The Choice of Formal Realities:
A Meta-Mathematical Argument for Explicit Foundational Context

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Abstract

Foundational discussions across mathematics, physics, and philosophy frequently proceed under an implicit assumption of neutrality with respect to formal frameworks. This paper argues that such neutrality is structurally impossible. Any sufficiently rich scientific or mathematical task presupposes a choice of a coherent formal reality—a package of axioms, logic, ontological commitments, and admissible inferential practices. We present four meta-mathematical results, stated theorem-style, establishing the inevitability, non-uniqueness, structural cost, and stability constraints of such choices. Together they yield a metatheorem: foundational completeness requires explicit disclosure of the chosen formal reality. The analysis is prescriptive rather than ontological: it formalizes constraints on foundational practice using standard results (independence, non-categoricity, reverse mathematics, and constructive/classical contrasts), without proposing a new foundational program.

Keywords: foundational pluralism; formal reality; methodological norm; reverse mathematics; interpretability; interdisciplinarity

1 Introduction

Scientific and mathematical work is routinely conducted against a background of implicit formal commitments. In mathematics, axiomatic and logical defaults are often treated as silent infrastructure; in physics, effective theories are adopted with tacit ontological and inferential norms; in philosophy, debates over realism and objectivity often proceed without explicit attention to the frameworks that fix truth conditions and admissible inferences. This manuscript argues that such implicitness constitutes a methodological deficiency whenever claims are meant to be foundational, transferable across contexts, or interdisciplinary.

The central claim is not that pluralism exists—this is widely acknowledged in logic, foundations, and philosophy of science—but that pluralism imposes nontrivial obligations on how foundational claims should be stated, compared, and applied. These obligations can be articulated as general constraints governing framework choice and disclosure.

Section 2 defines the notion of a formal reality. A short methodological stance statement precedes the theorem-style results in Section 3. Sections 3–6 present four results and a culminating metatheorem. Sections 7–9 give disciplinary corollaries, related work, and discussion.

2 Formal Realities

Definition 1. *A* formal reality $\mathcal{R}$ *is a coherent package consisting of:*

- *a formal language (or family of languages) adequate to state the task,*
- *a logical system (classical, intuitionistic, modal, etc.),*
- *background axioms or structural principles,*
- *ontological commitments regarding admissible entities and constructions,*
- *inferential and explanatory norms (what counts as a proof, witness, explanation, or admissible idealization).*

Typical examples include ZFC with classical logic, Bishop-style constructive analysis, categorical foundations (ETCS), and homotopy type theory. Each can support substantial mathematics, yet differs in admissible existence claims, proof norms, and ontological commitments.

3 Methodological Stance on "Theorem-Style" Results

The results below are stated in theorem–proof form as a disciplinary device: a compact way to present constraints on foundational practice with explicit premises and traceable support. They are *not* offered as derivations inside a single fixed background metatheory, and their "proofs" should be read as structured arguments showing how standard, widely accepted results in logic and foundations (independence, non-categoricity, reverse mathematics classifications, and constructive/classical contrasts) jointly enforce the advertised constraints. The point is methodological accountability rather than a claim to a new technical foundation.

4 Theorem 1: Inevitability of Choice

Theorem 1. *For any sufficiently rich mathematical or scientific task T—rich in the sense that changing logic or axioms can alter provability, truth conditions, or admissible existence claims about T—neutrality with respect to formal reality is impossible.*

Proof. A task T presupposes criteria of validity, existence, and admissible inference. These criteria are not invariant across frameworks: for example, classical existence theorems may assert objects without providing witnesses, whereas constructive settings treat "existence" as requiring explicit construction or realizability. If such criteria are not fixed, then the semantic and inferential status of central statements about T is underdetermined, and the task cannot have determinate content in the sense required for foundational discussion or interdisciplinary transfer. Hence some formal reality must be presupposed; suppressing the choice merely conceals it. □

5 Theorem 2: Non-Uniqueness of Formal Realities

Theorem 2. *No formal reality adequate for a sufficiently rich task T is uniquely determined by T (where T is understood at the level of subject-matter content, not as "work inside a named theory S").*

Proof. Independence phenomena show that widely shared core practice does not force a unique extension: Gödel–Cohen independence of the Continuum Hypothesis exhibits incompatible but internally coherent set-theoretic continuations of ZFC. Thus tasks in topology, measure, and functional analysis can be developed in distinct set-theoretic backgrounds that agree on large common parts while diverging on further claims. Separately, non-categoricity results (Skolem) show that even fixing a first-order theory does not pin down a unique model up to isomorphism; multiple non-isomorphic models may satisfy the same axioms. Therefore, even when T is stable in intent, more than one adequate formal reality may support it. $\square$

6 Theorem 3: Structural Cost of Formal Realities

Theorem 3. *For any formal reality $\mathcal{R}$ adequate for a sufficiently rich task T, there exists a non-zero structural cost vector*

$$\mathrm{Cost}(\mathcal{R}) = \langle C_{\mathrm{proof}}, C_{\mathrm{comp}}, C_{\mathrm{ont}} \rangle,$$

such that for some alternative adequate formal reality $\mathcal{R}'$, at least one component is strictly larger for $\mathcal{R}$ than for $\mathcal{R}'$.

Proof. The three components capture persistent trade-offs visible across foundational practice.

Proof-theoretic cost arises because expressive strength buys theorems at the price of stronger commitments and metatheoretic limitations: by Gödel's incompleteness, sufficiently strong systems cannot be complete, and in reverse mathematics many "ordinary" theorems are equivalent over RCA_0 to stronger subsystems such as WKL_0 or ACA_0 (as standardly classified in Simpson's development of the program). For instance, the Heine–Borel compactness of $[0, 1]$ is equivalent (over RCA_0) to WKL_0, illustrating that adopting compactness principles entails a specific increase in foundational strength.

Computational cost arises because classical principles (e.g., the Law of Excluded Middle and, in many settings, Choice) permit non-constructive existence claims, often forfeiting witness extraction or algorithmic content, whereas constructive or realizability-informed realities recover computational meaning at the expense of excluding some classical inferences.

Ontological cost arises because reducing element-level commitments via structural or categorical approaches can introduce higher-order entities (universes, large-cardinal-like size principles, homotopy levels) or shift the ontology from sets/elements to morphisms/structures.

These costs cannot, in general, be simultaneously minimized: reducing one dimension typically requires paying in another (strength vs. constructivity vs. ontology), and foundational practice reflects this persistent "no free lunch" structure. $\square$

7 Theorem 4: Stability Under Admissible Variation

Definition 2. *An* admissible variation *between formal realities for a task T is a translation or interpretation between theories that preserves the portion of language in which T is formulated and preserves the inferential role relevant to the intended application of T (e.g., by an interpretation/translation that transfers T-statements while respecting what counts as evidence or construction in the receiving framework).*

Theorem 4. *A result about T possesses structural generality (relative to the intended application of T) if and only if it is invariant under all admissible variations of formal reality for T.*

Proof. If a claim holds in $\mathcal{R}$ but fails in an admissible variation $\mathcal{R}'$, then the claim depends on framework-specific commitments that do not survive even translations preserving the relevant T-content; its generality is therefore limited to that reality's inferential or ontological regime. For example, compactness principles equivalent to WKL_0 may fail in weaker admissible settings and thus cannot be advertised as "framework-free" general truths about analysis. Conversely, claims that remain valid across admissible translations track the structure of T rather than artifacts of a particular foundation. This criterion functions as a regulative ideal: admissibility is context-sensitive in practice, but the invariance requirement captures what it means to claim structural generality responsibly. □

8 Metatheorem: Foundational Completeness

Definition 3. *A foundational claim is* foundationally complete *if it (i) specifies its background formal reality, (ii) acknowledges salient non-uniqueness and cost trade-offs relevant to the claim's use, and (iii) indicates (at least qualitatively) the stability status of its central assertions under admissible variation.*

Metatheorem 1. *Foundational completeness requires explicit specification of the formal reality within which claims are made.*

Proof. By Theorem 1, some framework choice is unavoidable. By Theorem 2, that choice is not uniquely forced by the task, so silence about it is informationally significant. By Theorem 3, choices carry structured costs that affect what kind of content is being delivered (proof strength, computational meaning, ontology). By Theorem 4, claims of generality require stability information. Therefore, omitting the chosen formal reality prevents the reader from assessing scope, transferability, and methodological legitimacy of the claim, and so violates foundational completeness. □

9 Corollaries

Mathematics. Statements intended to be foundational should disclose their background commitments. For example, Heine–Borel compactness of $[0,1]$ is equivalent (over RCA_0) to WKL_0, so presenting it as “basic analysis” without indicating the compactness/choice strength risks cross-framework misapplication.

Physics. Explanations are framework-relative: effective field theories used in condensed matter and low-energy QFT constitute formal realities with renormalization-based norms and cutoff-dependent admissible entities, whereas UV-complete proposals impose different ontological and inferential constraints. Treating these as interchangeable realities without disclosure invites category mistakes about what is being explained.

Philosophy. Many metaphysical disputes become tractable—or dissolve—once the operative formal reality is made explicit. Constructive or ultrafinitist shifts, for instance, can reframe debates about “existence” of non-computable entities, changing what counts as an acceptable resolution rather than merely changing vocabulary.

10 Related Work

This work engages with three strands of literature. Logical pluralism (Beall & Restall) and related debates establish that multiple logics may be legitimate. Set-theoretic multiversism (Hamkins) articulates non-uniqueness at the level of set-theoretic reality. Philosophy of mathematical practice (Mancosu; Corfield) emphasizes how mathematicians navigate frameworks informally, while scientific perspectivism (Giere) and Cartwright’s “dappled world” provide analogous positions in physics.

The present manuscript does not claim novelty in the existence of pluralism. Its contribution is a compact, prescriptive package that consolidates widely accepted results into theorem-style constraints, adds an explicit multidimensional cost framing, and turns “framework disclosure” into a methodological norm required for foundational completeness and interdisciplinary accountability.

11 Discussion

The objection that working mathematicians proceed without foundational commitments confuses tacit reliance with neutrality. Informal work can remain flexible while it stays within a shared vernacular; however, formalization, comparison, and application across contexts force choices about existence and inference to the surface. Claims that a single universal background (e.g., ZFC) can function as neutral infrastructure neglect the cost structure: even if ZFC is widely used, its non-constructive commitments and proof-theoretic profile are methodologically relevant when computational content or transferability is at issue.

The recommended norm is therefore not to abandon familiar frameworks, but to disclose them when a claim is presented as foundational, general, or interdisciplinary. Doing so does not weaken scientific results; rather, it clarifies their scope, stability, and legitimate modes of application.

12 Conclusions

This paper has argued that explicit specification of formal reality is not an optional philosophical refinement, but a methodological requirement for foundational completeness. By articulating four meta-level constraints—inevitability of framework choice, non-uniqueness of adequate frameworks, structural cost trade-offs, and stability under admissible variation—we have shown that silence about foundational context systematically obscures the scope and content of scientific claims.

The contribution of this work does not lie in proposing a new foundational system, nor in defending pluralism as a metaphysical doctrine. Instead, it consolidates widely accepted results from logic, foundations, and scientific practice into a compact, prescriptive framework that disciplines how foundational claims are formulated and communicated. The theorem-style presentation is intended to emphasize constraint, not derivation: these results function as regulative principles governing responsible foundational discourse.

The proposed norm is minimal but consequential. When claims are intended to be foundational, general, or transferable across disciplines, authors should specify the formal reality in which those claims are made, acknowledge relevant non-uniqueness and cost trade-offs, and indicate the degree of stability their results enjoy under admissible variation. This requirement neither restricts mathematical creativity nor mandates uniform foundations; it instead promotes transparency, prevents category mistakes, and facilitates meaningful comparison across frameworks.

In this sense, the choice of formal reality is not merely a background decision but an integral part of scientific content. Making that choice explicit is a necessary step toward methodological clarity in mathematics, physics, philosophy, and any domain where foundational claims are advanced.

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Author Biography

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