JOURNAL OF COMPUTER AND SYSTEM SCIENCES 11, 1–55 (1975)

Proving Containment of Bounded AFL*

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Received August 11, 1972; revised April 2, 1974

A study is made of necessary conditions in order for a bounded language V to be in the smallest [full] (semi-)AFL generated by a bounded language U. Two major classes of bounded languages are introduced. Specific necessary conditions are then derived for the case when U and V are both in one of these classes. For a wide variety of pairs (U, V) at least one of the necessary conditions is violated. The net result of these conditions is thus the establishment of techniques for proving that V is not in the smallest [full] (semi-)AFL generated by U. A typical example given is that the language $\{a^{n^2}b^{n^2} \mid n \geqslant 0\}$ is not in the smallest full AFL generated by the language $\{a^{(n+1)^2}b^{n^2} \mid n \geqslant 0\}$.

1. INTRODUCTION

The notion of an AFL (abstract family of languages) was introduced in [5] as a unifying concept for the study of formal languages. This abstraction not only eliminated the duplication of proofs of similar properties in different families of languages, but also permitted the formulation (and solution) of many new questions about established families. In these studies of AFL made so far, one very important and constantly reoccurring problem is that of

(*) determining for two given AFL whether the second is a subfamily of the first.

This is the problem to which we address ourselves in this paper.

In the theory of context-free languages, the subfamily of bounded context-free languages plays an interesting role. Besides being a tool in the general theory of context-free languages, the bounded context-free languages have been quite tractable and have displayed many desirable properties [8, 9]. Recently, efforts have been initiated

* This work was supported in part by the National Science Foundation under grant number GJ-28787. The results contained in this paper appeared in the author's Ph.D. Thesis presented to the University of Southern California, 1972.

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to see if families of bounded languages can play a similar role in the development of AFL theory. (To date, there has been a modest amount of success [10, 11, 18].) Motivated by this recent activity and by the difficulty of treating (*) in general, we consider (*) only for the case when each AFL is generated by a bounded language. Thus we direct our investigation toward the problem of determining for two given AFL, each generated by a bounded language, whether the second is a subfamily of the first. Equivalently, we are concerned with

(**) determining for two given bounded languages U and V whether V is in $\mathscr{F}(U)$, $\mathscr{F}(U)$ being the smallest AFL containing U.

(Actually, we are also interested in three other cases, namely, whether V is in the smallest full AFL generated by U, and whether V is in the smallest (full) semi-AFL generated by U.) In considering (**), experience has indicated that it is usually a relatively easy task to verify (if so) that V is in $\mathscr{F}(U)$, but it is a comparatively difficult feat to prove (if so) that V is not in $\mathscr{F}(U)$. We therefore devote most of our energy to the latter aspect. Thus, while we do establish one sufficiency condition for V to be in $\mathscr{F}(U)$, most of our results are necessary conditions. We are able to show that at least one of these necessary conditions is violated for a variety of languages U and V. Our results consequently enable us to demonstrate that V is not in $\mathscr{F}(U)$ (thus $\mathscr{F}(V) \not\subseteq \mathscr{F}(U)$) for a number of different pairs (U, V). (Two of these pairs were already in the literature, having been justified by rather special methods.) The most surprising of these is the case when $U = \{a^{(n+1)^2}b^{n^2} \mid n \geqslant 0\} \cup \{\epsilon\}$ and $V = \{a^{n^2}b^{n^2} \mid n \geqslant 0\}$.

The paper itself is divided into six sections. Section 2 reviews some of the basic concepts involving families of languages and a-transducers.

Section 3 introduces the notion of a loop which, together with that of a computation of an a-transducer, is basic to our proofs. In addition, the section is concerned with proving six lemmas, each dealing with either the loop structure of a computation or the replacement of a computation by one in which the loops are regulated in some manner. None of these lemmas is of interest in itself. However, the second, the third, and the sixth play essential roles in subsequent sections.

Section 4 considers languages essentially of the form

(***) $\{a_1^{f_1(x)} \cdots a_n^{f_n(x)} \mid x \geqslant 1\}$,

where $n \geqslant 2$ and all the f_i are strictly increasing.

The first major result is a necessary and sufficient condition for a language V of the above form to be in a (full)[semi-] AFL generated by a bounded language U. In order to obtain some practical necessary conditions, the generator U of a (semi-)AFL is also restricted to the form (***). The second major result of this section (Theorem 4.2) concerns languages U and V, both of the form (***). It expresses a necessary condition in terms of an arbitrary function whose variables are the number of occurrences of letters in words of U and V. Three consequences of Theorem 4.2 are then derived, the second applying to a large class of languages U and V of the form (***), with all the f_i polynomials. Several examples of application of these consequences are given, one being the specific example mentioned earlier.

The necessary conditions given in Section 4 are simple and easy to apply. However, the class of languages to which they apply is, on occasion, too small. Sections 5 and 6 treat a more general class of languages, the family of distance (f, I)-guided and I-uniformly infinite languages. As would be expected, the results are weaker than those in Section 4. Section 5 is a collection of lemmas needed for Section 6. The key results (none of interest in itself) are Lemmas 5.4–5.7.

Section 6 presents several necessary conditions for a distance (f, I)-guided language V to be in a (full)[semi-] AFL generated by a bounded language U. The main results are Theorems 6.1 and 6.3. These, in turn, give rise to Theorems 6.2 and 6.4, which are of a more practical nature. The methods developed here allow us to verify, for example, that V is not in $\mathscr{F}(U)$, where

$$U = \{a^m b^n c^k \mid n \geqslant 1, m \geqslant n^2, k \geqslant n^3\} \text{ and } V = \{a^m b^n c^k \mid m \geqslant k^3, n \geqslant k^2, k \geqslant 1\}.$$

2. Preliminaries

In this section we review some of the basic concepts involving families of languages and a-transducers. The reader is referred to [5] for further background and motivation.

For each set Σ let Σ^* be the free semigroup with identity ϵ generated by Σ, i.e., the set of all finite strings $a_1 \cdots a_m$, $m \geqslant 0$, each a_i in Σ. (By definition, $a_1 \cdots a_m = \epsilon$ when $m = 0$.) The elements of Σ are called *letters* and the elements of Σ^* are called *words* (over the alphabet Σ).

A word z in Σ^* is a *subword* of a word w in Σ^* if $w = xzy$ for some x and y in Σ^*.

For each word $w = a_1 \cdots a_m$, $m \geqslant 0$, each a_i in Σ, let $| w |$ be the length of w, i.e., $| w | = m$.

If A is a finite set then the number of elements in it is denoted by $| A |$.

There should be no confusion between the two uses of the symbol $| \ |$.

A *language* (over Σ) is a set $L \subseteq \Sigma^*$ such that $L \subseteq \Sigma_0{}^*$ for some finite subset Σ_0 of Σ. If ϵ is not in L then L is said to be *ϵ-free*.

We shall be concerned with certain classes of languages.

Definition. *A family of languages* is a pair $(\Sigma, \mathscr{L})$, or $\mathscr{L}$ when Σ is understood, where

(a) Σ is an infinite set of symbols,

(b) each L in $\mathscr{L}$ is a language (over Σ), and

(c) $L \neq \varnothing$ for some L in $\mathscr{L}$.

Henceforth, Σ will always denote a given infinite set of symbols, and Σ with a subscript a finite subset of Σ. All languages considered are assumed to be over Σ.

In the sequel we shall consider families of languages which are closed with respect to certain operations on languages. We now note some of them.

For languages S and T, the *concatenation* (*product*) $S \cdot T$, usually written as ST, is the set $\{uv \mid u \text{ in } S, v \text{ in } T\}$.

For a language S, let $S^0 = \{\epsilon\}$, $S^{i+1} = S^i \cdot S$, $i \geqslant 0$, $S^+ = \bigcup_{i>0} S^i$, and $S^* = \bigcup_{i\geqslant 0} S^i$. Note that $S^* = S^+ \cup \{\epsilon\}$.

If $S = \{w\}$, w in Σ^*, then the symbols $\{$ and $\}$ are usually omitted. Thus wT, w^+, and w^* denote $\{w\} \cdot T$, $\{w\}^+$, and $\{w\}^*$ respectively (similarly for $T = \{w\}$).

A *homomorphism h* is a mapping h from A^* into B^* (A and B sets) such that $h(uv) = h(u)\, h(v)$ for all u and v in A^*; h is *ϵ-free* if $h(u) = \epsilon$ implies $u = \epsilon$.

If h is a homomorphism from A^* into B^*, then the mapping h^{-1} of subsets of B^* into subsets of A^* defined by $h^{-1}(Y) = \{x \mid h(x) \text{ in } Y\}$ for all $Y \subseteq B^*$ is called an *inverse homomorphism.*

A set is said to be *regular* if it is contained in the least class closed under union, concatenation, and * which contains each finite set of words.

We shall be interested in four types of families of languages, now given.

Definition. A *(full) semi-AFL* $\mathscr{L}$ is a family of languages closed under the operations of union, ϵ-free (arbitrary) homomorphism, inverse homomorphism, and intersection with regular sets. If in addition $\mathscr{L}$ is closed under concatenation and $+$, then it is called a *(full) AFL.*

Historically, (full) AFL preceded (full) semi-AFL.

The operations mentioned so far have been either set-theoretic or algebraic in nature. An operation obtained by a different method depends on the notion of "machine" now given.

Definition. A *sequential transducer with accepting states*, abbreviated *a-transducer*, is a 6-tuple $M = (K, \Sigma_1, \Sigma_2, H, q_0, F)$, where

(i) K, Σ_1, and Σ_2 are finite sets (of *states*, *inputs*, and *outputs*, respectively),

(ii) H is a finite subset of $K \times \Sigma_1{}^* \times \Sigma_2{}^* \times K$ (the *moves*),

(iii) q_0 is in K (the *start* state), and

(iv) $F \subseteq K$ (the set of *accepting* states).

If $H \subseteq K \times \Sigma_1{}^* \times \Sigma_2{}^+ \times K$, then M is called *ϵ-free.*

To define how an *a*-transducer acts as a transformation device we recall some notation first used by Goldstine [10].

Notation. Let $M = (K, \Sigma_1, \Sigma_2, H, q_0, F)$ be an *a*-transducer. (In the sequel, unless otherwise stated, the symbol M denotes the *a*-transducer $(K, \Sigma_1, \Sigma_2, H, q_0, F)$.)

For i in $\{0, 1, 2, 3\}$, let pr_i be the homomorphism on H^* defined by $pr_i(h) = x_i$ for each $h = (x_0, x_1, x_2, x_3)$ in H.

For each γ in H^*, $pr_1(\gamma)$ thus represents the *input* of γ, and $pr_2(\gamma)$ the *output*.

DEFINITION. A *computation* of M is a word of the form $h_1 \cdots h_m$, where $m \geqslant 1$, each h_i is in H, $pr_0(h_i) = pr_3(h_{i-1})$ for $1 < i \leqslant m$, $pr_0(h_1) = q_0$, and $pr_3(h_m)$ is in F. In addition, if q_0 is in F then the empty word ϵ is a computation of M. Let Π_M be the set of all computations of M.

DEFINITION. For each word w in $\Sigma_1{}^*$ let $M(w) = pr_2(pr_1^{-1}(w) \cap \Pi_M)$. For every $S \subseteq \Sigma_1{}^*$, let $M(S) = \bigcup_{w \text{ in } S} M(w)$. The mapping from subsets of $\Sigma_1{}^*$ into subsets of $\Sigma_2{}^*$ so defined is called an *a-transducer mapping.*

Thus an a-transducer is a finite-state device with accepting states, which moves under the input words and transforms them (by parts, nondeterministically) into the output words. The output is recorded only if the device lands at an accepting state after processing the entire input word. Note that Π_M is regular [3].

In connection with a-transducers we shall use the following symbolism.

Notation. For each family of languages $\mathscr{L}$ let

$$\mathscr{M}(\mathscr{L}) = \{M(L) \mid L \text{ in } \mathscr{L},\ M \text{ an } \epsilon\text{-free } a\text{-transducer}\} \text{ and}$$

$$\hat{\mathscr{M}}(\mathscr{L}) = \{M(L) \mid L \text{ in } \mathscr{L},\ M \text{ an a-transducer}\}.$$

We shall be especially concerned with the case where $\mathscr{L} = \{L\}$ for some $L \neq \varnothing$. It was shown in [6] that $\mathscr{M}(L)(\hat{\mathscr{M}}(L))$ is the smallest (full) semi-AFL containing L. We write $\mathscr{M}(L)$ and $\hat{\mathscr{M}}(L)$ instead of $\mathscr{M}(\{L\})$ and $\hat{\mathscr{M}}(\{L\})$, respectively.

We now introduce a similar symbolism for AFL.

Notation. For each set $\mathscr{L}$ of languages, let $\mathscr{F}(\mathscr{L})$ $[\hat{\mathscr{F}}(\mathscr{L})]$ be the smallest [full] AFL containing $\mathscr{L}$. We write $\mathscr{F}(L)$ and $\hat{\mathscr{F}}(L)$ instead of $\mathscr{F}(\{L\})$ and $\mathscr{F}(\{L\})$, respectively.

For each $\mathscr{L}$, $\mathscr{F}(\mathscr{L})$ and $\hat{\mathscr{F}}(\mathscr{L})$ exist.

As in the a-transducer case, we shall be interested in the case when $\mathscr{L}$ contains exactly one element. This leads to the following

DEFINITION. A (full)[semi-] AFL $\mathscr{L}$ is said to be (*full*) *principal* if there exists a language L, called a (*full*)[*semi-*] *AFL generator* of $\mathscr{L}$, such that

$$\mathscr{L} = \mathscr{F}(L)\ (\mathscr{L} = \hat{\mathscr{F}}(L))\ [\mathscr{L} = \mathscr{M}(L)\ (\mathscr{L} = \hat{\mathscr{M}}(L))].$$

Since their inception several years ago, AFL and semi-AFL have developed an extensive literature. One constantly recurring problem is to

(*) determine for two given (semi-) AFL whether the second is contained in the first.

This is the problem we are concerned with in this paper. Because of the difficulty of (*) in general, we wish to restrict the type of the (semi-) AFL chosen. This leads to the following concept.

DEFINITION. A language L is said to be *bounded* if $L \subseteq w_1{}^* \cdots w_n{}^*$ for some words $w_1, \ldots, w_n$ in Σ^*.

Because of their simply described structure, bounded languages are frequently used in formal language theory for example and counterexample purposes. This has been especially notable in the study of context-free languages. There are many signs which indicate that the family of bounded languages will play an analogous role in the study of AFL [1, 4, 5, 7, 10–13, 17]. One of the drawbacks in the development of AFL theory has been the difficulty in showing whether $\mathscr{M}(V)$ [$\mathscr{F}(V)$] is a subset of $\mathscr{M}(U)$ [$\mathscr{F}(U)$] for given languages U and V, equivalently, whether V is in $\mathscr{M}(U)$ [$\mathscr{F}(U)$]. This difficulty also exists when U and V are bounded languages. By experience, it is usually a relatively easy task to prove (if so) that V is in $\mathscr{M}(U)$ [$\mathscr{F}(U)$], but comparatively difficult to show (if so) that V is not in $\mathscr{M}(U)$ [$\mathscr{F}(U)$]. The main goal of this document is to establish methods for showing that V is not in $\mathscr{M}(U)$ [$\mathscr{F}(U)$, $\hat{\mathscr{M}}(U)$, or $\hat{\mathscr{F}}(U)$] for bounded languages U and V. The idea is to derive necessary conditions for V to be in $\mathscr{M}(U)$ [$\mathscr{F}(U)$, etc.] which can readily be shown to be violated for a variety of different pairs (U, V). The hope is that techniques and methods such as are developed here will ultimately reduce the labor involved in determining that

$$\mathscr{M}(V) \not\subseteq \mathscr{M}(U) \, [\mathscr{F}(V) \not\subseteq \mathscr{F}(U),$$

etc.].

For technical reasons it is convenient only to consider those bounded languages of the form $L \subseteq a_1{}^* \cdots a_n{}^*$, the a_i distinct letters of Σ and $n \geqslant 2$. It is not clear how the results translate for arbitrary bounded languages.

Henceforth, $a_1, \ldots, a_n$, respectively $b_1, \ldots, b_m$, are always assumed to be distinct symbols in Σ.

To illustrate the difficulties that may arise, consider the following:

EXAMPLE 2.1. Let $U = \{a_1{}^n a_2^{f(n)} \mid n \geqslant 0\}$, where

$$f(n) = \begin{cases} 2n & \text{if } \; n = m^2 \text{ for some integer } m, \\ 2n + 1 & \text{otherwise,} \end{cases}$$

and let $V = \{b_1^{n^2} b_2^{n^2} \mid n \geqslant 0\}$. It is easy to see that there is an (ϵ-free) a-transducer M such that $V = M(U)$. (For, M just copies each a_1 into b_1, each $a_2 a_2$ into b_2, and blocks whenever the number of occurrences of a_2 is odd.)

The above example shows that no relationship between the function specifying the number of a_1 (i.e., n in this case) and the function specifying the number of b_1 (i.e. n^2 in this case) has to exist. Thus the method involving the "rate of growth," used by Goldstine [10] for languages over one letter, is not suitable for languages over two or more letters.

We conclude the section with some simple results, occasionally used.

In order to simplify AFL arguments, it is often convenient to modify the form of a-transducers. Each such modification is usually accompanied by a statement asserting that the modified a-transducers give rise to the same class of a-transducer mappings as the original a-transducers.

It is easy to see that there is no loss of generality in the following

Assumption. Each a-transducer $M = (K, \Sigma_1, \Sigma_2, H, q_0, F)$ considered in the sequel has the property that if $(p, \epsilon, \epsilon, q)$ is in H then $p = q_0$ and $q \neq q_0$.

We conclude by stating some simple properties of (semi-) AFL.

PROPOSITION 2.1. *Let $\mathscr{L}$ be a [semi-]* AFL.

(a) The following three statements are equivalent.

 (i) $\{\epsilon\}$ is in $\mathscr{L}$.

 (ii) Some language in $\mathscr{L}$ contains ϵ.

 (iii) If L is in $\mathscr{L}$, then $L \cup \{\epsilon\}$ is in $\mathscr{L}$.

(b) $\{L - \{\epsilon\} \mid L \text{ in } \mathscr{L}\}$ and $\{L, L \cup \{\epsilon\} \mid L \text{ in } \mathscr{L}\}$ are both [semi-] AFL.

(c) $\mathscr{L}$ contains each ϵ-free regular set.

(d) $\mathscr{L}$ contains each regular set if and only if $\{\epsilon\}$ is in $\mathscr{L}$.

PROPOSITION 2.2. *Let U and V be infinite languages and let U' and V' be finite sets. Then*

(i) *V is in $\hat{\mathscr{M}}(U)$ $[\hat{\mathscr{F}}(U)]$ implies $V - V'$ is in $\hat{\mathscr{M}}(U - U')$ $[\hat{\mathscr{F}}(U - U')]$.*

(ii) *If $V - V'$ is ϵ-free and V is in $\mathscr{M}(U)$ $[\mathscr{F}(U)]$, then $V - V'$ is in*

$$\mathscr{M}(U - U')\ [\mathscr{F}(U - U')].$$

3. LOOPS AND $\langle a_1^{f_1(x)} \cdots a_n^{f_n(x)} \rangle$-LANGUAGES

In deriving necessary conditions on bounded languages U and V so that V is in $\mathscr{M}(U)$ $[\hat{\mathscr{M}}(U)]$ we shall examine the manner in which an a-transducer processes a word, i.e., we shall examine the computations. In this section we introduce the notion of a loop and then study the occurrence of loops in computations. We shall be especially interested in replacing one computation by an "equivalent" one, the loops in the latter being regulated in some manner. We present six lemmas, none of importance in itself. The second, third, and sixth lemmas are the key ones and will be used in later sections.

We start with the definition of a loop.

DEFINITION. Let $h_1, \ldots, h_n$, $n \geqslant 1$, be in H. The word $\gamma = h_1 \cdots h_n$ is said to be a *loop in* M if (i) $pr_3(h_n) = pr_0(h_1)$; (ii) $pr_3(h_i) = pr_0(h_{i+1})$ for $1 \leqslant i < n$; (iii) $pr_1(\gamma)$ is in a^* and $pr_2(\gamma)$ is in b^* for some a in Σ_1 and b in Σ_2; and (iv) $pr_0(h_i) \neq pr_0(h_j)$ for $i \neq j$, $1 \leqslant i, j \leqslant n$. (The explicit reference to an a-transducer M is usually omitted when M is understood.) A loop γ with input in a^+ is said to be an *a-loop*.

Thus $(p_1, a, a, p_2)(p_2, a, a, p_2)(p_2, a, a, p_1)$ and (p_1, ab, a, p_1), where a and b are symbols, are not loops, since they do not satisfy (iv) and (iii), respectively.

No proper subword of a loop in M can be a loop in M. Thus, for each a-transducer M, there is only a finite number of loops in M.

Note that by the assumption on a-transducers, made in Section 2, there are no ϵ input ϵ output loops in M.

The output v of a computation α is an image of the input u of α. Clearly there may be some other computation β, with $pr_1(\beta) = u$ and $pr_2(\beta) = v$, which is in some way simpler than α. From the point of view of a-transducer mappings we can disregard α provided we consider β. Since some computations may be unpleasantly complicated we shall concentrate on a "good" subclass of all computations, containing enough computations "to preserve the a-transducer mapping." One way to find such class is to show that each computation can be replaced by (or transformed into) a computation with a "simple loop structure" which has the same input and output. Thus, we are concerned with replacing one computation by another, subject to certain constraints. The constraints will always involve limitations on the nature of loops occurring in the computation.

Consider an arbitrary loop γ in an a-transducer M, with $|\gamma| = j > 1$. There are $j - 1$ other loops, each a subword of γ^k $(k > 1)$, each occurring $k - 1$ times in γ^k, and each consisting of the same j 4-tuples as γ. These j loops are very similar to each other and lead to the following concept.

DEFINITION. A loop γ_1 is said to be *associated* with a loop γ_2, written $\gamma_1 \sim \gamma_2$, if γ_1 is a subword of $\gamma_2\gamma_2$. Loop γ_1 is said to be *nonassociated* with a loop γ_2, written $\gamma_1 \not\sim \gamma_2$, if not $\gamma_1 \sim \gamma_2$.

Our first lemma characterizes when γ_1 is associated with γ_2.

LEMMA 3.1. *$\gamma_1 \sim \gamma_2$ if and only if there are words w_1 and w_2 in H^* such that $\gamma_1 = w_1 w_2$ and $\gamma_2 = w_2 w_1$.*

Proof. If $\gamma_1 = w_1 w_2$ and $\gamma_2 = w_2 w_1$, then $\gamma_2\gamma_2 = w_2 w_1 w_2 w_1 = w_2 \gamma_1 w_1$, so that $\gamma_1 \sim \gamma_2$.

Now let $\gamma_1 \sim \gamma_2$, i.e., γ_1 is a subword of $\gamma_2\gamma_2$. Let $\gamma_1 = y_1 \cdots y_m$ and $\gamma_2 = x_1 \cdots x_n$, with each x_i and y_i in H. If $n = 1$, i.e., $\gamma_2 = x_1$, then $\gamma_1 = x_1$ and lemma holds with $w_1 = x_1$ and $w_2 = \epsilon$. Suppose $n \geqslant 2$. Since γ_1 is a subword of $\gamma_2\gamma_2$, there exists i such that $y_1 = x_i$. If $i = 1$ then $m = n$ and $\gamma_1 = \gamma_2$. [For otherwise condition (iv) of the definition of loop is violated.] Thus we can take $w_1 = \gamma_1$, $w_2 = \epsilon$. Suppose $i > 1$. Then $m > n - i + 1$. (For otherwise γ_1 would be a proper subword of γ_2, violating condition (iv) of the definition of loop.) Thus we have the following situation. $y_1 = x_i$, $y_2 = x_{i+1}, \ldots, y_{n-i+1} = x_n, y_{n-i+2} = x_1, \ldots, y_m = x_{m-n+i-1}$, with $i > 1$ and

$$1 \leqslant m - n + i - 1 \leqslant n.$$

From (i) of the definition of loop, $pr_0(y_1) = pr_3(y_m)$. This in turn implies $pr_0(x_i) = pr_3(x_{m-n-i-1})$. If $m \neq n$, the last equality violates (iv) of the definition of loop. Thus $m = n$ and the lemma holds, with $w_2 = x_1 \cdots x_{i-1}$ and $w_1 = x_i \cdots x_n$.

COROLLARY. *$\sim$ is an equivalence relation on the set of loops in M.*

Thus we can read "$\gamma_1 \sim \gamma_2$" as "loops γ_1 and γ_2 are associated" and "$\gamma_1 \not\sim \gamma_2$" as "loops γ_1 and γ_2 are nonassociated."

In view of the similarity between associated loops it seems reasonable to try to replace each computation by one in which all associated loops occur in clusters. More precisely:

DEFINITION. A computation α is said to be in *standard form* if

$$\alpha = w_1\gamma_1^{k_1}w_2\gamma_2^{k_2} \cdots w_q\gamma_q^{k_q}w_{q+1},$$

where $\gamma_1, \ldots, \gamma_q$ are pairwise nonassociated loops, no w_i, $1 \leqslant i \leqslant q + 1$, contains a loop, and $k_j \geqslant 1$ for all j, $1 \leqslant j \leqslant q$.

Our second lemma, the first of the three key ones in this section, asserts that each computation can be replaced by an "equivalent" one in standard form.

LEMMA 3.2. *Let M be an a-transducer, $L \subseteq a_1^* \cdots a_n^*$, and $M(L) \subseteq b_1^* \cdots b_m^*$. Then for every computation α in $\Pi_M \cap pr_1^{-1}(L)$ there is a computation α' in standard form such that $pr_1(\alpha') = pr_1(\alpha)$ and $pr_2(\alpha') = pr_2(\alpha)$.*

Proof. If α does not contain a loop, then $\alpha = \alpha'$ satisfies the conclusions of the lemma. Suppose α contains a loop. Then $\alpha = w_1\gamma_1u_1$, where γ_1 is a loop and w_1 does not contain a loop. If u_1 does not contain a loop let $w_2 = u_1$ and $\alpha = \alpha' = w_1\gamma_1w_2$ has the desired form.

Suppose u_1 contains a loop $\gamma_1' \sim \gamma_1$. By Lemma 3.1, there exist words v_1 and v_2 in H^* such that $\gamma_1 = v_1v_2$ and $\gamma_1' = v_2v_1$. Thus $\alpha = w_1v_1v_2u_1'v_2v_1u_1''$ for some u_1' and u_1'' in H^*. Suppose γ_1 (hence γ_1') has non-ϵ input and output. By property (iii) of the definition of loop, $pr_1(\gamma_1) = pr_1(\gamma_1')$ is in a_i^+ for some i. From this and the fact that $L \subseteq a_1^* \cdots a_n^*$ it follows that $pr_1(u_1')$ is in a_i^*. Since $pr_2(\gamma_1) \neq \epsilon$, from the definition of loop it follows that $pr_2(\gamma_1) = pr_2(\gamma_1')$ is in b_j^+ for some j. Consequently $pr_2(u_1')$ is in b_j^*. Thus $pr_1(\gamma_1 u_1' \gamma_1') = pr_1(v_1 v_2 u_1' \gamma_1') = pr_1(v_1 \gamma_1' v_2 u_1') = pr_1(v_1 v_2 v_1 v_2 u_1') = pr_1(\gamma_1 \gamma_1 u_1')$. Similarly, $pr_2(\gamma_1 u_1' \gamma_1') = pr_2(\gamma_1 \gamma_1 u_1')$. Clearly $\alpha_1 = w_1 \gamma_1^2 u_1' u_1''$ is a computation and $pr_1(\alpha) = pr_1(\alpha_1)$ and $pr_2(\alpha) = pr_2(\alpha_1)$. (The procedure just described formalizes what will be referred to as *moving the loop* γ_1' *to its associate* γ_1.) The case when both the input and the output of γ_1 are ϵ is excluded by the assumption on M (following Example 2.1). For the case of γ_1 with ϵ-input or ϵ-output the loop γ_1' is "moved" in an analogous way (with simpler considerations).

Now suppose u_1 contains no loop $\gamma_1' \sim \gamma_1$ but does contain a loop γ_2, $\gamma_2 \not\sim \gamma_1$. Then $\alpha = \alpha_1 = w_1 \gamma_1 w_2 \gamma_2 u_2$, where w_1 and w_2 do not contain a loop and $\gamma_1 \not\sim \gamma_2$.

We continue by induction. Let $\alpha_r = w_1 \gamma_1^{l_1} w_2 \gamma_2^{l_2} \cdots w_s \gamma_s^{l_s} u$, γ_i pairwise nonassociated, and no w_i containing a loop. Let $\Delta_r = \sum_{i=1}^{s} l_i$. (Note that the l_i and s depend on r.) If u does not contain a loop, let $\alpha_{r+1} = \alpha_r$. Otherwise let p be the smallest integer such that u contains a loop γ, $\gamma \sim \gamma_p$. If no such p exists, let

$$\alpha_{r+1} = w_1 \gamma_1^{l_1} w_2 \gamma_2^{l_2} \cdots w_s \gamma_s^{l_s} w_{s+1} \gamma_{s+1} u',$$

where $w_{s+1} \gamma_{s+1} u' = u$ and w_{s+1} does not contain a loop. (Note that $\Delta_{r+1} = \Delta_r + 1$.) If p exists, then α_{r+1} is obtained from α_r by moving loop γ to γ_p. Thus

$$\alpha_{r+1} = w_1 \gamma_1^{l_1} w_2 \cdots w_p \gamma_p^{l_p+1} w_{p+1} \cdots \gamma_s^{l_s} u' u'',$$

where $u' \gamma u'' = u$. As at the beginning of the proof, $pr_1(\alpha_{r+1}) = pr_1(\alpha_r)$ and $pr_2(\alpha_{r+1}) = pr_2(\alpha_r)$. Again $\Delta_{r+1} = \Delta_r + 1$. Since $|\alpha_r| = |\alpha|$ for all r, and since $|\gamma_i| \geq 1$ (by the definition of loop) for all i, $\Delta_r \leq |\alpha|$ for all r. Thus there exists t such that $\alpha_t = \alpha_{t+1}$. Let $\alpha' = \alpha_t$. Then α' satisfies the properties of the lemma and the proof is complete.

A common method for proving "$M(U) \neq V$ for all M" is to show that "$V \subseteq M(U)$ implies $V \not\subseteq M(U)$", i.e., to show that if M outputs all of the words in V then it must also give some "superfluous" words (i.e., some words not in V). For this purpose we shall want to replace a computation α in $\Pi_M \cap pr_1^{-1}(L)$, which has two nonassociated a_i-loops, by a computation α' in $\Pi_M \cap pr_1^{-1}(L)$ satisfying (a) $pr_1(\alpha) = pr_1(\alpha')$, (b) the loops in α and α' coincide, and (c) the number of occurrences of one of the two specified a_i-loops in α' is "small." The next lemma, the second of our key ones, asserts that this replacement is always possible.

LEMMA 3.3. *Let M be an a-transducer, $L \subseteq a_1^* \cdots a_n^*$, and $M(L) \subseteq b_1^* \cdots b_m^*$. Then there exists an integer $k \geq 2$ with the following property. For every computation $\alpha = w_1 \gamma_1^{k_1} w_2 \gamma_2^{k_2} w_3$ in $\Pi_M \cap pr_1^{-1}(L)$, where* (a) *γ_1 and γ_2 are nonassociated a_i-loops for some i, $1 \leq i \leq n$,* (b) *$k_1 \geq 1$, and* (c) *$k_2 \geq 1$; there exist nonnegative integers q_1 and q_2 such that for $l_1 = k_1 - q_1 \cdot |pr_1(\gamma_2)|$, $l_2 = k_2 + q_1 \cdot |pr_1(\gamma_1)|$, $p_1 = k_1 + q \cdot |pr_1(\gamma_2)|$, and $p_2 = k_2 - q \cdot |pr_1(\gamma_1)|$, the following holds.*

(i) $0 < l_1 < k$ *and* $0 < p_2 < k$,

(ii) $pr_1(w_1\gamma_1^{l_1}w_2\gamma_2^{l_2}w_3) = pr_1(w_1\gamma_1^{p_1}w_2\gamma_2^{p_2}w_3) = pr_1(\alpha)$.

Proof. Let $k = \max\{|pr_1(\gamma)| + 1, 2 \mid \gamma \text{ a loop in } M\}$. As noted earlier, there are only finitely many loops in M. Thus k exists. Clearly $k \geqslant 2$. Consider the proof for q_1; an analogous proof holds for q_2. Let $\alpha = w_1\gamma_1^{k_1}w_2\gamma_2^{k_2}w_3$. If $k_1 < k$, $q_1 = 0$ satisfies the lemma. Suppose $k_1 \geqslant k$. Let q_1 be the smallest nonnegative integer such that $0 < k_1 - q_1 \mid pr_1(\gamma_2)| < k$. (By the definition of k, such an integer q_1 clearly exists.) Deleting $q_1 \cdot |pr_1(\gamma_2)|$ of γ_1 loops and inserting $q_1 \cdot |pr_1(\gamma_1)|$ of γ_2 loops results in a computation $\alpha' = w_1\gamma_1^{l_1}w_2\gamma_2^{l_2}w_3$, satisfying (i). Since $L \subseteq a_1^* \cdots a_n^*$, with all a_j distinct, and both γ_1 and γ_2 are a_i-loops, $pr_1(w_2)$ is in a_i^*. From

$$|pr_1(\gamma_1^{l_1}w_2\gamma_2^{l_2})| = |pr_1(\gamma_1^{k_1}w_2\gamma_2^{k_2})|$$

it follows that $pr_1(\gamma_1^{l_1}w_2\gamma_2^{l_2}) = pr_1(\gamma_1^{k_1}w_2\gamma_2^{k_2})$. Thus $pr_1(\alpha') = pr_1(\alpha)$, so that (ii) is satisfied.

The computation α' in Lemma 3.3 was such that $pr_1(\alpha') = pr_1(\alpha)$. It could happen that $pr_2(\alpha') \neq pr_2(\alpha)$. From the informal discussion above one can guess that this is an unwanted phenomenon. In what follows some restrictions are placed on the output language $M(L)$ in order to overcome this problem. That enables us to continue the "replacement policy" on computations. The third of our three key lemmas, Lemma 3.6. asserts that under certain conditions α may be replaced by α', where $pr_1(\alpha') = pr_1(\alpha)$, $pr_2(\alpha') = pr_2(\alpha)$, and for each i, all but one of the a_i-loops occur just a small number of times.

Lemma 3.6 requires us to establish two preliminary results on the nature of loops occurring in a computation in the case of restricted output language $M(L)$.

Notation. Let $n \geqslant 2$ and let f_i be a strictly increasing[1] function from N into N for all i, $1 \leqslant i \leqslant n$. ($N$ is the set of nonnegative integers.) We write

$$\langle a_1^{f_1(x)} \cdots a_n^{f_n(x)} \rangle$$

for the set $\{a_1^{f_1(x)} \cdots a_n^{f_n(x)} \mid x \text{ in } N\}$.

The first preliminary lemma asserts that when $M(L)$ is of the form

$$\langle b_1^{g_1(x)} \cdots b_m^{g_m(x)} \rangle,$$

all loops occurring in a computation in $\Pi_M \cap pr_1^{-1}(L)$ are a_i-loops for some a_i.

[1] A function f is said to be *strictly increasing* if $x_1 < x_2$ implies $f(x_1) < f(x_2)$.

LEMMA 3.4. *Let M be an a-transducer and $L \subseteq a_1{}^* \cdots a_n{}^*$. Let*

$$M(L) = \langle b_1^{g_1(x)} \cdots b_m^{g_m(x)} \rangle.$$

Then for every loop γ occurring in a computation in $\Pi_M \cap pr_1^{-1}(L)$, $pr_1(\gamma) \neq \epsilon$.

Proof. Suppose the contrary. By the assumption on M, there are no ϵ input ϵ output loops in M (following Example 2.1). Thus there exist α in $\Pi_M \cap pr_1^{-1}(L)$ and a loop γ such that $\alpha = v_1 \gamma v_2$, with $v_1 v_2$ in H^*, $pr_1(\gamma) = \epsilon$, and $pr_2(\gamma) \neq \epsilon$. Clearly $\alpha' = v_1 \gamma\gamma v_2$ is in $\Pi_M \cap pr_1^{-1}(L)$, since γ is a loop and $pr_1(\alpha') = pr_1(\alpha)$. By the definition of loop, $pr_2(\gamma) = b_j{}^l$ for some j, $1 \leqslant j \leqslant m$, and some $l \geqslant 1$. If $pr_2(\alpha) = b_1^{k_1} \cdots b_m^{k_m}$, then $pr_2(\alpha') = b_1^{k_1} \cdots b_j^{k_j+l} \cdots b_m^{k_m}$. Since each g_i is strictly increasing, $m \geqslant 2$, and $k_j + l > k_j$, $pr_2(\alpha')$ is in $M(L)$ but not in $\langle b_1^{g_1(x)} \cdots b_m^{g_m(x)} \rangle$, a contradiction.

COROLLARY. *For M and L as in Lemma 3.4, $M(w)$ is finite for each w in L.*

The second preliminary lemma shows that M has actually very little choice in its selection of loops for a computation in standard form. For each i, $1 \leqslant i \leqslant n$, all a_i-loops must have (i) the output in the same $b_j{}^*$, and (ii) the same output–input ratio.

LEMMA 3.5. *Let M be an a-transducer and $L \subseteq a_1{}^* \cdots a_n{}^*$ such that*

$$M(L) = \langle b_1^{g_1(x)} \cdots b_m^{g_m(x)} \rangle.$$

Let $k \geqslant 2$ be the integer guaranteed by Lemma 3.3. Let $\alpha = v_1 \gamma_1^{h_1} \cdots v_p \gamma_p^{h_p} v_{p+1}$ in $\Pi_M \cap pr_1^{-1}(L)$ be a computation in standard form. Then for every s and t, $1 \leqslant s, t \leqslant p$, such that γ_s and γ_t are both a_i-loops for some i, $1 \leqslant i \leqslant n$, and $h_s \geqslant k$ the following hold.

(i) *$pr_2(\gamma_s)$ and $pr_2(\gamma_t)$ are both in $b_j{}^*$ for some j, $1 \leqslant j \leqslant m$; and*

(ii) $|pr_2(\gamma_s)|/|pr_1(\gamma_s)| = |pr_2(\gamma_t)|/|pr_1(\gamma_t)|$.

(Note that $|pr_1(\gamma_s)| \neq 0$ and $|pr_1(\gamma_t)| \neq 0$, since γ_s and γ_t are a_i-loops.)

Proof. By the definition of loop there exist j and l, $1 \leqslant j, l \leqslant m$, such that $pr_2(\gamma_s)$ is in $b_j{}^*$ and $pr_2(\gamma_t)$ is in $b_l{}^*$. If $s = t$, the lemma holds trivially. Suppose $s \neq t$. Either $s < t$ or $s > t$. We shall assume the former, an analogous argument holding if the latter. To simplify the notation we write γ, γ', k_1, and k_2 for γ_s, γ_t, h_s, and h_t respectively. Then for appropriate w_1, w_2, and w_3, $\alpha = w_1 \gamma^{k_1} w_2 \gamma'^{k_2} w_3$, with $k_1 = h_s \geqslant k$ and $k_2 = h_t \geqslant 1$.

By Lemma 3.3, there exists a nonnegative integer q_1 and consequently positive integers l_1 and l_2 such that (i) $0 < l_1 < k$, and (ii) $pr_1(\alpha') = pr_1(\alpha)$, where $\alpha' = w_1 \gamma^{l_1} w_2 \gamma'^{l_2} w_3$. Since $k_1 \geqslant k$, $q_1 > 0$ and consequently $l_2 > k_2$. Note that $pr_2(\alpha')$ is in $M(L)$. Four cases arise:

(α) $pr_2(\gamma) = \epsilon$ and $pr_2(\gamma') = \epsilon$. Clearly (i) and (ii) are satisfied.

(β) $pr_2(\gamma) = \epsilon$ and $pr_2(\gamma') \neq \epsilon$. Then $pr_2(\gamma') = b_l{}^q$ for some $q \geqslant 1$. Since $pr_2(\alpha)$ is in $M(L)$, $pr_2(\alpha) = b_1^{z_1} \cdots b_m^{z_m}$ for some $z_1, \ldots, z_m$. By assumption $pr_2(\gamma) = \epsilon$. Thus $pr_2(\alpha') = pr_2(w_1\gamma^{l_1}w_2\gamma'^{l_2}w_3) = pr_2(w_1w_2\gamma'^{l_2}w_3)$. Since $pr_2(\gamma') = b_l{}^q$ and $l_2 > k_2$, $pr_2(\alpha') = pr_2(w_1w_2\gamma'^{k_2}\gamma'^{l_2-k_2}w_3) = pr_2(w_1w_2\gamma'^{k_2})pr_2(\gamma'^{l_2-k_2})pr_2(w_3) = b_1^{z_1} \cdots b_l^{z_l}b_l^{q\cdot(l_2-k_2)}b_{l+1}^{z_{l+1}} \cdots b_m^{z_m}$. Then $b_1^{z_1} \cdots b_m^{z_m}$ and $b_1^{z_1} \cdots b_l^{z_l+q\cdot(l_2-k_2)} \cdots b_m^{z_m}$ are both in $M(L)$. Since $l_2 - k_2 > 0$ and $q \geqslant 1$ this contradicts the fact that the g_i are strictly increasing. Hence this case cannot occur.

(γ) $pr_2(\gamma) \neq \epsilon$ and $pr_2(\gamma') = \epsilon$. By an argument analogous to that in (β), this case cannot occur.

(δ) $pr_2(\gamma) \neq \epsilon$ and $pr_2(\gamma') \neq \epsilon$. Thus $pr_2(\gamma) = b_j^r$ and $pr(\gamma') = b_l^q$ for some $q, r \geqslant 1$. Let $pr_2(\alpha) = b_1^{z_1} \cdots b_m^{z_m}$.

Consider (i). Suppose $j \neq l$. Then $j < l$, since γ occurs to the left of γ'. Then

$$pr_2(\alpha') = pr_2(w_1\gamma^{l_1}w_2\gamma'^{l_2}w_3) = pr_2(w_1\gamma^{k_1-(k_1-l_1)}w_2\gamma'^{k_2+(l_2-k_2)}w_3)$$

$$= b_1^{z_1} \cdots b_j^{z_j-(k_1-l_1)r} \cdots b_l^{z_l+(l_2-k_2)q} \cdots b_m^{z_m}.$$

This is a contradiction, since all g_i are strictly increasing, $pr_2(\alpha)$ and $pr_2(\alpha')$ are in $M(L)$, $l_2 - k_2 > 0$, $k_1 - l_1 > 0$, $q > 0$, and $r > 0$. Hence $j = l$ and (i) holds.

Consider (ii). Let $\bar{\alpha}$ be the computation obtained from α by deleting $|pr_1(\gamma')|$ occurrences of γ and inserting $|pr_1(\gamma)|$ occurrences of γ', i.e.

$$\bar{\alpha} = w_1\gamma^{k_1-|pr_1(\gamma')|}w_2\gamma'^{k_2+|pr_1(\gamma)|}w_3.$$

(Note that $k_1 - |pr_1(\gamma')| \geqslant 0$ since $k_1 \geqslant k$.) Because of (i), $pr_2(w_2)$ is in $b_j{}^*$. Then $pr_2(\bar{\alpha}) = b_1^{z_1} \cdots b_j^{z_j+Q} \cdots b_m^{z_m}$, where $Q = |pr_1(\gamma)|\,|pr_2(\gamma')| - |pr_1(\gamma')|\,|pr_2(\gamma)|$. Since γ and γ' are a_i-loops and $L \subseteq a_1{}^* \cdots a_n{}^*$, $pr_1(w_2)$ is in $a_i{}^*$. Clearly $pr_1(\alpha) = pr_1(\bar{\alpha})$. Hence $pr_2(\bar{\alpha})$ is in $M(L)$. Now all g_i are strictly increasing. Thus, if $Q \neq 0$, then $pr_2(\bar{\alpha})$ is not in $\langle b_1^{g_1(x)} \cdots b_m^{g_m(x)} \rangle$, a contradiction. Hence $Q = 0$, i.e.,

$$|pr_1(\gamma)|\,|pr_2(\gamma')| - |pr_1(\gamma')|\,|pr_2(\gamma)| = 0.$$

By Lemma 3.4, $|pr_1(\gamma)| \neq 0$ and $|pr_1(\gamma')| \neq 0$. Thus (ii) holds.

We are now ready for the third key lemma.

LEMMA 3.6. *Let M be an a-transducer, $L \subseteq a_1{}^* \cdots a_n{}^*$, and $M(L) = \langle b_1^{g_1(x)} \cdots b_m^{g_m(x)} \rangle$. Then there exists an integer $\bar{k}$ with the following property. For each α in $\Pi_M \cap pr_1^{-1}(L)$*

there exists a computation $\alpha' = v_1\gamma_1^{k_1} \cdots v_n\gamma_n^{k_n}v_{n+1}$ *in* $\Pi_M \cap pr_1^{-1}(L)$, *all* $k_i \geqslant 0$, *such that*

(i) $pr_1(\alpha') = pr_1(\alpha)$ *and* $pr_2(\alpha') = pr_2(\alpha)$;

(ii) γ_i *is an* a_i*-loop for all* i, $1 \leqslant i \leqslant n$; *and*

(iii) *for each loop* γ *in* M *and each* i, $1 \leqslant i \leqslant n+1$, v_i *contains at most* $\bar{k} - 1$ *occurrences of* γ.

Proof. By Lemma 3.2, we may assume that α is in standard form,

$$\alpha = w_1\gamma_1'^{h_1} \cdots w_r\gamma_r'^{h_r}w_{r+1}\,.$$

By Lemma 3.4, all γ_i', $1 \leqslant i \leqslant r$, have non-$\epsilon$ input. Let R_α be the set of all i such that γ_j' is an a_i-loop for some j, $1 \leqslant j \leqslant r$. For each i in R_α let $j(i)$ be the smallest integer such that $\gamma'_{j(i)}$ is an a_i-loop. Furthermore, define $j(i+1) = j(i)$ for i not in R_α and $j(n+1) = r+1$. Let $Q_\alpha = \{j(i) | \, i \text{ in } R_\alpha\}$. Let $Q_\alpha' = \{j \text{ not in } Q_\alpha \mid h_j \geqslant k\}$, where k is an integer guaranteed by Lemma 3.3. Let $s_\alpha = |\, Q_\alpha' \,|$. For each i in R_α let $\gamma_i = \gamma'_{j(i)}$. For each i in $R_\alpha' = \{1,..., n\} - R_\alpha$ let γ_i be an arbitrary a_i-loop in M. (We can add a new state p and for each i, $1 \leqslant i \leqslant n$, a 4-tuple (p, a_i, b_1, p) to M, to guarantee the existence of at least one a_i-loop.) Thus γ_i is defined for each i, $1 \leqslant i \leqslant n$. Let $\bar{k} = k + 2k'$, where k' is the number of loops in M.

The proof is by induction on s_α. Suppose $s_\alpha = 0$. Let $k_i = h_{j(i)}$ for i in R_α and $k_i = 0$ for i in R_α'. Clearly α has the form $\alpha = v_1\gamma_1^{k_1} \cdots v_n\gamma_n^{k_n}v_{n+1}$ and $\alpha' = \alpha$ satisfies (i) and (ii). Consider (iii). We can obviously choose $v_1 = w_1$, hence v_1 does not contain a loop. For each i, $1 < i \leqslant n+1$, let $t(i) = j(i-1) + 1$ and $t'(i) = j(i) - t(i)$. Now for each i, with $i - 1$ in R_α, define $v_i = w_{t(i)}$ if $t'(i) = 0$ and

$$v_i = w_{t(i)}\gamma'^{h_{t(i)}}_{t(i)} \cdots \gamma'^{h_{t(i)+t'(i)-1}}_{t(i)+t'(i)-1} w_{t(i)+t'(i)}$$

if $t'(i) > 0$. Define $v_i = \epsilon$, otherwise. Let γ be a loop in M. Suppose γ occurs in v_i for some i. Since no w_j contains a loop, there are just three possibilities: (a) γ occurs in $\gamma_j'^{h_j}$ for some j, $t(i) \leqslant j \leqslant t(i) + t'(i) - 1$; (b) γ consists of a terminal[2] subword of some w_j followed by an initial subword of γ_j'; or (c) γ consists of a terminal subword of some γ_j' followed by an initial subword of w_{j-1}. Since $s_\alpha = 0$, $h_j < k$ for $t(i) \leqslant j \leqslant t(i) + t'(i) - 1$. Clearly γ cannot occur as a subword of $\gamma_j'^{h_j}$ and $\gamma_l'^{h_l}$ for $j \neq l$, because α is in standard form. Thus at most $k - 1$ occurrences of γ result in case (a). It is easy to see that at most $t'(i)$ occurrences of γ result in each of the cases (b) and (c). Hence

[2] A word z in Σ^* is an *initial* (*terminal*) *subword* of a word w in Σ^* if $w = zx(w = xz)$ for some x in Σ^*.

there are at most $z = k - 1 + 2t'(\mathrm{i})$ occurrences of γ in v_i. Since $t'(\mathrm{i}) \leqslant r \leqslant k'$, $z < k + 2k' = \bar{k}$ and (iii) holds.

Continuing by induction suppose the lemma holds for all computations α satisfying $s_\alpha \leqslant s$. Let $s_\alpha = s + 1$. Let j be in Q_α' and let γ_j' be an a_i-loop. Hence $\alpha = u_1\gamma_i u_2 \gamma_j'^{h_j} u_3$ for some u_1, u_2, and u_3. By Lemma 3.3, there exists $\bar{\alpha}$ in $\Pi_M \cap pr_1^{-1}(L)$ such that (1) $\bar{\alpha} = u_1\gamma_i^{p_1}u_2\gamma_j'^{p_2}u_s$, $p_1 > 1$, $0 < p_2 < k$, and (2) $pr_1(\bar{\alpha}) = pr_1(\alpha)$. Clearly

$$s_{\bar{\alpha}} = s_\alpha - 1 = s.$$

It thus suffices to prove that $pr_2(\bar{\alpha}) = pr_2(\alpha)$.

By Lemma 3.3, $p_1 = 1 + q \cdot |pr_1(\gamma_j')|$ and $p_2 = h_j - q \cdot |pr_1(\gamma_i)|$, where q is the smallest integer such that $0 < h_j - q \mid pr_1(\gamma_i)| < k$. Suppose $pr_2(\gamma_i) = pr_2(\gamma_j') = \epsilon$. Then $pr_2(\bar{\alpha}) = pr_2(u_1\gamma_i^{p_1}u_2\gamma_j'^{p_2}u_3) = pr_2(u_1u_2u_3) = pr_2(u_1\gamma_i u_2\gamma_j'^{h_j}u_3) = pr_2(\alpha)$. Suppose $pr_2(\gamma_i) \neq \epsilon$ or $pr_2(\gamma_j') \neq \epsilon$. By (ii) of Lemma 3.5, $pr_2(\gamma_i) \neq \epsilon$ and $pr_2(\gamma_j') \neq \epsilon$. By (i) of Lemma 3.5, $pr_2(\gamma_i) = b_l^{z_1}$ and $pr_2(\gamma_j') = b_l^{z_2}$ for some l, $1 \leqslant l \leqslant m$, $z_1 \geqslant 1$, and $z_2 \geqslant 1$. Thus $pr_2(u_2) = b_l^{z_3}$ for some $z_3 \geqslant 0$. Then

$$pr_2(\bar{\alpha}) = pr_2(u_1)\, pr_2(\gamma_i^{p_1})\, pr_2(u_2)\, pr_2(\gamma_j'^{p_2})\, pr_2(u_3) = pr_2(u_1)\, b_l^{z_1p_1+z_2p_2+z_3} pr_2(u_3).$$

Similarly $pr_2(\alpha) = pr_2(u_1)\, b_l^{z_1+z_2h_j+z_3} pr_2(u_3)$. Let $y_1 = |pr_1(\gamma_i)|$ and $y_2 = |pr_1(\gamma_j')|$. By (ii) of Lemma 3.5, $z_1/y_1 = z_2/y_2$. Then

$$\begin{aligned} p_1z_1 + p_2z_2 &= (1 + qy_2)\, z_1 + (h_j - qy_1)\, z_2 \\ &= z_1 + h_jz_2 + q(y_2z_1 - y_1z_2) = z_1 + h_jz_2 . \end{aligned}$$

Hence $pr_2(\bar{\alpha}) = pr_2(\alpha)$ and the induction is extended.

4. Necessary Conditions on $\langle a_1^{f_1(x)} \cdots a_n^{f_n(x)}\rangle$-Languages

In the present section we use the properties of computations, established in Section 3, to prove a necessary and sufficient condition on a language

$$V = \langle b_1^{g_1(x)} \cdots b_m^{g_m(x)}\rangle$$

in order to be in $\mathscr{M}(U)$, $\mathscr{F}(U)$, $\hat{\mathscr{M}}(U)$, or $\hat{\mathscr{F}}(U)$ for some given bounded language U (Theorem 4.1). We then derive several necessary conditions which are consequences of Theorem 4.1 One of these consequences is applicable to a large class of polynomials.

First consider the following property of $\langle a_1^{f_1(x)} \cdots a_n^{f_n(x)}\rangle$-languages.

Lemma 4.1. *Let $L_1 = \langle b_1^{g_1(x)} \cdots b_m^{g_m(x)}\rangle$ and let L_2 be a nonempty language. Then L_1 is in $\mathscr{F}(L_2)$ $[\hat{\mathscr{F}}(L_2)]$ if and only if L_1 is in $\mathscr{M}(L_2)$ $[\hat{\mathscr{M}}(L_2)]$.*

Proof (cf. Example 3.1 in [14]). Since $\mathscr{M}(L_2) \subseteq \mathscr{F}(L_2)$ $[\hat{\mathscr{M}}(L_2) \subseteq \hat{\mathscr{F}}(L_2)]$, it suffices to consider the "only if." Let L_1 be in $\mathscr{F}(L_2)$ $[\hat{\mathscr{F}}(L_2)]$. As proved in [5], L_1 is expressible as a regular expression of languages in $\mathscr{M}(L_2)$ $[\hat{\mathscr{M}}(L_2)]$. It is easy to see that L_1 cannot contain a subset uv^*w for $v \neq \epsilon$ or a subset UV for both U and V infinite. Thus the regular expression for L_1 cannot involve * and in any subexpression $V_1 \cdots V_k$, at most one V_i is infinite. Hence, distributing concatenation over union, L_1 must be a finite union of languages in $\mathscr{M}(L_2)$ $[\hat{\mathscr{M}}(L_2)]$. Since $\mathscr{M}(L_2)$ $[\hat{\mathscr{M}}(L_2)]$ is closed under union, L_1 is in $\mathscr{M}(L_2)$ $[\hat{\mathscr{M}}(L_2)]$ and the proof is complete.

To express our necessary and sufficient condition we need the following

Notation. Let $\psi_{\langle a_1,\ldots,a_n\rangle}$ be the mapping from $a_1^* \cdots a_n^*$ into N^n defined by $\psi_{\langle a_1,\ldots,a_n\rangle}(z) = (\#_{a_1}(z),\ldots, \#_{a_n}(z))$, where $\#_{a_i}(z)$ is the number of occurrences of a_i in z. (N^n is the set of all n-tuples of nonnegative integers.)

Notation. Let $c = (c_1,\ldots, c_n)$ and $l = (l_1,\ldots, l_n)$ be in N^n and let $L \subseteq a_1^* \cdots a_n^*$. Let

$$\mathscr{K}(L, c, l) = \{(k_1,\ldots, k_n) \mid a_1^{c_1+k_1l_1} \cdots a_n^{c_n+k_nl_n} \text{ in } L,\ k_i \geqslant 0 \text{ for each } i\}.$$

THEOREM 4.1. *Let* $U \subseteq a_1^* \cdots a_n^*$ *and* $V = \langle b_1^{g_1(x)} \cdots b_m^{g_m(x)}\rangle$[*with* ϵ *in* U *if* ϵ *is in* V]. *Then* V *is in* $\hat{\mathscr{F}}(U)$ $[\mathscr{F}(U)]$ *if and only if*:

(*) $V = \bigcup_{i=1}^{q} V_i$ *for some* $q \geqslant 1$, *each* V_i *of the form*

$$\psi^{-1}_{\langle b_1,\ldots,b_m\rangle}\left[\left\{\left(d_{i1} + \sum_{j=z_{i1}}^{z_{i2}-1} k_jp_{ij},\ldots, d_{im} + \sum_{j=z_{im}}^{z_{i(m+1)}-1} k_jp_{ij}\right) \,\middle|\, (k_1,\ldots, k_n) \textit{ is in } \mathscr{K}(U, c_i, l_i)\right\}\right],$$

where c_i, $l_i = (l_{i1},\ldots, l_{in})$, $(p_{i1},\ldots, p_{in})$ *are in* N^n, *all* $l_{ij} > 0$, [*all* $p_{ij} > 0$], $(d_{i1},\ldots, d_{im})$ *in* N^m, *and* $(z_{i1},\ldots, z_{i(m+1)})$ *in* N^{m+1}, *with* $1 = z_{i1} < \cdots < z_{i(m+1)} = n + 1$.

Proof. We shall only discuss the proof for $\mathscr{F}(U)$, the other case being analogous.

Suppose (*) holds. It suffices to show that $V = M(U)$ for some ϵ-output bounded a-transducer[3] M. Such an a-transducer can be constructed in the following way. For each i, $1 \leqslant i \leqslant q$, let

$$U_i = \{a_1^{c_{i1}+k_1l_{i1}} \cdots a_n^{c_{in}+k_nl_{in}} \text{ in } U \mid (k_1,\ldots, k_n) \text{ in } N^n\},$$

[3] An a-transducer M is said to be *ϵ-output bounded* if there exists k in N such that $r \leqslant k$ for every word $h_1 \cdots h_r$, each h_i in H, satisfying $pr_2(h_1 \cdots h_r) = \epsilon$ and $pr_0(h_{i+1}) = pr_3(h_i)$, $1 \leqslant i < r$. It was shown in [3] that $M(L) - \{\epsilon\}$ is in $\mathscr{M}(L)$ for every ϵ-output bounded a-transducer M. Thus $V - \{\epsilon\} = M(U) - \{\epsilon\}$ is in $\mathscr{M}(U)$. If ϵ is not in V, $V - \{\epsilon\} = V$. Suppose ϵ is in V. By the assumption ϵ is in U and by (a) of Proposition 2.1, it follows that $(V - \{\epsilon\}) \cup \{\epsilon\} = V$ is in $\mathscr{M}(U)$.

where $(c_{i1}, \ldots, c_{in}) = c_i$. Let s_i be a state corresponding to U_i for each i. Beginning in start state q_0, M nondeterministically goes to one of the s_i via 4-tuple $(q_0, \epsilon, \epsilon, s_i)$. At s_i, M checks whether the input word is of the form

$$a_1^{c_{i1}+k_1 l_{i1}} \cdots a_n^{c_{in}+k_n l_{in}}$$

and maps a_h's onto appropriate b_j's using the following 4-tuples:

(a) $(s_i, a_1^{c_{i1}}, b_1^{d_{i1}}, s_{i1})$;

(b) $(s_{ij}, a_{j+1}^{c_{i(j+1)}}, b_l^r, s_{i(j+1)})$ for $1 \leqslant j < n$, where l is the unique integer such that $z_{il} \leqslant j+1 < z_{i(l+1)}$, $r = d_{il}$ if $j+1 = z_{il}$ and $r = 0$ otherwise; and s_{ij} are new states; and

(c) $(s_{ij}, a_j^{l_{ij}}, b_l^{p_{ij}}, s_{ij})$ for $1 \leqslant j \leqslant n$, where l is the unique integer such that $z_{il} \leqslant j < z_{i(l+1)}$.

Clearly M is ϵ-output bounded if each $p_{ij} > 0$. Note that once in state s_i, M blocks on each word in U which is not in U_i. Letting $\{s_{in} \mid 1 \leqslant i \leqslant q\}$ be the set of accepting states of M, it clearly follows that $M(U_i) = V_i$ and $M(U) = V$.

Suppose V is in $\mathscr{F}(U)$. By Lemma 4.1, V is in $\mathscr{M}(U)$. Then there exists an ϵ-free a-transducer M such that $V = M(U)$. By Lemma 3.4, there are no ϵ-input loops in $\Pi_M \cap pr_1^{-1}(U)$. For each i, $1 \leqslant i \leqslant n$, let $Q_i = \{\gamma \mid \gamma \text{ an } a_i\text{-loop in } M\}$. (As in Lemma 3.6, we may assume that at least one a_i-loop exists for each i.) Since the number of loops is finite, each Q_i is finite. Let $\bar{k}$ be the integer guaranteed by Lemma 3.6. Let W be the set of all subwords w of words in Π_M such that (1) w contains at most $\bar{k} - 1$ occurrences of each loop in M, (2) $pr_1(w)$ is in $a_1^* \cdots a_n^*$, and (3) $pr_2(w)$ is in $b_1^* \cdots b_m^*$. Clearly W is finite. Let e be a new symbol. Consider $Q = WeQ_1eWeQ_2e \cdots eWeQ_neW$. Since the Q_i and W are finite, Q is finite. For each $\alpha = w_{\alpha 1}e\gamma_{\alpha 1}ew_{\alpha 2} \cdots w_{\alpha n}e\gamma_{\alpha n}ew_{\alpha(n+1)}$ in Q, let

$$Z_\alpha = \{w_{\alpha 1}\gamma_{\alpha 1}^{k_1}w_{\alpha 2} \cdots w_{\alpha n}\gamma_{\alpha n}^{k_n}w_{\alpha(n+1)} \mid (k_1, \ldots, k_n) \text{ in } N^n\} \cap pr_1^{-1}(U) \cap \Pi_M .$$

By Lemma 3.6, for every computation β in $\Pi_M \cap pr_1^{-1}(U)$ there exists a computation β' such that β' is in Z_α for some α in Q, $pr_1(\beta) = pr_1(\beta')$, and $pr_2(\beta) = pr_2(\beta')$. Thus $V = \bigcup_{\alpha \text{ in } Q} pr_2(Z_\alpha)$.

Since V is infinite, there exists α_0 in Q with $pr_2(Z_{\alpha_0})$ infinite. Thus

$$\{\#b_{i_0}(w) \mid w \text{ in } pr_2(Z_{\alpha_0})\}$$

is infinite for some i_0. Since $pr_2(Z_{\alpha_0}) \subseteq V$ and $V = \langle b_1^{g_1(x)} \cdots b_m^{g_m(x)} \rangle$, the set $\{\# b_i(w) \mid w$ in $pr_2(Z_{\alpha_0})\}$ is infinite for each i, $1 \leqslant i \leqslant m$. Thus for each i, $1 \leqslant i \leqslant m$, there exists j, $1 \leqslant j \leqslant n$, such that $pr_2(\gamma_{\alpha_0 j})$ is in b_i^+. Hence $m \leqslant n$.

Let α be an arbitrary element in Q. Two cases occur:

(i) Suppose $pr_2(Z_\alpha)$ contains less than two elements. Let $(z_{\alpha 1},..., z_{\alpha(m+1)}$ be any $(m + 1)$-tuple of integers satisfying $1 = z_{\alpha 1} < \cdots < z_{\alpha(m+1)} = n + 1$. Since $m \leqslant n$, such an $(m + 1)$-tuple always exists.

(ii) Suppose $pr_2(Z_\alpha)$ contains at least two elements. All g_i are strictly increasing. Hence for each i, $1 \leqslant i \leqslant m$, b_i must occur in the output of some loop $\gamma_{\alpha j}$. From the definition of the loop it then follows that for each i, $1 \leqslant i \leqslant m$, there exists j, $1 \leqslant j \leqslant n$, such that $pr_2(\gamma_{\alpha j})$ is in b_i^+. Let $z_{\alpha 1} = 1$, $z_{\alpha(m+1)} = n + 1$, and for $1 < i \leqslant m$ let $z_{\alpha i}$ be the smallest j with the property that $pr_2(\gamma_{\alpha j})$ is in b_i^+. Clearly

$$1 = z_{\alpha 1} < \cdots < z_{\alpha(m+1)} = n + 1.$$

Then

$$pr_2(Z_\alpha) = \psi^{-1}_{\langle b_1,...,b_m\rangle} \Big[\Big\{\Big(d_{\alpha 1} + \sum_{j=z_{\alpha 1}}^{z_{\alpha 2}-1} k_j \mid pr_2(\gamma_{\alpha j})|,..., d_{\alpha m} + \sum_{j=z_{\alpha m}}^{z_{\alpha(m+1)}-1} k_j \mid pr_2(\gamma_{\alpha j})|\Big) \Big| (k_1,..., k_n) \text{ in } \mathscr{K}(U, c_\alpha, t_\alpha)\Big\}\Big],$$

where $t_\alpha = (|pr_1(\gamma_{\alpha 1})|,..., |pr_1(\gamma_{\alpha n})|)$, $c_\alpha = (c_{\alpha 1},..., c_{\alpha n})$, $c_{\alpha i} = \#_{a_i}(pr_1(w))$ and $d_{\alpha j} = \#_{b_j}(pr_2(w))$ if $Z_\alpha = \{w\}$, and $c_{\alpha i} = \#_{a_i}(pr_1(w_{\alpha 1} w_{\alpha 2} \cdots w_{\alpha(n+1)}))$ and $d_{\alpha j} = \#_{b_j}(pr_2(w_{\alpha 1} w_{\alpha 2} \cdots w_{\alpha(n+1)}))$, otherwise.

Let q be the number of elements in Q. Let μ be a one-to-one map from Q onto $\{1,..., q\}$. Let $V_{\mu(\alpha)} = pr_2(Z_\alpha)$, $l_{\mu(\alpha)j} = |pr_1(\gamma_{\alpha j})|$, and $p_{\mu(\alpha)j} = |pr_2(\gamma_{\alpha j})|$. Since each $\gamma_{\alpha j}$ is an a_j-loop, $l_{\mu(\alpha)j} > 0$. Since M is ϵ-free, $p_{\mu(\alpha)j} > 0$. Thus (*) holds and the proof is complete.

Theorem 4.1 is rather awkward to apply in practice. In order to obtain more practical conditions we shall restrict the form of the language U. At the same time we shall allow the language V to be of a slightly more general form. In particular, both U and V shall be of the form $\langle a_1^{f_1(x)} \cdots a_n^{f_n(x)}\rangle_\rho$, defined below. Our major result for these languages is Theorem 4.2. Its importance stems from the numerous applications which follow from it.

Notation. Let $n \geqslant 2$, ρ in N, and let f_i be a function from N into N, strictly increasing[4] on $\{x \text{ in } N \mid x \geqslant \rho\}$, for each i, $1 \leqslant i \leqslant n$. We write $\langle a_1^{f_1(x)} \cdots a_n^{f_n(x)}\rangle_\rho$ for the set $\{a_1^{f_1(x)} \cdots a_n^{f_n(x)} \mid x \geqslant 0\}$.

[4] A function is said to be *strictly increasing on a set* $A \subseteq N$ if for each x_1 and x_2 in A, $x_1 < x_2$ implies $f(x_1) < f(x_2)$.

Note that $\langle a_1^{f_1(x)} \cdots a_n^{f_n(x)} \rangle = \langle a_1^{f_1(x)} \cdots a_n^{f_n(x)} \rangle_0$.

We start with the following lemma, which is of modest interest in itself.

LEMMA 4.2. *Let* $U = \langle a_1^{f_1(x)} \cdots a_n^{f_n(x)} \rangle_{\rho_1}$ *and* $V = \langle b_1^{g_1(x)} \cdots b_m^{g_m(x)} \rangle_{\rho_2}$, *with* V *in* $\mathscr{F}(U)$ [$\hat{\mathscr{F}}(U)$]. *Then there exist nonnegative integers* $\bar{c}_i$, *positive integers* $\bar{l}_i$, *positive* [*nonnegative*] *integers* $\bar{p}_i$, $1 \leqslant i \leqslant n$, *and nonnegative integers* $\bar{d}_j, \bar{z}_j$, $1 \leqslant j \leqslant m$, $1 = \bar{z}_1 < \cdots < \bar{z}_{m+1} = n + 1$, *such that given the Diophantine system*

$$(1) \quad \bar{c}_i + k_i \bar{l}_i = f_i(x), \qquad 1 \leqslant i \leqslant n,$$

$$(2) \quad \bar{d}_j + \sum_{s=\bar{z}_j}^{\bar{z}_{j+1}-1} k_s \bar{p}_s = g_j(y), \qquad 1 \leqslant j \leqslant m,$$

(a) *For each* $x = \max(\rho_1, \rho_2) + 1$ *in* N *there exists at most one* $(n + 2)$-*tuple* $(k_1, \ldots, k_n, x, y)$ *in* N^{n+2} *which is a solution of* (1) *and* (2).

(b) *The set of all* y *such that* $(k_1, \ldots, k_n, x, y)$ *in* N^{n+2} *is a solution of* (1) *and* (2) *for some* $k_1, \ldots, k_n, x$, *with* $x \geqq \max(\rho_1, \rho_2) + 1$ *is infinite.*

Proof. We prove the lemma for $\mathscr{F}(U)$, the full case being analogous. By Lemma 4.1, V is in $\mathscr{M}(U)$.

First suppose that $\rho_1 = \rho_2 = 0$.

Consider (a). Since all $\bar{l}_i$ are positive and all f_i are strictly increasing, for each $x \geqslant 1$ in N there exists at most one n-tuple $(k_1, \ldots, k_n)$ in N^n such that $(k_1, \ldots, k_n, x)$ is a solution of (1). Since all g_i are strictly increasing, this implies that for each $x \geqslant 1$ in N there exists at most one $(n + 2)$-tuple $(k_1, \ldots, k_n, x, y)$ in N^{n+2}, which is a solution of (1) and (2).

Consider (b). By Theorem 4.1, there exist $q \geqslant 1$; $c_i, l_i = (l_{i1}, \ldots, l_{in})$, and $(p_{i1}, \ldots, p_{in})$ in N^n, with all l_{ij} and p_{ij} positive; $(d_{i1}, \ldots, d_{im})$ in N^m; and $(z_{i1}, \ldots, z_{i(m+1)})$ in N^{m+1} such that $V = \bigcup_{i=1}^{q} V_i$, where each V_i is as in Theorem 4.1. For each i, $1 \leqslant i \leqslant q$, let

$$U_i = \{a_1^{c_{i1}+k_1 l_{i1}} \cdots a_n^{c_{in}+k_n l_{in}} \text{ in } U \mid (k_1, \ldots, k_n) \text{ in } N^n\}.$$

Since V is infinite, at least one of the V_i, say V_t, is infinite. Now $\psi^{-1}_{\langle b_1, \ldots, b_m \rangle}$ maps each $(n_1, \ldots, n_m)$ in N^m onto a unique word in $b_1^* \cdots b_m^*$. Thus $\mathscr{K}((U, c_t, l_t))$ is infinite and consequently U_t is infinite. Let $\bar{c}_i = c_{ti}$, $\bar{l}_i = l_{ti}$, $\bar{p}_i = p_{ti}$, $\bar{d}_j = d_{tj}$, and $\bar{z}_k = z_{tk}$ for all i, j, and k, where $c_t = (c_{t1}, \ldots, c_{tn})$. Clearly U_t is infinite if and only if the system (1) has infinitely many solutions $(k_1, \ldots, k_n, x)$, $x \geqslant 1$, in N^{n+1}. By the definition of V_t, each such solution $(k_1, \ldots, k_n, x)$ gives rise to a word in V_t (in the obvious manner), hence in V. Thus there exists a nonnegative integer y such that $(k_1, \ldots, k_n, x, y)$ is a solution of (1) and (2). Hence there are infinitely many solutions $(k_1, \ldots, k_n, x, y)$ of (1) and (2) in N^{n+2}. This, by (a), proves (b).

Now suppose one of ρ_1 and ρ_2 is nonzero. Let $\rho = \max\{\rho_1, \rho_2\} + 1$, $f_i'(x) = f_i(x + \rho)$, $1 \leqslant i \leqslant n$, and $g_j'(x) = g_j(x + \rho)$, $1 \leqslant j \leqslant m$. Then f_i' and g_j' are strictly increasing on N. Let $U' = \langle a_1^{f_1'(x)} \cdots a_n^{f_n'(x)} \rangle$ and $V' = \langle b_1^{g_1'(x)} \cdots b_m^{g_m'(x)} \rangle$. Then $U' = U - U''$ and $V' = V - V''$ for some finite sets U'' and V''. Clearly ϵ is not in $U' \cup V'$. By Proposition 2.2, V' is in $\mathscr{F}(U')$ $[\hat{\mathscr{F}}(U')]$. By the first part of the proof there exist nonnegative integers $\bar{c}_i$, positive integers $\bar{l}_i$, positive [nonnegative] integers $\bar{p}_i$, $1 \leqslant i \leqslant n$, and nonnegative integers $\bar{d}_j$ and $\bar{z}_j$, $1 \leqslant j \leqslant m$, $1 = z_1 < \cdots < z_{m+1} = n + 1$, such that the Diophantine system

$$\text{(i)} \quad \bar{c}_i + k_i \bar{l}_i = f_i'(x), \qquad 1 \leqslant i \leqslant n,$$

$$\text{(ii)} \quad \bar{d}_j + \sum_{s=z_j}^{z_{j+1}-1} k_s \bar{p}_s = g_j'(y), \qquad 1 \leqslant j \leqslant m,$$

satisfies (a) and (b). It is readily seen that to each solution $(k_1, \ldots, k_n, x', y')$ of (i) and (ii), with $x', y' \geqslant \rho$, there corresponds a solution $(k_1, \ldots, k_n, x' - \rho, y' - \rho)$ of

$$(1) \quad \bar{c}_i + k_i \bar{l}_i = f_i(x), \qquad 1 \leqslant i \leqslant n,$$

$$(2) \quad \bar{d}_j + \sum_{s=z_j}^{z_{j+1}-1} k_s \bar{p}_s = g_j(y), \qquad 1 \leqslant j \leqslant m.$$

Clearly (a) and (b) of Lemma 4.2 hold. Hence the proof is complete.

Using Lemma 4.2, we now establish our major result for languages U and V, each of the form $\langle a_1^{f_1(x)} \cdots a_n^{f_n(x)} \rangle_\rho$.

THEOREM 4.2. *Let* $U = \langle a_1^{f_1(x)} \cdots a_n^{f_n(x)} \rangle_{\rho_1}$ *and* $V = \langle b_1^{g_1(x)} \cdots b_m^{g_m(x)} \rangle_{\rho_2}$, *with* V in $\hat{\mathscr{F}}(U)$ $[\mathscr{F}(U)]$. *Then there exist a set* $Q \subseteq \{1, \ldots, n\}$ $[Q = \varnothing]$ *and integers* $z_1, \ldots, z_{m+1}$, $1 = z_1 < \cdots < z_{m+1} = n + 1$, *with the following two properties.*

(i) $Q_i = \{s \mid z_i \leqslant s < z_{i+1}, s \text{ not in } Q\} \neq \varnothing$ *for all* i, $1 \leqslant i \leqslant m$.

(ii) *There exist real numbers* t_j *and positive real numbers* t_{js}, $1 \leqslant j \leqslant m$ *and* s *in* Q_j, *such that for every* m-ary-*real-valued function*[5] $F(x_1, \ldots, x_m)$,

$$\lim_{x \to \infty} F(g_1(x), \ldots, g_m(x)) = \lim_{x \to \infty} F\left(t_1 + \sum_{Q_1} t_{1s} f_s(x), \ldots, t_m + \sum_{Q_m} t_{ms} f_s(x)\right),$$

if both limits exist.

[5] An m-ary *real-valued* function F is a partial function mapping m-tuples of real numbers into real numbers.

Proof. Since V is in $\hat{\mathscr{F}}(U)$ $[\mathscr{F}(U)]$, by (b) of Lemma 4.2, the Diophantine system

$$(1)\quad c_i + k_i l_i = f_i(x), \qquad 1 \leqslant i \leqslant n,$$

$$(2)\quad d_j + \sum_{s=z_j}^{z_{j+1}-1} k_s p_s = g_j(y), \qquad 1 \leqslant j \leqslant m,$$

has infinitely many solutions $(k_1, \ldots, k_n, x, y)$ in N^{n+2} for some nonnegative integers c_i, l_i, p_i, d_j, with $l_i > 0$ [and $p_i > 0$], and $1 = z_1 < \cdots < z_{m+1} = n + 1$. By (a) of Lemma 4.2, for each x in N there exists at most one $(n+2)$-tuple $(k_1, \ldots, k_n, x, y)$ in N^{n+2}, which is a solution of (1) and (2). Let A be the set of all x in N for which a solution $(k_1(x), \ldots, k_n(x), x, y(x))$ of (1) and (2) in N^{n+2} exists. Since the system (1) and (2) has infinitely many solutions, A is infinite. By (b) of Lemma 4.2, the set of y which occurs in some solution is infinite. Since each g_j is strictly increasing, for each j there exists s_j, $z_j \leqslant s_j < z_{j+1}$, such that $p_{s_j} > 0$. Let $Q = \{i \mid p_i = 0\}$. Then (i) clearly holds.

Consider (ii). Let $t_j = d_j - \sum_{Q_j} (c_s p_s / l_s)$ and $t_{js} = p_s / l_s$ for all j, $1 \leqslant j \leqslant m$, and all s in Q_j. Let $F(x_1, \ldots, x_m)$ be an m-ary real-valued function such that

$$\lim_{x\to\infty} F(g_1(x), \ldots, g_m(x))$$

and $\lim_{x\to\infty} F(t_1 + \sum_{Q_1} t_{1s} f_s(x), \ldots, t_m + \sum_{Q_m} t_{ms} f_s(x))$ exist. Due to the fact that $l_i > 0$ and f_i is strictly increasing,

$$\lim_{\substack{x\to\infty \\ x \text{ in } A}} k_i(x) = \lim_{\substack{x\to\infty \\ x \text{ in } A}} \frac{f_i(x) - c_i}{l_i} = \infty$$

for all i. This and the fact that $Q_j \neq \varnothing$ imply that

$$\lim_{\substack{x\to\infty \\ x \text{ in } A}} g_j(y(x)) = \infty$$

for all j and consequently

$$\lim_{\substack{x\to\infty \\ x \text{ in } A}} y(x) = \infty.$$

Then

$$\begin{aligned}
&\lim_{y\to\infty} F(g_1(y), \ldots, g_m(y)) \\
&\quad = \lim_{\substack{x\to\infty \\ x \text{ in } A}} F(g_1(y(x)), \ldots, g_m(y(x))) \\
&\quad = \lim_{\substack{x\to\infty \\ x \text{ in } A}} F\left(d_1 + \sum_{Q_1} k_s(x)\, p_s, \ldots, d_m + \sum_{Q_m} k_s(x)\, p_s\right)
\end{aligned}$$

$$= \lim_{\substack{x\to\infty \\ x \text{ in } A}} F\left(d_1 + \sum_{Q_1} \frac{f_s(x) - c_s}{l_s} p_s ,..., d_m + \sum_{Q_m} \frac{f_s(x) - c_s}{l_s} p_s\right)$$

$$= \lim_{\substack{x\to\infty \\ x \text{ in } A}} F\left(d_1 - \sum_{Q_1} \frac{c_s p_s}{l_s} + \sum_{Q_1} \frac{p_s}{l_s} f_s(x) ,..., d_m - \sum_{Q_m} \frac{c_s p_s}{l_s} + \sum_{Q_m} \frac{p_s}{l_s} f_s(x)\right)$$

$$= \lim_{\substack{x\to\infty \\ x \text{ in } A}} F\left(t_1 + \sum_{Q_1} t_{1s} f_s(x) ,..., t_m + \sum_{Q_m} t_{ms} f_s(x)\right)$$

$$= \lim_{x\to\infty} F\left(t_1 + \sum_{Q_1} t_{1s} f_s(x) ,..., t_m + \sum_{Q_m} t_{ms} f_s(x)\right),$$

since the last limit exists. This completes the proof.

COROLLARY. *Let* $U = \langle a_1^{f_1(x)} \cdots a_n^{f_n(x)} \rangle_{\rho_1}$ *and* $V = \langle b_1^{g_1(x)} \cdots b_n^{g_n(x)} \rangle_{\rho_2}$, *with* V *in* $\hat{\mathscr{F}}(U)$. *Then there exist real numbers* t_j *and positive real numbers* t_j', $1 \leqslant j \leqslant n$, *such that for every* n-ary *real-valued function* $F(x_1 ,..., x_n)$,

$$\lim_{x\to\infty} F(g_1(x) ,..., g_n(x)) = \lim_{x\to\infty} F(t_1 + t_1' f_1(x) ,..., t_n + t_n' f_n(x)),$$

if both limits exist.

Proof. The proof follows immediately from Theorem 4.2 for $n = m$.

We shall now exhibit three consequences of the above theorem; each is then applied to at least one example.

THEOREM 4.3. *Let* $U = \langle a_1^{f_1(x)} \cdots a_n^{f_n(x)} \rangle_{\rho_1}$ *and* $V = \langle b_1^{g_1(x)} \cdots b_m^{g_m(x)} \rangle_{\rho_2}$, *with* V in $\hat{\mathscr{F}}(U)$ $[\mathscr{F}(U)]$. *Then there exist a set* $Q \subseteq \{1,..., n\}$ $[Q = \varnothing]$ *and integers* $z_1 ,..., z_{m+1}$, $1 = z_1 < \cdots < z_{m+1} = n + 1$, *with the following two properties.*

(i) $Q_i = \{s \mid z_i \leqslant s < z_{i+1} , s \text{ not in } Q\} \neq \varnothing$, *for* $1 \leqslant i \leqslant m$.

(ii) *For all integers* i *and* j, $1 \leqslant i, j \leqslant m$, *and for all positive real numbers* k *and* l *the following holds. Suppose there exist a largest element*[6] f_i' *of* $\{f_s \mid s \text{ in } Q_i\}$ *and a largest element* f_j' *of* $\{f_s \mid s \text{ in } Q_j\}$. *Then*

$$(*) \quad \lim_{x\to\infty} [(g_i(x))^k/(g_j(x))^l] > 0$$

if and only if

$$\lim_{x\to\infty} \frac{(f_i'(x))^k}{(f_j'(x))^l} > 0,$$

if both limits exist.

[6] The function f_i, $1 \leqslant i \leqslant q$, is a *largest* element of $\{f_s \mid 1 \leqslant s \leqslant q\}$ if $\lim_{x\to\infty}(f_i(x)/f_j(x))$ exists and is nonzero for all j, $1 \leqslant j \leqslant q$.

Proof. Let $Q \subseteq \{1,\ldots, n\}$, $z_1, \ldots, z_{m+1}$, $t_1, \ldots, t_m$, and t_{js}, $1 \leqslant j \leqslant m$ and s in Q_j, be as guaranteed by Theorem 4.2. Let i and j be in $\{1,\ldots, m\}$ and let k and l be positive real numbers. Let $F(x_1, \ldots, x_m) = x_i^k/x_j^l$. Suppose $f_i'(x), f_j'(x)$, $\lim_{x\to\infty} [(g_i(x))^k/(g_j(x))^l]$, and $\lim_{x\to\infty} [(f_i'(x))^k/(f_j'(x))^l]$ exist. Then

$$\begin{aligned}
&\lim_{x\to\infty} F\Big(t_1 + \sum_{Q_1} t_{1s}f_s(x), \ldots, t_m + \sum_{Q_m} t_{ms}f_s(x)\Big) \\
&\quad = \lim_{x\to\infty} \Big[\Big(t_i + \sum_{Q_i} t_{is}f_s(x)\Big)^k \Big/ \Big(t_j + \sum_{Q_j} t_{js}f_s(x)\Big)^l\Big] \\
&\quad = \lim_{x\to\infty} \frac{(f_i'(x))^k}{(f_j'(x))^l} \cdot \Big[\lim_{x\to\infty} \Big(\frac{t_i}{f_i'(x)} + \sum_{Q_i} t_{is} \frac{f_s(x)}{f_i'(x)}\Big)\Big]^k \Big/ \Big[\lim_{x\to\infty} \Big(\frac{t_j}{f_j'(x)} + \sum_{Q_j} t_{js} \frac{f_s(x)}{f_j'(x)}\Big)\Big]^l \\
&\quad = \frac{p}{q} \cdot \lim_{x\to\infty} \frac{(f_i'(x))^k}{(f_j'(x))^l},
\end{aligned}$$

for appropriate positive real numbers p and q. Hence $\lim_{x\to\infty} F(t_1 + \sum_{Q_1} t_{1s}f_s(x), \ldots, t_m + \sum_{Q_m} t_{ms}f_s(x))$ exists. By Theorem 4.2,

$$\lim_{x\to\infty} [(g_i(x))^k/(g_j(x))^l] = (p/q) \lim_{x\to\infty} [(f_i'(x))^k/(f_j'(x))^l],$$

and (*) follows.

Corollary. *Let $U = \langle a_1^{f_1(x)} \cdots a_n^{f_n(x)}\rangle_{\rho_1}$ and $V = \langle b_1^{g_1(x)} \cdots b_n^{g_n(x)}\rangle_{\rho_2}$, with V in $\mathscr{F}(U)$. Then for every pair of integers i, j, $1 \leqslant i, j \leqslant n$, and for all positive real numbers k and l*

$$\lim_{x\to\infty} \frac{[g_i(x)]^k}{[g_j(x)]^l} > 0 \qquad \textit{if and only if} \quad \lim_{x\to\infty} \frac{[f_i(x)]^k}{[f_j(x)]^l} > 0,$$

if both limits exist.

Proof. Since $z_i = i$, $1 \leqslant i \leqslant n$, $Q = \varnothing$ in Theorem 4.3 (i). Then $f_i'(x) = f_i(x)$ and the result follows from (ii) of Theorem 4.3.

Example 4.1. Let $U = \langle a_1^{2^{3x}} a_2^{2^{2x}} a_3^{2^x}\rangle$ and $V = \langle b_1^{2^{4x}} b_2^{2^{3x}}\rangle$. Suppose V is in $\mathscr{F}(U)$. Using Theorem 4.3, $Q = \varnothing$ and there are only two possible choices for z_1, z_2, and z_3:

(a) $z_1 = 1$, $z_2 = 2$, and $z_3 = 4$. Then $f_1'(x) = 2^{3x}$ and $f_2'(x) = 2^{2x}$. For $k = 4$, $l = 3$, $i = 2$, and $j = 1$,

$$\lim_{x\to\infty} \frac{[g_i(x)]^k}{[g_j(x)]^l} = \lim_{x\to\infty} \frac{[2^{3x}]^4}{[2^{4x}]^3} = 1 \qquad \text{and} \qquad \lim_{x\to\infty} \frac{[f_i'(x)]^k}{[f_j'(x)]^l} = \lim_{x\to\infty} \frac{[2^{2x}]^4}{[2^{3x}]^3} = 0.$$

Thus (*) of Theorem 4.3 does not hold.

(b) $z_1 = 1$; $z_2 = 3$, and $z_3 = 4$. Then $f_1'(x) = 2^{3x}$ and $f_2'(x) = 2^x$. For $k = 4$, $l = 3$, $i = 2$, and $j = 1$,

$$\lim_{x\to\infty} \frac{[g_i(x)]^k}{[g_j(x)]^l} = \lim_{x\to\infty} \frac{[2^{3x}]^4}{[2^{4x}]^3} = 1 \quad \text{and} \quad \lim_{x\to\infty} \frac{[f_i'(x)]^k}{[f_j'(x)]^l} = \lim_{x\to\infty} \frac{[2^x]^4}{[2^{3x}]^3} = 0.$$

Thus (*) of Theorem 4.3 does not hold. Hence V is not in $\mathscr{F}(U)$.

EXAMPLE 4.2. Let[7] $U = \langle a_1^{x^2} a_2^{x^2 \log x} a_3^{x} \rangle_{10}$ and $V = \langle b_1^{x \log x} b_2^{x} \rangle_{10}$. Suppose V is in $\mathscr{F}(U)$. Using Theorem 4.3, one has only two possible choices for z_1, z_2, and z_3, and in each case only three possible choices for a set Q:

(1) $z_1 = 1$, $z_2 = 2$, and $z_3 = 4$.

(a) $Q = \varnothing$. Then $f_1'(x) = x^2$ and $f_2'(x) = x^2 \log x$. For $j = 2$ and $i = l = k = 1$,

$$\lim_{x\to\infty} \frac{(g_i(x))^k}{(g_j(x))^l} = \lim_{x\to\infty} \frac{x \log x}{x} = \infty \quad \text{and} \quad \lim_{x\to\infty} \frac{(f_i'(x))^k}{(f_j'(x))^l} = \lim_{x\to\infty} \frac{x^2}{x^2 \log x} = 0.$$

Thus (*) of Theorem 4.3 does not hold.

(b) $Q = \{2\}$. Then $f_1'(x) = x^2$ and $f_2'(x) = x$. For $k = i = 1$ and $l = j = 2$,

$$\lim_{x\to\infty} \frac{(g_i(x))^k}{(g_j(x))^l} = \lim_{x\to\infty} \frac{x \log x}{(x)^2} = 0 \quad \text{and} \quad \lim_{x\to\infty} \frac{(f_i'(x))^k}{(f_j'(x))^l} = \lim_{x\to\infty} \frac{x^2}{(x)^2} = 1.$$

Thus (*) of Theorem 4.3 does not hold.

(c) $Q = \{3\}$. Then $f_1'(x) = x^2$ and $f_2'(x) = x^2 \log x$. As in case (1)(a), (*) of Theorem 4.3 does not hold.

(2) $z_1 = 1$, $z_2 = 3$, and $z_3 = 4$.

(a) $Q = \varnothing$. Then $f_1'(x) = x^2 \log x$ and $f_2'(x) = x$. For $k = i = 1$ and $l = j = 2$,

$$\lim_{x\to\infty} \frac{(g_i(x))^k}{(g_j(x))^l} = \lim_{x\to\infty} \frac{x \log x}{(x)^2} = 0 \quad \text{and} \quad \lim_{x\to\infty} \frac{(f_i'(x))^k}{(f_j'(x))^l} = \lim_{x\to\infty} \frac{x^2 \log x}{(x)^2} = \infty.$$

Thus (*) of Theorem 4.3 does not hold.

(b) $Q = \{1\}$. Then $f_1'(x) = x^2 \log x$ and $f_2'(x) = x$. As in case (2)(a), (*) of Theorem 4.3 does not hold.

(c) $Q = \{2\}$. Then $f_1'(x) = x^2$ and $f_2'(x) = x$. As in case (1)(b), (*) of Theorem 4.3 does not hold.

Hence $\langle b_1^{x \log x} b_2^{x} \rangle_{10}$ is not in $\mathscr{F}(\langle a_1^{x^2} a_2^{x^2 \log x} a_3^{x} \rangle_{10})$.

[7] If a real-valued function f_i is written in $\langle a_1^{f_1(x)} \cdots a_n^{f_n(x)} \rangle_p$, it is interpreted as function $\bar{f}_i$ from N into N, where $\bar{f}_i(x)$ is the integral part of $f_i(x)$. In what follows $\log_{10}$ is abbreviated to log. By definition, $\log x = 0$ when $x = 0$.

EXAMPLE 4.3. Using a special argument it was shown in [7] that $\{a^n b a^{2^{2^n}} b a^n \mid n \geqslant 1\}$ is not in $\mathscr{F}(\{a^n b a^{2^{2^n}} b a^n b a^{2^{2^n}} \mid n \geqslant 1\} \cup \{\epsilon\})$. This is clearly equivalent to showing that $\langle b_1{}^n b_2^{2^{2^n}} b_3{}^n \rangle$ is not in $\mathscr{F}(\langle a_1{}^n a_2^{2^{2^n}} a_3{}^n a_4^{2^{2^n}} \rangle)$. Using Theorem 4.3 with $k = l = 1$ we have the following. Suppose $\langle b_1{}^n b_2^{2^{2^n}} b_3{}^n \rangle$ is in $\mathscr{F}(\langle a_1{}^n a_2^{2^{2^n}} a_3{}^n a_4^{2^{2^n}} \rangle)$. Then $Q = \varnothing$ and there are three possible choices for z_1, z_2, z_3, and z_4:

(a) $z_1 = 1$, $z_2 = 2$, $z_3 = 3$, and $z_4 = 5$. Then $f_1'(n) = n$, $f_2'(n) = 2^{2^n}$, and $f_3'(n) = 2^{2^n}$. For $i = 1$ and $j = 3$,

$$\lim_{n\to\infty} \frac{g_i(n)}{g_j(n)} = \lim_{n\to\infty} \frac{n}{n} = 1 \qquad \text{and} \qquad \lim_{n\to\infty} \frac{f_i'(n)}{f_j'(n)} = \lim_{n\to\infty} \frac{n}{2^{2^n}} = 0.$$

Thus (*) of Theorem 4.3 does not hold.

(b) $z_1 = 1$, $z_2 = 2$, $z_3 = 4$, and $z_4 = 5$. Then $f_1'(n) = n$, $f_2'(n) = 2^{2^n}$, and $f_3'(n) = 2^{2^n}$. As in (a), (*) of Theorem 4.3 does not hold.

(c) $z_1 = 1$, $z_2 = 3$, $z_3 = 4$, and $z_4 = 5$. Then $f_1'(n) = 2^{2^n}$, $f_2'(n) = n$, and $f_3'(n) = 2^{2^n}$. For $i = 2$ and $j = 3$,

$$\lim_{n\to\infty} \frac{g_i(n)}{g_j(n)} = \lim_{n\to\infty} \frac{2^{2^n}}{n} = \infty \qquad \text{and} \qquad \lim_{n\to\infty} \frac{f_i'(n)}{f_j'(n)} = \lim_{n\to\infty} \frac{n}{2^{2^n}} = 0.$$

Thus (*) of Theorem 4.3 does not hold.

Hence $\langle b_1{}^n b_2^{2^{2^n}} b_3{}^n \rangle$ is not in $\mathscr{F}(\langle a_1{}^n a_2^{2^{2^n}} a_3{}^n a_4^{2^{2^n}} \rangle)$.

We now establish a necessary condition for a large class of languages

$$U = \langle a_1^{f_1(x)} \cdots a_n^{f_n(x)} \rangle_{\rho_1}$$

and

$$V = \langle b_1^{g_1(x)} \cdots b_n^{g_n(x)} \rangle_{\rho_2},$$

with f_i and g_i polynomials.

THEOREM 4.4. *Let* $U = \langle a_1^{f_1(x)} \cdots a_n^{f_n(x)} \rangle_{\rho_1}$ *and* $V = \langle b_1^{g_1(x)} \cdots b_n^{g_n(x)} \rangle_{\rho_2}$, *with* V *in* $\hat{\mathscr{F}}(U)$ *and all* $f_i(x)$ *and* $g_i(x)$ *polynomials.*

(a) *If* $\deg(f_i) = \deg(g_i)$ *for some* i, $1 \leqslant i \leqslant n$, *then* $\deg(f_j) = \deg(g_j)$ *for all* j, $1 \leqslant j \leqslant n$. *(For each polynomial* $f(x) = c_0 + c_1 x + \cdots + c_p x^p$, *with* $c_p \neq 0$, *let* $\deg(f) = p$.*)*

(b) *If* $f_i(x) = cx$ *and* $g_i(x) = dx$ *for some* i, $1 \leqslant i \leqslant n$, c, *and* d, *then there exist real numbers* p, r, $q_1, \ldots, q_n$, $q_1', \ldots, q_n'$, p *and each* q_j *positive, such that* $g_j(px + r) = q_j f_j(x) + q_j'$ *for each* j, $1 \leqslant j \leqslant n$.

Proof. Consider (a). Suppose the contrary. Then there exist i and j, $1 \leqslant i, j \leqslant n$, satisfying $\deg(f_i) = \deg(g_i)$ and $\deg(f_j) \neq \deg(g_j)$. Suppose $\deg(f_j) < \deg(g_j)$, an analogous argument holding if $\deg(f_j) > \deg(g_j)$. Note that $\deg(f_i) \geqslant 1$, since f_i is strictly increasing for $x \geqslant \rho_1$. Now for $k = \deg(f_j)/\deg(f_i)$ and $l = 1$,

$$\lim_{x\to\infty} \frac{(f_i(x))^k}{(f_j(x))^l} = s \qquad \text{and} \qquad \lim_{x\to\infty} \frac{(g_i(x))^k}{(g_j(x))^l} = 0,$$

where s is an appropriate positive real number. This contradicts the Corollary to Theorem 4.3. Hence (a) holds.

Consider (b). By (a), $\deg(g_j) = \deg(f_j)$ for each j. Let j, $1 \leqslant j \leqslant n$, be arbitrary, $g_j(x) = d_h x^h + \cdots + d_1 x + d_0$ and $f_j(x) = c_h x^h + \cdots + c_1 x + c_0$, with $d_h > 0$ and $c_h > 0$. Let t_s and t_s' be real numbers guaranteed by the Corollary to Theorem 4.2. Let $F(x_1, \ldots, x_n) = (1/x_i)(x_j - g_j(x_i/d))$. Clearly $\lim_{x\to\infty} F(g_1(x), \ldots, g_n(x)) = 0$ and

$$\begin{aligned}
&\lim_{x\to\infty} F(t_1 + t_1' f_1(x), \ldots, t_n + t_n' f_n(x)) \\
&\quad = \lim_{x\to\infty} [1/(t_i + t_i' cx)]\{t_j + t_j' f_j(x) - g_j[(t_i + t_i' cx)/d]\} \\
&\quad = \lim_{x\to\infty} [1/(t_i + t_i' cx)](t_j' f_j(x) - g_j(px + r))
\end{aligned}$$

exists, with $p = t_i' c/d > 0$ and $r = t_i/d$. By the Corollary to Theorem 4.2,

$$\lim_{x\to\infty} F(t_1 + t_1' f_1(x), \ldots, t_n + t_n' f_n(x)) = \lim_{x\to\infty} F(g_1(x), \ldots, g_n(x)) = 0.$$

This implies that $\deg(t_j' f_j(x) - g_j(px + r)) < 1$. Thus $g_j(px + r) = t_j' f_j(x) + q_j'$ for some real number q_j'. The theorem then holds with $q_j = t_j'$.

EXAMPLE 4.4. Let $U = \langle a_1^x a_2^{x^3+x} a_3^{x^3+x^2} \rangle$ and $V = \langle b_1^x b_2^{x^3+x^2+x} b_3^{x^3+5x^2+x} \rangle$. Suppose V is in $\hat{\mathscr{F}}(U)$. By (b) of Theorem 4.4, there exist real numbers p and r, $p > 0$, such that

(1) $(px + r)^3 + (px + r)^2 + (px + r) = q_2(x^3 + x) + q_2'$ and

(2) $(px + r)^3 + 5(px + r)^2 + (px + r) = q_3(x^3 + x^2) + q_3'$ for some real numbers q_2, q_2', q_3, and q_3', with q_2 and q_3 positive. Comparing coefficients of x^2 in (1) yields $r = -\frac{1}{3}$. Comparing coefficients of x in (2) (with $r = -\frac{1}{3}$) yields $p = 0$, a contradiction. Thus V is not in $\hat{\mathscr{F}}(U)$.

Examples 4.1–4.4 indicate that Theorems 4.3 and 4.4 can be used to show that V is not in $\mathscr{F}(U)$ $[\hat{\mathscr{F}}(U)]$ for a wide class of languages

$$U = \langle a_1^{f_1(x)} \cdots a_n^{f_n(x)} \rangle_{\rho_1}$$

and

$$V = \langle b_1^{g_1(x)} \cdots b_m^{g_m(x)} \rangle_{\rho_2} .$$

The following example illustrates the fact that Theorems 4.3 and 4.4 fail in some cases of interest.

EXAMPLE 4.5. Let $U = \langle a^{(x-1)^2} b^{x^2} \rangle$ and $V = \langle a^{x^2} b^{x^2} \rangle$. Then

$$\lim_{x\to\infty} \frac{[(x+1)^2]^k}{[x^2]^l} > 0 \quad \text{if and only if} \quad \lim_{x\to\infty} \frac{[x^2]^k}{[x^2]^l} > 0,$$

and

$$\lim_{x\to\infty} \frac{[x^2]^k}{[(x+1)^2]^l} > 0 \quad \text{if and only if} \quad \lim_{x\to\infty} \frac{[x^2]^k}{[x^2]^l} > 0.$$

Thus the necessary condition of Theorem 4.3 (actually its Corollary) and that of (a) of Theorem 4.4 are satisfied. The condition (b) of Theorem 4.4 is not applicable. Nevertheless, it will be shown shortly that V is not in $\hat{\mathscr{F}}(U)$.

To cover cases similar to that of the above example we establish the following theorem.

THEOREM 4.5. *Let* $U = \langle a_1^{f_1(x)} \cdots a_n^{f_n(x)} \rangle_{\rho_1}$ *and* $V = \langle b_1^{g_1(x)} \cdots b_m^{g_m(x)} \rangle_{\rho_2}$, *with* V in $\hat{\mathscr{F}}(U)$ $[\mathscr{F}(U)]$. *Then there exist a set* $Q \subseteq \{1,\ldots, n\}$ $[Q = \varnothing]$ *and integers* $z_1,\ldots, z_{m+1}$, $1 = z_1 < \cdots < z_{m+1} = n + 1$, *with the following two properties.*

(i) $Q_i = \{s \mid z_i \leqslant s < z_{i+1},\ s \text{ not in } Q\} \neq \varnothing$ *for* $1 \leqslant i \leqslant m$.

(ii) *Let* i *in* $\{1,\ldots, m\}$ *and real numbers* $q_1,\ldots, q_m$, l, $l > 0$, *be such that*

   (a) *there exists a largest element* f_i' *of* $\{f_s \mid s \text{ in } Q_i\}$,

   (b) $\lim_{x\to\infty} [(\sum_{s=1}^m q_s g_s(x))/(g_i(x))^l]$ *exists, and*

   (c) $\lim_{x\to\infty} [(\sum_{s=1}^n r_s f_s(x))/(f_i'(x))^l]$ *exists for all real numbers* r_s *which satisfy* $\operatorname{sgn}(r_s) = \operatorname{sgn}(q_j)$ *for all* s in Q_j, $1 \leqslant j \leqslant m$, *and* $r_s = 0$ *for all* s in Q. (*For all real numbers* x, $\operatorname{sgn}(x)$ *is* -1, 0 *or* 1 *if* $x < 0$, $x = 0$, *or* $x > 0$, *respectively.*)

Then there exist real numbers $v_1,\ldots, v_n$ ($\operatorname{sgn}(v_s) = \operatorname{sgn}(q_j)$ *for all* s in Q_j, $1 \leqslant j \leqslant m$, *and* $v_s = 0$ *for* s in Q) *for which*

$$\lim_{x\to\infty} \left[\left(\sum_{s=1}^m q_s g_s(x) \right) \Big/ (g_i(x))^l \right] = \lim_{x\to\infty} \left[\left(\sum_{s=1}^n v_s f_s(x) \right) \Big/ (f_i'(x))^l \right] .$$

Proof. Let $Q \subseteq \{1,\ldots, n\}$, $z_1,\ldots, z_{m+1}$, $t_1,\ldots, t_m$, and t_{js}, $1 \leqslant j \leqslant m$ and s in Q_j, be as guaranteed by Theorem 4.2. Let $F(x_1,\ldots, x_m) = (1/x_i^l) \sum_{j=1}^m q_j x_j$.

Then

$$
\begin{aligned}
&\lim_{x\to\infty} F\Big(t_1 + \sum_{Q_1} t_{1s} f_s(x), \ldots, t_m + \sum_{Q_m} t_{ms} f_s(x)\Big) \\
&\quad = \lim_{x\to\infty} \left\{\left[\sum_{j=1}^{m} q_j \Big(t_j + \sum_{Q_j} t_{js} f_s(x)\Big)\right] \Big/ \Big(t_i + \sum_{Q_i} t_{is} f_s(x)\Big)^l\right\} \\
&\quad = \lim_{x\to\infty} \left\{\left[\sum_{j=1}^{m} q_j t_j + \sum_{j=1}^{m} \sum_{Q_j} q_j t_{js} f_s(x)\right] \Big/ \left[(f_i'(x))^l \Big(\frac{t_i}{f_i'(x)} + \sum_{Q_i} \frac{t_{is} f_s(x)}{f_i'(x)}\Big)^l\right]\right\} \\
&\quad = \lim_{x\to\infty} \left[\Big(\sum_{j=1}^{m} q_j t_j\Big) \Big/ k(f_i'(x))^l\right] + \lim_{x\to\infty} \left[\Big(\sum_{j=1}^{m} \sum_{Q_j} q_j t_{js} f_s(x)\Big) \Big/ k(f_i'(x))^l\right] \\
&\quad = \lim_{x\to\infty} \left[\Big(\sum_{s=1}^{m} v_s f_s(x)\Big) \Big/ k(f_i'(x))^l\right],
\end{aligned}
$$

where $k > 0$ is an appropriate real number, $v_s = 0$, for s in Q, and $v_s = q_j t_{js}/k$ for s in Q_j, $1 \leqslant j \leqslant m$. Since all $t_{js} > 0$ (s in Q_j), $\operatorname{sgn}(v_s) = \operatorname{sgn}(q_j)$ for all s in Q_j. By (c), the last limit exists. By (b) and Theorem 4.2,

$$
\lim_{x\to\infty} \left[\Big(\sum_{j=1}^{m} q_j g_j(x)\Big) \Big/ (g_i(x))^l\right] = \lim_{x\to\infty} \left[\Big(\sum_{s=1}^{n} v_s f_s(x)\Big) \Big/ (f_i'(x))^l\right],
$$

and the theorem holds.

Example 4.5. (continued). Using Theorem 4.5, $Q = \varnothing$, $z_1 = 1$, $z_2 = 2$, and $z_3 = 3$. Let $q_1 = 1$, $q_2 = -1$, $i = 2$, and $l = \frac{1}{2}$. Clearly (a), (b), and (c) of (ii) of Theorem 4.5 hold. However,

$$
\lim_{x\to\infty} \frac{g_1(x) - g_2(x)}{(g_2(x))^{1/2}} = \lim_{x\to\infty} \frac{x^2 - x^2}{[x^2]^{1/2}} = 0
$$

and

$$
\lim_{x\to\infty} \frac{v_1 f_1(x) + v_2 f_2(x)}{(f_2'(x))^{1/2}} = \lim_{x\to\infty} \frac{v_1(x+1)^2 + v_2 x^2}{[x^2]^{1/2}} \neq 0
$$

for every choice of nonzero real numbers v_1 and v_2. Thus V is not in $\hat{\mathscr{F}}(U)$.

It is known that if $\mathscr{L}$ is a family of languages closed under reversal[8], then $\mathscr{M}(\mathscr{L})$, $[\hat{\mathscr{M}}(\mathscr{L})$, $\mathscr{F}(\mathscr{L})$, and $\hat{\mathscr{F}}(\mathscr{L})]$ is closed under reversal [3]. Hence $\mathscr{M}(L)$, $[\hat{\mathscr{M}}(L)$, $\mathscr{F}(L)$,

[8] The *reversal* of the word $w = x_1 \cdots x_n$, $n \geqslant 0$, each x_i in Σ, is the word $w^{\mathrm{R}} = x_n \cdots x_1$. The reversal of the language L is the language $L^{\mathrm{R}} = \{w^{\mathrm{R}} \mid w \text{ in } L\}$. Note that $\epsilon^{\mathrm{R}} = \epsilon$.

and $\hat{\mathscr{F}}(L)$] is closed under reversal if and only if L^R is in $\mathscr{M}(L)$, [$\hat{\mathscr{M}}(L)$, $\mathscr{F}(L)$, and $\hat{\mathscr{F}}(L)$]. Theorems 4.2–4.5 give us a variety of principal (semi-) AFL not closed under reversal. For example $\mathscr{M}(\langle a^n b^{n^2}\rangle)$, $\mathscr{F}(\langle a^n b^{n^2} c^{n^3}\rangle)$, and $\hat{\mathscr{F}}(\langle a^{(n+1)^2} b^{n^2}\rangle)$.

We conclude the section with some remarks on a possible extension of the results obtained.

Remark 4.1. Let $\mathscr{L}$ be a family of languages, $L = \langle b_1^{g_1(x)} \cdots b_m^{b_m(x)}\rangle_\rho$, and let L be in the smallest [full](semi-) AFL generated by $\mathscr{L}$. Then similarly as in the proof of Lemma 4.1 (replacing L_1 by L and L_2 by $\mathscr{L}$) it can be shown that

$$L = M_1(U_1) \cup \cdots \cup M_r(U_r)$$

for some $r \geqslant 1$, $U_1, \ldots, U_r$ in $\mathscr{L}$, and $M_1, \ldots, M_r$ ϵ-free [arbitrary] a-transducers. Thus L is in $\mathscr{F}(\mathscr{L})$ [$\hat{\mathscr{F}}(\mathscr{L})$] if and only if L is in $\mathscr{M}(\mathscr{L})$ [$\hat{\mathscr{M}}(\mathscr{L})$].

Remark 4.1 shows that Theorem 4.1 can easily be modified for the case of [full] AFL generated by a family of bounded languages $\mathscr{L}$.

From the definition of $\langle b_1^{g_1(x)} \cdots b_m^{g_m(x)}\rangle_\rho$ we have the following.

Remark 4.2. Let $L_1 = \langle b_1^{g_1(x)} \cdots b_m^{g_m(x)}\rangle_\rho$ and let L_2 be an infinite subset of L_1. Then $L_2 = \langle b_1^{g_1'(x)} \cdots b_m^{g_m'(x)}\rangle_\rho$ for some $g_1', \ldots, g_m'$, and $\lim_{x\to\infty} F(g_1(x), \ldots, g_m(x)) = \lim_{x\to\infty} F(g_1'(x), \ldots, g_m'(x))$ for every F such that the first limit exists.

Remarks 4.1 and 4.2 clearly imply the following.

Remark 4.3. Let $V = \langle b_1^{g_1(x)} \cdots b_m^{g_m(x)}\rangle_{\rho_1}$ be in the smallest [full] (semi-) AFL generated by a family of languages $\mathscr{L}$, each L in $\mathscr{L}$ being of the form

$$\langle a_1^{f_1(x)} \cdots a_n^{f_n(x)}\rangle_{\rho_2}.$$

Then there exists U in $\mathscr{L}$ such that Theorems 4.2–4.5 hold for U and V.

We now exhibit a family of languages $\mathscr{L}$ such that $\hat{\mathscr{F}}(\mathscr{L})$ is closed under reversal but $\mathscr{F}(\mathscr{L})$ is not closed under reversal.

EXAMPLE 4.6. Let $\mathscr{L} = \{\langle a^n b^{n^2}\rangle, \langle a^{n^2} b^n c^{n^3}\rangle, \langle a^{n^3} b^n c^{n^2}\rangle\}$. From the above discussion on reversal it is clear that $\hat{\mathscr{F}}(\mathscr{L})$ is closed under reversal. To show that $\mathscr{F}(\mathscr{L})$ is not closed under reversal it suffices to show that $V = \langle a^{n^2} b^n\rangle$ is not in $\mathscr{F}(\mathscr{L})$. This readily follows from Remark 4.3 and the fact that Theorem 4.3 does not hold for any U in $\mathscr{L}$.

5. LOOPS AND (f, I)-GUIDED LANGUAGES

So far we have studied languages of the form $\langle a_1^{f_1(x)} \cdots a_n^{f_n(x)}\rangle$. We now turn our attention to a different family of languages, the distance (f, I)-guided and I-uniformly infinite languages, defined below. In the present section we introduce the basic definitions and notation. We then establish seven lemmas. The first five deal with computations and are somewhat parallel to those in Section 3. The last two lemmas are independent of the first five and are strictly technical results about (f, I)-guided and I-uniformly infinite languages. In Section 6 we use the last four of these lemmas to derive some necessary conditions on languages U and V, U bounded and V distance (f, I)-guided and I-uniformly infinite.

We first present some notation and give the definition of distance (f, I)-guided languages.

Notation. Let I be a nonempty subset of $\{1,..., m\}$. Let $I = \{i_1 ,..., i_r\}$, $i_1 < \cdots < i_r$, and let $\psi_{I\langle b_1,...,b_m\rangle}$ be the mapping from $b_1{}^* \cdots b_m{}^*$ into N^r defined by

$$\psi_{I\langle b_1,...,b_m\rangle}(z) = (\#_{b_{i_1}}(z),..., \#_{b_{i_r}}(z)).$$

Whenever $\langle b_1 ,..., b_m\rangle$ is understood, ψ and ψ_I are used instead of $\psi_{\langle b_1,...,b_m\rangle}$ and $\psi_{I\langle b_1,...b_m\rangle}$. (See Section 4 for the definition of $\#_{b_i}$ and $\psi_{\langle b_1,...,b_m\rangle}$.)

DEFINITION. A language $L \subseteq b_1{}^* \cdots b_m{}^*$ is said to be *(f, I)-guided* if

(i) $\varnothing \neq I = \{i_1 ,..., i_r\} \subsetneqq \{1,..., m\}$;

(ii) $f: N^r \rightarrow N$ is increasing and unbounded in each variable[9]; and

(iii) for each $(k_1 ,..., k_m)$ in $\psi(L)$ and j in $\{1,..., m\} - I$, $k_j \geqslant f(k_{i_1} ,..., k_{i_r})$.

L is said to be *distance (f, I)-guided* if, in addition,

(iv) $| j - i | \geqslant 2$ for each j and i in I, $j \neq i$.

By condition (i), $m \geqslant 2$ if $L \subseteq b_1{}^* \cdots b_m{}^*$ is (f, I)-guided. Note that by condition (iii) and the fact that f is increasing and unbounded in each variable, the set $\psi_I(\{w$ in $L \mid \#_{b_j}(w) = k\})$ is finite for each j in $\{1,..., m\} - I$ and each k in N.

The notion of a (distance)(f, I)-guided language is slightly inadequate for our purposes. To see this consider an arbitrary language $L \subseteq b_1{}^* \cdots b_m{}^*$. Then

$$L \subseteq b_1{}^* \cdots b_{m+1}^*$$

and L is distance (f, I)-guided for $f(x) = x$ and $I = \{m + 1\}$. Thus the restriction that the output language of an a-transducer is distance (f, I)-guided is meaningless by itself. In order to render the notions of (f, I)-guided and distance (f, I)-guided useful, the following condition will be added.

DEFINITION. Let $I = \{i_1 ,..., i_r\}$ be a nonempty subset of $\{1,..., m\}$. A bounded

[9] A function $f: N^r \rightarrow N$ is said to be *increasing and unbounded in each variable* if for each i, $k_i \leqslant k_i'$ implies $f(k_1 ,..., k_i ,..., k_r) \leqslant f(k_1 ,..., k_i',..., k_r)$ and for each k in N there exist $\bar{k}$ in N such that $f(0,..., 0, \bar{k}, 0,..., 0) > k$, $\bar{k}$ at the i-th coordinate. This implies that $f(k_1 ,..., k_{i-1}$, $\bar{k}$, $k_{i+1} ,..., k_r) > k$ for all $k_1 ,..., k_{i-1}$, $k_{i+1} ,..., k_r$ in N.

language $L \subseteq b_1^* \cdots b_m^*$ is said to be *I-uniformly infinite* if the set $\{(k_1, \ldots, k_r)$ in $\psi_I(L) \mid$ each $k_i > t\}$ is infinite for each $t \geqslant 0$.

To simplify the notation we introduce the following

Notation. For each I and m, $I \subset \{1, \ldots, m\}$, let $\bar{I}_m = \{1, \ldots, m\} - I$.

EXAMPLE 5.1.

(a) $\{a^i b^j \mid 1 \leqslant i \leqslant j\}$ is a distance $(x, \{1\})$-guided language.

(b) $\{a^i b^j c^k d^l \mid i, k \geqslant 1,\ j,\ l \geqslant (i+k)^2\}$ is a distance $((x+y)^2, \{1, 3\})$-guided language.

(c) $\{a^i b^j c^k d^l \mid i,\ j \geqslant 1,\ k,\ l \geqslant i+j\}$ is a $(x+y, \{1, 2\})$-guided language. However the language is not distance (f, I)-guided for any choice of f and I.

We shall frequently use the following constants in connection with an a-transducer M.

Notation. Let $m_1 = \max\{|pr_2(v)| \mid v \text{ in } H\}$, $m_2 = |H|$, and $m_3 = \max\{|S| \mid S$ a set of nonassociated loops in $M\}$.

We now turn to the lemmas on computations. The first four deal with the character of loops in computations of an a-transducer with a (distance)(f, I)-guided output language. The final result of this sequence of lemmas, Lemma 5.4, is used in Section 6.

We start with a lemma which is used to establish the existence of a loop γ with at least k occurrences in a computation α.

LEMMA 5.1. *Let $L \subseteq a_1^* \cdots a_n^*$ and $M(L) \subseteq b_1^* \cdots b_m^*$. Let*

$$\alpha = v_1 \gamma_1^{h_1} v_2 \cdots v_p \gamma_p^{h_p} v_{p+1}$$

be a computation in standard form in $\Pi_M \cap pr_1^{-1}(L)$ and let $k \geqslant 1$ be an arbitrary integer. Then for each j, $1 \leqslant j \leqslant m$, the following holds. If $\#_{b_j}(pr_2(\alpha)) > 3km_1m_2m_3$, then there exists i, $1 \leqslant i \leqslant p$, such that $pr_2(\gamma_i)$ is in b_j^+ and $h_i > k$.

Proof. Suppose the contrary. Hence there exists j, $1 \leqslant j \leqslant m$, such that

$$\#_{b_j}(pr_2(\alpha)) > 3km_1m_2m_3$$

and for each i, $1 \leqslant i \leqslant p$, if $pr_2(\gamma_i)$ is in b_j^+, then $h_i \leqslant k$. Therefore

$$\begin{aligned} \#_{b_j}(pr_2(\alpha)) &= \#_{b_j}(v_1 \gamma_1^{h_1} v_2 \cdots v_p \gamma_p^{h_p} v_{p+1}) = \#_{b_j}(v_1 v_2 \cdots v_{p-1}) + \#_{b_j}(\gamma_1^{h_1} \cdots \gamma_p^{h_p}) \\ &\leqslant (p+1)\, m_1 m_2 + \#_{b_j}(\gamma_1^k \gamma_2^k \cdots \gamma_p^k) \leqslant (p+1)\, m_1 m_2 - pkm_1m_2 \\ &\leqslant m_1 m_2(m_3 + 1 + m_3 k) \leqslant m_1 m_2(m_3 k + m_3 k + m_3 k) = 3km_1m_2m_3, \end{aligned}$$

a contradiction.

The second lemma shows that (for each "long" computation) to each letter b_i, i in $\bar{I}_m$, there corresponds at least one letter a_j such that all a_j-loops have output in b_i^+. It is a major step in our effort to establish Lemma 5.4.

LEMMA 5.2. *Let M be an a-transducer, $L \subseteq a_1^* \cdots a_n^*$, and $M(L) \subseteq b_1^* \cdots b_m^*$. Let $M(L)$ be an (f, I)-guided language for some f and I. Then there exists an integer $\bar{k} \geqslant 0$ with the following property. For every computation $\alpha = v_1\gamma_1^{h_1}v_2 \cdots v_p\gamma_p^{h_p}v_{p+1}$ in standard form in $\Pi_M \cap pr_1^{-1}(L)$, with $f(\psi_I(pr_2(\alpha))) \geqslant \bar{k}$, there exists a set*

$$\{l(j) \mid j \text{ in } \bar{I}_m,\ 1 \leqslant l(j) \leqslant n\}$$

of integers such that

(i) *for each j in $\bar{I}_m$, $pr_1(\gamma_s)$ is* in $a_{l(j)}^+$ *for some s, $1 \leqslant s \leqslant p$;*

(ii) *for all t and j, $1 \leqslant t \leqslant p$ and j* in $\bar{I}_m$, *$pr_2(\gamma_t)$ is in b_j^+ if $pr_1(\gamma_t)$ is* in $a_{l(j)}^+$; *and*

(iii) *$i_1 < i_2$ implies $l(i_1) < l(i_2)$ for each i_1 and i_2* in $\bar{I}_m$.

Proof. Note that every set $\{l(j) \mid 1 \leqslant l(j) \leqslant n,\ j \text{ in } \bar{I}_m\}$ satisfying (i) and (ii) also satisfies (iii). [For, all a_i and b_i are distinct, $L \subseteq a_1^* \cdots a_n^*$, and $M(L) \subseteq b_1^* \cdots b_m^*$.] It thus suffices to show that for each computation α in standard form in

$$\Pi_M \cap pr_1^{-1}(L)$$

for which $f(\psi_I(pr_2(\alpha))) \geqslant \bar{k}$, a set of integers $\{l(j) \mid 1 \leqslant l(j) \leqslant n,\ j \text{ in } \bar{I}_m\}$ satisfying (i) and (ii) can be found.

Let k be the integer guaranteed by Lemma 3.3 and let $m_4 = 3km_1m_2m_3$. By Lemma 5.1, $\#_{b_j}(pr_2(\alpha)) \leqslant m_4$ for every b_j and every computation α in standard form having no loop γ, $pr_2(\gamma)$ in b_j^+, with at least k occurrences. Let $\bar{k}$ be any integer greater than m_4.

Let $\alpha = v_1\gamma_1^{h_1}v_2 \cdots v_p\gamma_p^{h_p}v_{p+1}$ be an arbitrary computation in standard form in $\Pi_M \cap pr_1^{-1}(L)$, with $f(\psi_I(pr_2(\alpha))) \geqslant \bar{k}$. To show that a set

$$\{l(j) \mid 1 \leqslant l(j) \leqslant n,\ j \text{ in } \bar{I}_m\}$$

satisfying (i) and (ii) exists for α it suffices to prove

(iv) There is no j in $\bar{I}_m$ with the following property.

(*) For each i, $1 \leqslant i \leqslant n$, and each s, $1 \leqslant s \leqslant p$, for which $pr_1(\gamma_s)$ is in a_i^+ and $pr_2(\gamma_s)$ is in b_j^+, there exist $t \neq s$ and $q \neq j$ such that $1 \leqslant t \leqslant p$, $1 \leqslant q \leqslant m$, $pr_1(\gamma_t)$ is in a_i^+, and $pr_2(\gamma_t)$ is in b_q^*.

For suppose (iv) holds. This can be expressed (for $1 \leqslant i \leqslant n$, j in $\bar{I}_m$, $1 \leqslant s, t \leqslant p$, and $1 \leqslant q \leqslant m$) as

(1) $\neg(\exists j)[(\forall_i)(\forall s)((pr_1(\gamma_s) \text{ in } a_i^+ \wedge pr_2(\gamma_s) \text{ in } b_j^+) \Rightarrow (\exists t \neq s)(\exists q \neq j)(pr_1(\gamma_t) \text{ in } a_i^+ \wedge pr_2(\gamma_t) \text{ in } b_q^*))]$. It is easy to see that (1) is equivalent to

(2) $(\forall j)(\exists i)(\exists s)[pr_1(\gamma_s) \text{ in } a_i^+ \wedge pr_2(\gamma_s) \text{ in } b_j^+ \wedge (\forall t \neq s)\ (pr_1(\gamma_t) \text{ in } a_i^+ \Rightarrow (\forall q \neq j)\ pr_2(\gamma_t) \text{ not in } b_q^*)]$.

Clearly the statement $(\forall q \neq j)\, pr_2(\gamma_t)$ not in b_q^* is mathematically equivalent to the statement $pr_2(\gamma_t)$ in b_j^+. Thus (2) is equivalent to

(3) $(\forall j)(\exists i)(\exists s)[pr_1(\gamma_s) \text{ in } a_i^+ \wedge pr_2(\gamma_s) \text{ in } b_j^+ \wedge (\forall t \neq s)(pr_1(\gamma_t) \text{ in } a_i^+ \Rightarrow pr_2(\gamma_t) \text{ in } b_j^-)]$.

For each j in $\bar{I}_m$ let $l(j)$ be a specific i whose existence is guaranteed by (3). The set of all such $l(j)$ clearly satisfies (i) and (ii).

Assume (iv) is false. Hence there exists an integer j in $\bar{I}_m$ satisfying (*). Consider the following statement.

(v) For every computation of the form $\beta_\alpha = v_1\gamma_1^{r_1}v_2 \cdots v_p\gamma_p^{r_p}v_{p+1}$ in

$$\Pi_M \cap pr_1^{-1}(L),$$

with $r_i \geqslant 1$ for each i, there exists a computation $\bar{\beta}_\alpha = v_1\gamma_1^{l_1}v_2 \cdots v_p\gamma_p^{l_p}v_{p+1}$ for which $pr_1(\bar{\beta}_\alpha) = pr_1(\beta_\alpha)$, $pr_2(\bar{\beta}_\alpha) = b_1^{g_1} \cdots b_m^{g_m}$, $g_j \leqslant m_4$, and $f(\psi_l(pr_2(\bar{\beta}_\alpha))) \geqslant f(\psi_l(pr_2(\beta_\alpha)))$.

This in particular implies that there exists a computation $\bar{\alpha}$ corresponding to α, with $pr_1(\bar{\alpha}) = pr_1(\alpha)$, $\#_{b_j}(pr_2(\bar{\alpha})) \leqslant m_4$, and $f(\psi_l(pr_2(\bar{\alpha}))) \geqslant f(\psi_l(pr_2(\alpha)))$. Since

$$f(\psi_l(pr_2(\alpha))) \geqslant \bar{k}$$

and $\bar{k} > m_4$, $\#_{b_j}(pr_2(\bar{\alpha})) < f(\psi_l(pr_2(\bar{\alpha})))$. This contradicts the fact that j is in $\bar{I}_m$. Thus (v) is false.

To complete the proof of the lemma we now use the assumption that (iv) is false to show that (v) is true. The argument is by induction on the number of indices s', $1 \leqslant s' \leqslant p$, such that (a) $pr_2(\gamma_{s'})$ is in b_j^+, (b) $pr_1(\gamma_{s'})$ is in a_i^+ for some i, $1 \leqslant i \leqslant n$, and (c) $\gamma_{s'}$ occurs at least k times in β_α. If there is no such s' let $\bar{\beta}_\alpha$ be a computation obtained from β_α by deleting all (if any) occurrences of loops γ_s, $1 \leqslant s \leqslant p$, with $pr_1(\gamma_s) = \epsilon$ and $pr_2(\gamma_s)$ in b_j^+. It is readily seen that $\bar{\beta}_\alpha$ satisfies (v). Suppose $\bar{\beta}_\alpha$ exists for each computation β_α having at most n_0 distinct s', $1 \leqslant s' \leqslant p$, each of which satisfies (a), (b), and (c). Let β_α be a computation having exactly $n_0 + 1$ distinct s', $1 \leqslant s' \leqslant p$, each of which satisfies (a), (b), and (c). Let s be the smallest such integer. Then $pr_1(\gamma_s)$ is in a_i^+ for some i. Since (*) holds for j, there exist $t \neq s$ and $q \neq j$ such that $1 \leqslant t \leqslant p$, $1 \leqslant q \leqslant m$, $pr_1(\gamma_t)$ is in a_i^+ and $pr_2(\gamma_t)$ is in b_q^*. Suppose $q < j$, an analogous argument holding for $q > j$. Then $t < s$, since all b_i are distinct and $M(L) \subseteq b_1^* \cdots b_m^*$, and $\beta_\alpha = w_1\gamma_t^{r_t}w_2\gamma_s^{r_s}w_3$ for appropriate w_1, w_2, and w_3 not containing any occurrence of γ_t or γ_s, by the definition of standard form. By (c), $r_s \geqslant k$.

Thus by Lemma 3.3, there exists $\beta_\alpha' = w_1\gamma_t^{r_t'}w_2\gamma_s^{r_s'}w_3$ such that $0 < r_s' < k$, $r_t' > r_t$, and $pr_1(\beta_\alpha') = pr_1(\beta_\alpha)$. Let $pr_2(\beta_\alpha) = b_1^{k_1} \cdots b_m^{k_m}$ and $pr_2(\beta_\alpha') = b_1^{k_1'} \cdots b_m^{k_m'}$. Clearly $k_i' = k_i$ for any $i \neq q$ and $i \neq j$, $k_j' < k_j$ (since $pr_2(\gamma_s)$ is in b_j^+), and $k_q' \geqslant k_q$. Suppose q is in $\bar{I}_m$. Since j is also in $\bar{I}_m$, $f(\psi_I(pr_2(\beta_\alpha'))) = f(\psi_I(pr_2(\beta_\alpha)))$. Suppose q is in $I = \{i_1, ..., i_r\}$. Then $q = i_u$ for some u, $1 \leqslant u \leqslant r$. From the fact that f is increasing in each variable, $f(k_{i_1}', ..., k_{i_u}', ..., k_{i_r}') = f(k_{i_1}, ..., k_{i_u}', ..., k_{i_r}) \geqslant f(k_{i_1}, ..., k_{i_u}, ..., k_{i_r})$. Thus $f(\psi_I(pr_2(\beta_\alpha'))) \geqslant f(\psi_I(pr_2(\beta_\alpha)))$ whether q is in $\bar{I}_m$ or in I. By construction, the number of indices s', $1 \leqslant s' \leqslant p$, satisfying (a), (b), and (c) in β_α' is n_0. Thus by induction, there exists $\bar{\beta}_\alpha'$ such that $pr_1(\bar{\beta}_\alpha') = pr_1(\beta_\alpha')$, $pr_2(\bar{\beta}_\alpha') = b_1^{g_1} \cdots b_m^{g_m}$, with $f(\psi_I(pr_2(\bar{\beta}_\alpha'))) \geqslant f(\psi_I(pr_2(\beta_\alpha')))$ and $g_j \leqslant m_4$. Since $pr_1(\beta_\alpha') = pr_1(\beta_\alpha)$ and $f(\psi_I(pr_2(\beta_\alpha'))) \geqslant f(\psi_I(pr_2(\beta_\alpha)))$, $pr_1(\bar{\beta}_\alpha') = pr_1(\beta_\alpha)$ and $f(\psi_I(pr_2(\bar{\beta}_\alpha'))) \geqslant f(\psi_I(pr_2(\beta_\alpha)))$. Then $\bar{\beta}_\alpha = \bar{\beta}_\alpha'$ satisfies (v) and the proof is complete.

The above lemma states some properties of loops with output in b_j^+ for some j in $\bar{I}_m$. The next lemma deals with those loops having output in b_j^+ for some j in I.

LEMMA 5.3. *Let M be an a-transducer, $L \subseteq a_1^* \cdots a_n^*$, and $M(L) \subseteq b_1^* \cdots b_m^*$. Let $M(L)$ be an (f, I)-guided language for some f and I. Then for every computation α in $\Pi_M \cap pr_1^{-1}(L)$, for every j in I, and for every loop γ occurring in α, $pr_2(\gamma)$ in b_j^+ implies $pr_1(\gamma) \neq \epsilon$.*

Proof. The proof is similar to that of Lemma 3.4 and is omitted.

We are now in a position to prove the major lemma of this section on the character of loops in a computation.

LEMMA 5.4. *Let M be an a-transducer, $L \subseteq a_1^* \cdots a_n^*$, and $M(L) \subseteq b_1^* \cdots b_m^*$. Let $M(L)$ be distance (f, I)-guided for some f and I. Then there exists an integer $k' \geqslant 0$ with the following property. For every computation $\alpha = v_1\gamma_1^{h_1}v_2 \cdots v_p\gamma_p^{h_p}v_{p+1}$ in standard form in $\Pi_M \cap pr_1^{-1}(L)$, having $f(\psi_I(pr_2(\alpha))) \geqslant k'$ and $\#_{b_j}(pr_2(\alpha)) \geqslant k'$ for each j in I, the following holds. There exist $q_1, ..., q_m$ in $\{1, ..., p\}$ and $t_1, ..., t_m$ in $\{1, ..., n\}$ such that $t_1 < t_2 < \cdots < t_m$, $pr_1(\gamma_{q_i})$ is in $a_{t_i}^+$, and $pr_2(\gamma_{q_i})$ is in b_i^+ for each i, $1 \leqslant i \leqslant m$.*

Proof. Let $\bar{k}$ be the integer guaranteed by Lemma 5.2 and let

$$k' = \max\{\bar{k}, 3m_1m_2m_3 + 1\}.$$

By Lemma 5.1, for every j, $1 \leqslant j \leqslant m$, and for every computation α in standard form in $\Pi_M \cap pr_1^{-1}(L)$, if $\#_{b_j}(pr_2(\alpha)) > 3m_1m_2m_3$, then α contains two occurrences of some loop γ, $pr_2(\gamma)$ in b_j^+. Let $\alpha = v_1\gamma_1^{h_1}v_2 \cdots v_p\gamma_p^{h_p}v_{p+1}$ be a computation in standard form in $\Pi_M \cap pr_1^{-1}(L)$ for which $f(\psi_I(pr_2(\alpha))) \geqslant k'$ and $\#_{b_j}(pr_2(\alpha)) \geqslant k'$ for each j in I. Then for each j in I there exists an integer $s(j)$, $1 \leqslant s(j) \leqslant p$, such that $pr_2(\gamma_{s(j)})$ is in b_j^+. By Lemma 5.3, $pr_1(\gamma_{s(j)}) \neq \epsilon$. For each j in I let $l(j)$, $1 \leqslant l(j) \leqslant n$, be the integer such that $pr_1(\gamma_{s(j)})$ is in $a^+_{l(j)}$. By Lemma 5.2 there exists a set $\{l(j) | j \text{ in } \bar{I}_m, 1 \leqslant l(j) \leqslant n\}$ having the following property. For each j in $\bar{I}_m$ there exists $s(j)$, $1 \leqslant s(j) \leqslant p$, such that $pr_1(\gamma_{s(j)})$ is in $a^+_{l(j)}$ and $pr_2(\gamma_{s(j)})$ is in $b_j{}^+$. Furthermore, by (iii) of Lemma 5.2, $l(j) \neq l(i)$ for $i \neq j$ in $\bar{I}_m$; and by (ii) of Lemma 5.2, $l(j) \neq l(i)$ for j in $\bar{I}_m$ and i in I. By assumption, $| i - j | \geqslant 2$ for $i \neq j$ in I. Thus for each i and j in I, $i < j$, there exists l in $\bar{I}_m$, $i < l < j$. Consequently, $l(i) \neq l(j)$ for $i \neq j$ in I. It readily follows that $l(1) < l(2) < \cdots < l(m)$. Let $t_i = l(i)$ and $q_i = s(i)$, $1 \leqslant i \leqslant m$. Then $t_1 < t_2 < \cdots < t_m$. Clearly $pr_1(\gamma_{q_i})$ is in $a^+_{t_i}$ and $pr_2(\gamma_{q_i})$ is in $b_i{}^+$ for all i, $1 \leqslant i \leqslant m$.

The following lemma is the last lemma on computations in this section. Together with Lemma 5.4 it plays an essential role in the proofs of Section 6.

LEMMA 5.5. *Let M be an a-transducer, $L \subseteq a_1{}^* \cdots a_n{}^*$, and $M(L) \subseteq b_1{}^* \cdots b_m{}^*$. Then there exists an integer k'' with the following property. Let $\alpha = v_1\gamma_1^{h_1}v_2 \cdots v_p\gamma_p^{h_p}v_{p+1}$ be a computation in standard form in $\Pi_M \cap pr_1^{-1}(L)$ and let $B(\alpha) = \{j \mid \gamma_i$ is an a_j-loop for some i, $1 \leqslant i \leqslant p\}$. Let $| B(\alpha)| = q$ and let $B'(\alpha) \subseteq \{1,..., p\}$ be a set of indices such that $\{\gamma_j \mid j$ in $B'(\alpha)\}$ is a set of a_i-loops over q distinct letters. Then there exists a computation $\alpha' = w_1\delta_1^{k_1}w_2 \cdots w_n\delta_n^{k_n}w_{n+1}$ in $\Pi_M \cap pr_1^{-1}(L)$, all $k_i \geqslant 0$, such that*

(i) $pr_1(\alpha') = pr_1(\alpha)$;

(ii) *δ_i is an a_i-loop for all i, $1 \leqslant i \leqslant n$;*

(iii) *for each i in $B(\alpha)$, $\delta_i = \gamma_j$ for some j in $B'(\alpha)$; and*

(iv) *for each loop γ in M and each i, $1 \leqslant i \leqslant n + 1$, w_i contains at most $k'' - 1$ occurrences of γ.*

Proof. Let k be the integer guaranteed by Lemma 3.3 and let $k'' = k + 2m_3$. For each i in $B(\alpha)$ let $\delta_i = \gamma_{j(i)}$, where $j(i)$ is in $B'(\alpha)$ and $\gamma_{j(i)}$ is an a_i-loop. For each i in $\{1,..., n\} - B(\alpha)$ let δ_i be arbitrary a_i-loop in M. (As in Lemma 3.6, we may assume that at least one a_i-loop exists for each i.) The argument is by induction on the number $s(\alpha)$ of indices i in $\{1,..., p\} - B'(\alpha)$ for which $h_i \geqslant k$.

Suppose $s(\alpha) = 0$. Let $k_i = h_{j(i)}$ for i in $B(\alpha)$ and $k_i = 0$ for i in $\{1,..., n\} - B(\alpha)$. Clearly α has the form $\alpha = w_1\delta_1^{k_1}w_2 \cdots w_n\delta_n^{k_n}w_{n+1}$ and $\alpha' = \alpha$ satisfies (i), (ii), and (iii). Consider (iv). Suppose a loop γ in M occurs in w_i for some i. Clearly w_i is of the form $w_i = v_r\gamma_r^{h_r}v_{r+1} \cdots v_{r'}\gamma_{r'}^{h_{r'}}v_{r'+1}$ or of the form $w_i = v_r$ for some $1 \leqslant r \leqslant r' \leqslant p$. Since no v_j contains a loop, there are just three possibilities: (a) γ occurs in $\gamma_j^{h_j}$ for some j, $r \leqslant j \leqslant r'$; (b) γ consists of a terminal subword of some v_j followed by an initial subword of γ_j; or (c) γ consists of a terminal subword of some γ_j followed by an initial subword of v_{j+1}. Since $s(\alpha) = 0$, $h_j < k$ for j, $r \leqslant j \leqslant r'$. Clearly γ cannot occur as a subword of $\gamma_j^{h_j}$ and $\gamma_l^{h_l}$ for $j \neq l$, because α is in standard form. Thus at most $k - 1$ occurrences of γ result in case (a). It is easy to see that at most $r' - r + 1$ occurrences of γ result in each of cases (b) and (c).

Hence there are at most

$$k - 1 + 2(r' - r + 1) \leqslant k - 1 + 2p \leqslant k - 1 + 2m_3 = k'' - 1$$

occurrences of γ in w_i and (iv) holds.

Continuing by induction suppose the lemma holds for all computations α satisfying $s(\alpha) \leqslant s$. Let $s(\alpha) = s + 1$. If $h_j \geqslant k$ and $pr_1(\gamma_j) = \epsilon$ for some j in $\{1,..., p\} - B'(\alpha)$, let $\bar{\alpha}$ be obtained from α by deleting all of the occurrences of γ_j. Then $pr_1(\bar{\alpha}) = pr_1(\alpha)$, $s(\bar{\alpha}) = s$, and the induction is extended. Now suppose none of the indices in $\{1,..., p\} - B'(\alpha)$ is an index of a loop with ϵ input. Let t be the smallest of the indices in $\{1,..., p\} - B'(\alpha)$ for which $h_t \geqslant k$. Then $pr_1(\gamma_t)$ is in a_i^+ for some i in $B(\alpha)$. Suppose $t < j(i)$, an analogous argument holding if $t > j(i)$. Then $\alpha = u_1\gamma_t^{h_t}u_2\gamma_{j(i)}^{h_{j(i)}}u_3$ for appropriate u_1, u_2, and u_3 not containing any occurrence of γ_t or $\gamma_{j(i)}$. By Lemma 3.3, there exists a computation $\bar{\alpha} = u_1\gamma_t^{h'}u_2\gamma_{j(i)}^{h''}u_3$, with $pr_1(\bar{\alpha}) = pr_1(\alpha)$, $h'' \geqslant 1$, and $0 < h' < k$. Hence $s(\bar{\alpha}) = s$ and the induction is extended.

The next lemma exhibits a certain property of languages which are (f, I)-guided and I-uniformly infinite. It is used in the proof of Theorem 6.1.

LEMMA 5.6. *Let $L \subseteq b_1^* \cdots b_m^*$ be (f, I)-guided for some f and I. If L is I-uniformly infinite, then for each $t \geqslant 0$ the set $R(t) = \{b_1^{k_1} \cdots b_m^{k_m}$ in $L \mid k_i > t$ for each i in $I\}$ is not regular.*

Proof. Suppose the contrary. Let $I = \{i_1,..., i_r\}$, $i_1 < \cdots < i_r$. Then the set $S(t) = \{(k_1,..., k_r)$ in $\psi_I(L)\mid$ each $k_i > t\}$ is infinite for each t and $R(t_0)$ is regular for some t_0. By Theorem 1.2 of [9], $R(t_0) = \bigcup_{i=1}^n A_{i1}A_{i2} \cdots A_{im}$ for some integer n and nonempty regular sets A_{ij}, with $A_{ij} \subseteq b_j^*$ for each i and j. Let $s = i_q$ be in I. Since $S(t) \subseteq S(t')$ for $t \geqslant t'$, $\psi_{\{s\}}(R(t_0))$ is infinite. Thus A_{i_0s} is infinite for some i_0, $1 \leqslant i_0 \leqslant n$. Let $b_1^{k_1},..., b_m^{k_m}$ be some words in $A_{i_01},..., A_{i_0m}$ respectively. Then

$$b_1^{k_1} \cdots b_{s-1}^{k_{s-1}}A_{i_0s}b_{s+1}^{k_{s+1}} \cdots b_m^{k_m} \subseteq R(t_0) \subseteq L.$$

Let $k = \max\{k_i \mid 1 \leqslant i \leqslant m\}$. Now f is increasing and unbounded in each variable. Thus $f(0,..., 0, p, 0,..., 0) > k$ (p in the qth coordinate) for some $p \geqslant 0$. Since A_{i_0s} is infinite, $b_s^{k'}$ is in A_{i_0s} for some $k' \geqslant p$. Then $w = b_1^{k_1} \cdots b_{s-1}^{k_{s-1}}b_s^{k'}b_{s+1}^{k_{s+1}} \cdots b_m^{k_m}$ is in $R(t_0)$ and hence in L. Let s' be in $\bar{I}_m$. Then

$$f(\psi_I(w)) \geqslant f(0,..., 0, p, 0,..., 0) > k \geqslant k_{s'},$$

contradicting the fact that L is (f, I)-guided.

For $t = 0$ the above lemma yields

COROLLARY. *Let $L \subseteq b_1{}^* \cdots b_m{}^*$ be (f, I)-guided and I-uniformly infinite for some f and I. Then L is not regular.*

In Section 4 we showed that a language $\langle b_1^{g_1(x)} \cdots b_m^{g_m(x)} \rangle$ is in $\mathscr{M}(L)$ $[\hat{\mathscr{M}}(L)]$ if and only if it is in $\mathscr{F}(L)$ $[\hat{\mathscr{F}}(L)]$. This enabled us essentially to consider only $\mathscr{M}(L)$ and $\hat{\mathscr{M}}(L)$. As we shall see, a similar result holds in most cases for languages which are distance (f, I)-guided and I-uniformly infinite. The key point in the argument is the following theorem [6].

THEOREM (Ginsburg and Greibach). *Let L be a language over an alphabet not containing the symbol c. Then*

$$\mathscr{F}(L \cup \{\epsilon\}) = \mathscr{M}((Lc)^*) \qquad \text{and} \qquad \hat{\mathscr{F}}(L) = \hat{\mathscr{M}}((Lc)^*).$$

If $L \neq \varnothing$, then $\mathscr{F}(L - \{\epsilon\}) = \mathscr{M}((Lc)^+)$.

LEMMA 5.7. *Let $U \subseteq a_1{}^* \cdots a_n{}^*$ and $V \subseteq b_1{}^* \cdots b_m{}^*$, with V in $\mathscr{F}(U)$ $[\hat{\mathscr{F}}(U)]$. Let V be (f, I)-guided and I-uniformly infinite for some f and I, with $\bar{I}_m \cap \{1, m\} \neq \varnothing$ if $|\bar{I}_m| = 1$. Then there exists an (f, I)-guided and I-uniformly infinite language $T \subseteq V$ in $\mathscr{M}(U)$ $[\hat{\mathscr{M}}(U)]$.*

Proof. We shall prove the lemma for $\mathscr{F}(U)$, an analogous argument holding for $\hat{\mathscr{F}}(U)$.

By the Corollary to Lemma 5.6, V is not regular. Since V is in $\mathscr{F}(U)$, U is not regular. Thus $U \neq \varnothing$. Let c be a new symbol. By the above theorem of Ginsburg and Greibach, $\mathscr{F}(U) = \mathscr{M}((Uc)^+)$ if ϵ is not in U and $\mathscr{F}(U) = \mathscr{M}((Uc)^*)$ if ϵ is in U. We shall assume the latter an analogous but simpler argument holding for the other case. Thus $V = M((Uc)^*)$ for some ϵ-free a-transducer $M = (K, \{a_1, \ldots, a_n, c\}, \{b_1, \ldots, b_m\}, H, p_0, F)$. For each p and q in K let M_{pq} be the a-transducer $(K, \{a_1, \ldots, a_n, c\}, \{b_1, \ldots, b_m\}, H, p, \{q\})$ and let $L(p, q) = M_{pq}(Uc)$. Let Q be the set of all finite sequences $p_0, \ldots, p_j$ of states in K such that $j \geqslant 1$, p_j is in F, and $L(p_i, p_{i+1}) \neq \varnothing$, $0 \leqslant i < j$. Clearly

$$(1) \quad V = \bigcup_{p_0, \ldots, p_j \text{ in } Q} L(p_0, p_1) \cdots L(p_{j-1}, p_j) \cup R,$$

where $R = M(\{\epsilon\})$ is regular.

We now establish the following.

(2) Let $p_0, \ldots, p_j$ be a sequence in Q and let z, $0 \leqslant z < j$, be such that $L(p_z, p_{z+1})$ contains a word with an occurrence of b_i for some i in I. Then there is no finite sequence $q_1, \ldots, q_s$ of states in K having $q_1 = p_{z+1}$, $q_s = p_z$, $s \geqslant 1$, and

$$L(q_t, q_{t+1}) \neq \varnothing, \quad 1 \leqslant t < s.$$

(3) Let $p_0, \ldots, p_j$ be a sequence in Q and let z, $0 \leqslant z < j$, be such that

$$\psi_{\{i\}}(L(p_z, p_{z+1}))$$

is infinite for some i in I. Then $\psi_{\{t\}}(L(p_z, p_{z+1}))$ is infinite for each t in $\bar{I}_m$.

(4) Let $p_0, \ldots, p_j$ be a sequence in Q. Then there exists at most one t, $0 \leqslant t < j$, having $\psi_{\{i\}}(L(p_t, p_{t+1}))$ infinite for some i in I.

Consider (2). Suppose the contrary. Let $q_1, \ldots, q_k$ be a shortest sequence having $q_1 = p_{z+1}$, $q_k = p_z$, and $L(q_t, q_{t+1}) \neq \varnothing$, $1 \leqslant t < k$. Clearly

$$L(p_0, p_1) \cdots L(p_z, p_{z+1})(L(q_1, q_2) \cdots L(q_{k-1}, q_k) L(p_z, p_{z+1}))^* \cdot L(p_{z+1}, p_{z+2}) \cdots L(p_{j-1}, p_j) \subseteq M((Uc)^*) = V.$$

Since $V \subseteq b_1^* \cdots b_m^*$, with all b_t distinct, and $L(p_z, p_{z+1})$ contains a word with an occurrence of b_i, $L(q_1, q_2) \cdots L(p_z, p_{z+1}) \subseteq b_i^*$ and it contains a non-ϵ word b_i^s, $s \geqslant 1$. Let w_1 be in $L(p_0, p_1) \cdots L(p_z, p_{z+1})$ and w_2 in

$$L(p_{z+1}, p_{z+2}) \cdots L(p_{j-1}, p_j).$$

(w_2 is assumed to be ϵ if $z + 2 > j$. Similarly, in analogous cases to appear.) Then $w_1(b_i^s)^* w_2 \subseteq V$. Let i_0 be in $\bar{I}_m$. Since $s \geqslant 1$ and f is increasing and unbounded in each variable, there exists $l \geqslant 0$ for which

$$\#_{b_{i_0}}(w_1 w_2) = \#_{b_{i_0}}(w_1 b_i^{ls} w_2) < f(\psi_I(w_1 b_i^{ls} w_2)).$$

This contradicts the fact that V is (f, I)-guided. Hence (2) holds.

Consider (3). Suppose the contrary. Then there exists i_0 in $\bar{I}_m$ and a positive integer s with the property that $\#_{b_{i_0}}(w) < s$ for each w in $L(p_z, p_{z+1})$. Let w_1 be in

$$L(p_0, p_1) \cdots L(p_{z-1}, p_z)$$

and w_2 in $L(p_{z+1}, p_{z+2}) \cdots L(p_{j-1}, p_j)$. Then $w_1 L(p_z, p_{z+1}) w_2 \subseteq V$. Since

$$\psi_{\{i\}}(L(p_z, p_{z+1}))$$

is infinite and f is increasing and unbounded in each variable, there exists a word v in $L(p_z, p_{z+1})$ for which $f(\psi_I(w_1 v w_2)) > \#_{b_{i_0}}(w_1 w_2) + s$. Then

$$\#_{b_{i_0}}(w_1 v w_2) = \#_{b_{i_0}}(w_1 w_2) + \#_{b_{i_0}}(v) < \#_{b_{i_0}}(w_1 w_2) + s < f(\psi_I(w_1 v w_2)),$$

contradicting the fact that V is (f, I)-guided. Hence (3) holds.

Consider (4). Suppose the contrary. Then there exist t_1 and t_2, $0 \leqslant t_1 < t_2 < j$, and i_1, i_2 in I for which $\psi_{\{i_1\}}(L(p_{t_1}, p_{t_1+1}))$ and $\psi_{\{i_2\}}(L(p_{t_2}, p_{t_2+1}))$ are both infinite. By (3), $\psi_{\{k\}}(L(p_{t_1}, p_{t_1+1}))$ and $\psi_{\{k\}}(L(p_{t_2}, p_{t_2+1}))$ are both infinite for each k in $\bar{I}_m$. Let w_1 be in $L(p_0, p_1) \cdots L(p_{t_1-1}, p_{t_1})$, w_2 in $L(p_{t_1+1}, p_{t_1+2}) \cdots L(p_{t_2-1}, p_{t_2})$, and w_3 in $L(p_{t_2+1}, p_{t_2+2}) \cdots L(p_{j-1}, p_j)$. Two cases arise:

(a) $|\bar{I}_m| = 1$. By the hypothesis, $\bar{I}_m = \{1\}$ or $\bar{I}_m = \{m\}$. Assume the former, an analogous argument holding if the latter. Now $L(p_{t_2}, p_{t_2+1})$ contains a word v_2 with an occurrence of b_1 and $L(p_{t_1}, p_{t_1+1})$ contains a word v_1 with an occurrence of b_{i_1}. Since i_1 is in I, $i_1 > 1$. But $w_1 v_1 w_2 v_2 w_3$ is in V, contradicting the fact that $V \subseteq b_1{}^* \cdots b_m{}^*$.

(b) $|\bar{I}_m| > 1$. Thus there exist j_1 and j_2 in $\bar{I}_m$, $j_1 < j_2$. By (3), $L(p_{t_1}, p_{t_1+1})$ contains a word v_1 with an occurrence of b_{j_2} and $L(p_{t_2}, p_{t_2+1})$ contains a word v_2 with an occurrence of b_{j_1}. But $w_1 v_1 w_2 v_2 w_3$ is in V, contradicting the fact that $V \subseteq b_1{}^* \cdots b_m{}^*$. Thus (4) holds.

A sequence of states $q_1, \ldots, q_k$ is said to be a *cycle*, if $q_1 = q_k$ and $k \geqslant 2$. A sequence $p_0, \ldots, p_j$ in Q is said to *contain a cycle*, if $p_i = p_k$ for some i and k, $0 \leqslant i < k \leqslant j$. By (2),

(5) If a sequence $p_0, \ldots, p_j$ in Q contains a cycle $p_i, p_{i+1}, \ldots, p_k$, then no word in $L(p_i, p_{i+1}) \cdots L(p_{k-1}, p_k)$ contains an occurrence of a letter b_t, t in I.

Let

$$Q' = \{p_0, \ldots, p_j \text{ in } Q \mid p_i \neq p_k \text{ for } i \neq k\}.$$

By (5),

$$(6) \quad \psi_I \left\{ \bigcup_{p_0, \ldots, p_j \text{ in } Q} L(p_0, p_1) \cdots L(p_{j-1}, p_j) \right\} = \psi_I \left\{ \bigcup_{p_0, \ldots, p_j \text{ in } Q'} L(p_0, p_1) \cdots L(p_{j-1}, p_j) \right\}.$$

By (6), (1), Lemma 5.6, and the fact that V is I-uniformly infinite,

$$(7) \quad \left\{ (k_1, \ldots, k_r) \text{ in } \psi_I \left(\bigcup_{p_0, \ldots, p_j \text{ in } Q'} L(p_0, p_1) \cdots L(p_{j-1}, p_j) \right) \middle| \text{ each } k_i > t \right\}$$

is infinite for each $t \geqslant 0$. Since K is finite, Q' is finite. Thus by (4) and (7),

(8) There exist a sequence $p_0, \ldots, p_j$ in Q' and an integer z, $0 \leqslant z < j$, having $\{(k_1, \ldots, k_r) \text{ in } \psi_I(L(p_z, p_{z+1})) \mid \text{each } k_i > t\}$ infinite for each $t \geqslant 0$. By (3),

$$\psi_{(k)}(L(p_z, p_{z+1}))$$

is infinite for each k in $\bar{I}_m$. Since $V \subseteq b_1^* \cdots b_m^*$, $L(p_0, p_1) \cdots L(p_{z-1}, p_z) \subseteq b_1^*$ and $L(p_{z+1}, p_{z+2}) \cdots L(p_{j-1}, p_j) \subset b_m^*$. Let w_1 and w_2 be words in $L(p_0, p_1) \cdots L(p_{z-1}, p_z)$ and $L(p_{z+1}, p_{z+2}) \cdots L(p_{j-1}, p_j)$, respectively. Since $w_1 L(p_z, p_{z+1}) w_2 \subseteq V$ and V is (f, I)-guided, $w_1 L(p_z, p_{z+1}) w_2$ is (f, I)-guided. By definition, $L(p_z, p_{z+1})$ is in $\mathscr{M}(Uc)$. Thus $L(p_z, p_{z+1})$ is in $\mathscr{M}(U)$. It readily follows that $w_1 L(p_z, p_{z+1}) w_2$ is in $\mathscr{M}(U)$. By (8), $w_1 L(p_z, p_{z+1}) w_2$ is I-uniformly infinite. Hence $T = w_1 L(p_z, p_{z+1}) w_2$ satisfies the lemma and the proof is complete.

Remark. In case V is distance (f, I)-guided, the condition $\bar{I}_m \cap \{1, m\} \neq \varnothing$ if $|\bar{I}_m| = 1$ is equivalent to the condition that if $m = 3$ then $I \neq \{1, 3\}$.

We conclude the section by pointing out that, in general, Lemma 5.7 does not hold in the case $|\bar{I}_m| = 1$ and $\bar{I}_m \cap \{1, m\} = \varnothing$.

EXAMPLE 5.2. Let $U = \{a_1^i a_2^i \mid i \geqslant 1\}$ and $V = \{b_1^i b_2^j b_3^k \mid i, k \geqslant 1, j \geqslant i + k\}$. For $f(x, y) = x + y$ and $I = \{1, 3\}$, V is the distance (f, I)-guided and I-uniformly infinite. It is easy to see that $L_1 = \{b_1^i b_2^j \mid j \geqslant i \geqslant 1\}$ and $L_2 = \{b_2^i b_3^j \mid i \geqslant j \geqslant 1\}$ are both in $\mathscr{M}(U)$. Since $V = L_1 L_2$, V is in $\mathscr{F}(U)$. It will be shown shortly (Corollary of Theorem 6.1) that if $L \subseteq b_1^* \cdots b_m^*$ is a distance (f, I)-guided and I-uniformly infinite language in $\mathscr{M}(U)$, then $m \leqslant 2$. Thus no distance (f, I)-guided and I-uniformly infinite language $T \subseteq V$ exists in $\mathscr{M}(U)$. Hence Lemma 5.7 does not hold in this case.

6. Necessary Conditions on (f, I)-Guided and I-Uniformly Infinite Languages

In the present section we employ the lemmas and notation of the previous section to establish some necessary conditions on U and V, V distance (f, I)-guided and I-uniformly infinite and U bounded, in order that V be in $\mathscr{M}(U)$, $\hat{\mathscr{M}}(U)$, $\mathscr{F}(U)$, or $\hat{\mathscr{F}}(U)$. The principal results are Theorems 6.1 and 6.3. Their consequences, Theorems 6.2 and 6.4, respectively, are of a more practical nature.

We now turn to the first theorem. The basic underlying idea here (as well as for Theorem 6.3) is the following. If $V = M(U)$ and L is a subset of U, chosen so that M has an output for an infinite number of words in L, then there exists a corresponding subset L' of V which is obtained from L using computations of the type α' described in Lemma 5.5. More precisely we have

Notation. Let $1 \leqslant m \leqslant n$, $c = (c_1, \ldots, c_n)$ and $l = (l_1, \ldots, l_n)$ be in N^n, and

$z = (z_1, \ldots, z_{m+1})$ in N^{m+1}, with $1 = z_1 < z_2 < \cdots < z_{m+1} = n + 1$. For $L \subseteq a_1^* \cdots a_n^*$, let

$$\mathcal{N}(L, c, l, z) = \Big\{(k_1, \ldots, k_n) \mid a_1^{c_1+k_1l_1} \cdots a_n^{c_n+k_nl_n} \text{ is in } L,\ k_i \geqslant 0 \text{ for each } i,$$
$$\text{and } \sum_{s=z_j}^{z_{j+1}-1} k_s \geqslant 1 \text{ for each } j,\ 1 \leqslant j \leqslant m\Big\}.$$

(Compare with $\mathcal{K}(L, c, l)$ in Section 4.)

THEOREM 6.1. *Let $U \subseteq a_1^* \cdots a_n^*$, $V \subseteq b_1^* \cdots b_m^*$, and let V be* in $\mathcal{M}(U)$ $[\mathcal{F}(U)]$. *Let V be distance (f, I)-guided and I-uniformly infinite for some f and I [and let $I \neq \{1, 3\}$ if $m = 3$]. Then for each subset $L \subseteq U$ with the property that $U - R$ is regular for every regular set R containing L, there exist n-*tuples *$c = (c_1, \ldots, c_n)$, $l = (l_1, \ldots, l_n)$, $p = (p_1, \ldots, p_n)$* in *N^n, an m-*tuple *$d = (d_1, \ldots, d_m)$* in *N^m, and an $(m + 1)$-*tuple *$z = (z_1, \ldots, z_{m+1})$* in *$N^{m+1}$ satisfying the following.*

(i) $1 = z_1 < \cdots < z_{m+1} = n + 1$;

(ii) $p_i > 0$ *and* $l_i > 0$, $1 \leqslant i \leqslant n$;

(iii) $a_1^{c_1} \cdots a_n^{c_n}$ *is in* L;

(iv) $b_1^{d_1} \cdots b_m^{d_m}$ *is in* V;

(v) $\mathcal{N}(L, c, l, z)$ *is infinite*; *and*

(vi) $\{(d_1 + \sum_{i=z_1}^{z_2+1} k_ip_i, \ldots, d_m + \sum_{i=z_m}^{z_{m+1}-1} k_ip_i) \mid k_1, \ldots, k_n)$ in $\mathcal{N}(L, c, l, z)\} \subseteq \psi(V)$.

Proof. Suppose V is in $\mathcal{M}(U)$. Then there exists an ϵ-free a-transducer M, with $M(U) = V$. Let $I = \{i_1, \ldots, i_r\}$ and $L \subseteq U$ satisfy the hypothesis of the theorem.

For each $t \geqslant 0$ let

$$S(t) = \{(k_1, \ldots, k_r) \text{ in } \psi_I(pr_2(\Pi_M \cap pr_1^{-1}(L))) \mid \text{each } k_i > t\}.$$

Then

(1) $S(t)$ is infinite for each $t \geqslant 0$.

For, suppose the contrary. Then $S(t) = \varnothing$ for some $t \geqslant 0$. The set

$$R(t) = \{b_1^{k_1} \cdots b_m^{k_m} \mid k_j > t \text{ for all } j \text{ in } I\}$$

is clearly regular. Thus $pr_2^{-1}(R(t)) \cap \Pi_M$ and $R'(t) = pr_1(pr_2^{-1}(R(t)) \cap \Pi_M)$ are both regular. $R'(t)$ is the set of all inputs for which there is at least one computation with an output in $R(t)$. There may be some words in L on which M "blocks." It is convenient to eliminate these words. Let $\bar{L} = pr_1(\Pi_M \cap pr_1^{-1}(L))$ and

$$\bar{R} = a_1^* \cdots a_n^* - pr_1(\Pi_M).$$

Since $S(t) = \varnothing$, $\bar{L} \subseteq \bar{\bar{R}}$, where $\bar{\bar{R}} = a_1^* \cdots a_n^* - R'(t)$ is the regular set of all inputs for which there is no computation with an output in $R(t)$. Clearly $L \subseteq \bar{R} \cup \bar{\bar{R}}$ and, by the assumption on L, $U - (\bar{R} \cup \bar{\bar{R}})$ is regular. It easily follows that the set

$$pr_2(\Pi_M \cap pr_1^{-1}(U - (\bar{R} \cup \bar{\bar{R}}))) \cap R(t) = \{b_1^{k_1} \cdots b_m^{k_m} \text{ in } V \mid k_j > t \text{ for each } j \text{ in } I\}$$

is regular. By Lemma 5.6, this contradicts the fact that V is I-uniformly infinite.

For each $t \geqslant 0$ let $A(t)$ be the set of all computations α in standard form in $\Pi_M \cap pr_1^{-1}(L)$, with $\#_{b_j}(pr_2(\alpha)) > t$ for each j in I. Then $A(t)$ is infinite by (1). Furthermore,

(2) The set $pr_1(A(t))$ is infinite for each $t \geqslant 0$.

For, suppose $pr_1(A(t_0))$ is finite for some $t_0 \geqslant 0$. Since $pr_1(A(t_1)) \subseteq pr_1(A(t_2))$ for $t_1 \leqslant t_2$, $pr_1(A(t)) = \varnothing$ for some $t \geqslant 0$. By Lemma 3.2, every computation in $\Pi_M \cap pr_1^{-1}(U)$ can be replaced by a computation in standard form having the same input and output. Hence $L \subseteq \bar{R} \cup \bar{\bar{R}}$. As above, this yields a contradiction of the fact that V is I-uniformly infinite.

Let W be the set of all subwords w of words in Π_M such that w does not contain a loop, $pr_1(w)$ is in $a_1^* \cdots a_n^*$, and $pr_2(w)$ is in $b_1^* \cdots b_m^*$. Clearly W is finite. Let Q be the set of all loops in M. As noted in Section 3, Q is finite. Let

$$Q' = \bigcup_{p=1}^{m_3} WeQ_1eWeQ_2eW \cdots WeQ_peW \cup W,$$

where $Q_i = Q$ for each i and e is a new symbol. Then Q' is finite. Let k' be the integer guaranteed by Lemma 5.4. Let j_0 be an integer such that $j_0 \geqslant k'$ and $f(j_0, ..., j_0) \geqslant k'$. Since f is increasing and unbounded in each variable, j_0 exists. For each

$$\alpha = w_{\alpha 1}e\gamma_{\alpha 1}ew_{\alpha 2} \cdots w_{\alpha p(\alpha)}e\gamma_{\alpha p(\alpha)}w_{\alpha(p(\alpha)+1)} \text{ in } Q'$$

and each $t \geqslant 0$, let

$$Q(\alpha) = \{w_{\alpha 1}\gamma_{\alpha 1}^{k_1}w_{\alpha 2} \cdots w_{\alpha p(\alpha)}\gamma_{\alpha p(\alpha)}^{k_{p(\alpha)}}w_{\alpha(p(\alpha)+1)} \mid \text{each } k_i \geqslant 1\} \cap A(j_0)$$

and

$$S(\alpha, t) = \{(k_1, ..., k_r) \text{ in } \psi_I(pr_2(Q(\alpha))) \mid \text{each } k_i > t\}.$$

Clearly $A(j_0) = \bigcup_{\alpha \text{ in } Q'} Q(\alpha)$. Then

(3) There exists β in Q' for which $pr_1(Q(\beta))$ is infinite and the set $S(\beta, t)$ is infinite for each $t \geqslant 0$.

For, suppose the contrary. From the definition of $A(j_0)$, $\psi_I(pr_2(A(j_0))) = S(j_0)$. By (1), $S(j_0)$ is infinite. Thus $\psi_I(pr_2(A(j_0)))$ is infinite. Since Q' is finite, there exists β_0 in Q'

for which $S(\beta_0, t)$ is infinite for each $t \geqslant 0$. Since (3) is assumed to be false, $pr_1(Q(\beta))$ is finite for each β in Q' having $S(\beta, t)$ infinite for each $t \geqslant 0$. Let $t_0 \geqslant j_0$ be an integer with the property that if $S(\beta, t_0) \neq \varnothing$ then $S(\beta, t)$ is infinite for each $t \geqslant 0$. Clearly t_0 exists. [For, if $S(\beta, t)$ is finite for some t, then there exists $t(\beta)$ satisfying

$$S(\beta, t(\beta)) = \varnothing.$$

Let $t_0 = \max(\{t(\beta) \mid \beta \text{ in } Q',\ S(\beta, t) \text{ finite for some } t\} \cup \{j_0\})$. Since Q' is finite, t_0 exists.] Since $t_0 \geqslant j_0$, $A(t_0) \subseteq A(j_0)$. Thus

$$A(t_0) \subseteq \bigcup_{\substack{S(\beta, t_0) \neq \varnothing \\ \beta \text{ in } Q'}} Q(\beta).$$

Then

$$pr_1(A(t_0)) \subseteq pr_1\Big(\bigcup_{\substack{S(\beta, t_0) \neq \varnothing \\ \beta \text{ in } Q'}} Q(\beta)\Big)$$

is finite, contradicting (2). Hence (3) holds.

Let $\beta = w_{\beta 1} e \gamma_{\beta 1} e w_{\beta 2} \cdots w_{\beta p(\beta)} e \gamma_{\beta p(\beta)} e w_{\beta(p(\beta)+1)}$ be in Q' and let β satisfy (3). By Lemma 5.4 there exist $q_1, \ldots, q_m$ in $\{1, \ldots, p(\beta)\}$ and $t_1, \ldots, t_m$ in $\{1, \ldots, n\}$, $t_1 < t_2 < \cdots < t_m$, such that $pr_1(\gamma_{q_i})$ is in $a_{t_i}^+$ and $pr_2(\gamma_{q_i})$ is in $b_i{}^+$ for each i, $1 \leqslant i \leqslant m$. Let $B(\beta) = \{j \mid pr_1(\gamma_i) \text{ in } a_j{}^- \text{ for some } i,\ 1 \leqslant i \leqslant p(\beta)\}$. Clearly $\{t_1, \ldots, t_m\} \subseteq B(\beta)$. Let $q = |B(\beta)|$. Let $B'(\beta) = \{i_1, \ldots, i_q\} \subseteq \{1, \ldots, p(\beta)\}$ be any set such that (α) $\{q_1, \ldots, q_m\} \subseteq B'(\beta)$, ($\beta$) $pr_1(\gamma_j) \neq \epsilon$ for each j in $B'(\beta)$, and (γ) γ_{i_r} and γ_{i_t} have input over distinct letters for all $r \neq t$ in $B'(\beta)$.

We shall apply Lemma 5.5 to computations in $Q(\beta)$. Let k'' be the integer guaranteed by Lemma 5.5. Let W' be the set of all subwords w of words in Π_M such that w contains at most $k'' - 1$ occurrences of each loop in M, $pr_1(w)$ is in $a_1{}^* \cdots a_n{}^*$, and $pr_2(w)$ is in $b_1{}^* \cdots b_m{}^*$. Obviously W' is finite. Let c be a new symbol and let

$$W'' = W'cQ_1cW'cQ_2cW' \cdots W'cQ_ncW',$$

where for each i, Q_i is the set of all a_i-loops in M. (As in Lemma 3.6, we may assume that at least one a_i-loop exists for each i.) Since W' and all Q_i are finite, W'' is finite. For each $v = w_{v1} c \delta_{v1} c w_{v2} \cdots w_{vn} c \delta_{vn} c w_{v(n+1)}$ in W'' and each $t \geqslant 0$ let

$$P(v) = \{w_{v1} \delta_{v1}^{k_1} w_{v2} \cdots w_{vn} \delta_{vn}^{k_n} w_{v(n+1)} \mid \text{all } k_i \geqslant 0\} \cap pr_1^{-1}(L) \cap \Pi_M$$

and

$$S'(v, t) = \{(k_1, \ldots, k_r) \text{ in } \psi_I(pr_2(P(v))) \mid \text{each } k_i > t\}.$$

By Lemma 5.5, for each computation ω in $Q(\beta)$ there exists v in W'' and ω' in $P(v)$ satisfying $pr_1(\omega') = pr_1(\omega)$. Since $S(\beta, t)$ is infinite for each $t \geqslant 0$ and W'' is finite,

(4) There exists u in W'' having $S'(u, t)$ infinite for each $t \geqslant 0$.

Since $pr_1(\delta_{ui}) \neq \epsilon$, $1 \leqslant i \leqslant n$, (4) implies $pr_1(P(u))$ is infinite.

Let g the mapping from N^n onto the set of computations

$$\{w_{u1}\delta_{u1}^{k_1}w_{u2} \cdots w_{un}\delta_{un}^{k_n}w_{u(n+1)} \mid \text{each } k_i \geqslant 0\}$$

defined by $g((k_1, ..., k_n)) = w_{u1}\delta_{u1}^{k_1}w_{u2} \cdots w_{un}\delta_{un}^{k_n}w_{u(n+1)}$. The function g is one-to-one since the δ_{ui} are nonassociated. The set $P(u)$ is infinite, because $pr_1(P(u))$ is infinite. Thus the set of n-tuples $g^{-1}(P(u))$ is infinite. Let $\prec$ be the partial ordering on N^n defined by $(k_1, ..., k_n) \prec (k_1', ..., k_n')$ if $k_i < k_i'$ for each i, $1 \leqslant i \leqslant n$. Then the set of minimal elements of $g^{-1}(P(u))$ is finite [15]. This and the fact that $S'(u, t)$ is infinite for each $t \geqslant 0$ imply that there exists an infinite chain Z in $g^{-1}(P(u))$, with $S''(t)$ infinite for each $t \geqslant 0$, where

$$S''(t) = \{(k_1, ..., k_r) \text{ in } \psi_l(pr_2(g(Z))) \mid \text{each } k_i > t\}.$$

Let $z = (z_1, ..., z_m, n + 1)$, where z_i, $1 \leqslant i \leqslant m$, is the smallest integer such that $pr_2(\delta_{uz_i})$ is in b_i^+. Since $pr_2(\gamma_{q_i})$ is in b_i^+ and γ_{q_i} is in $\{\delta_{uj} \mid 1 \leqslant j \leqslant n\}$ by the definition of $B'(\beta)$, z_i exists for each i, $1 \leqslant i \leqslant m$. Let $z_{m+1} = n + 1$. Obviously

$$1 = z_1 < \cdots < z_{m+1} = n + 1.$$

Let

$$Z' = \left\{(k_1, ..., k_n) \text{ in } N^n \mid \tau + (k_1, ..., k_n) \text{ in } Z, \sum_{s=z_j}^{z_{j+1}-1} k_s \geqslant 1, 1 \leqslant j \leqslant m\right\},$$

where τ is the smallest element of Z. Since $S''(t)$ is infinite for each $t \geqslant 0$, Z' is infinite. Let $c_i = \#_{a_i}(pr_1(g(\tau)))$, $d_j = \#_{b_j}(pr_2(g(\tau)))$, $l_i = |pr_1(\delta_{ui})|$, and $p_i = |pr_2(\delta_{ui})|$ for each i and j, $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant m$. Let $c = (c_1, ..., c_n)$, $d = (d_1, ..., d_m)$, $l = (l_1, ..., l_n)$, and $p = (p_1, ..., p_n)$. Now $pr_1(\delta_{ui}) \neq \epsilon$ for each i, because δ_{ui} is an a_i-loop. Since M is ϵ-free, $pr_2(\delta_{ui}) \neq \epsilon$ for each i. Thus $l_i > 0$ and $p_i > 0$ for each i. Since $g(\tau)$ is in $g(T) \subseteq P(u) \subseteq pr_1^{-1}(L) \cap \Pi_M$, $pr_1(g(\tau))$ is in L and $pr_2(g(\tau))$ is in $M(L) \subseteq M(U) = V$. Since $g(Z) \subseteq pr_1^{-1}(L)$, $Z' \subseteq \mathscr{N}(L, c, l, z)$. Hence $\mathscr{N}(L, c, l, z)$ is infinite. For each $(c_1 + k_1l_1, ..., c_n + k_nl_n)$ having $(k_1, ..., k_n)$ in $\mathscr{N}(L, c, l, z)$, $\alpha = g(\tau + (k_1, ..., k_n))$ is in $P(u)$ and is such that $pr_1(\alpha)$ is in L,

$$\psi(pr_1(\alpha)) = (c_1 + k_1l_1, ..., c_n + k_nl_n),$$

and

$$\psi(pr_2(\alpha)) = \left(d_1 + \sum_{i=z_1}^{z_2-1} k_ip_i, ..., d_m + \sum_{i=z_m}^{z_{m+1}-1} k_ip_i\right).$$

Thus

$$\left(d_1 + \sum_{i=z_1}^{z_2-1} k_i p_i ,..., d_m + \sum_{i=z_m}^{z_{m+1}-1} k_i p_i\right)$$

is in $\psi(V)$ for each $(k_1 ,..., k_n)$ in $\mathscr{N}(L, c, l, z)$, and the theorem holds.

Now suppose V is in $\mathscr{F}(U)$. By Lemma 5.7, there exists a distance (f, I)-guided and I-uniformly infinite language $T \subseteq V$ in $\mathscr{M}(U)$. By the first part of the argument, c, l, p, d, and z exist for T satisfying (i)–(vi) of the theorem. Since $T \subseteq V$, $\psi(T) \subseteq \psi(V)$. Thus it is readily seen that these c, l, p, d, and z satisfy (i)–(vi) of the theorem, with V. Hence the proof is complete.

COROLLARY. *Let $U \subseteq a_1^* \cdots a_n^*$, $V \subseteq b_1^* \cdots b_m^*$, and let V be in $\mathscr{M}(U)$ $[\mathscr{F}(U)]$. Let V be distance (f, I)-guided and I-uniformly infinite for some f and I [and let $I \neq \{1, 3\}$ if $m = 3\}$. Then $m \leqslant n$.*

Theorem 6.1 provides a necessary condition in terms of the existence of a subset L' of V obtained from certain computations of a subset L of U. Our next result (as well as Theorem 6.4), gives a technique for showing that the above-mentioned subset L' cannot exist. The net effect of this theorem is to demonstrate that V is not in $\mathscr{M}(U)$ $[\mathscr{F}(U)]$ for a number of pairs (U, V).

THEOREM 6.2. *Let $U \subseteq a_1^* \cdots a_n^*$, $V \subseteq b_1^* \cdots b_m^*$,* and let V be in $\mathscr{M}(U)$ $[\mathscr{F}(U)]$. *Let V be distance (f, I)-guided and I-uniformly infinite for some f and I [and $I \neq \{1, 3\}$ if $m = 3$]. Let L be a subset of U such that $U - R$ is regular for each regular set R containing L. Then integers $z_1 ,..., z_{m+1}$, $1 = z_1 < \cdots < z_{m+1} = n + 1$, exist satisfying the following. For every pair of integers i and j $(1 \leqslant i, j \leqslant m)$ and for all positive real numbers p and q, positive real numbers r_{ij} and r'_{ij} can be found such that for $\rho_j = \sum_{s=z_j}^{z_{j+1}-1} t_s$,*

$$\text{(a)} \qquad \liminf_{\substack{(t_1,...,t_m) \text{ in } \psi(V) \\ t_j \neq 0}} \frac{(t_i)^p}{(t_j)^q} \leqslant r_{ij} \limsup_{\substack{(t_1,...,t_n) \text{ in } \psi(L) \\ \rho_j \neq 0}} \frac{(\rho_i)^p}{(\rho_j)^q};$$

and

$$\text{(b)} \qquad \liminf_{\substack{(t_1,...,t_n) \text{ in } \psi(L) \\ \rho_j \neq 0}} \frac{(\rho_i)^p}{(\rho_j)^q} \leqslant r'_{ij} \limsup_{\substack{(t_1,...,t_m) \text{ in } \psi(V) \\ t_j \neq 0}} \frac{(t_i)^p}{(t_j)^q}.$$

Proof. We prove the theorem for $\mathscr{M}(U)$ and (a), the remaining cases being analogous. By Theorem 6.1, there exist $c = (c_1 ,..., c_n)$, $l = (l_1 ,..., l_n)$, $p = (p_1 ,..., p_n)$, $d = (d_1 ,..., d_m)$, and $z = (z_1 ,..., z_{m+1})$ satisfying (i)–(vi) of Theorem 6.1. Let

$$L_1 = \{a_1^{c_1+k_1 l_1} \cdots a_n^{c_n+k_n l_n} \mid (k_1 ,..., k_n) \text{ in } \mathscr{N}(L, c, l, z)\}$$

and

$$L_2 = \left\{ b_1^{h_1} \cdots b_m^{h_m} \,\middle|\, h_i = d_i + \sum_{s=z_i}^{z_{i+1}-1} k_s p_s \text{ for each } i, (k_1, ..., k_n) \text{ in } \mathcal{N}(L, c, l, z) \right\}.$$

(Note that $L_1 \subseteq L$ and $L_2 \subseteq b_1^+ \cdots b_m^+ \cap V$.) To simplify the notation let $\sum_i$ denotes $\sum_{s=z_i}^{z_{i+1}-1}$ for each i. Then

$$\liminf_{\substack{(t_1,...,t_m) \text{ in } \psi(V) \\ t_j \neq 0}} \frac{(t_i)^p}{(t_j)^q} \leqslant \liminf_{(t_1,...,t_m) \text{ in } \psi(L_2)} \frac{(t_i)^p}{(t_j)^q},$$

since $L_2 \subseteq V \cap b_1^+ \cdots b_m^+$,

$$= \liminf_{(k_1,...,k_n) \text{ in } \mathcal{N}(L,c,l,z)} \frac{(d_i + \sum_i k_s p_s)^p}{(d_j + \sum_j k_s p_s)^q},$$

by the definition of L_2.

Now, for each $a \geqslant 0$, $b \geqslant 1$, $x \geqslant 1$, and positive real number r, $(a + bx)^r \leqslant (ax + bx)^r = (a + b)^r x^r$ and $x^r \leqslant (a + bx)^r$. Since all $d_s \geqslant 0$, all $p_s \geqslant 1$, and $\sum_i k_s \geqslant 1$ for each i, $1 \leqslant i \leqslant m$,

$$\Big(d_i + \sum_i k_s p_s\Big)^p \Big/ \Big(d_j + \sum_j k_s p_s\Big)^q \leqslant \Big(d_i + p_{\max} \sum_i k_s\Big)^p \Big/ \Big(\sum_j k_s\Big)^q$$

$$\leqslant (d_i + p_{\max})^p \left[\Big(\sum_i k_s\Big)^p \Big/ \Big(\sum_j k_s\Big)^q\right],$$

where $p_{\max} = \max\{p_i \mid 1 \leqslant i \leqslant m\}$. Hence

$$\liminf_{(k_1,...,k_n) \text{ in } \mathcal{N}(L,c,l,z)} \left[\Big(d_i + \sum_i k_s p_s\Big)^p \Big/ \Big(d_j + \sum_j k_s p_s\Big)^q\right]$$

$$\leqslant (d_i + p_{\max})^p \liminf_{(k_1,...,k_n) \text{ in } \mathcal{N}(L,c,l,z)} \left[\Big(\sum_i k_s\Big)^p \Big/ \Big(\sum_j k_s\Big)^q\right].$$

Similarly,

$$\limsup_{\substack{(t_1,...,t_n) \text{ in } \psi(L) \\ \sum_j t_s \neq 0}} \left[\Big(\sum_i t_s\Big)^p \Big/ \Big(\sum_j t_s\Big)^q\right]$$

$$\geqslant \limsup_{(t_1,...,t_n) \text{ in } \psi(L_1)} \left[\Big(\sum_i t_s\Big)^p \Big/ \Big(\sum_j t_s\Big)^q\right]$$

$$= \limsup_{(k_1,...,k_n) \text{ in } \mathcal{N}(L,c,l,z)} \left[\Big(\sum_i (c_s + k_s l_s)\Big)^p \Big/ \Big(\sum_j (c_s + k_s l_s)\Big)^q\right],$$

by the definition of L_1. Since all $c_s \geqslant 0$, all $l_s \geqslant 1$, and $\sum_i k_s \geqslant 1$ for each i, $1 \geqslant i \geqslant m$,

$$\Big(\sum_i (c_s + k_s l_s)\Big)^p \Big/ \Big(\sum_j (c_s + k_s l_s)\Big)^q \geqslant \Big(\sum_i k_s\Big)^p \Big/ \Big(\sum_j c_{\max} + l_{\max} \sum_j k_s\Big)^q$$

$$\geqslant \Big[1 \Big/ \Big(\sum_j c_{\max} + l_{\max}\Big)^q\Big] \cdot \Big[\Big(\sum_i k_s\Big)^p \Big/ \Big(\sum_j k_s\Big)^q\Big],$$

where $c_{\max} = \max\{c_i \mid 1 \leqslant i \leqslant n\}$ and $l_{\max} = \max\{l_i \mid 1 \leqslant i \leqslant n\}$. Hence

$$\limsup_{(k_1,\ldots,k_n) \text{ in } \mathscr{N}(L,c,l,z)} \Big[\Big(\sum_i (c_s + k_s l_s)\Big)^p \Big/ \Big(\sum_j (c_s + k_s l_s)\Big)^q\Big]$$

$$\geqslant \Big[1 \Big/ \Big(\sum_j c_{\max} + l_{\max}\Big)^q\Big] \limsup_{(k_1,\ldots,k_n) \text{ in } \mathscr{N}(L,c,l,z)} \Big[\Big(\sum_i k_s\Big)^p \Big/ \Big(\sum_j k_s\Big)^q\Big].$$

Since

$$\liminf_{(k_1,\ldots,k_n) \text{ in } \mathscr{N}(L,c,l,z)} \Big[\Big(\sum_i k_s\Big)^p \Big/ \Big(\sum_j k_s\Big)^q\Big] \leqslant \limsup_{(k_1,\ldots,k_n) \text{ in } \mathscr{N}(L,c,l,z)} \Big[\Big(\sum_i k_s\Big)^p \Big/ \Big(\sum_j k\Big)^q\Big],$$

it follows that

$$\liminf_{\substack{(t_1,\ldots,t_m) \text{ in } \psi(V) \\ t_j \neq 0}} \frac{(t_i)^p}{(t_j)^q} \leqslant r_{ij} \limsup_{\substack{(t_1,\ldots,t_n) \text{ in } \psi(L) \\ \Sigma_j t_s \neq 0}} \Big[\Big(\sum_i t_s\Big)^p \Big/ \Big(\sum_j t_s\Big)^q\Big],$$

where $r_{ij} = (d_i + p_{\max})^p (\sum_j c_{\max} + l_{\max})^q$. Clearly r_{ij} is positive, and the proof is complete.

An immediate consequence of Theorem 6.2 for the case $m = n$ is the following.

COROLLARY. *Let $U \subseteq a_1^* \cdots a_n^*$, $V \subseteq b_1^* \cdots b_n^*$, and let V be* in $\mathscr{M}(U)$ $[\mathscr{F}(U)]$. *Let V be distance (f, I)-guided and I-uniformly infinite for some f and I [and let $I \neq \{1, 3\}$ if $n = 3$]. Then for every $L \subseteq U$ with the property that $U - R$ is regular for each regular set R containing L, for every pair of integers i and j $(1 \leqslant i, j \leqslant n)$, and for all positive real numbers p and q, positive real numbers r_{ij} and r'_{ij} can be found satisfying*

(a) $$\liminf_{\substack{(t_1,\ldots,t_n) \text{ in } \psi(V) \\ t_j \neq 0}} \frac{(t_i)^p}{(t_j)^q} \leqslant r_{ij} \limsup_{\substack{(t_1,\ldots,t_n) \text{ in } \psi(L) \\ t_j \neq 0}} \frac{(t_i)^p}{(t_j)^q}, \qquad \textit{and}$$

(b) $$\liminf_{\substack{(t_1,\ldots,t_n) \text{ in } \psi(L) \\ t_j \neq 0}} \frac{(t_i)^p}{(t_j)^q} \leqslant r'_{ij} \limsup_{\substack{(t_1,\ldots,t_n) \text{ in } \psi(V) \\ t_j \neq 0}} \frac{(t_i)^p}{(t_j)^q}.$$

EXAMPLE 6.1. Let $U = \{a^m b^n c^k \mid n \geqslant 1,\ m \geqslant n^2,\ k \geqslant n^3\}$ and

$$V = \{a^m b^n c^k \mid m \geqslant k^3,\ \ n \geqslant k^2,\ \ k \geqslant 1\}.$$

Then V is distance (f, I)-guided and I-uniformly infinite for $f(x) = x^2$ and $I = \{3\}$. Let $L = U$. Then $U - R = \varnothing$ for every regular set R containing L. Hence L satisfies the hypothesis of Theorem 6.2. The only possible values for z_1, z_2, z_3 and z_4 in the conclusion of Theorem 6.2 are $z_1 = 1$, $z_2 = 2$, $z_3 = 3$, and $z_4 = 4$. Then for $i = 2$, $j = 3$, and $p = q = 1$,

$$\liminf_{(t_1,t_2,t_3)\ \mathrm{in}\ \psi(V)} \frac{t_2}{t_3} = \lim_{n\to\infty} \frac{n^2}{n} = \infty,$$

and

$$\limsup_{(t_1,t_2,t_3)\mathrm{in}\ \psi(L)} \frac{t_2}{t_3} = \limsup_{(t_1,t_2,t_3)\ \mathrm{in}\ \psi(U)} \frac{t_2}{t_3} = \lim_{n\to\infty} \frac{n}{n^3} = 0.$$

Thus no positive real number r_{ij} satisfying (a) of Theorem 6.2 exists. Hence V is not in $\mathscr{M}(U)$. Since $I \neq \{1, 3\}$, V is not in $\mathscr{F}(U)$.

The following example indicates that having $L = U$ in Theorem 6.2 does not always lead to a violation of the necessary condition.

EXAMPLE 6.2. Let $U = \{a^m b^n \mid n \geqslant 1,\ m \leqslant n\}$ and $V = \{a^m b^n \mid m \geqslant 1,\ n \leqslant m\}$. Then V is distance (f, I)-guided and I-uniformly infinite for $f(x) = x$ and $I = \{2\}$. For arbitrary positive p, q and for $i = 1, j = 2$,

$$\liminf_{(m,n)\ \mathrm{in}\ \psi(V)} \frac{m^p}{n^q} = \lim_{n\to\infty} \frac{n^p}{n^q} = \limsup_{(m,n)\ \mathrm{in}\ \psi(U)} \frac{m^p}{n^q},$$

$$\liminf_{(m,n)\mathrm{in}\ \psi(U)} \frac{m^p}{n^q} = 0, \qquad \text{and} \qquad \limsup_{(m,n)\ \mathrm{in}\ \psi(V)} \frac{m^p}{n^q} = \infty.$$

For $i = 2$ and $j = 1$ a symmetric result is obtained. Thus the necessary condition expressed by Theorem 6.2 holds with $L = U$. However, it will be shown shortly that V is not in $\mathscr{F}(U)$. (The fact that V is not in $\mathscr{F}(U)$ is well known [5], though the proof was not published.)

Due to the nature of the languages considered in Section 4, the application of the necessary conditions there was straightforward. This is not the case here. Some ingenuity is required to choose an "appropriate" set L. It is usually easy to verify if the language V in question is distance (f, I)-guided and I-uniformly infinite for some f and I. However, it may be difficult to show that a chosen set L satisfies the hypothesis of Theorem 6.1 or 6.2. In many cases the following proposition assists in this task.

Notation. For all positive integers n, t, and k $(1 \leqslant k \leqslant t)$, all $\bar{\imath} = (i_1, \ldots, i_n)$ in N^n, and all $B \subseteq \{1, \ldots, n\}$, let $R(B, \bar{\imath}, k, t) = \{a_1^{k_1} \cdots a_n^{k_n} \mid k_j \equiv i_j \pmod{k}$ for each j,

$k_j > t$ for each j not in B, and $k_j = i_j$ for each j in $B\}$. Note that $R(B, \bar{\imath}, k, t)$ is regular for each B, $\bar{\imath}$, k, and t.

PROPOSITION 6.1. *Let $U \subseteq a_1^* \cdots a_n^*$ and $L \subseteq U$ be infinite. Suppose the following holds for all integers k and t $(1 \leqslant k \leqslant t)$, all $\bar{\imath} = (i_1, \ldots, i_n)$ in N^n $(i_j \leqslant t$ for each $j)$, and all $B \subseteq \{1, \ldots, n\}$: If $R(B, \bar{\imath}, k, t) \cap U$ is infinite [not regular], then*

$$R(B, \bar{\imath}, k, t) \cap L \neq \varnothing .$$

Then $U - R$ is finite [regular] for every regular set R containing L.

Proof. Let R be a regular set containing L. Since $L \subseteq R \cap a_1^* \cdots a_n^*$ and $U - R = U - (R \cap a_1^* \cdots a_n^*)$, there is no loss of generality in assuming that $R \subseteq a_1^* \cdots a_n^*$. By Theorem 1.3 of [9], $R = \bigcup_{i=1}^{m} R_i$ for some $m \geqslant 0$, where for each i,

$$R_i = a_1^{c_{i1}}(a_1^{l(i,1,1)})^* \cdots (a_1^{l(i,1,s_{i1})})^* a_2^{c_{i2}} \cdots (a_n^{l(i,n,s_{in})})^*$$

$$(c_{ij} \geqslant 0,\ l(i,j,p) \geqslant 0,\ s_{ij} \geqslant 1,\ 1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant n,\ 1 \leqslant p \leqslant s_{ij}).$$

Let $t = \max\{k, 1 + c_{ij} \mid \text{all } i \text{ and } j\}$, where k is the product of all nonzero $l(i, j, p)$. Note that there exists at least one nonzero $l(i, j, p)$, since R is infinite.

Suppose $R(B, \bar{\imath}, k, t) \cap U$ is finite [regular] for some B and $\bar{\imath}$. Clearly $U - R$ is finite [regular] if and only if $[U - R(B, \bar{\imath}, k, t)] - R$ is finite [regular]. Let

$$U' = U - \cup \{R(B, \bar{\imath}, k, t) \mid R(B, \bar{\imath}, k, t) \cap U \text{ is finite [regular]}\}.$$

Then $U - R$ is finite [regular] if and only if $U' - R$ is. Clearly $R(B, \bar{\imath}, k, t) \cap U'$ is infinite [not regular] if $R(B, \bar{\imath}, k, t) \cap U' \neq \varnothing$. Suppose $R(B_0, \bar{\imath}_0, k, t) \cap U' \neq \varnothing$ for some B_0 and $\bar{\imath}_0$. Then $R(B_0, \bar{\imath}_0, k, t) \cap U'$ is infinite [not regular]. By hypothesis, $R(B_0, \bar{\imath}_0, k, t) \cap L \neq \varnothing$. Let $\prec$ be partial ordering on $a_1^* \cdots a_n^*$ defined by

$$a_1^{m_1} \cdots a_n^{m_n} \prec a_1^{l_1} \cdots a_n^{l_n} \qquad \text{if} \quad m_j \leqslant l_j \text{ for each } j.$$

Let A be the set of all minimal elements of $R(B_0, \bar{\imath}_0, k, t) \cap L$. By [15], A is finite. Let $v = a_1^{h_1} \cdots a_n^{h_n}$ be some element in A. Since $L \subseteq \bigcup_{j=1}^{m} R_j$, v is in

$$R(B_0, \bar{\imath}_0, k, t) \cap R_{q_v}$$

for some q_v. By the definition of $R(B_0, \bar{\imath}_0, k, t)$, $h_j > t$ for each j not in B_0. Since $| R(\{1, \ldots, n\}, \bar{\imath}, k, t)| = 1$ for each $\bar{\imath}$, k, and t and $R(B_0, \bar{\imath}_0, k, t) \cap U'$ is infinite [not regular], $B_0 \neq \{1, \ldots, n\}$. Thus there exists j in $\{1, \ldots, n\} - B_0$. Since $t > c_{q_v p} (1 \leqslant p \leqslant n)$ for each p not in B_0, there exists some s_p such that $l(q_v, p, s_p) \neq \varnothing$. Now each such $l(q_v, p, s_p)$ divides k. Hence $\{u \text{ in } R(B_0, \bar{\imath}_0, k, t) \mid v \prec u\} \subseteq R_{q_v}$. This holds for each v in A. Hence $R(B_0, \bar{\imath}_0, k, t) - R \subseteq R(B_0, \bar{\imath}_0, k, t) - \bigcup_{v \text{ in } A} R_{q_v}$ is finite. Thus $R(B, \bar{\imath}, k, t) = R$ is finite for each B and $\bar{\imath}$ having $R(B, \bar{\imath}, k, g) \cap U' \neq \varnothing$. Clearly there are only finitely many possible choices for B and $\bar{\imath}$ for every t and k, and

$$\bigcup_{\substack{\text{all } B \\ \text{all } \bar{\imath}}} R(B, \bar{\imath}, k, t) = a_1{}^* \cdots a_n{}^*.$$

Since

$$U' - R = \Big(\bigcup_{\substack{\text{all } B \\ \text{all } \bar{\imath}}} R(B, \bar{\imath}, k, t) \cap U'\Big) - R$$

$$\subseteq \bigcup \{R(B, \bar{\imath}, k, t) \mid R(B, \bar{\imath}, k, t) \cap U' \neq \varnothing\} - R,$$

$U' - R$ is finite. Thus $U - R$ is finite [regular] and the proof is complete.

EXAMPLE 6.2 (continued). Let $L = \{a^i b^j \mid j \geqslant i^2 \geqslant 1\}$. For arbitrary k and t, $k, t \geqslant 1$, and for $B = \{2\}$, or $B = \{1, 2\}$, the set $\{a^{k_1} b^{k_2}$ in $U \mid k_j \equiv i_j \pmod{k}$, $k_j > t$ for j not in B, $k_j = i_j$ for j in $B\}$ is finite for all $\bar{\imath} = (i_1, i_2)$, each $i_j \leqslant t$. For $B = \varnothing$ or $B = \{1\}$ the set $\{a^{k_1} b^{k_2}$ in $L \mid k_j \equiv i_j \pmod{k}$ for each j, $k_j > t$ for each j not in B, $k_j = i_j$ for j in $B\}$ is nonempty. Thus the hypothesis of Proposition 6.1 is satisfied. Hence $U - R$ is finite for every regular set R containing L. Thus the hypothesis of Theorem 6.2 is satisfied. However, for $j = p = 1$ and $i = q = 2$,

$$\liminf_{(m,n) \text{ in } \psi(L)} \frac{n}{(m)^2} = \lim_{m \to \infty} \frac{m^2}{(m)^2} = 1 \qquad \text{and} \qquad \limsup_{(m,n) \text{ in } \psi(V)} \frac{n}{(m)^2} = \lim_{n \to \infty} \frac{n}{(n)^2} = 0.$$

Thus no positive real number r'_{12} satisfying (b) of Theorem 6.2 exists. Hence V is not in $\mathscr{F}(U)$.

EXAMPLE 6.3. Let $U = \{a^h b^m c^n \mid m \geqslant 1, h \geqslant m^2, n \geqslant m^2\}$ and

$$V = \{a^m b^n \mid n \geqslant m^2\}.$$

It is easy to see that V is in $\hat{\mathscr{M}}(U)$ and $\hat{\mathscr{F}}(U)$. Consider V with respect to $\mathscr{F}(U)$. Clearly V is distance $(x, \{1\})$-guided and $\{1\}$-uniformly infinite. Let

$$L = \{a^h b^m c^n \mid m \geqslant 1, m^2 \leqslant h \leqslant m^3, m^2 \leqslant n \leqslant m^3\}.$$

It is readily seen that if $B \neq \varnothing$, then $R(B, \bar{\imath}, k, t) \cap U$ is regular for every $\bar{\imath}$, k, and t. If $B = \varnothing$, then $R(B, \bar{\imath}, k, t) \cap L \neq \varnothing$ for every $\bar{\imath}$, k, and t. Thus by Proposition 6.1, $U - R$ is regular for every regular set R containing L. Hence the hypothesis of Theorem 6.2 is satisfied. There are two possible choices for z_1, z_2, and z_3 :

(i) $z_1 = 1$, $z_2 = 2$, and $z_3 = 4$. Then for $i = 1, j = 2, p = 3$, and $q = 2$,

$$\liminf_{(h,m,n)\ \text{in}\ \psi(L)} \frac{h^3}{(m+n)^2} = \lim_{m\to\infty} \frac{(m^2)^3}{(m+m^3)^2} = 1$$

and

$$\limsup_{(m,n)\ \text{in}\ \psi(V)} \frac{(m)^3}{(n)^2} = \lim_{m\to\infty} \frac{(m)^3}{(m^2)^2} = 0.$$

Thus no positive real number r'_{12} satisfying (b) of Theorem 6.2 exists.

(ii) $z_1 = 1$, $z_2 = 3$, and $z_3 = 4$. Then for $i = 2$, $j = 1$, $p = 1$, and $q = 2$,

$$\liminf_{(m,n)\ \text{in}\ \psi(V)} \frac{n}{(m)^2} = \lim_{m\to\infty} \frac{m^2}{(m)^2} = 1$$

and

$$\limsup_{(h,m,n)\ \text{in}\ \psi(L)} \frac{n}{(h+m)^2} = \lim_{m\to\infty} \frac{m^3}{(m^2+m)^2} = 0.$$

Thus no positive real number r_{21} satisfying (a) of Theorem 6.2 exists. Since for no z_1, z_2, and z_3 do properties (a) and (b) of Theorem 6.2 hold, V is neither in $\mathscr{M}(U)$ nor in $\mathscr{F}(U)$.

In Theorems 6.1 and 6.2 we established some results for principal AFL and semi-AFL. We now prove the corresponding results for full principal AFL and semi-AFL. In order to do this, we need the "full" equivalent of $\mathscr{N}(L, c, l, z)$.

Notation. Let $1 \leqslant m \leqslant n$, $c = (c_1, ..., c_n)$ and $l = (l_1, ..., l_n)$ be in N^n and $z = (z_1, ..., z_{m+1})$ in N^{m+1}, with $1 = z_1 < \cdots < z_{m+1} = n + 1$. Let $E \subseteq \{1, ..., n\}$ be such that for each i, $1 \leqslant i \leqslant m$, $E_i = \{j \mid z_i \leqslant j < z_{i+1}, j \text{ not in } E\} \neq \varnothing$. For $L \subseteq a_1^* \cdots a_n^*$, let $\hat{\mathscr{N}}(L, c, l, z, E) = \{(k_1, ..., k_n) \mid a_1^{c_1+k_1l_1} \cdots a_n^{c_n+k_nl_n}$ is in L, $k_i \geqslant 0$ for each i, and $\sum_{s\ \text{in}\ E_j} k_s \geqslant 1$ for each j, $1 \leqslant j \leqslant m\}$.

THEOREM 6.3. *Let $U \subseteq a_1^* \cdots a_n^*$, $V \subseteq b_1^* \cdots b_m^*$, and let V be in $\hat{\mathscr{M}}(U)$ $[\hat{\mathscr{F}}(U)]$. Let V be distance (f, I)-guided and I-uniformly infinite for some f and I [and let $I \neq \{1, 3\}$ if $m = 3$]. Then for each subset $L \subseteq U$ with the property that $U - R$ is regular for every regular set R containing L, there exist n-tuples $c = (c_1, ..., c_n)$, $l = (l_1, ..., l_n)$, $p = (p_1, ..., p_n)$ in N^n, m-tuple $d = (d_1, ..., d_m)$ in N^m $(m+1)$-tuple $z = (z_1, ..., z_{m+1})$ in N^{m+1} and a set $E \subseteq \{1, ..., n\}$ satisfying the following.*

(i) $1 = z_1 < \cdots < z_{m+1} = n + 1$;

(ii) *$l_i > 0$, $1 \leqslant i \leqslant n$, and $p_j = 0$ if and only if j is in E;*

(iii) *$a_1^{c_1} \cdots a_n^{c_n}$ is in L;*

(iv) $b_1^{d_1} \cdots b_m^{d_m}$ *is* in V;

(v) $\mathscr{N}(L, c, l, z, E)$ *is infinite*;

(vi) $\{(d_1 + \sum_{i=z_1}^{z_2-1} k_i p_i, ..., d_m + \sum_{i=z_m}^{z_{m+1}-1} k_i p_i) \mid (k_1, ..., k_n)$ in $\mathscr{N}(L, c, l, z, E)\} \subseteq \psi(V)$; *and*

(vii) $\{j \mid z_i \leqslant j < z_{i+1}, j$ *not* in $E\} \neq \varnothing$ *for each* i, $1 \leqslant i \leqslant m$.

Proof. The argument is essentially the same as that for Theorem 6.1. We shall therefore only indicate the changes.

Let M be an arbitrary a-transducer having $M(U) = V$. Let $I, L, S(t), R(t), R'(t), A(t), W, Q, Q', l, k', j_0, Q(\alpha), S(\alpha, t), \beta, t_0, e, B(\beta), B'(\beta), W', W'', P(v), S'(v, t), u, \delta_{ui}, g, Z, S''(t), z, \tau, c, p$, and d be as in the proof of Theorem 6.1. Let

$$Z' = \left\{(k_1, ..., k_n) \text{ in } N^n \mid \tau + (k_1, ..., k_n) \text{ in } Z, \sum_{\substack{i=z_j \\ i \text{ in } E}}^{z_{j+1}-1} k_i \geqslant 1, 1 \leqslant j \leqslant m\right\}$$

and let $E = \{i \mid pr_2(\delta_{ui}) = \epsilon, 1 \leqslant i \leqslant n\}$. From the definition of p, z, $B'(\beta)$, and the fact that $pr_2(\gamma_{q_i})$ is in b_i^+, $1 \leqslant i \leqslant n$, it follows that (iii) and (vii) hold for E and p. Since $pr_2(\delta_{ui}) = \epsilon$ for i in E and $S''(t)$ is infinite for each $t \geqslant 0$, Z' is infinite. The rest of the proof then follows with $\mathscr{N}(L, c, l, z)$ replaced by $\mathscr{N}(L, c, l, z, E)$.

COROLLARY. *Let* $U \subseteq a_1^* \cdots a_n^*$, $V \subseteq b_1^* \cdots b_m^*$, *and let* V *be in* $\hat{\mathscr{M}}(U)$ $[\hat{\mathscr{F}}(U)]$. *Let* V *be distance* (f, I)*-guided and* I*-uniformly infinite for some* f *and* I [*and let* $I \neq \{1, 3\}$ *if* $m = 3$]. *Then* $m \leqslant n$.

As in the case of Theorem 6.1, the above theorem is rather awkward to apply directly in many cases. The following analog of Theorem 6.2 for the full case is easier to use.

THEOREM 6.4. *Let* $U \subseteq a_1^* \cdots a_n^*$, $V \subset b_1^* \cdots b_m^*$ *and let* V *be* in $\hat{\mathscr{M}}(U)$ $[\hat{\mathscr{F}}(U)]$. *Let* V *be distance* (f, I)*-guided and* I*-uniformly infinite for some* f *and* I [*and let* $I \neq \{1, 3\}$ *if* $m = 3$]. *Let* $L \subseteq U$ *be such that* $U - R$ *is regular for each regular set* R *containing* L. *Then integers* $z_1, ..., z_{m+1}$, $1 = z_1 < \cdots < z_{m+1} = n + 1$, *and a set* $E \subseteq \{1, ..., n\}$, *with* $E_i = \{j \mid z_i \leqslant j < z_{i+1}, j$ *not* in $E\} \neq \varnothing$ *for each* i, *exist satisfying the following. For every pair of integers* i *and* j $(1 \leqslant i, j \leqslant m)$ *and for all positive real numbers* p *and* q, *positive real numbers* r_{ij} *and* r'_{ij} *can be found such that for* $\rho_j = \sum_{s \text{ in } E_j} t_s$,

$$\text{(a)} \quad \liminf_{\substack{(t_1, ..., t_m) \text{ in } \psi(V) \\ t_j \neq 0}} \frac{(t_i)^p}{(t_j)^q} \leqslant r_{ij} \limsup_{\substack{(t_1, ..., t_n) \text{ in } \psi(L) \\ \rho_j \neq 0}} \frac{(\rho_i)^p}{(\rho_j)^q}; \qquad \textit{and}$$

$$\text{(b)} \quad \liminf_{\substack{(t_1, ..., t_n) \text{ in } \psi(L) \\ \rho_j \neq 0}} \frac{(\rho_i)^p}{(\rho_j)^q} \leqslant r'_{ij} \limsup_{\substack{(t_1, ..., t_m) \text{ in } \psi(V) \\ t_j \neq 0}} \frac{(t_i)^p}{(t_j)^q}.$$

Proof. The proof is an obvious modification of that of Theorem 6.2 and is therefore omitted.

An immediate consequence of Theorem 6.4 for the case $m = n$ is the following

COROLLARY. *Let $U \subseteq a_1^* \cdots a_n^*$, $V \subseteq b_1^* \cdots b_n^*$, and let V be in $\hat{\mathscr{M}}(U)$ [$\hat{\mathscr{F}}(U)$]. Let V be distance (f, I)-guided and I-uniformly infinite for some f and I [and let $I \neq \{1, 3\}$ if $n = 3$]. Then for every $L \subseteq U$ with the property that $U - R$ is regular for each regular set R containing L, for every pair of integers i and j $(1 \leqslant i, j \leqslant n)$, and for all positive real numbers p and q, positive real numbers r_{ij} and r'_{ij} can be found satisfying*

$$\text{(a)} \qquad \liminf_{\substack{(t_1,\ldots,t_n)\ \mathrm{in}\ \psi(V)\\ t_j\neq 0}} \frac{(t_i)^p}{(t_j)^q} \leqslant r_{ij} \limsup_{\substack{(t_1,\ldots,t_n)\ \mathrm{in}\ \psi(L)\\ t_j\neq 0}} \frac{(t_i)^p}{(t_j)^q}, \qquad \textit{and}$$

$$\text{(b)} \qquad \liminf_{\substack{(t_1,\ldots,t_n)\ \mathrm{in}\ \psi(L)\\ t_j\neq 0}} \frac{(t_i)^p}{(t_j)^q} \leqslant r'_{ij} \limsup_{\substack{(t_1,\ldots,t_n)\ \mathrm{in}\ \psi(V)\\ t_j\neq 0}} \frac{(t_i)^p}{(t_j)^q}.$$

EXAMPLE 6.4. Let $U = \{a_1{}^k a_2{}^l a_3{}^t \mid t \text{ divides } l,\ l \text{ divides } k,\ k, l, t \geqslant 1\}$ and $V = \{b_1{}^k b_2{}^l \mid k^2 \text{ divides } l,\ k, l \geqslant 1\}$. Suppose V is in $\hat{\mathscr{M}}(U)$ [$(\hat{\mathscr{F}}(U)$]. Clearly V is distance $(x^2, \{1\})$-guided and $\{1\}$-uniformly infinite. Let $L = U$. Then the hypothesis of Theorem 6.4 is satisfied. Consider the conclusion of Theorem 6.4. There are only two possible choices for z_1, z_2, and z_3, with three possible choices for E in each case. These are

(1) $z_1 = 1$, $z_2 = 2$, and $z_3 = 4$;
   (a) $E = \varnothing$, (b) $E = \{2\}$, and (c) $E = \{3\}$; and

(2) $z_1 = 1$, $z_2 = 3$, and $z_3 = 4$;
   (a) $E = \varnothing$, (b) $E = \{1\}$, and (c) $E = \{2\}$.

It is an easy matter to verify that in each case, for $i = 2$ and $j = p = q = 1$, no positive real number r_{12} exists satisfying (a) of Theorem 6.4. Hence the conclusion of the Theorem 6.4 does not hold. Consequently V is not in $\hat{\mathscr{F}}(U)$.

We conclude the section with a weak necessary condition that applies to (f, I)-guided languages which are not necessarily distance (f, I)-guided.

PROPOSITION 6.2. *Let $U \subseteq a_1^* \cdots a_n^*$, $V \subseteq b_1^* \cdots b_m^*$, and let V be in $\mathscr{F}(U)$[$\hat{\mathscr{F}}(U)$]. Let V be an (f, I)-guided language for some f and $I = \{i_1, \ldots, i_r\}$, with $\psi_I(V)$ infinite [and $\bar{I}_m \cap \{1, m\} \neq \varnothing$ if $|\bar{I}_m| = 1$]. Then $n \geqslant m - r + 1$.*

Proof. Suppose V is in $\mathscr{M}(U)$ [$\hat{\mathscr{M}}(U)$]. Then there exists an ϵ-free [arbitrary] a-transducer M such that $V = M(U)$. Let $\prec$ be the partial ordering on N^r defined by $(k_1, \ldots, k_r) \prec (l_1, \ldots, l_r)$ if $k_i \leqslant l_i$ for each i, $1 \leqslant i \leqslant r$. It is known [15] that each subset of N^r has only finitely many minimal elements. Thus there exists an infinite chain C in $\psi_I(V)$. Since f is increasing and unbounded in each variable, for each k in N there exists u in C such that $f(u) \geqslant k$. This holds in particular for $k = \bar{k}$, where $\bar{k}$ is an integer guaranteed by Lemma 5.2. Furthermore $f(v) \geqslant f(u)$ for each v in C, $u < v$. By Lemma 5.1, for each j, $1 \leqslant j \leqslant m$, and for each computation α in standard form in $\Pi_M \cap pr_1^{-1}(U)$, with $\#_{b_j}(pr_2(\alpha)) > 3m_1m_2m_3$, α contains two occurrences of some loop γ, $pr_2(\gamma)$ in b_j^+. (See Section 5 for the definition of m_1, m_2, and m_3.) Since C is an infinite chain, there exists $u' = (k_1, \ldots, k_r)$ in C, with $f(u') \geqslant \bar{k}$ and

$$k_i > 3m_1m_2m_3$$

for some i, $1 \leqslant i \leqslant r$. Hence there exists a computation α in standard form in

$$\Pi_M \cap pr_1^{-1}(U), \qquad f(\psi_I(pr_2(\alpha))) \geqslant \bar{k},$$

containing an occurrence of at least one loop γ such that $pr_2(\gamma)$ is in b_j^+ for some j in I. Let $\{l(i) \mid 1 \leqslant l(i) \leqslant n, i \text{ in } \bar{I}_m\}$ be a set satisfying (i), (ii), and (iii) of Lemma 5.2. By (iii) of Lemma 5.2, $l(s) \neq l(i)$ for $s \neq i$. By Lemma 5.3, $pr_1(\gamma) \neq \epsilon$. Hence there exists t, $1 \leqslant t \leqslant n$, such that $pr_1(\gamma)$ is in a_t^+. By (ii) of Lemma 5.2, t is not in

$$\{l(i) \mid i \text{ in } \bar{I}_m,\ 1 \leqslant l(i) \leqslant n\}.$$

Thus the number of distinct indices s, $1 \leqslant s \leqslant n$, is at least

$$|\{l(i) \mid i \text{ in } \bar{I}_m, \quad 1 \leqslant l(i) \leqslant n\}| + 1 = |\bar{I}_m| + 1 = m - r + 1.$$

Suppose V is in $\mathscr{F}(U)$ $[\hat{\mathscr{F}}(U)]$. By Lemma 5.7, there exists an (f, I)-guided and I-uniformly infinite language $T \subseteq b_1^* \cdots b_m^*$ in $\mathscr{M}(U)$ $[\hat{\mathscr{M}}(U)]$. Thus by the first part of the argument, $n \geqslant m - r + 1$ and the proof is complete.

Acknowledgments

The author acknowledges his gratitude to his thesis adviser, Professor Seymour Ginsburg, for many invaluable comments and priceless discussions. The author also thanks the referee for an extremely careful review and helpful suggestions.

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