HOMEWORK 4

1. Prove that the ideal in $\mathbb{Z}[\sqrt{-5}]$ generated by 2 and $1+\sqrt{-5}$ is not principal.

2. Determine whether the ring $\mathbb{Z}[\sqrt{5}]$ is a PID.

3. Let $R = \mathbb{Z}[i]$ be the ring of Gauss integers and let p be a prime integer such that $p \equiv 3$ modulo 4. Prove that p is prime in R.

4. Let $R = \mathbb{Z}[i]$ and let p be a prime integer such that $p \equiv 1$ modulo 4.
   a) Prove that p is not prime in R. (Hint: use HW 5, Problem 9 in 110AH).
   b) Prove that there are integers a and b such that $p = a^2 + b^2$.

5. Let R be a PID and let a be a prime element in R. Prove that the ideal pR is maximal.

6. Prove that the product of two Noetherian rings is also Noetherian.

7. An integral domain in which every ideal generated by two elements is principal is called a *Bezout domain*. Prove that a ring R is a PID if and only if R is a Noetherian Bezout domain.

8. Let $R_1 \subset R_2 \subset R_3 \subset \ldots$ be a chain of countably many subrings of a ring R such that $R = \cup R_i$. Suppose that all the R_i are UFD and any prime element in every R_i is prime in R_{i+1}. Prove that R is a UFD.

9. Prove that the polynomial ring $\mathbb{Z}[x_1, x_2, x_3, \ldots]$ in countably many variables is a UFD but not a Noetherian ring.

10. Let $R = \mathbb{Z}[\sqrt{-5}]$. Prove that the product of the two ideals $2R+(1+\sqrt{-5})R$ and $3R + (1 + \sqrt{-5})R$ in R is the principal ideal $(1 + \sqrt{-5})R$.