# The Principle of Definition-Dependent Provability: Provability as a Function of Definition

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## Abstract

This paper formulates and defends a metatheoretical principle governing formal reasoning: provability is a function of definition. Statements commonly regarded as "undecidable" or "in principle unprovable" fall into two distinct categories that philosophical discourse routinely conflates. The first category comprises genuinely undecidable statements—those that remain underivable even after complete formalization within a consistent, sufficiently strong system. The second category comprises statements that appear undecidable only because they lack the definitional precision required to enter formal discourse at all. This paper concerns the second category exclusively. We propose the Principle of Definition-Dependent Provability (DDP) as a methodological tool that clarifies when claims of unprovability are legitimate and when they merely disguise incomplete specification. The principle does not extend the reach of formal proof; it disciplines the conditions under which proof becomes applicable.

## 1. Introduction

In logic and metamathematics, provability is studied relative to fixed formal systems. Provability logics, such as the Gödel–Löb system GL, analyze the behavior of provability predicates once a system and its axioms are given. Gödel's incompleteness theorems establish that any consistent, recursively axiomatizable system containing sufficient arithmetic harbors true but unprovable sentences. These results are precise, rigorous, and inescapable.

However, philosophical discourse frequently applies the label "undecidable" or "unprovable" to claims that have never been formalized at all. When a philosopher asserts that some metaphysical thesis is "beyond formal demonstration," or that certain ethical claims "cannot be proven," the assertion typically precedes—and often substitutes for—any serious attempt at formalization. This paper argues that such claims conflate two fundamentally different phenomena: genuine logical undecidability, which is a technical property of well-formed sentences within specified formal systems, and definitional indeterminacy, which is a failure to specify what is being claimed with sufficient precision to admit proof or refutation.

The concern is not new. Carnap's distinction between genuine theoretical questions and pseudo-questions arising from linguistic confusion (1928, 1950), Quine's arguments about the indeterminacy of translation (1960), and the early Wittgenstein's demarcation of what can be meaningfully said (1921) all address related territory. More recently, work on conceptual engineering (Cappelen, 2018) and explication (Beaney, 2004) has examined how pre-theoretic concepts are refined for theoretical purposes. The present paper contributes to this tradition by proposing a specific principle—the Principle of Definition-Dependent Provability—that makes the relationship between definition and provability explicit and formally tractable.

### 1.1 Thesis and Scope

**The Principle of Definition-Dependent Provability (DDP):** For any statement whose formulation depends on terms admitting multiple admissible definitions, the provability status of that statement is a function of the definition chosen. A statement that varies in provability across admissible definitions is not undecidable; it is underdefined.

DDP is a methodological principle, not a completeness result. It does not claim that all statements become provable once defined, nor does it challenge Gödel's incompleteness theorems. Its target is narrower: claims of "in principle unprovability" that are advanced without specifying the formal system or the precise formulation of the statement at issue. DDP diagnoses such claims as incomplete—neither true nor false, but malformed.

DDP differs from the familiar thesis that "all proof is framework-relative." That thesis concerns the relativity of proof to axioms and inference rules within a fixed language. DDP concerns the prior question of whether the statement to be proven has been rendered determinate at all. Framework-relativity presupposes a well-formed statement; DDP asks whether we have one.

## 2. Formal Framework

We work within a metatheory capable of representing formal systems via standard arithmetization. The framework is purely syntactic and requires no semantic commitments beyond those implicit in ordinary metamathematics.

### 2.1 Primitive Predicates

- **System(S):** S codes a formal system (language, axioms, inference rules).
- **Def(d, S):** d is an admissible definition in S, introducing a new symbol via explicit equivalence or contextual definition.
- **Formula(φ, S):** φ is a well-formed formula in the language of S.
- **DependsOn(φ, d):** φ contains a symbol whose meaning is fixed by definition d.
- **Ext(S, d) = S′:** S′ is the conservative extension of S obtained by adding definition d.
- **Prov(S, φ):** φ is derivable in system S.

### 2.2 Background Assumptions

**A1. Conservativity of Definitions.** If S′ = Ext(S, d) is a definitional extension of S, then for any formula ψ in the language of S: Prov(S′, ψ) ↔ Prov(S, ψ). This is the standard result that definitions, properly constructed, do not add new theorems in the base language (Suppes, 1957).

**A2. Extension Existence.** For any System(S) and admissible Def(d, S), the extension Ext(S, d) exists and is unique up to notational variance.

**A3. Definitional Dependence.** If DependsOn(φ, d), then φ is not expressible in the base language of S without d or some equivalent definition.

### 2.3 Scope

The machinery above applies directly to conservative definitional extensions of a fixed base theory. Section 9 discusses extensions to non-conservative cases.

## 3. The Provability Function

For a fixed base system S and formula schema φ[T] containing a defined term T, we define the provability function:

**$f : D \rightarrow \{0, 1\}$**, where D is the space of admissible definitions for T.

**$f(d) = 1$ iff $Prov(Ext(S, d), \varphi[T/d])$**

where $\varphi[T/d]$ denotes the formula $\varphi$ with T interpreted according to definition d.

This function is well-defined because Ext(S, d) exists by A2, and $Prov(Ext(S, d), \varphi[T/d])$ is a determinate syntactic property.

### 3.1 A Simple Formal Example

Consider the arithmetic language L with base system PA (Peano Arithmetic). Let T be a predicate symbol to be defined, and let $\varphi[T]$ be the sentence $\exists x(T(x) \wedge x > 2)$.

**Definition $d_1$:** $T(x) \equiv$ (x is even), i.e., $\exists y(x = 2y)$.

Under $d_1$, $\varphi$ becomes $\exists x(Even(x) \wedge x > 2)$. This is provable in PA: witness $x = 4$. $\therefore f(d_1) = 1$.

**Definition $d_2$:** $T(x) \equiv$ (x is both even and odd), i.e., $\exists y(x = 2y) \wedge \exists z(x = 2z + 1)$.

Under $d_2$, $\varphi$ becomes $\exists x(EvenAndOdd(x) \wedge x > 2)$. Since no natural number is both even and odd, this is refutable in PA. $\therefore f(d_2) = 0$.

This trivial example illustrates the core point: the provability of $\varphi[T]$ varies with the definition of T. The question "Is $\varphi[T]$ provable?" is indeterminate until T is defined.

**Core claim:** The provability function f is, in general, non-constant across the space of admissible definitions.

## 4. Non-Triviality

One might object that the dependency of provability on definition is trivially true. The response is that while the dependency is elementary, its systematic neglect in philosophical discourse makes DDP non-trivial in practice.

Claims such as "The existence of God cannot be proven," "Moral truths are not formally derivable," and "Consciousness is beyond physical explanation" are routinely advanced without specifying which formalization is under discussion. DDP makes explicit that these claims are incomplete: they assert that some vaguely indicated proposition-schema fails to be derivable under all admissible precisifications—a claim that is either false (if some precisification yields a derivable statement) or trivial (if no precisification is genuinely admissible).

The situation parallels what Gallie (1956) termed "essentially contested concepts." DDP draws a further consequence: where a concept is essentially contested, claims about provability involving that concept are correspondingly indeterminate.

## 5. Relation to Genuine Undecidability

DDP does not challenge Gödel's incompleteness theorems. The Gödel sentence G for a system S is a fully defined, syntactically precise formula. Its undecidability within S is a structural property of S itself, not a matter of definitional latitude.

The distinction:

- **Gödelian undecidability:** A precisely defined sentence φ is neither provable nor refutable in a specified consistent system S.
- **Definitional indeterminacy:** A sentence-schema φ[T] varies in provability depending on how T is defined; the question of its provability is malformed absent a specified definition.

DDP concerns the second phenomenon. It contracts the domain of illegitimate claims of unprovability by requiring that such claims specify their target with sufficient precision.

## 6. The Problem of Admissibility

A central challenge for DDP concerns the boundaries of admissible definitions. We propose the following criteria as methodological guidelines:

**C1. Inferential Role Preservation.** An admissible definition must preserve the core inferential connections that competent users of the pre-theoretic term recognize.

**C2. Extensional Adequacy.** An admissible definition must yield correct verdicts on clear cases.

**C3. Theoretical Fruitfulness.** An admissible definition should facilitate rather than obstruct inquiry.

### 6.1 Toward a More Formal Characterization

To make these criteria more precise, we introduce the notion of a **conceptual profile**:

**Definition:** The conceptual profile of a term T relative to a linguistic community is the tuple $P(T) = \langle I, E, R \rangle$, where: - I is the set of inferential connections (entailments, incompatibilities) that competent users accept - E is the set of clear positive and negative instances (the extension on paradigm cases) - R is the set of theoretical roles the term plays in inquiry

**Definition:** Two definitions $d_1$ and $d_2$ are **admissibility-equivalent** iff they yield the same verdicts on I and E (though they may differ on R).

**Criterion (Semi-formal):** A definition d is admissible for T iff: - d preserves a substantial subset of I (C1) - d agrees with E on paradigm cases (C2) - d enables non-trivial inquiry relative to R (C3)

This characterization does not eliminate all indeterminacy—there may be multiple admissible definitions that differ on peripheral inferences or borderline cases—but it provides a framework for comparing definitions and adjudicating disputes.

### 6.2 Meta-Controversy: When Admissibility Itself Is Disputed

In actual philosophical disputes, the criteria C1–C3 may themselves become contested. Different communities may disagree about which inferential connections are "core," which cases are "clear," or what counts as "facilitating inquiry." This generates a meta-level controversy about definitional admissibility.

DDP applies recursively to such meta-controversies. A claim like "Definition $d_1$ is inadmissible" is itself subject to DDP: it depends on how "admissibility" is formalized. If disputants operate with different standards of admissibility, their disagreement about whether $d_1$ is admissible may reflect definitional divergence at the meta-level rather than substantive disagreement.

This recursion is not vicious; it terminates when disputants share enough common ground to adjudicate. The practical consequence is that DDP requires not only specifying definitions but also, when necessary, specifying the standards by which definitions are evaluated. In most cases, shared linguistic competence provides sufficient common ground. In hard cases, the meta-level standards must be made explicit.

### 6.3 Application: Formalizing "Free Will"

Consider formalizing "free will" as compatibilist free will (Fc) vs. libertarian free will (Fl):

- Fc: An agent acts freely iff the agent acts on desires formed through rational deliberation without external coercion.
- Fl: An agent acts freely iff the agent is the uncaused cause of the action.

Both satisfy C1–C3 relative to different conceptual profiles. "Humans have free will" is provable (given empirical premises) under Fc and unprovable under Fl. DDP clarifies that claims about free will's provability are indeterminate until the formalization is specified.

## 7. Illustrative Cases

### 7.1 Technical Example: Formalizing "Randomness"

**Question:** Is the sequence S = 01010101… random?

Consider three admissible definitions, each applied in distinct disciplinary contexts:

**Definition $R_1$ (Frequency):** A binary sequence is random iff the limiting frequency of 1s equals 1/2. *Applied in: classical probability theory, early statistical mechanics.*

Under $R_1$: S is random (frequency of 1s is exactly 1/2). $\therefore f(R_1) = 1$.

**Definition $R_2$ (Kolmogorov Complexity):** A binary sequence is random iff it is incompressible. *Applied in: algorithmic information theory, cryptography, data compression.*

Under $R_2$: S is not random (compressible as "repeat 01"). $\therefore f(R_2) = 0$.

**Definition $R_3$ (Martin-Löf):** A binary sequence is random iff it passes all effective statistical tests. *Applied in: algorithmic randomness, statistical hypothesis testing.*

Under $R_3$: S is not random (periodic sequences form an effectively null set). ∴ $f(R_3) = 0$.

| Definition | Context | f(d) |
|---|---|---|
| $R_1$ (Frequency) | Probability theory | 1 |
| $R_2$ (Kolmogorov) | Cryptography, compression | 0 |
| $R_3$ (Martin-Löf) | Statistical testing | 0 |

All three definitions satisfy C1–C3 within their respective domains. The question "Is S random?" is not undecidable; it is underdefined.

**7.2 Moral Realism: A Semi-Formal Analysis**

Consider the metaethical claim: "The existence of objective moral facts cannot be proven."

We formalize three definitions of "objective moral fact" (OMF):

**Definition $M_1$ (Naturalist-Reductionist):** OMF(p) ≡ p is a fact about states of affairs that maximize aggregate well-being.

Under $M_1$, "There exist OMFs" reduces to "There exist facts about well-being maximization"—provable given empirical premises. ∴ $f(M_1) = 1$.

**Definition $M_2$ (Non-Naturalist Platonist):** OMF(p) ≡ p is a fact about abstract normative entities existing independently of natural facts.

Under $M_2$, the claim requires metaphysical axioms beyond naturalistic base theories. ∴ $f(M_2)$ = indeterminate (varies with metaphysical commitments).

**Definition $M_3$ (Constructivist):** OMF(p) ≡ p is a fact that would be endorsed under idealized rational deliberation.

Under $M_3$, provability depends on whether the constructivist procedure converges. ∴ $f(M_3)$ = depends on convergence assumptions.

| Definition | f(d) |
|---|---|
| $M_1$ (Naturalist) | 1 |
| $M_2$ (Platonist) | Indeterminate |
| $M_3$ (Constructivist) | Conditional |

The persistent disagreement in metaethics reflects competing implicit formalizations. DDP clarifies that “Can moral realism be proven?” is malformed until the operative definition is specified.

### 7.3 The Continuum Hypothesis: Why CH Is Not a Counterexample

The continuum hypothesis (CH) is independent of ZFC. One might object: CH exhibits “formal relativity” across models—true in some (e.g., L), false in others. Isn’t this analogous to definitional variance?

The objection fails because CH involves no definitional indeterminacy. The terms “set,” “cardinality,” “integers,” and “real numbers” are fully specified within ZFC. The independence is a structural feature of the axiom system, not a matter of vague terms admitting multiple precisifications.

The relativity of CH across models is model-theoretic, not definitional. Different models interpret the same fully specified sentence differently; this is distinct from different definitions yielding different sentences. DDP concerns the latter. Thus CH exemplifies genuine undecidability: DDP correctly identifies it as outside its scope.

## 8. Related Work

### 8.1 Logical Pluralism and Proof-Theoretic Semantics

Logical pluralism holds that multiple logics are legitimate for different purposes (Beall & Restall, 2006). Proof-theoretic semantics grounds meanings in inferential roles (Prawitz, 1965; Schroeder-Heister, 2018). Both emphasize that consequence depends on the logical framework.

DDP is complementary: logical pluralism concerns the choice of logic; DDP concerns the choice of definitions for non-logical terms within a fixed logic. A full account of provability-relativity incorporates both.

### 8.2 Meta-Ontological Deflationism

Meta-ontological deflationists argue that many metaphysical disputes are framework-relative (Carnap, 1950; Hirsch, 2011). DDP shares structural similarities but is more narrowly focused: it concerns provability claims specifically, not truth or existence claims generally. One might accept DDP while rejecting deflationism about ontology.

### 8.3 Conceptual Engineering

Conceptual engineering evaluates and improves concepts for theoretical purposes (Cappelen, 2018; Burgess & Plunkett, 2013). DDP intersects with this project in two ways:

**Complementarity:** Conceptual engineering asks which concept we *should* use; DDP asks what follows from each choice. DDP provides a formal tool for mapping the provability consequences of different conceptual choices.

**Challenge:** Conceptual engineering sometimes assumes a privileged standpoint for evaluating concepts. DDP suggests that such evaluations are themselves relative to standards that may require specification. There is no view from nowhere on admissibility.

**Integration:** DDP can serve as a diagnostic within conceptual engineering: before recommending a concept revision, engineers should map the provability profiles of competing formalizations. DDP disciplines this mapping.

### 8.4 Logic Pedagogy and Proof Practices

Recent empirical work on logic education emphasizes that students struggle with the role of definitions in proof (Inglis & Alcock, 2012; Weber, 2001). Instructors often treat definitions as given, obscuring their constitutive role in determining what can be proven. DDP articulates explicitly what experienced logicians know tacitly: definitional choice shapes provability. This has pedagogical applications for teaching the relationship between definition and proof.

## 9. Extensions to Non-Conservative Cases

The formal framework of Section 2 focuses on conservative definitional extensions. Many philosophically interesting formalizations, however, require non-conservative extensions—adding new axioms, primitives, or theoretical postulates.

### 9.1 Theoretical Embedding

When formalizing a pre-theoretic concept requires enriching the base theory, we model this as **theoretical embedding**: the concept is formalized not by a conservative definition but by a translation into an extended theory $S^+$ that includes new axioms.

In this case, provability becomes relative to both the embedding and the enriched system:

**$f(d, S^+) = 1$ iff $Prov(S^+, \varphi[T/d])$**

The provability function now has two arguments: the formalization d and the background theory $S^+$. DDP generalizes: provability is a function of how concepts are formally rendered *and* which theoretical resources are available.

### 9.2 Example: Formalizing "Consciousness"

Formalizing "consciousness" plausibly requires non-conservative extension. A physicalist formalization might embed consciousness-talk into a completed neuroscience; a dualist formalization might add primitive phenomenal properties.

Under physicalist embedding ($S^+$ = completed physics + bridge laws): "Consciousness exists" may be provable. Under dualist embedding ($S^+$ = physics + phenomenal primitives): "Consciousness is irreducible" may be provable.

The question "Is consciousness reducible?" is doubly underdetermined: it depends on both the definition of "consciousness" and the background theory into which it is embedded.

### 9.3 Framework Change

When formalization involves replacing rather than extending a base theory, DDP applies to the new framework. Claims about provability must specify which framework is operative. This is continuous with the core principle: provability is always relative to a specified formal context.

## 10. Methodological Consequences

DDP functions as a methodological filter for assessing provability claims.

**Consequence 1: Burden of Specification.** Anyone claiming that a statement is unprovable bears the burden of specifying the formal system and the precise formulation. Claims omitting this specification are incomplete.

**Consequence 2: Dissolution of Pseudo-Problems.** Many philosophical debates persist because central terms admit multiple formalizations yielding different provability outcomes. DDP diagnoses such debates as underdetermined rather than undecidable.

**Consequence 3: Diagnostic Checklist.** When confronted with a claim of the form "X cannot be proven," apply:

1. **Is the formal system specified?** If not, the claim is incomplete.
2. **Is the statement precisely formulated?** If not, the claim is incomplete.
3. **Do the key terms admit multiple admissible definitions?** If so, specify which is operative.
4. **Under the specified definition, is the statement provable, refutable, or independent?** Only now is the claim well-formed.
5. **If independent, is this Gödelian undecidability or definitional indeterminacy?** DDP applies only to the latter.

Claims failing steps 1–3 are not claims about provability; they are gestures toward a claim not yet made.

## 11. Objections and Replies

**Objection 1:** DDP is trivial.

**Reply:** The principle is elementary but systematically violated in practice. Its value is diagnostic: identifying where arguments trade on definitional ambiguity.

**Objection 2:** DDP allows gerrymandering definitions to secure desired conclusions.

**Reply:** Admissibility is constrained by C1–C3. Not every stipulation is admissible. Section 6.1 provides a semi-formal framework for adjudicating disputes.

**Objection 3:** Some concepts are essentially contested and admit no canonical formalization.

**Reply:** DDP accommodates this: where no canonical formalization exists, provability claims are correspondingly indeterminate. This is DDP's point, not its limitation.

**Objection 4:** The restriction to conservative extensions is too narrow.

**Reply:** Section 9 extends DDP to non-conservative cases. The core insight—provability is a function of how concepts are formally rendered—survives the extension.

**Objection 5:** CH shows that fully defined statements exhibit relativity across models.

**Reply:** CH's independence is model-theoretic, not definitional. The sentence is fully specified; models interpret it differently. DDP concerns definitional variance, not model-theoretic variance. See Section 7.3.

## 12. Conclusion

The Principle of Definition-Dependent Provability is not a discovery about the limits of reason. It is a reminder about the conditions of rational discourse. Provability is a relation between formal systems and well-formed sentences. Where sentences depend on terms without determinate formal content, questions of provability do not arise.

There are no undecidable statements prior to definition. There are only statements that have not yet entered the formal space where proof is possible. Provability does not fail because reality resists reason; it fails to apply because language has not yet accepted formal responsibility for what it asserts.

The philosophical utility of DDP consists in its refusal to let this failure masquerade as profundity.

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