Undecidability as a Methodological Signal: System-Relative Provability and Explanatory Adequacy

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Abstract

It is a standard fact that provability is relative to a formal system: a statement may be provable in one system and unprovable in another. This paper advances a methodological interpretation of this fact. When a statement is true but persistently unprovable within a given sound system, this non-provability should be interpreted not as an epistemic defect of the statement, but as a signal of the system's explanatory inadequacy with respect to that truth. The central claim is a heuristic principle: undecidability rationally motivates the search for an alternative framework in which the same truth becomes provable through principled means. While the logical background relies on elementary proof-theoretic results, the novelty lies in their interpretation as a general guide for theory choice and conceptual progress in the foundations of arithmetic.

1 Introduction

Independence and non-provability results are a familiar feature of logic and the foundations of mathematics. Traditionally, such results are treated descriptively: they mark the limits of derivability relative to a fixed axiom system. Philosophically, however, undecidability raises a further question: how inquiry ought to proceed when a statement that is independently well-motivated and true resists proof within an accepted framework.

Common responses include restricting attention to weaker systems, accepting pluralism among incompatible frameworks, or treating undecidability as an epistemic boundary. This paper articulates a different response. Under appropriate conditions, persistent non-provability functions as a methodological signal that the current system lacks the resources required to render a truth explanatorily transparent. In such cases, rational inquiry is directed toward identifying or developing a more adequate framework.

The aim of this work is not to introduce a new technical result in proof theory, but to make explicit and defend this heuristic as a general methodological principle.

2 Systems, Truth, and Provability

We restrict attention to arithmetic in order to work with a fixed intended semantics.

Definition 1 (Formal System). *A formal system S consists of a language L_S, a set of axioms A_S, and inference rules R_S.*

Definition 2 (Truth). *A statement φ is true if it holds in the standard model of arithmetic $\mathbb{N}$.*

Definition 3 (Provability). *A statement φ is provable in S if there exists a finite derivation of φ from A_S using R_S.*

Throughout, systems are assumed to be sound with respect to $\mathbb{N}$.

3 A Structural Principle of System-Relative Provability

Principle 1 (System-Relative Provability of Truth). *For each system in which a truth is provable, there exists a system in which the same truth is not provable. Conversely, the absence of provability in a given system does not exclude the existence of another system in which that same truth is provable.*

This principle expresses a symmetry: truth is invariant, while provability is system-relative and non-transferable. On its own, this observation is logically elementary; its significance lies in its methodological interpretation.

4 Adequacy and Non-Trivial Extension

Definition 4 (Adequacy). *Let S be a sound system and φ a true statement. A system S' is adequate for φ relative to S if:*

(i) S' extends S by principled axioms or rules;

(ii) φ is provable in S' without being introduced as a bare axiom;

(iii) the extension provides structural insight into why φ holds.

5 Logical Background

Lemma 1 (Finite Dependence of Proofs). *Any formal proof depends on finitely many axioms and inference rules.*

Lemma 2 (Consistent Extendability). *If S is sound and φ is true in $\mathbb{N}$, then $S + \varphi$ is consistent.*

These facts guarantee extendability but do not determine which extensions are explanatorily adequate.

6 Undecidability as a Heuristic Principle

Principle 2 (Undecidability as a Methodological Signal). *Let φ be a true arithmetical statement and S a sound formal system. If φ is not provable in S but is provable in some adequate extension S', then the non-provability of φ in S should be interpreted as evidence of the explanatory inadequacy of S with respect to φ, directing rational inquiry toward identifying or developing such an adequate system S'.*

7 Example: Goodstein's Theorem

Goodstein's theorem illustrates the heuristic. Goodstein's original proof employed transfinite induction up to ε_0. Kirby and Paris later showed that this induction strength exceeds that of Peano Arithmetic (PA), establishing unprovability in PA. The result did not undermine PA but revealed limits of its inductive resources, motivating ordinal analysis and calibrated extensions.

8 Structural Parallel with Scale-Specific Principles

A closely related rational structure appears in theoretical physics.

Law of Scale-Specific Principles. In effective field theory and renormalization-group analysis, when extrapolation across physical scales fails while stable observables persist, consistency requires the introduction of new scale-specific organizing principles, determined by fixed-point structure and the relevance or irrelevance of operators.

This formulation is due to the present author and was developed as a methodological synthesis of standard results in effective field theory and renormalization-group formalism. A detailed justification is given in *No Final Theory: Law of Scale-Specific Principles* (Kriger 2025). In the present work, the law is treated as an established constraint and employed as an operative analogy rather than rederived.

The analogy is methodological rather than ontological. In both cases, persistent failure within a framework signals a mismatch between descriptive resources and the structure of the phenomenon.

9 Relation to Alternative Positions

The heuristic complements Maddy's naturalism by providing a criterion for when extrinsic justification of stronger axioms is rationally warranted. It diverges from strict constructivism, which may treat non-provability as grounds for rejection rather than extension. It is compatible with pluralist views, including set-theoretic multiverse perspectives, by articulating why explanatory extensions remain rational even when pluralism is accepted.

10 Conclusion

Undecidability is often treated as an endpoint or a purely technical classification. This paper argues that, under appropriate conditions, it functions as a methodological signal indicating limits of explanatory adequacy. Recognizing this signal redirects inquiry toward more adequate frameworks capable of rendering stable truths transparent.

References

[1] R. L. Goodstein, *On the Restricted Ordinal Theorem*, Journal of Symbolic Logic, 1944.

[2] L. Kirby and J. Paris, *Accessible Independence Results for Peano Arithmetic*, Bulletin of the London Mathematical Society, 1982.

[3] G. Gentzen, *Consistency Proof for Arithmetic*, Mathematische Annalen, 1943.

[4] M. Rathjen, *Ordinal Analysis*, Bulletin of Symbolic Logic, 2005.

[5] S. Simpson, *Subsystems of Second Order Arithmetic*, Cambridge University Press, 2009.

[6] S. Feferman, *Turing in the Land of O(z)*, Proceedings of the Herbrand Symposium, 1988.

[7] R. Batterman, *Asymptotics and the Role of Minimal Models*, British Journal for the Philosophy of Science, 2002.

[8] J. Butterfield and N. Bouatta, *Renormalization for Philosophers*, Physics in Perspective, 2012.

[9] B. Kriger, *No Final Theory: Law of Scale-Specific Principles*, Altaspera Publishing, 2025.