Math 110BH Homework 6

Sample Student

February 17, 2023

For all of this assignment R denotes an arbitrary ring unless stated otherwise.

Problem 1*

a) Let M be an R-module. Let $x \in M$. Define the *annihilator of* m to be the set

$$\mathrm{ann}_R(m) = \{r \in R \mid r \cdot m = 0\}.$$

Prove that $\mathrm{ann}_R(m)$ is a left ideal of R.

b) Suppose M is a cyclic R-module. Prove that there is a left ideal I of R such that M is isomorphic to the quotient module R/I.

Problem 2

Let R be a commutative ring.

a) Let M, N be two R-modules. Prove that the set $\mathrm{Hom}_R(M, N)$ is an R-module under pointwise addition and R-action $(r \cdot f)(x) = r \cdot f(x)$.

b) Prove that $\mathrm{Hom}_R(R, M)$ and M are isomorphic as left R-modules.

c) (*You do not need to turn in this part of the exercise, but you should still convince yourself this is true!*) Verify that $\mathrm{End}_R(M)$ is a ring where multiplication is given by function composition and addition is still pointwise. We call this the *endomorphism ring of M*.

Problem 3

Let M, N be R-modules and suppose $\phi : M \to N$ is an R-module homomorphism.

a) Let A be a submodule of M. Prove that $\phi(A) = \{\phi(a) \mid a \in A\}$ is a submodule of N.

b) Let B be a submodule of N. Prove that $\phi^{-1}(B) = \{x \in M \mid \phi(x) \in B\}$ is a submodule of M.

Problem 4

Prove the *Second Isomorphism Theorem*. That is, prove that if M is an R-module and A, N are submodules that $A/(A \cap N)$ is isomorphic to $(A + N)/N$.

Problem 5

Prove the *Correspondence Theorem*. That is, prove that if M, Q are R-modules and $f : M \to Q$ is a surjective R-module homomorphism then function

$$\Phi\colon \{\text{submodules of } M \text{ containing } \ker(f)\} \to \{\text{submodules of } Q\}$$

given by $A \mapsto f(A)$ is an order-preserving bijection.

Problem 6*

Let N be a submodule of an R-module M. Prove that if N and M/N are finitely generated, then M is finitely generated.

Problem 7

An R-module M is called *simple* or *irreducible* if $M \neq 0$ and its only submodules are 0 and M. Prove that M is simple if and only if M is a cyclic R-module and every nonzero element of M generates M.

Problem 8

Let M be an R-module. Prove that if M is simple, then $\mathrm{End}_R(M)$ is a division ring.

Problem 9

Let A, B, M be R-modules. Note that $A \times B$ is again an R-module under the diagonal R-action, i.e. $r \cdot (a, b) = (r \cdot a, r \cdot b)$. Prove that we have the following isomorphisms of R-modules:

a) $\mathrm{Hom}_R(A \times B, M) \cong \mathrm{Hom}_R(A, M) \times \mathrm{Hom}_R(B, M)$, and

b) $\mathrm{Hom}_R(M, A \times B) \cong \mathrm{Hom}_R(M, A) \times \mathrm{Hom}_R(M, B)$.

Problem 10*

Let M be the $\mathbb{Z}$-module given the infinite direct product $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times \cdots = \prod_{i=1}^{\infty} \mathbb{Z}$. That is, M consists of all infinite tuples of integers $(a_1, a_2, a_3, \dots)$. Let R denote the endomorphism ring $R = \mathrm{End}_{\mathbb{Z}}(M)$. Consider the $\varphi_1, \varphi_2 \in R$ defined by

$$\varphi_1((a_1, a_2, a_3, \dots)) = (a_1, a_3, a_5, \dots),$$
$$\varphi_2((a_1, a_2, a_3, \dots)) = (a_2, a_4, a_6 \dots).$$

a) Prove that $\{\varphi_1, \varphi_2\}$ is a basis of the free R-module R.

b) Use (a) to show $R^2 \cong R$. Then show that $R^n \cong R$ for all $n \in \mathbb{Z}^+$.

Note: Problem 10 shows that "dimension" is not a well-defined notion for non-commutative rings. In particular, you just found a free module with the property that for all $n \in \mathbb{Z}^+$ there exists a basis with n elements!