Let $(V, \mathcal{A})$ be an operator, hence V is an $F[X]$-module.

Let $f(x) = b_m X^m + b_{m-1} X^{m-1} + \ldots + b_0$. We have for $v \in V$,

$$f \cdot v = f(\mathcal{A})(v)$$

where $f(\mathcal{A})$ is the operator

$$b_m \mathcal{A}^m + b_{m-1} \mathcal{A}^{m-1} + \ldots + b_0 \, Id.$$

Clearly, $f \cdot v = 0 \; \forall v \in V$ iff $f(\mathcal{A}) = 0$ is the zero operator.

Consider the ideal

$$I = \{ f \in F[x] : f \cdot v = 0 \;\forall\, v \in V \} = \{ f : f(\mathcal{A}) = 0 \} \subset F[x].$$

Thus There is unique monic polynomial m s.t. $I = m \cdot F[x]$. $m_{\mathcal{A}} = m$ is the monic polynomial of the least degree such that $m(\mathcal{A}) = 0$; m is called the <u>minimal polynomial</u> of $\mathcal{A}$.

Clearly, $f(\mathcal{A}) = 0 \iff m \mid f$. Moreover, m divides any poly f such that $f(\mathcal{A}) = 0$

<u>Example.</u> Let $V = F[x] / f \cdot F[x]$ cyclic (f is monic).

Then $m = f$.

Now $V = F[x]/f_1 F[x] \oplus \ldots \oplus F[x]/f_k F[x]$ with monic $f_1 \mid f_2 \mid \ldots \mid f_k$. Clearly, $f_k = m$ is the minimal polynomial.

Since $f_k \mid P_{\mathcal{A}}$, we have $P_{\mathcal{A}}(\mathcal{A}) = 0$, Cayley-Hamilton theorem.

<u>Example</u> Classify up to similarity all 3×3 matrices A over $\mathbb{Q}$ such that $A^2 + 2A^3 + A^4 = 0$ but $A + A^2 \neq 0$. <u>Solution:</u> $IF(A) = \{f_1, \ldots, f_k\}$, $f_1 \mid f_2 \mid \ldots \mid f_k = m$. By assumption $m \mid x^2 + 2x^3 + x^4 = x^2(x+1)^2$, but $m \nmid x + x^2 = x(x+1)$. Also $\Sigma \deg(f_i) = 3$. Hence $m = x^2$, or $x^2(x+1)$, or $(x+1)^2$, or $x(x+1)^2$

Hence we have the following choices for $IF(A)$: $\{x, x^2\}$, $\{x^2(x+1)\}$, $\{x+1, (x+1)^2\}$, $\{x(x+1)^2\}$

$$RCF(A) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \text{ or } \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix} \text{ or } \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & -2 \end{bmatrix} \text{ or } \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & -2 \end{bmatrix}$$

Proposition. The following conditions are equivalent:

(1) V is a cyclic $F[X]$-module

(2) The matrix of $\mathcal{A}$ is $C(f)$ for some monic f in some basis.

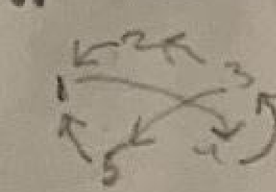
(3) The list of invariant factors consists of one polynomial.

(4) $m_{\mathcal{A}} = \mathcal{P}_{\mathcal{A}}$. (5) The elementary divisors are pairwise coprime

Proof. Clearly $(3) \Rightarrow (2) \Rightarrow (1)$

$(1) \Rightarrow (4)$; $V = F[X]/fF[X] \Rightarrow m_{\mathcal{A}} = f \mid P_{\mathcal{A}}$, $\deg m_{\mathcal{A}} = \deg P_{\mathcal{A}} \Rightarrow m_{\mathcal{A}} = P_{\mathcal{A}}$.

$(4) \Rightarrow (3)$; $\mathcal{P}_{\mathcal{A}} = d_1 d_2 \cdots d_k$, $m_{\mathcal{A}} = d_k \Rightarrow k = 1$. ∎



The invariant divisor form of the main theorem implies that

$$V \simeq F[X]/p_1^{\alpha_1} \cdot F[X] \oplus \ldots \oplus F[X]/p_s^{\alpha_s} \cdot F[X]$$

where the p_i are irreducible monic polynomials (with the $p_i^{\alpha_i}$ being the divisors of invariant factors and hence the divisors of $\mathcal{P}_{\mathcal{A}}$).

Assume that p is linear, $p = X - \lambda$, $\lambda \in F$.

Consider $V = F[X]/p^{\alpha} F[X]$. In the basis $\bar{1}, \bar{X}-\lambda, \ldots, (\bar{X}-\lambda)^{\alpha-1}$ of V. Since

$$X \cdot (\bar{X}-\lambda)^i = \lambda \cdot (\bar{X}-\lambda)^i + (\bar{X}-\lambda)^{i+1},$$

the matrix of $\mathcal{A}$ in this basis looks as follows:

$$\begin{array}{cc} \alpha & \begin{pmatrix} \lambda & & & & 0 \\ 1 & \lambda & & & \\ & 1 & \lambda & & \\ & & \ddots & \ddots & \\ 0 & & & 1 & \lambda \end{pmatrix} \\ & \alpha \end{array}$$

called <u>Jordan block</u>.

If $\alpha = 1$, $\boxed{\lambda}$.

Theorem (Jordan Canonical Form for Linear Operator)

Let $\mathcal{A}$ be a linear operator on a vector space V, $\dim V < \infty$. Assume that $\mathcal{P}_{\mathcal{A}}$ factors into a product of linear polynomials. Then there is a basis of V with respect to which the matrix A of $\mathcal{A}$ is of the form

$$\begin{pmatrix} \boxed{J_1} & & 0 \\ & \boxed{J_2} & \\ 0 & & \boxed{J_s} \end{pmatrix}$$

called Jordan Canonical Form

where the J_i are Jordan blocks, uniquely determined up to a permutation along the diagonal.

Proof. All elementary divisors are divisors of $\mathcal{P}_{\mathcal{A}}$, hence are equal to $(x-\lambda)^{\alpha}$, $\lambda \in F$. ■

Theorem (Jordan Canonical Form for matrices)

Let A be an $n \times n$ matrix such that $\mathcal{P}_A$ factors into a product of linear polynomials. Then A is similar to a matrix

$$\begin{pmatrix} \boxed{J_1} & & 0 \\ & \boxed{J_2} & \\ 0 & & \boxed{J_s} \end{pmatrix}$$

called Jordan Canonical Form

where the J_i are Jordan blocks, uniquely determined up to a permutation along the diagonal. ■

Remark. Diagonal elements are the roots of $\mathcal{P}_A$.

Remark. The condition for S_A holds over $F = \mathbb{C}$.

How to find Jordan Canonical form:

(1) Find the invariant factors;

(2) Find the elementary divisors $(x-\lambda_1)^{\alpha_1} \cdots (x-\lambda_s)^{\alpha_s}$.

Then the Jordan Canonical Form is:

$$\begin{pmatrix} J_1 & & & 0 \\ & J_2 & & \\ & & \ddots & \\ 0 & & & J_s \end{pmatrix}$$

where J_i is the Jordan block of $(x-\lambda_i)^{\alpha_i}$.

Example. Let the invariant factors of a matrix A are:

$d_1 = x+1$, $d_2 = (x+1)(x-2)$, $d_3 = (x+1)^2(x-2)^3$. Then A is similar to

$$\begin{pmatrix} \boxed{-1} & & & & & & & \\ & \boxed{-1} & & & & & & 0 \\ & & \boxed{2} & & & & & \\ & & & -1 & & & & \\ & & & 1 & -1 & & & \\ & 0 & & & & 2 & & \\ & & & & & 1 & 2 & \\ & & & & & & 1 & 2 \end{pmatrix}.$$

Let A be an $(n \times n)$-matrix. An element $\lambda \in F$ is called an eigenvalue of A if $AX = \lambda X$ for some nonzero $(1 \times n)$ column X. Any column X with this property is called an eigenvector.

Proposition. An element $\lambda \in F$ is an eigenvalue of A if and only if λ is a root of $\wp_A$.

Proof. λ is an eigenvalue of $A \iff AX = \lambda X$ for $X \neq 0$
$\iff (\lambda I_n - A)X = 0$, $X \neq 0 \iff \det(\lambda I_n - A) = 0 \iff \wp_A(\lambda) = 0$. ∎

Let $(V, \mathcal{A})$ be a linear operator. An element $\lambda \in F$ is called an eigenvalue of $\mathcal{A}$ if $\mathcal{A}(v) = \lambda v$ for some $v \in V$, $v \neq 0$. All $v \in V$ with this property are called the eigenvectors. Similarly, eigenvalues are the roots of $\wp_{\mathcal{A}}$.

Proposition. Let $\mathcal{A}$ be a linear operator on a vector space V. Then the following conditions are equivalent:

(1) There matrix of $\mathcal{A}$ is diagonal with respect to some basis;

(2) There exists a basis consisting of eigenvectors;

(3) V is the direct sum of eigenspaces;

(4) All elementary divisors are linear;

(5) All invariant factors split into a product of distinct linear polynomials;

(6) The minimal polynomial splits into a product of distinct linear polynomials.

Proof $(1) \iff (2)$, $(2) \iff (3)$ LA

$(2) \iff (4)$; basis consists of eigenvectors $\iff V \simeq F[X]/(X-\lambda_1)F[X] \oplus$
$\oplus F[X]/(X-\lambda_n)F[X] \iff X-\lambda_1, \dots, X-\lambda_n$ are the elementary divisors.

(4) $\Leftrightarrow$ (5) the elementary divisors are the primary divisors of the invariant factors.

(5) $\Leftrightarrow$ (6) $m_A = d_k$ and all $d_i \mid d_k$. ∎

<u>Example</u>. Find the inv. factors and elementary divisors of

$$\begin{pmatrix} -1 & & & & \\ 1 & -1 & & O & \\ & & 3 & & \\ & O & 1 & 3 & \\ & & & & 3 \end{pmatrix}$$

$\{(x+1)^2, (x-3)^2, x-3\}$ elem. divisors

$\{(x-3), (x+1)^2(x-3)^2\}$ inv. factors.