MATH554: Astrophysical Fluids

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1 Magnetohydrodynamic Equations (MHD)

Hydrodynamics:

$$
\begin{aligned}
t &\gg \tau \text{ (collisional)} \\
L &\gg l \text{ (large-scale)}
\end{aligned}
$$

Mass-conservation:

$$
\begin{aligned}
\rho &= \rho(x, y, z, t) \\
\vec{v} \cdot \nabla &= v_x \frac{\partial}{\partial x} + v_y \frac{\partial}{\partial y} + v_z \frac{\partial}{\partial z}
\end{aligned}
$$

$$
\begin{aligned}
\frac{dp}{dt} &= \frac{\partial \rho}{\partial t} + \vec{v} \cdot \nabla \rho \\
&= -\rho \nabla \cdot \vec{v}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \rho}{\partial t} + \vec{v} \cdot \nabla \rho + \rho \nabla \cdot \vec{v} &= 0 \\
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) &= 0
\end{aligned}
$$

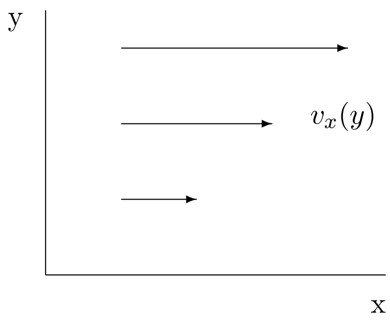
Incompressibility:

$$
\frac{dp}{dt} = 0 \implies \nabla \cdot \vec{v} = 0
$$

Equation of Motion:

$$\begin{aligned}
\vec{a} = \frac{d\vec{v}}{dt} &= \frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\vec{v} \\
\rho\frac{d\vec{v}}{dt} &= -\nabla p + \vec{F}_l + \vec{F}_g + \vec{F}_v \\
\vec{F}_g &= \rho\vec{g} \\
\vec{F}_l &= \frac{1}{c}\vec{J}\times\vec{B} \text{ (but no electric force } \vec{F}_E = \rho^q\vec{E}) \\
\vec{F}_v &= \text{viscous force}
\end{aligned}$$

Shear force:



$$\begin{aligned}
S_{xy} &= \mu\frac{dv_x}{dy} \\
\text{force } F_x &= \frac{dS_{xy}}{dy} = \mu\frac{d^2v_x}{dy^2} \\
F_{v,i} &= \frac{\partial S_{ij}}{\partial x_j}
\end{aligned}$$

Newtonian fluid: S_{ij} is a linear combination of $\frac{\partial v_i}{\partial x_j}$. $\vec{v} = \vec{\Omega}\times\vec{r} \implies S_{ij} = 0$.

$$S_{ij} = a\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) + b\delta_{ij}(\nabla\cdot\vec{v}).$$