Math 110BH Homework 5

Sample Student

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Problem 1

Let R be a nontrivial commutative ring. Prove that if $f(t) \in R[t]$ is a zero divisor then there exists a nonzero element $b \in R$ such that $bf(t) = 0$.

Problem 2

Let F be a field and $p(t) \in F[t]$. Describe all ideals of the quotient ring $F[t]/(p(t))$ in terms of the factorization of $p(t)$.

Problem 3*

Let R be a commutative ring. Prove that a polynomial $f(t) \in R[t]$ is nilpotent if and only if all of the coefficients from $f(t)$ are nilpotent in R.

Problem 4

Determine if the following statements are true or false. If true, provide brief justification (you can cite a result(s) from class or the textbook). If false, provide a concrete counterexample showing why it is false.

a) If R is an integral domain, then $R[t]$ is an integral domain.

b) If R is a UFD, then $R[t]$ is a UFD.

c) If R is a PID, then $R[t]$ is a PID.

d) If R is a Euclidean domain, then $R[t]$ is a Euclidean domain.

Problem 5*

Let R be an integral domain. Let $f(t) \in R[t]$ be a monic polynomial of degree 2 or degree 3. Prove that $f(t)$ is reducible in $R[t]$ if and only if $f(t)$ has a root in R.

Problem 6

Give an example of a reducible polynomial in $\mathbb{Q}[t]$ that has no roots in $\mathbb{Q}$.

Problem 7

Decide if the following statements are true or false. If true, provide brief justification (you can cite a result(s) from class or the textbook). If false, provide a concrete counterexample showing why it is false.

a) Let $f(t)$ be a monic polynomial in $\mathbb{Z}[t]$. If $f(t)$ is irreducible over $\mathbb{Z}[t]$, then $f(t)$ is irreducible over $\mathbb{Q}[t]$.

b) Let $f(t)$ be a monic polynomial in $\mathbb{Q}[t]$. If $f(t)$ is irreducible over $\mathbb{Q}[t]$, then $f(t)$ is irreducible over $\mathbb{R}[t]$.

Problem 8*

Consider the subring $R = \mathbb{Z} + t\mathbb{Q}[t]$ of the ring $\mathbb{Q}[t]$. In words R is the ring of all polynomials in $\mathbb{Q}[t]$ whose constant term is an integer.

a) Show that the irreducibles in R are $\pm p$ where $p \in \mathbb{Z}$ is prime in $\mathbb{Z}$ and the polynomials $f(t) \in R$ with constant term ± 1 that are irreducible in $\mathbb{Q}[t]$.

b) Prove that t cannot be written as a product of irreducibles in R (in particular, t is not irreducible). Conclude that R is not a UFD.

Problem 9

Let R be a ring and M be a left R-module. Prove for all $m \in M$ that $0m = 0$ and $(-1)m = -m$.

Problem 10

Let R be a ring and M be a left R-module. Prove that if $rm = 0$ for some $r \in R$ and some nonzero $m \in M$ then r is not a unit in R.