ON THE POSSIBILITY OF SELF-SUFFICIENT SYSTEMS

Fixed Points and Cyclical Closure

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Abstract

We provide a rigorous mathematical characterization of cyclically closed hierarchical structures. A foundational hierarchy is defined as a tuple $(S, \leq, F)$ where S is a state space, $\leq$ is a grounding relation, and F is a determination function. We prove that such hierarchies admit self-consistent configurations if and only if an associated composite operator possesses a fixed point. Existence conditions are established via classical fixed-point theorems (Knaster-Tarski, Banach, Brouwer) under precisely specified hypotheses. Topological characterization using compactness, connectedness, and first homology group provides invariant criteria for essential cyclicity. A concrete example from quantum physics—self-consistent configurations of entangled subsystems—demonstrates the framework. The results establish that cyclically closed structures are mathematically possible, providing precise criteria for a question traditionally addressed through informal metaphysical debate.

Keywords: fixed-point theorems, cyclical hierarchy, well-foundedness, topological closure, grounding relation

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1 Introduction

The assumption that hierarchical structures must be well-founded—that every descending chain of dependencies terminates in a primary element—pervades foundational discourse across mathematics, philosophy, and physics. From Aristotle's unmoved mover to contemporary debates about grounding in analytic metaphysics [Schaffer, 2009, Bliss & Priest, 2018], the demand for a terminating foundation has been treated as a logical requirement rather than a contingent feature of certain structures.

This paper investigates the complementary case: hierarchical structures that are cyclically closed, where determination relations form loops rather than terminating chains. Such structures have appeared in discussions of autopoiesis in biology [Maturana & Varela, 1980], organizational closure in systems theory [Moreno & Mossio, 2015], and non-well-founded sets in mathematical logic [Aczel, 1988]. However, precise existence conditions have remained underspecified.

We formalize such structures and prove that their existence reduces to a fixed-point problem. Specifically, if states $S_1, \ldots, S_n$ satisfy mutual determination relations $S_i =$

$F_i(S_{i+1})$ with $S_{n+1} = S_1$, then $S_1 = \Phi(S_1)$ where $\Phi = F_1 \circ F_2 \circ \cdots \circ F_n$. The existence of such a configuration is equivalent to the existence of a fixed point of Φ.

The contribution of this paper is threefold: (1) precise definitions of foundational hierarchies and cyclical closure as mathematical objects; (2) existence theorems specifying conditions under which cyclically closed hierarchies exist; (3) topological characterization providing invariant criteria for essential cyclicity.

2 Definitions

We begin with the fundamental definitions.

Definition 2.1. A *foundational hierarchy* is a triple $(S, \leq, F)$ where:

(i) S is a non-empty set (the state space);

(ii) $\leq$ is a binary relation on S (the grounding relation);

(iii) $F : S \to S$ is a function (the determination function) such that $x \leq y$ implies $F(y)$ is defined and determines x.

Definition 2.2. A foundational hierarchy $(S, \leq, F)$ is *well-founded* (or *open*) if $(S, \leq)$ contains no infinite descending chains: there is no sequence $(x_n)_{n\in\mathbb{N}}$ with $x_{n+1} < x_n$ for all n.

Definition 2.3. A foundational hierarchy is *cyclically closed* of order n if there exist elements $S_1, S_2, \ldots, S_n \in S$ and functions $F_1, F_2, \ldots, F_n : S \to S$ such that:

(i) $S_i = F_i(S_{i+1})$ for $i = 1, \ldots, n-1$;

(ii) $S_n = F_n(S_1)$;

(iii) $S_i \leq S_{i+1}$ for all i (indices mod n).

Proposition 2.4. *A cyclically closed hierarchy of order n exists if and only if the composite operator $\Phi = F_1 \circ F_2 \circ \cdots \circ F_n$ has a fixed point.*

Proof. By substitution: $S_1 = F_1(S_2) = F_1(F_2(S_3)) = \cdots = F_1(F_2(\cdots F_n(S_1)\cdots)) = \Phi(S_1)$. Conversely, if $\Phi(S_1) = S_1$, define $S_{i+1} = (F_{i+1} \circ \cdots \circ F_n)(S_1)$ to recover the cycle. □

3 Existence Theorems

Proposition 2.4 reduces the existence of cyclically closed hierarchies to the existence of fixed points. We now state classical theorems providing sufficient conditions.

Theorem 3.1 (Knaster-Tarski). *Let $(L, \leq)$ be a complete lattice and let $\Phi : L \to L$ be order-preserving (monotonic). Then the set of fixed points of Φ is non-empty and forms a complete lattice under the induced order.*

Corollary 3.2. *If the state space S admits a complete lattice structure and each F_i is monotonic, then a cyclically closed hierarchy exists.*

Theorem 3.3 (Banach Contraction Principle)**.** *Let (X, d) be a complete metric space and let $\Phi : X \to X$ be a contraction, i.e., there exists $k \in [0, 1)$ such that $d(\Phi(x), \Phi(y)) \leq k \cdot d(x, y)$ for all $x, y \in X$. Then Φ has a unique fixed point $x^* \in X$, and for any $x_0 \in X$, the sequence $(\Phi^n(x_0))_n$ converges to x^*.*

Corollary 3.4. *If (S, d) is a complete metric space and $\Phi = F_1 \circ \cdots \circ F_n$ is a contraction, then a unique cyclically closed hierarchy exists.*

Theorem 3.5 (Brouwer)**.** *Let $K \subset \mathbb{R}^n$ be non-empty, compact, and convex. Let $\Phi : K \to K$ be continuous. Then Φ has at least one fixed point.*

Corollary 3.6. *If $S \subset \mathbb{R}^n$ is non-empty, compact, and convex, and each F_i is continuous with $\Phi(S) \subset S$, then a cyclically closed hierarchy exists.*

Remark 3.7. Recall that a topological space X is *compact* if every open cover of X has a finite subcover. In metric spaces, this is equivalent to sequential compactness (every sequence has a convergent subsequence). Compactness is strictly stronger than closedness; a closed subset of $\mathbb{R}^n$ need not be compact (e.g., $\mathbb{R}$ itself is closed but not compact).

4 Topological Characterization

We now characterize the essential cyclicity of hierarchical structures using algebraic topology. The key question is: when is the cyclic structure an intrinsic feature that cannot be removed by continuous deformation?

Definition 4.1. Let X be a topological space encoding a hierarchical structure (e.g., a simplicial complex with vertices as states and edges as grounding relations). We say X is *essentially cyclic* if:

(i) X is compact;

(ii) X is connected;

(iii) The first homology group $H_1(X; \mathbb{Z}) \neq 0$.

Proposition 4.2. *If X is essentially cyclic, then there exists no continuous retraction $r : X \to \{x_0\}$ that preserves the cyclic structure. Equivalently, the cycle cannot be continuously contracted to a point.*

Proof. If such a retraction existed, it would induce a homomorphism $r_* : H_1(X; \mathbb{Z}) \to H_1(\{x_0\}; \mathbb{Z}) = 0$. But retractions induce surjections on homology, contradicting $H_1(X; \mathbb{Z}) \neq 0$. □

Example 4.3. The circle S^1 has $H_1(S^1; \mathbb{Z}) \cong \mathbb{Z}$. Any encoding of a cyclical hierarchy as a circle-like structure inherits this non-trivial homology. The first Betti number $b_1 = \text{rank}(H_1)$ counts the number of independent cycles; $b_1 > 0$ is a necessary condition for essential cyclicity.

5 Application: Self-Consistent Quantum States

We demonstrate the framework with a concrete example from quantum physics: self-consistent configurations in mutually entangled subsystems.

Setup. Consider a tripartite quantum system with subsystems A, B, C and Hilbert space $\mathcal{H} = \mathcal{H}^A \otimes \mathcal{H}^B \otimes \mathcal{H}^C$, where each subsystem has dimension d. The reduced density matrices satisfy mutual determination via partial traces:

$$\rho^A = \mathrm{Tr}_{BC}(\rho^{ABC}), \quad \rho^B = \mathrm{Tr}_{AC}(\rho^{ABC}), \quad \rho^C = \mathrm{Tr}_{AB}(\rho^{ABC}).$$

For certain global states $|\Psi\rangle \in \mathcal{H}$, the reduced states form a cyclically determined structure: the properties of each subsystem are fixed by its correlations with the others, with no subsystem serving as an independent "foundation."

Formal structure. Let S be the space of valid density matrices on $\mathcal{H}$ (positive semidefinite, trace one). This is a compact convex subset of the space of Hermitian operators. Define $F : S \to S$ by $F(\rho) = \Phi(\rho)$ where Φ represents the constraint that the global state must be consistent with quantum mechanics (e.g., arising from unitary evolution and environmental decoherence).

Proposition 5.1. *If $\Phi : S \to S$ is continuous (which holds for typical quantum channels), then a self-consistent configuration exists.*

Proof. The set S of density matrices on a finite-dimensional Hilbert space is non-empty, compact (closed and bounded in the operator norm), and convex. If Φ is continuous and maps S into S, then by Theorem 3.5 (Brouwer), Φ has a fixed point $\rho^* = \Phi(\rho^*)$. □

Remark 5.2. This quantum example illustrates the framework's applicability to physics: entangled subsystems exist in self-consistent configurations without any subsystem being ontologically "primary." The question "which subsystem grounds the others?" has no distinguished answer—each is determined by its correlations with the rest, forming a cyclically closed structure.

Remark 5.3. In certain approaches to emergent spacetime [Van Raamsdonk, 2010, Ryu & Takayanagi, 2006], geometry arises from entanglement structure. If spacetime and quantum fields mutually determine each other—fields require a spacetime arena for their definition, while spacetime geometry emerges from field entanglement—the resulting structure could, in principle, exhibit cyclical closure of the type formalized here. Whether this possibility is realized remains a question for quantum gravity research.

6 Discussion

The results establish that cyclically closed hierarchies are mathematically coherent objects whose existence is governed by fixed-point conditions. This provides a precise criterion replacing the informal question of "primary foundation" with the well-posed question of fixed-point existence.

What the theorems establish: (1) Cyclically closed hierarchies exist under specifiable conditions (Theorems 3.1, 3.3, 3.5). (2) Essential cyclicity is a topological invariant detectable via homology (Proposition 4.2). (3) Cyclical grounding is mathematically coherent: the theorems demonstrate the possibility of self-sufficient structures, though they do not by themselves adjudicate whether foundational questions are appropriate for any particular domain.

What the theorems do not establish: (1) That any particular empirical system is cyclically closed (this requires verifying the hypotheses). (2) That all self-referential structures have fixed points (many do not—see Lawvere's diagonal theorem for criteria). (3) Any claims about physics, cosmology, or metaphysics beyond the mathematical framework.

Relation to well-foundedness: Standard set theory (ZFC) includes the axiom of foundation, which asserts that the membership relation is well-founded. The structures studied here violate this axiom, belonging instead to the realm of non-well-founded set theory [Aczel, 1988]. Our results may be viewed as providing existence conditions for non-well-founded structures in more general contexts.

Historical and philosophical context: The question of whether all explanatory or ontological hierarchies must terminate in a primary element has a long history. Aristotle's argument for an unmoved mover, Aquinas's cosmological arguments, and Leibniz's principle of sufficient reason all assume that explanatory chains cannot form closed loops. Contemporary grounding theory [Schaffer, 2009, Rosen, 2010] typically axiomatizes grounding as asymmetric, which precludes cycles by definition. The present results do not resolve these philosophical debates; they establish only that cyclically closed structures are mathematically possible. Whether such possibility has implications for particular metaphysical or scientific questions remains a matter for domain-specific inquiry.

Relation to grounding theory: It should be emphasized that this paper does not propose an alternative axiomatization of grounding, nor does it stand in contradiction to standard grounding theory. The cyclical hierarchies formalized here are not about the relation between wholes and parts (as in priority monism), but about abstract dependency structures that satisfy certain closure conditions. In correspondence, Schaffer (2026) noted that asymmetry is an axiom within grounding theory; our results concern a different subject matter—mathematical structures satisfying fixed-point conditions—and make no claim about whether grounding relations in metaphysics should or should not be asymmetric.

Scope and limitations: This paper provides mathematical existence conditions, not claims about which empirical or metaphysical structures instantiate them. Whether physical, biological, or cosmological systems exhibit cyclical closure in the relevant sense requires domain-specific investigation that goes beyond the present framework. The contribution is to provide precise mathematical criteria for structures that philosophical tradition has often addressed informally.

7 Conclusion

We have provided a rigorous mathematical framework for cyclically closed hierarchical structures. The main results are:

1. Definition 2.3 formalizes cyclical closure as a fixed-point condition (Proposition 2.4).

2. Theorems 3.1, 3.3, and 3.5 provide sufficient conditions for existence under lattice-theoretic, metric, and topological hypotheses respectively.

3. Proposition 4.2 establishes that essential cyclicity is a topological invariant.

4. Proposition 5.1 demonstrates applicability to self-consistent quantum configurations, with implications for emergent spacetime approaches.

The framework reduces questions about "primary foundations" in cyclical structures to the mathematically precise question of fixed-point existence, which is decidable under the hypotheses of the classical theorems.

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References

Aczel, P. (1988). *Non-Well-Founded Sets*. CSLI Lecture Notes, Stanford.

Banach, S. (1922). Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales. *Fundamenta Mathematicae*, 3, 133–181.

Bliss, R., & Priest, G. (Eds.). (2018). *Reality and Its Structure: Essays in Fundamentality*. Oxford University Press.

Brouwer, L. E. J. (1911). Über Abbildung von Mannigfaltigkeiten. *Mathematische Annalen*, 71, 97–115.

Hatcher, A. (2002). *Algebraic Topology*. Cambridge University Press.

Knaster, B. (1928). Un théorème sur les fonctions d'ensembles. *Annales de la Société Polonaise de Mathématique*, 6, 133–134.

Lawvere, F. W. (1969). Diagonal arguments and Cartesian closed categories. *Lecture Notes in Mathematics*, 92, 134–145.

Maturana, H. R., & Varela, F. J. (1980). *Autopoiesis and Cognition: The Realization of the Living*. D. Reidel.

Moreno, A., & Mossio, M. (2015). *Biological Autonomy: A Philosophical and Theoretical Enquiry*. Springer.

Rosen, G. (2010). Metaphysical dependence: Grounding and reduction. In B. Hale & A. Hoffmann (Eds.), *Modality: Metaphysics, Logic, and Epistemology* (pp. 109–136). Oxford University Press.

Ryu, S., & Takayanagi, T. (2006). Holographic derivation of entanglement entropy from AdS/CFT. *Physical Review Letters*, 96, 181602.

Schaffer, J. (2009). On what grounds what. In D. Chalmers, D. Manley, & R. Wasserman (Eds.), *Metametaphysics* (pp. 347–383). Oxford University Press.

Tarski, A. (1955). A lattice-theoretical fixpoint theorem and its applications. *Pacific Journal of Mathematics*, 5, 285–309.

Van Raamsdonk, M. (2010). Building up spacetime with quantum entanglement. *General Relativity and Gravitation*, 42, 2323–2329.