Galois Theory

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1 Resources

- The Basic Graduate Year In Algebra
- Richard Elman, Lectures on Abstract Algebra, Chapters 11 and 12

2 Definitions and Basic Properties

Definition 1. A **Fermat prime** is $f_n = 2^{2^n} + 1$ such that f_n is prime.

f_5 is not prime. f_1, f_2, f_3, f_4 are prime. It's unknown whether we have infinitely many non Fermat primes or Fermat primes. It's an extremely difficult number theoretical problem.

If $n = 2^n p_1...p_k$ where p_i is a Fermat prime, we can construct an n-gon.

2.1 Applications of Galois Theory

Langlands Program: Galois groups of extensions of $\mathbb{Q}$ have "something to do" with modular forms. Proving bits of "something to do" leads to striking results. This is also used in the proof of Fermat's Last Theorem.

The Galois group turns out to be related to the fundamental group in algebraic topology. Field extensions end up corresponding to covering spaces. Algebraic closures end up corresponding to universal covering spaces.

Inverse Problem: Given a finite group G, is there a Galois extension K of $\mathbb{Q}$ such that Galois Group of K over $\mathbb{Q}$ is G? This is an open problem though it's been proven for specific types of groups such as solvable and Abelian groups.

3 Classical Straightedge and Compass Constructions

Construction by ruler and compass. You can't measure.

The ruler just allows you to draw straight lines between two points on the plane.

The compass allows you to draw a circle with some radius r.

Euclid

Here are some problems they considered:

1. Constructing a sqaure with the area of a circle

2. Trisecting an angle

3. Constructing a regular polygon

4 Finite Fields