Hypothesis Testing

The scientific method in action

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Confidence Intervals

Being confident is important

Confidence Intervals

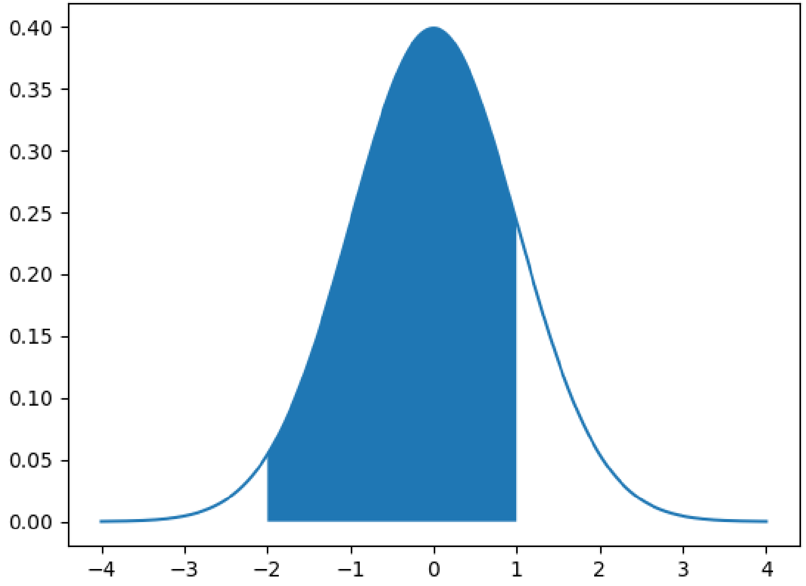
- In an experiment, we can't observe the variables' true values directly
  - We observe other values
  - We make assumptions as to how they are distributed
  - We can estimate the true value
    - **Law of large numbers**: when our sample is big enough, the sample parameters approach the population parameters
- With continuous values, it's useless to say that the mean is equal to a certain value (why?)
- **Confidence interval** – a range of values that we're fairly sure contains the true value
  - **How confident?** A matter of choice
- **Confidence level** – the probability that the value falls within the interval

Confidence Intervals – Interpretation

- Similar to the probability interpretations
- To illustrate these, let's take a confidence interval $[5;\ 7{,}3]$ and a 70% confidence level
- Frequency
  - If we perform the experiment many times, 70% of the values will fall in the interval $[5;\ 7{,}3]$ and 30% – outside it
- Certainty of next trial
  - Next time we perform the experiment, we are 70% certain that the value will fall within $[5;\ 7{,}3]$
  - Note that this is a statement **about the interval**, not about the value
- Typically used confidence levels
  - 50%; 90%; 95%; 99,7%

Confidence Intervals and Z-Scores

- Observe the Z-distribution (Gaussian, $\mu = 0,\ \sigma = 1$)
- What's the probability that a value drawn from it $x \in [-2; 1]$?
  - This corresponds to the shaded area in the graph
  - The cumulative function gives us the area to the left of some value
  - $\text{Shaded area} = cdf(1) - cdf(-2) = 0{,}819 = 81{,}9\%$
- Interpretations
  - If we draw many random numbers from the Z-distribution, we expect that 81,9% of them will be in $[-2; 1]$
  - If we draw one random number, there is 81,9% chance of it being in $[-2; 1]$
- Commonly used intervals
  - $1\sigma \rightarrow 68{,}27\%;\ 2\sigma \rightarrow 95{,}45\%; 3\sigma \rightarrow 99{,}73\%$
  - Also $1{,}96\sigma \rightarrow 95\%$



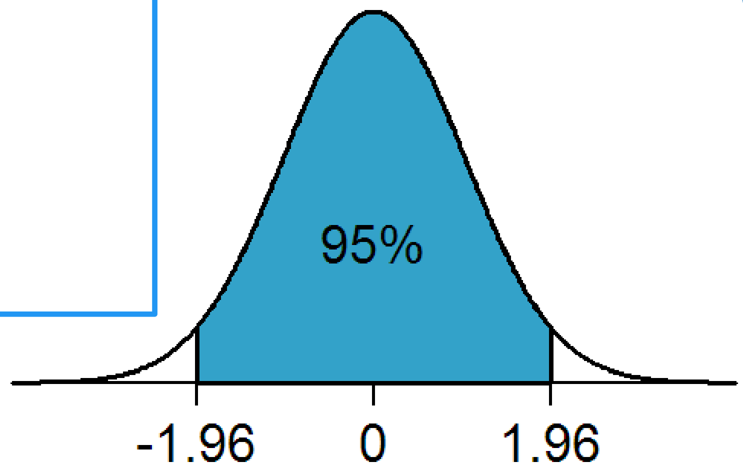
Confidence Intervals: Example

- In the dataset `heights.csv` you're given the measured heights (in cm) of 351 elderly women (from an osteoporosis study)
    - Plot a histogram and / or boxplot to see what the distribution is
    - Print the mean $\bar{x}$ and standard deviation s of the sample
    - Assume that the population follows a normal distribution
        - Real parameters – unknown; our best guess: $\mu = \bar{x}, \sigma = s$
    - What are the confidence intervals of
        - 50%, 90%, 95%
- To calculate the confidence intervals, we need to calculate the Z-scores
    - To do this, we'll use the percent point function, ppf
        - Inverse of the cdf
        - Returns the value at which the probability is less than or equal to the given probability
        - Example: Z-distribution
            - $ppf(0) = -\infty$; $ppf(1) = \infty$; $ppf(0{,}5) = 0$; $ppf(0{,}975) = 1{,}96$

Confidence Intervals Example (2)

- Note that once again we need to subtract the left white region
  - Area of shaded region: p (e.g., $p = 0{,}95$)
  - Area of both tails: $1 - p$
  - Percentage point of left tail: $\frac{1-p}{2}$
  - Percentage point of right tail: $\frac{1-p}{2} + p = \frac{1-p+2p}{2} = \frac{1+p}{2}$

```
import scipy.stats as st
def get_real_confidence_interval(probability, mean, std):
  lower_area = (1 - probability) / 2
  upper_area = (1 + probability) / 2
  return [
    st.norm.ppf(lower_area, mean, std),
    st.norm.ppf(upper_area, mean, std)]
```



Testing Hypotheses

The scientific method in action

Hypotheses

- After performing an experiment and getting data,
  the scientific method requires that we form a hypothesis
  - Fact, law, theory and hypothesis are different terms
- In the simplest case, we have two hypotheses
  - **Null hypothesis** (H_0) – the status quo is real,
    "nothing interesting happens"
  - **Alternate hypothesis** (H_1) – what we're trying to demonstrate
- Types of hypotheses
  - Attributive – something exists and can be measured
  - Associative – there is a relationship between two behaviors
  - Causal – differences in the amount / kind of one behavior cause differences in other behaviors

Hypotheses – Examples

- Examples of hypotheses – study of Disneyland visitors
  - Attributive
    - Most of the population has heard of Disneyland
    - Disneyland visitors are diverse in demographics
  - Associative
    - Income level is correlated with visiting Disneyland
    - People who live closer to Disneyland are more apt to visit Disneyland
  - Causal
    - Frequent exposure to Disneyland advertising results in increased attendance
    - Discounting tickets for local residents produces an increase in visitor numbers
- Note that attributive hypotheses involve one variable (univariate) while associative and causal hypotheses involve two variables (bivariate)

Testing a Hypothesis

- In random experiments, we have error sources
  - Human error, systematic error, random errors, etc.
- We cannot prove (or reject) a hypothesis with complete certainty
- The errors we can make are two types
  - **Type I error** – reject H_0 while it's true (false positive)
  - **Type II error** – accept H_0 while H_1 is true (false negative)
- The possible results can be summarized in the following truth table
  - Also called **confusion matrix**

		Action	
		Don't reject H_0	**Reject H_0**
Reality	**H_0 true**	**TN** true negative	**FP** (type I error) false positive
	H_0 false	**FN** (type II error) false negative	**TP** true positive

Testing a Hypothesis (2)

- To measure the probability of producing a wrong hypothesis, we use a **test statistic** – measure of deviations from H_0
  - Different tests produce different measures (statistics)
  - **We accept or reject the null hypothesis based on the value of the test statistic**
- Let's denote the probability of getting a type I error with α
  - Each value of the selected test statistic has a corresponding alpha-value
  - We perform the experiment, get data and calculate the test statistic value
  - From that, we calculate the corresponding alpha-value
  - We reject the null hypothesis if $\alpha < \alpha_c$, where α_c is a **critical confidence level**

Z-test

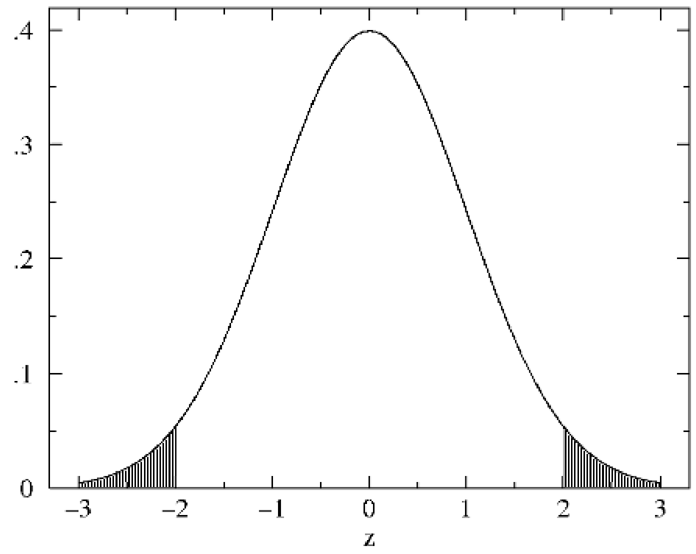
- A Z-test uses the Z-statistic
- H_0: standard normal distribution
- Example: light bulb factory
  - A factory produces light bulbs with lifetime $X \sim N(\mu = 500h,\ \sigma = 50h)$
  - A sample of 25 bulbs has a mean lifetime $\bar{x} = 480h$
  - Is there something wrong with the production line?
- Forming hypotheses
  - H_0: The production line works normally; the observed deviation of the sample mean from the population mean is due to chance
  - H_1: The production line is broken

Z-test (2)

- Suppose we take a lot of samples from the entire population
  - Each sample mean will be different
  - The distribution of sample means will be more or less Gaussian
    - Parameters (our best estimate): $\mu_{\bar{x}} = \mu, \sigma_{\bar{x}} = \sigma/\sqrt{n}$
    - Here's why the parameters are chosen like this
- If H_0 is correct, we assume that $\bar{x} \sim N(\mu, \sigma/\sqrt{n})$
- Z-statistic
  - $Z = \frac{\bar{x}-\mu}{\sigma_{\bar{x}}} = \frac{480-500}{50/\sqrt{25}} = -2$
- We can see that we are 2 std's below the mean
- How extreme is that?
  - What's the probability that we get results as extreme or more extreme than we observed, assuming the null hypothesis is true?
    - Less than 5%

Two-tailed Z-test

- We can get the confidence interval from the Z-statistic
- We are looking for **more extreme** values
  - Values **outside** the confidence interval
  - What's the probability $\mathrm{P}(|Z| \geq 2)$?
  - We're looking for a value different than the mean
    - We can't assume whether it's smaller or larger
    - Therefore, we have to look at both "tails" of the distribution



- If we assume a critical value (also called a p-value) of 5%, **the results are significant**
  - $P(|Z| > 2) \approx 0{,}0455 = 4{,}55\%$
- We can reject H_0 at the 5% level
  - Even at lower levels, up to 4,55%

One-tailed Z-test

- The same logic applies, but now we're looking at one tail only
- Question: Is the lifespan **significantly lower** than it should be? Cutoff point: $\alpha_c = 5\%, Z = -2$
  - $P(Z \leq -2) = \frac{0{,}00455}{2} - 0{,}02275 = 2{,}275\% < \alpha_c$
  - Answer: Yes, at the given significance level
- Question: Is the lifespan **significantly higher** than it should be?
  - $P(Z \geq -2) = 97{,}725\% \gg \alpha_c$
  - Answer: No, at the given significance level

t-test

- The Z-test requires that we know the standard deviation of the population
  - Usually not available
- We can use another test statistic, called **t**
- Advantages over the Z-test
  - We don't need to know the population σ
  - It's better when we have very small sample sizes (e.g., $n < 30$)
  - It can be used for testing the mean of a sample against a standard, but also, for comparing two means
    - We can see whether two sets of data are significantly different from each other
- Null hypothesis: The test statistic follows Student's t-distribution
  - Similar to Gaussian distribution, with "fatter" tails

One-Sample t-test

- The details of the calculation are fairly complex but we can do this in code
  - Using `scipy.stats`
- First, we generate 100 random numbers with $\mu = 5, \sigma = 10$
- Then we ask whether the sample mean is equal to the true mean (and other values, just for testing)
- We get the p-value – probability of the null hypothesis being true
  - I.e., probability that the mean is equal to the given mean

```
sample_data = st.norm.rvs(5, 10, 100)

print(st.ttest_1samp(sample_data, 5).pvalue) # 0.9301
print(st.ttest_1samp(sample_data, 4).pvalue) # 0.3352
print(st.ttest_1samp(sample_data, 0).pvalue) # 1.104e-6
```

Independent Two-Sample t-test

- We compare two independent distributions
  - We want to see whether they have the same mean
  - We assume equal variances (`scipy` can also do tests with unequal variances – important when sample sizes differ)
- Example: Grain size
  - We are given data (in `grain_data.csv`) of grain sizes from two different farms
  - Do they differ significantly (at the 95% level)?
  - * We can also plot histograms to see what the distributions look like

```
grain_data = ...
st.ttest_ind(grain_data.GreatNorthern, grain_data.BigFour)
# Ttest_indResult(statistic=1.3123367064875 64,
# pvalue=0.20792200785311768)
```

Paired Two-Sample t-test

- We compare two distributions
  - Observations in samples can be paired
  - Examples – before / after observations; comparison between two different treatments applied to the same subjects
- Example: Drinking water
  - We are given data (in `water_data.csv`) of Zn concentration in surface and bottom water at 10 different locations
  - Does the true average concentration in bottom water exceed that of top water?
  - We use a paired t-test because the samples are from the same locations
  - It reduces experimental error (and provides stronger evidence)

```
water_data = ...
# We use a one-tailed t-test
st.ttest_rel(water_data.surface, water_data.bottom).pvalue / 2
# 0.00044555772891127738
```

Generalizations to More Variables

- Sometimes it's not enough to compare two distributions
  - We may want to compare multiple distributions against the same null hypothesis
  - E.g., how is the percentage of smokers distributed by income and age?
- Other times, we create a model and want to evaluate it
  - E.g., a linear regression
  - We can explain some of the variance in the sample
- There are other tests to perform these "checks"
  - **ANOVA** (Analysis of Variance) – useful for grouped data
    - Observe the variance inside groups and between groups
  - **Chi-square(d) test** – can be applied to categorical data
    - Two common types
      - How good a model is (goodness of fit)
      - Whether two variables are independent

Analysis of Variance (ANOVA)

- We want to compare several **groups**
- H_0: The means of the groups are the same
- Method (`scipy.stats.f_oneway()`)
  - For each group ⇒ group mean
    - In-group variance: distances from an individual point to the group mean
    - Between-group variance: distances between the means of two groups
  - For the entire data ⇒ total mean (mean of all data)
    - Also equal to the mean of all group means
    - Total variance: in-group + between-group
- F-statistic (Fisher)
  - $F = \dfrac{\text{variance between groups}}{\text{variance within groups}}$
    - F – large ⇒ the variance between groups dominates
    - For each value of F, there's a corresponding p-value
      - If $p \leq p_c$, we can reject H_0

Chi-Squared (χ^2) Test

- Compares expected (predicted) and observed frequencies
  - Is there a significant difference between these?
  - Used to compare **categories** (one against another)
    - Compare to ANOVA – numbers w.r.t. categories
  - May also be used as a goodness-of-fit measure
    - How well were we able to predict
- Statistic: $\chi^2 = \frac{(f_{\text{observed}} - f_{\text{estimated}})^2}{f_{\text{estimated}}}$
- H_0: No significant difference between observed and estimated frequencies among the categories (groups)
  - The test returns the value of the statistic and the p-value corresponding to it
  - Works the same as any other test
  - Python: `scipy.stats.chisquare()`

Common Misconceptions

Because everyone can be wrong

Some p-value Misconceptions

- Goodman, S. (2008), source
- "If $p = 0{,}05$, H_0 has 5% chance of being true"
  - **The data alone can't tell us how likely we are to be wrong**
  - p is calculated under H_0, so it can't be the probability of H_0 being false
- "$p = 0{,}05$ means that if we reject H_0, the probability of type I error (false positive) is only 5%"
  - I.e., seeing a difference where there isn't any
  - ⇒ 5% chance of false rejection = 5% chance H_0 is true
    - Wrong, see first bullet
- "If $p = 0{,}05$, we have observed data that will occur **only** 5% of the time assuming H_0"
  - The p-value is the probability of observing data **as extreme or more extreme** under H_0

Some p-value Misconceptions (2)

- "A nonsignificant difference means the groups are the same"
  - It only means we don't have enough data to reject H_0
- "A scientific conclusion or treatment policy must be based on whether or not the p-value is significant"
  - **The results have to be checked** against prior data
- Failing to reject H_0 means that H_0 is true
  - It means that we don't have enough evidence to reject it
  - **We can't accept (or reject) any other hypothesis**
  - *"Absence of evidence is not evidence of absence"*
- *https://xkcd.com/882/*
- *https://www.xkcd.com/1478/*
- "Still. Not. Significant" article

Summary

- Confidence intervals
  - Confidence level
- Hypothesis tests
  - Z-test
  - t-test (one-sample, two-sample)
- Hypothesis tests of many variables
  - ANOVA
  - Chi-squared
- p-value misconceptions

Questions?