Then $\mathrm{d}\mathbf{P}/\mathrm{d}t = 0$ for internal interactions.
Interpretation: No unilateral changes occur. Internal interactions redistribute momentum without changing its total. This abstracts Newton's third law: action equals reaction.

4 The Incompatibility Theorem

Theorem 4.1 (Incompatibility of First-Order Dynamics)**.** *Let* $(\mathbf{S}, g)$ *be a Riemannian manifold with* $\dim(\mathbf{S}) \geq 1$*. Any first-order dynamical law on configuration space* $\mathbf{S}$ *satisfying Persistence, Additivity, and Compensation must be trivial (no dynamics).*

The Core Insight: Before the formal proof, we state the essential observation:

> *In any first-order system $\mathrm{d}x/\mathrm{d}t = f(x, \mathcal{F})$, the future trajectory is fully determined once position is specified. There is no room for an external interaction to modify acceleration independently of current velocity, because velocity itself is not a free variable—it is a derived quantity, slaved to position.*

This is the structural reason why first-order dynamics cannot support Additivity as we have defined it.

Proof. We proceed through a series of lemmas.

Definition 4.2 (First-Order Dynamics, General Form)**.** A first-order dynamical law on $\mathbf{S}$ is any rule of the form:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x, \mathcal{F}) \tag{3}$$

where $x \in \mathbf{S}$ is position and $\mathcal{F}$ represents any external parameters, fields, or "interactions." The key feature is that velocity $v = \mathrm{d}x/\mathrm{d}t$ is *determined by* this equation, not specified independently.

Lemma 4.3 (Velocity is Slaved)**.** *Under any first-order law, once x and $\mathcal{F}$ are specified, the velocity $v = f(x, \mathcal{F})$ is uniquely determined. There is no freedom to specify v independently of x.*

Proof. Immediate from the form of the equation. The state space is $\mathbf{S}$, not $T\mathbf{S}$. □

Lemma 4.4 (Acceleration Depends on Velocity)**.** *Under any first-order law, the acceleration is:*

$$a = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t} f(x, \mathcal{F}) = \frac{\partial f}{\partial x} \cdot v + \frac{\partial f}{\partial \mathcal{F}} \cdot \dot{\mathcal{F}} \tag{4}$$

In particular, when $\mathcal{F}$ is constant or absent:

$$a = \frac{\partial f}{\partial x} \cdot f(x, \mathcal{F}) \tag{5}$$

The acceleration depends on $v = f(x, \mathcal{F})$ through the chain rule. It cannot be specified independently of the current velocity.

Proof. Direct differentiation. □

Lemma 4.5 (No Velocity-Independent Response)**.** *In a first-order system, there exists no map $L : \mathcal{F} \mapsto a$ such that:*

1. *L is well-defined (same $\mathcal{F}$ produces same a)*
2. *L is independent of the current velocity v*

Proof. Suppose such an L exists. Consider two trajectories passing through the same point x_0 at the same instant, subject to the same interaction $\mathcal{F}$. Under a first-order law, both trajectories have the same velocity at x_0 (namely $v = f(x_0, \mathcal{F})$), so they are in fact the same trajectory. There is no way to have "two systems at the same position with different velocities"—the concept is undefined in a first-order framework.

But Additivity requires precisely this: a universal map from interaction to acceleration that applies regardless of the system's velocity. The acceleration response to a given force should be the same whether the system is moving fast or slow, left or right. In a first-order system, this question is meaningless because velocity is not a free parameter. □

Main Argument:

Step 1 (Additivity Requires Velocity-Independence): Postulate 3.2 (Additivity) states that there exists a linear map $L : \mathcal{F} \mapsto a$ from interactions to accelerations. Crucially, this map must be *universal*: the same interaction produces the same acceleration regardless of the system's state.

In second-order dynamics, this is straightforward: $a = F/m$ where F encodes $\mathcal{F}$. Two systems at the same x with different v both experience the same acceleration under the same force.

Step 2 (First-Order Dynamics Cannot Support This): By Lemma 4.5, first-order dynamics cannot provide a velocity-independent response. The acceleration at any point depends on the velocity at that point, which is itself determined by the first-order law—there is no freedom.

More precisely: in a first-order system, asking "what acceleration does $\mathcal{F}$ produce at (x, v)?" is ill-posed because v is not a free variable. The only meaningful question is "what acceleration does $\mathcal{F}$ produce at x?"—but then the answer necessarily depends on $v = f(x, \mathcal{F})$, violating universality.

Step 3 (Compensation Independently Forces Constraints): Even setting aside Additivity, Compensation requires momentum conservation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_i m_i v_i\right) = 0 \quad \text{for internal interactions.} \tag{6}$$

Under first-order dynamics $v_i = f(x_i, \mathcal{F})$, this becomes:

$$\sum_i m_i \frac{\partial f}{\partial x}\bigg|_{x_i} \cdot f(x_i, \mathcal{F}) + \sum_i m_i \frac{\partial f}{\partial \mathcal{F}}\bigg|_{x_i} \cdot \dot{\mathcal{F}} = 0 \tag{7}$$

For this to hold for all configurations $(x_1, \ldots, x_n)$ and all masses $(m_1, \ldots, m_n)$, the terms must vanish individually. Combined with Step 2, this leaves no room for non-trivial dynamics.

Conclusion: First-order dynamics on configuration space cannot satisfy Additivity (which requires velocity-independent response to interaction) because velocity is not an independent variable. Only by promoting velocity to an independent state variable—yielding second-order dynamics on $\mathbf{S}$—can both postulates be satisfied. □

Remark 4.6 (The Essential Point)**.** The theorem does not depend on how interaction is "encoded" (as δf, as external parameter, etc.). It depends only on the structural fact that first-order systems have velocity slaved to position, making velocity-independent response impossible. This is the logical core of the incompatibility.

5 Emergence of Second-Order Dynamics

Theorem 5.1 (Necessity of Second-Order Form)**.** *Under Postulates 3.1–3.3, the evolution of a system is governed by a second-order law of the form:*

$$\nabla_v v = \frac{F(x, v)}{m(x)} \tag{8}$$

where $m : \mathbf{S} \to \mathbb{R}_+$ is positive and $F : T\mathbf{S} \to T\mathbf{S}$ is the interaction field.

Proof. **Step 1 (Persistence implies acceleration-based laws):** Persistence requires that unperturbed trajectories be geodesics: $\nabla_v v = 0$ when $F = 0$. This means the dynamical law must relate *acceleration* (not velocity) to interaction:

$$\nabla_v v = G(x, v, F) \tag{9}$$

for some function G that vanishes when $F = 0$.

Step 2 (Additivity implies linearity in F): Let $L : F \mapsto \nabla_v v$ be the map from interaction to acceleration. Additivity states L is linear:

$$G(x, v, F) = L_{x,v}(F) \tag{10}$$

where $L_{x,v}$ is a linear map from forces to accelerations at each (x, v).

In full generality, $L_{x,v} : T_x\mathbf{S} \to T_x\mathbf{S}$ could be any linear operator. The most general form is:

$$L_{x,v}(F) = M(x)^{-1}F \tag{11}$$

where $M(x) : T_x\mathbf{S} \to T_x\mathbf{S}$ is a positive-definite linear operator (a "mass tensor"). This yields:

$$M(x)\nabla_v v = F(x, v) \tag{12}$$

Axiom (Isotropy): If the system exhibits no preferred directions, then $M(x) = m(x) \cdot I$ for some scalar function $m(x) > 0$, yielding:

$$m(x)\nabla_v v = F(x, v) \tag{13}$$

Remark (Tensor Mass Extension): The full derivation holds for the tensorial case $M(x)\nabla_v v = F$ with M positive-definite. The essential point—that dynamics must be second-order—is independent of whether mass is scalar or tensorial.

Step 3 (Compensation constrains F): For internal interactions, conservation of total momentum requires:

$$\sum_i m_i \nabla_{v_i} v_i = \sum_i F_i = 0 \tag{14}$$

This is satisfied if $F_{ij} = -F_{ji}$ (Newton's third law). □

5.1 On the Positivity of $m(x)$

We require $m(x) > 0$ to ensure:

1. **Well-definedness:** The equation is non-singular
2. **Finite response:** Finite forces produce finite accelerations
3. **Non-degeneracy and well-posedness:** The initial value problem has a unique solution depending continuously on initial data
4. **Hyperbolicity:** The equation preserves causal structure

6 Newtonian Dynamics as Special Case

6.1 Euclidean Specialization

Let $\mathbf{S} = \mathbb{R}^3$ with Euclidean metric $g = \delta_{ij}$. Then:

- Christoffel symbols vanish: $\Gamma^i_{jk} = 0$

- Covariant derivative reduces to ordinary derivative: $\nabla_v v = \mathrm{d}v/\mathrm{d}t = \mathrm{d}^2x/\mathrm{d}t^2$
- Geodesics are straight lines at constant velocity

The equation of motion becomes:

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = F(x, v) \tag{15}$$

With $F = -\nabla\Phi$ for gravitational potential $\Phi = -GM/r$, this is Newton's law of gravitation.

6.2 What Remains Empirical

Our derivation establishes the *form* of the dynamical law. The following remain empirical:

- The dimensionality of space (3)
- The specific metric (Euclidean vs. curved)
- The value of constants (G, m, c, $\hbar$)
- The form of potentials (inverse-square vs. other)
- Whether the postulates hold for a given physical system

6.3 Relation to Relativity and Quantum Theory

General Relativity: Replaces Euclidean $\mathbf{S} = \mathbb{R}^3$ with curved spacetime. The geodesic equation still has second-order character, with curvature encoding "gravitational force." Our framework is compatible: curved metric spaces satisfy Axiom 2.1.

Quantum Theory: The Schrödinger equation is first-order in time: $i\hbar\partial\psi/\partial t = H\psi$. However, this describes evolution in Hilbert space (a different state space), and the dynamics of expectation values $\langle x\rangle$ follows second-order equations via Ehrenfest's theorem. In Bohmian mechanics, particle trajectories are explicitly governed by second-order equations [6], suggesting our framework may apply directly to the guidance equation.

7 Example: Belief Dynamics in Epistemic Space

To illustrate potential application beyond physics, consider a model of belief revision.

State space: Let $\mathbf{S}$ be the probability simplex $\Delta^n = \{p \in \mathbb{R}^{n+1}_+ : \sum_i p_i = 1\}$, representing probability distributions over $n+1$ hypotheses.

Metric: The Fisher information metric:

$$g_{ij}(p) = \sum_k \frac{1}{p_k}\frac{\partial p_k}{\partial\theta^i}\frac{\partial p_k}{\partial\theta^j} \tag{16}$$

where θ are coordinates on Δ^n. This metric has deep connections to statistical inference and information geometry [2, 4].

Under the postulates, belief evolution would satisfy a second-order equation:

$$m(p)\nabla_v v = -\nabla D_{\mathrm{KL}}(p\|q) \tag{17}$$

where D_{KL} is Kullback–Leibler divergence from a target distribution q determined by evidence, and $m(p)$ represents epistemic inertia.

Remark 7.1. This is an exploratory sketch, not a proof that belief systems satisfy the axioms. The invocation of "epistemic momentum conservation" is analogical; standard Bayesian theory does not obviously imply such a constraint. A complete theory would require deriving the postulates from information-theoretic first principles.

8 Discussion

8.1 Scope and Limitations

Our theorem applies to systems satisfying all axioms:

- **S** is a Riemannian manifold (Axiom 2.1)
- Trajectories are smooth (twice-differentiable)
- Persistence, Additivity, and Compensation hold

Systems violating any axiom fall outside our scope: discrete systems, stochastic systems, strongly nonlinear interactions, and open systems.

Remark 8.1 (Stochastic Extensions)**.** While stochasticity violates deterministic persistence, the second-order *form* can be preserved. Langevin dynamics is second-order but stochastic:

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -\nabla\Phi - \gamma\frac{\mathrm{d}x}{\mathrm{d}t} + \sqrt{2\gamma k_B T}\,\xi(t) \tag{18}$$

where $\xi(t)$ is white noise. Stochasticity enters the *content* of F without altering the second-order *form*.

8.2 Philosophical Considerations

One potential objection: our axioms embody a classical conception of causation (forces as causes of acceleration) that may not apply in all domains. In thermodynamics, evolutionary dynamics, or economic systems, the notions of "persistence," "additivity," and "compensation" require non-trivial reinterpretation.

We acknowledge this limitation. The theorem's power lies in its conditional structure: *if* a system satisfies the postulates, *then* second-order dynamics follows.

8.3 Open Questions

1. **Minimal structure:** Can Axiom 2.1 be weakened?
2. **Stochastic extension:** Can persistence be reformulated probabilistically?
3. **Quantization:** Is there a quantum analogue?
4. **Empirical tests:** Can simulations test whether Newtonian-form dynamics emerges when the postulates hold?

9 Conclusion

We have shown that, under specific axioms (Riemannian state space, persistence, additivity, compensation), second-order dynamics is not a contingent feature but a structural necessity.

The essential insight: In a first-order system, velocity is slaved to position—it is a derived quantity, not an independent variable. This structural feature makes it impossible to define velocity-independent response to interaction, which Additivity requires. The emergence of second-order dynamics is precisely the promotion of velocity to an independent state variable, necessary for coherent additive interaction to exist.

This is not merely a technical result about differential equations. It reveals *why* the Newtonian form arises: not as an empirical discovery about the physical world, but as a structural necessity for any system where interactions combine additively and act universally.

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Velocity must be promoted to an independent state variable for interaction to exist in a coherent additive system.

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Within the specified structural framework—smooth Riemannian state spaces satisfying Persistence, Additivity, and Compensation—Newtonian-form second-order dynamics is not a choice but a necessity.

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