MATH554: Astrophysical Fluids

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1 Magnetohydrodynamic Equations (MHD)

Hydrodynamics:

$$\begin{aligned} t &\gg \tau \text{ (collisional)} \\ L &\gg l \text{ (large-scale)} \end{aligned}$$

Mass-conservation:

$$\begin{aligned} \rho &= \rho(x, y, z, t) \\ \vec{v} \cdot \nabla &= v_x \frac{\partial}{\partial x} + v_y \frac{\partial}{\partial y} + v_z \frac{\partial}{\partial z} \end{aligned}$$

$$\begin{aligned} \frac{dp}{dt} &= \frac{\partial \rho}{\partial t} + \vec{v} \cdot \nabla \rho \\ &= -\rho \nabla \cdot \vec{v} \end{aligned}$$

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \vec{v} \cdot \nabla \rho + \rho \nabla \cdot \vec{v} &= 0 \\ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) &= 0 \end{aligned}$$

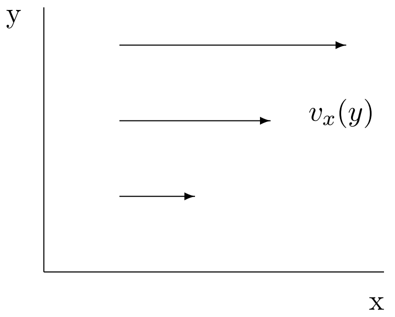
Incompressibility:

$$\frac{dp}{dt} = 0 \implies \nabla \cdot \vec{v} = 0$$

Equation of Motion:

$$\begin{aligned}
\vec{a} = \frac{d\vec{v}}{dt} &= \frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\vec{v} \\
\rho\frac{d\vec{v}}{dt} &= -\nabla p + \vec{F}_l + \vec{F}_g + \vec{F}_v \\
\vec{F}_g &= \rho\vec{g} \\
\vec{F}_l &= \frac{1}{c}\vec{J}\times\vec{B} \text{ (but no electric force } \vec{F}_E = \rho^q\vec{E}) \\
\vec{F}_v &= \text{viscous force}
\end{aligned}$$

Shear force:



$$\begin{aligned}
S_{xy} &= \mu\frac{dv_x}{dy} \\
\text{force } F_x &= \frac{dS_{xy}}{dy} = \mu\frac{d^2v_x}{dy^2} \\
F_{v,i} &= \frac{\partial S_{ij}}{\partial x_j}
\end{aligned}$$

Newtonian fluid: S_{ij} is a linear combination of $\frac{\partial v_i}{\partial x_j}$. $\vec{v} = \vec{\Omega}\times\vec{r} \implies S_{ij} = 0$.

$$S_{ij} = a\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) + b\delta_{ij}(\nabla\cdot\vec{v}).$$

If $S_{ij} = 0$ (not pressure):

$$
\begin{aligned}
0 &= 2a\nabla \cdot \vec{v} + 3b\nabla \cdot \vec{v} \\
a &= \mu \\
b &= \frac{-2}{3}\mu \\
S_{\text{total}_{ij}} &= -p\delta_{ij} + S_{ij} \\
\vec{F}_{v,i} &= \mu \left[\nabla^2 v_i + \frac{\partial}{\partial x_i}(\nabla \cdot \vec{v}) \right] - \frac{2}{3}\mu \frac{\partial}{\partial x_i}(\nabla \cdot \vec{v}) \\
\vec{F}_v &= \mu \nabla^2 \vec{v} + \frac{1}{3}\mu \nabla(\nabla \cdot \vec{v})
\end{aligned}
$$