MATH110BH HW9 Problem 1

Find the IFs of the quotient group $\mathbb{Z}^3/N$, where N is generated by $(-4,4,2)$, $(16,-4,-8)$, $(12,0,-6)$ and $(8,4,2)$.

We'll consider $\mathbb{Z}^3/N$ as a $\mathbb{Z}$-module.

Then, the matrix with column vectors $(-4,4,2)$, $(16,-4,-8)$, $(12,0,-6)$ and $(8,4,2)$ is a representation of $\mathbb{Z}^3/N$. We can therefore use the IF algorithm for EDs.

$$\begin{bmatrix} -4 & 16 & 12 & 8 \\ 4 & -4 & 0 & 4 \\ 2 & -8 & -6 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 & -8 & -6 & 2 \\ 4 & -4 & 0 & 4 \\ -4 & 16 & 12 & 8 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 12 & 12 & 0 \\ 0 & 0 & 0 & 12 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 12 & 0 & 0 \\ 0 & 0 & 12 & 0 \end{bmatrix}$$

Thus, $IF(\mathbb{Z}^3/N) = \{2,12,12\}$.

MATH110BH HW9 Problem 2

Find the RCF form over $\mathbb{Q}$ of

$$\begin{bmatrix} -2 & 0 & 0 \\ -1 & -4 & -1 \\ 2 & 4 & 0 \end{bmatrix}$$

Clever way

Notice that this is almost in RCF form and do some ETs.

$$\begin{bmatrix} -2 & 0 & 0 \\ -1 & -4 & -1 \\ 2 & 4 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} -2 & 0 & 0 \\ 0 & -4 & -1 \\ 0 & 4 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & -4 \\ 0 & 0 & 4 \end{bmatrix}$$

Long, stupid way

$$\begin{bmatrix} x+2 & 0 & 0 \\ 1 & x+4 & 1 \\ -2 & -4 & x \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & x+4 & 1 \\ x+2 & 0 & 0 \\ -2 & -4 & x \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & (x+4)(x+2) & x+2 \\ 0 & 2(x+2) & x+2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & (x+2)^2 & 0 \\ 0 & 0 & x+2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & x+2 & 0 \\ 0 & 0 & (x+2)^2 \end{bmatrix}$$

We then get the same matrix as the clever way.

MATH110BH HW9 Problem 3

Find the RCF form over $\mathbb{Z}/2\mathbb{Z}$ of

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

long way

$$\begin{bmatrix} x-1 & -1 & 0 \\ 0 & x-1 & -1 \\ 0 & 0 & x-1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & x+1 & 0 \\ x+1 & 0 & 1 \\ 0 & 0 & x+1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & (x+1)^2 & 1 \\ 0 & 0 & x+1 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & (x+1)^3 & x+1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & x+1 & (x+1)^3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & (x+1)^3 \end{bmatrix}$$

$$(x+1)^3 = x^3 + 3x^2 + 3x + 1$$

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

MATH110BH HW9 Problem 4

Let $V \subset \mathbb{R}[x,y]$ be the subspace of all polynomials of the form $ax + by + c$, where $a, b, c \in \mathbb{R}$. Let A be a linear operator in V defined by

$$A(ax+by+c) = a(x+1) + b(y-1) + c$$

Find the EDs and JF.

Let $1, x, y$ be a basis for V. Then,

$$\begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Then,

$$\begin{bmatrix} x-1 & -1 & 1 \\ 0 & x-1 & 0 \\ 0 & 0 & x-1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1 & x-1 \\ 0 & x-1 & 0 \\ x-1 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & x-1 & 0 \\ 0 & x-1 & -(x-1)^2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & x-1 & 0 \\ 0 & 0 & (x-1)^2 \end{bmatrix}$$

Then, the RCF is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}$$

MATH110BH HW9 Problem 5

Find the JF over $\mathbb{C}$ of

$$A = \begin{bmatrix} 2i & 1 \\ 1 & 0 \end{bmatrix}$$

$tr(A) = 2i$ and $\det(A) = -1$, so the eigenvalues are i twice.

Then,

$$\begin{bmatrix} 2i & 1 \\ 1 & 0 \end{bmatrix} \to \begin{bmatrix} i & 1 \\ 1 & 0 \end{bmatrix} \to \begin{bmatrix} i & 1 \\ 0 & i \end{bmatrix}$$

HOMEWORK 9

1. Find the invariant factors of the quotient group $\mathbb{Z}^3/N$, where N is generated by $(-4,4,2)$, $(16,-4,-8)$, $(12,0,-6)$ and $(8,4,2)$.

2. Find the rational canonical form over $\mathbb{Q}$ of the matrix

$$\begin{pmatrix} -2 & 0 & 0 \\ -1 & -4 & -1 \\ 2 & 4 & 0 \end{pmatrix}$$

3. Find the rational canonical form over $\mathbb{Z}/2\mathbb{Z}$ of the matrix

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

4. Let $V \subset \mathbb{R}[x,y]$ be the subspace of all polynomials of the form $ax+by+c$, where $a,b,c \in \mathbb{R}$. Let $\mathcal{A}$ be a linear operator in V defined by

$$\mathcal{A}(ax+by+c) = a(x+1) + b(y-1) + c.$$

Find the elementary divisors and the canonical form of $\mathcal{A}$.

5. Find the Jordan canonical form over $\mathbb{C}$ of the matrix

$$\begin{pmatrix} 2i & 1 \\ 1 & 0 \end{pmatrix}$$

6. Prove that two 2×2 matrices over a field that are not scalar matrices are similar if and only if they have the same characteristic polynomials.

7. Prove that two 3×3 matrices are similar if and only if they have the same characteristic and the same minimal polynomials.

8. Show that the minimal polynomial of an $n \times n$-matrix A has the same irreducible divisors as the characteristic polynomial of A.

9. Let A be a nilpotent $n \times n$-matrix (that is $A^N = 0$ for some $N > 0$). Show that the invariant factors of A are powers of X. Prove that $A^n = 0$.

10. Prove that any $n \times n$-matrix A is similar to its transpose A^t.