Math 110BH Homework 3

Topics: Chinese remainder theorem, primes/irreducibles, Euclidean domains, PIDs, UFDs

Due: Wednesday, February 1st at 11:59pm

For all of this assignment R denotes a commutative ring.

Problem 1

An element $e \in R$ is called an *idempotent* if $e^2 = e$. Let e be an idempotent of R.

a) Show that the ideals Re and $R(1-e)$ are rings with identities e and $1-e$, respectively. (Note in general these are not "subrings" of R because they do not contain $1 \in R$.)

b) Prove that $R/Re \cong R(1-e)$ as rings.

c) Use the (b) and the Chinese Remainder Theorem to prove that $R \cong Re \times R(1-e)$.

Problem 2

Prove that a quotient of a PID by a prime ideal is again a PID.

Problem 3

Let n be a positive, square free integer. Let $R = \mathbb{Z}[\sqrt{-n}]$ where $\mathbb{Z}[\sqrt{-n}]$ denotes the subring of $\mathbb{C}$ given by $\{a + b\sqrt{-n} \mid a, b \in \mathbb{Z}\}$.

a) Let $\alpha \in R$. Prove there exists unique $a, b \in \mathbb{Z}$ such that $\alpha = a + b\sqrt{-n}$. That is, show that if $a + b\sqrt{-n} = c + d\sqrt{-n}$ then $a = c$ and $b = d$.

b) Define a function $N : R \to \mathbb{Z}^+ \cup \{0\}$ by $N(a + b\sqrt{-n}) = a^2 + nb^2$. Check that this function is multiplicative (i.e. for all $\alpha, \beta \in R$, $N(\alpha\beta) = N(\alpha)N(\beta)$) and $N(\alpha) = 0$ if and only if $\alpha = 0$.

Problem 4*

Let n be a square free integer that is greater than 3.

a) Prove that 2, $\sqrt{-n}$, $1 + \sqrt{-n}$ are irreducibles in R.

b) Show that at least one of the elements from (a) is not prime. Deduce that R is not a UFD.

Problem 5*

Let R be an integral domain. Prove that if the following two conditions hold then R is a principal ideal domain:

i) any two nonzero elements $a, b \in R$ have a greatest common divisor g which can be expressed as $g = ra + sb$ for some $r, s \in R$, and

ii) if $a_1, a_2, a_3, \ldots$ are nonzero elements of R such that $a_{i+1} \mid a_i$ for all i, then there is a positive integer N such that a_n is a unit times a_N for all $n \geq N$.

Problem 6

Prove that in a PID any two ideals (a) and (b) are comaximal if and only if a greatest common divisor of a and b is 1.

Problem 7*

Let F be a field. Prove that $F[t]$ is a Euclidean domain (and hence a PID and UFD).

Problem 8

Show the following three conditions are equivalent in a ring R:

i) R satisfies the *ascending chain condition* (ACC). That is, every increasing sequence of ideals of R

$$I_1 \subseteq I_2 \subseteq I_3 \subseteq \dots$$

eventually stabilizes, i.e. there exists an N such that for all $n \geq N$, $I_n = I_N$.

ii) Every ideal of R is finitely generated.

iii) R satisfies the *maximal condition*, meaning any every nonempty collection of ideals contains a maximal element (note this might not be a maximal ideal, it's just maximal for that particular collection of ideals).

You may use Zorn's Lemma or the Axiom of Choice.

Problem 9

Let a, b be nonzero elements from a ring R. A least common multiple of a and b is an element $m \in R$ such that

i) $a \mid m$ and $b \mid m$, and

ii) if $a \mid m'$ and $b \mid m'$ for some $m' \in R$, then $m \mid m'$.

Prove that if a least common multiple of a and b exists, then it is a generator for the unique largest principal ideal contained in $(a) \cap (b)$. Deduce that in a PID any two nonzero elements have a least common multiple that is unique up to multiplication by a unit.

Problem 10

Decide if the following statements are true or false. If the statement is true, prove it. If the statement is false, provide a counterexample (make sure to justify why your counterexample is a counterexample!).

a) If R is a PID, then the polynomial ring $R[t]$ is a PID.

b) Any subring of a UFD is again a UFD.