# MATH110BH Homework 7

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## 1 Problem 1

**Lemma 1.1.** Let $R$ be a UFD, The intersection of two principal ideals $aR$ and $bR$ is a principal ideal generated by $lcm(a, b)$.

*Proof.* Let $a = p_1^{\alpha_1} p_2^{\alpha_2} ... p_n^{\alpha_n}$ and $b = p_1^{\beta_1} p_2^{\beta_2} ... p_n^{\beta_n}$ where $\alpha_i$ and $\beta_i$ are non-negative integers. Let $m_i = \max\{\alpha_i, \beta_i\}$. Let $c := lcm(a, b) = p_1^{m_1} ... p_n^{m_n}$. We'll prove that $aR \cap bR = cR$.

Notice that $aR \mid cR$ and $bR \mid cR$, so $cR \subseteq aR \cap bR$. Now, let $x \in aR \cap bR$. Notice that $p_i^{m_i} \mid x$, so $c \mid x \implies x \in cR$. Thus, $aR \cap bR \subseteq cR$ and we conclude the proof. $\square$

## 2 Problem 2

Let $F$ be a field. Then, $R = F[x_1, x_2, ...]$ is a Euclidean domain. Notice that $R$ is a free $R$-module since it's generated by 1. Now, let $N$ be the submodule generated by $\{x_1, x_2, ...\}$. Clearly, finitely many elements can't generate $N$ since every $x_i$ has to be in any generating set. Now, notice that if $x_i$ and $x_j$ are in some set $S$, $x_i \cdot x_j = x_j \cdot x_i$, so $S$ is not independent. Therefore, $N$ is not a free module.

Recall (2,x) not principal in Z[x] and use prev hw problem

## 3 Problem 3

**Lemma 3.1.** Let $R$ be a PID and $M$ be a $R$-module generated by $n$ elements. Let $N$ be a submodule of $M$. Then, $N$ can also be generated by $n$ elements.

*Proof.* Let $M$ be an $R$-module generated by $\{x_1, ..., x_n\}$. Let $f : R^n \to M$ defined by mapping $e_i \mapsto x_i$. This is a surjective module homomorphism. Let $P = f^{-1}(N)$. Since $N$ is a submodule of $M$, $P$ is a submodule of $R^n$. Then, by the theorem proven in class, $P$ is generated by at most $n$ elements. Then, let $\{y_1, ..., y_m\}$ be the generating set for $P$ where $m \leq n$. Then, $\{f(y_1), ..., f(y_m)\}$ is a generating set for $N$. $\square$

## 4 Problem 4

**Lemma 4.1.** The group $\mathbb{Z}^n$ can't be generated by $n - 1$ elements.

*Proof.* $\mathbb{Z}^n$ has a standard basis consisting of $n$ elements. We showed in Problem 8 of Homework 6 that every free module generated by $m$ elements has a basis consisting of at most $m$ elements. In Problem 9, we showed that every two bases for a free finitely generated R-module have the same number of elements. Putting these two facts together, any generating set of $\mathbb{Z}^n$ should have at least $n$ elements.

BRING IT DOWN TO THE EMPTY SET.

USE Q INSTEAD. $\square$

# 5 Problem 5

Let $M = \mathbb{Z}/2\mathbb{Z}$ and $N = \mathbb{Z}/3\mathbb{Z}$. Clearly, these aren't free $\mathbb{Z}/6\mathbb{Z}$ modules since $2 \cdot x = 0$ for any $x \in M$ and $3 \cdot x = 0$ for any $x \in N$. However, $(1, 1)$ generates $M \bigoplus N$, so $M \bigoplus N$ is free (in fact, cyclic).

# 6 Problem 6

**Lemma 6.1.** *Let $R$ be a PID and $M$ be a torsion finitely generated R-module with the invariant factors $d_1 \mid d_2 \mid ... \mid d_k$. Let $I = \{a \in R : aM = 0\}$. Then, $I = d_kR$.*

*Proof.* We prove both inclusions. By the representation for finitely generated modules, we have that $M \cong R/d_1R \bigoplus ... \bigoplus R/d_kR$. Let $x \in I$ and $s \in R/d_kR$. Then, $xs \in d_kR$ for any $s \in R$ since $x \cdot (s + d_kR) = d_kR$ for any $s \in R$. Letting $s = 1$, $x \in d_kR$. Now, let $s \in d_kR$. Then, $d_k \mid s \implies d_i \mid s$ for any $i$. Then, $s \in d_iR$ for any $i$. Then, $s \cdot (x + d_iR) = d_iR$ for any $i$, so $s \in I$. We thus conclude the proof. □

# 7 Problem 7

Let $A$ be an Abelian group of order 300. By the primary decomposition theorem, $A \cong A_1 \times A_2 \times A_3$ where $|A_1| = 2^2$, $|A_2| = 3$ and $|A_3| = 5^2$. Notice that $A_1 \cong C_4$ or $A_1 \cong C_2 \times C_2$, $A_2 \cong C_3$ and $A_3 \cong C_{25}$ or $A_3 \cong C_5 \times C_5$.

Therefore, we have the following 4 possible groups:

$$C_4 \times C_3 \times C_{25}$$
$$C_2 \times C_2 \times C_3 \times C_{25}$$
$$C_4 \times C_3 \times C_5 \times C_5$$
$$C_2 \times C_2 \times C_3 \times C_5 \times C_5$$

# 8 Problem 10

**Lemma 8.1.** *Let $R$ be a PID and $M$ be an R-module. $M$ is cyclic if and only if $M \cong R/(a)$ for some $a \in R$.*

*Proof.* Assume $M$ is cyclic. Then, there's some $x \in M$ such that $x$ generates $M$. Consider the module homomorphism $\phi_x : R \to M$ given by $\phi(r) = rx$. Since $x$ generates $M$, $\phi$ is surjective. Since $R$ is a PID, $ker(\phi_x) = (a)$ for some $a \in R$.

Then, by the first isomorphism theorem for modules, $M \cong R/(a)$.

For the converse, notice that $a$ is a generator for the module $R/(a)$. □

**Lemma 8.2.** *Let $M$ be a finitely generated torsion module over a PID $R$. $M$ is cyclic if and only if every two elementary divisors of $M$ are relatively prime.*

*Proof.* Assume $M$ is cyclic. Then, $M \cong R/(a)$. Since $R$ is a UFD, we have that a has a unique prime factorization.

$$a = p_1^{\alpha_1} ... p_n^{\alpha_n}$$

Then, by the Chinese Remainder Theorem,

$$R/aR \cong R/p_1^{\alpha_1}R \oplus ... \oplus R/p_n^{\alpha_n}R$$

Clearly, $p_i$ and $p_j$ are relatively prime. To see the converse, just apply the Chinese Remainder theorem again and see that $M \cong R/aR$ for some $a \in R$.

$\square$