Math 110BH Homework 8

Topics: direct sums, exact sequences, torsion

March 2, 2023

Problem 1

Let R be a ring and $M_1, \dots, M_k$ be R-modules. Suppose $N_1, \dots, N_k$ are submodules of $M_1, \dots, M_k$, respectively.

a) Prove that $N_1 \oplus \cdots \oplus N_k$ is a submodule of $M_1 \oplus \cdots \oplus M_k$.

b) Prove that as R-modules

$$(M_1 \oplus \cdots \oplus M_k)/(N_1 \oplus \cdots \oplus N_k) \cong M_1/N_1 \oplus \cdots \oplus M_k/N_k.$$

Problem 2

True or False? Let R be a ring and $M_1, \dots, M_k$ be R-modules. Every submodule of $M_1 \oplus \cdots \oplus M_k$ is of the form $N_1 \oplus \cdots \oplus N_k$ for some submodules $N_i \subseteq M_i$ for $i = 1, \dots, k$.

Problem 3

Let A, B, C be R-modules and $g : A \to B$, $f : B \to C$ be R-linear maps. We say that

$$A \xrightarrow{g} B \xrightarrow{f} C$$

is an *exact* sequence if $\ker(f) = \operatorname{im}(g)$. More generally, for modules M_n and R-linear maps $f_n : M_n \to M_{n-1}$, we say that a sequence

$$\cdots \longrightarrow M_{n+1} \xrightarrow{f_{n+1}} M_n \xrightarrow{f_n} M_{n-1} \longrightarrow \dots$$

is *exact* if $\ker(f_n) = \operatorname{im}(f_{n-1})$ for all n.

a) Prove that the sequence of three modules given by

$$0 \longrightarrow A \xrightarrow{g} B$$

is exact if and only if g is injective.

b) Prove that a sequence $B \xrightarrow{f} C \longrightarrow 0$ is exact if and only if f is surjective.

Problem 4

Let R be a ring. Prove that M is a finitely generated R-module if and only if there exists a surjective R-linear map $R^n \to M$ for some nonnegative integer n.

Problem 5

Let R be a ring and M be an R-module. Define the torsion submodule of M to be the subset

$$\operatorname{Tor}(M) := \{x \in M \mid rx = 0 \text{ for some } r \in R\}.$$

Prove that $\operatorname{Tor}(M)$ is a submodule of M.

Problem 6

For an R-module M, we say M is *torsion-free* if $\operatorname{Tor}(M) = \{0\}$. Let R be an integral domain. Prove that every ideal is torsion-free.

Problem 7*

Let R be an integral domain.

a) Prove that an ideal is a free submodule if and only if it is a principal ideal.

b) Prove that R is a PID if every submodule of a free module is again a free module. (Note we proved the converse of this statement in class.)

Problem 8*

Let R be an integral domain. Suppose M is a torsion R-module, i.e. $\operatorname{Tor}(M) = M$. Prove that if M is finitely generated, then

$$\operatorname{Ann}(M) = \{r \in R \mid r \cdot x = 0 \text{ for all } x \in M\}$$

is nonzero.

Problem 9

Give an example to show that finitely generated is a necessary hypothesis from the previous problem. That is, give an example of an integral domain R and an R-module N such that N is a torsion module but $\operatorname{Ann}(N) = \{0\}$.

Problem 10*

Let R be an integral domain. Prove the following claims.

a) Any free module is torsion-free.

b) For any R-module M the quotient module $M/\operatorname{Tor}(M)$ is torsion-free.