HOMEWORK 8

1. Let F be a free (left) R-module with basis $\{x_1, x_2, \ldots, x_n\}$ and let M be an R-module. Prove that for any elements $m_1, m_2, \ldots, m_n \in M$ there is a unique R-module homomorphism $f : F \to M$ such that $f(x_i) = m_i$ for all i.

2. Let $f : M \to N$ be surjective homomorphism of (left) R-modules. Prove that if N is free, there is a homomorphism of (left) R-modules $g : N \to M$ such that $f \circ g$ is the identity of N.

3. Let f be a linear operator in a vector space V over $\mathbb{R}$ such that $f(f(v)) = -v$ for all $v \in V$. Prove that V has the structure of a vector space over $\mathbb{C}$ such that $iv = f(v)$ for all $v \in V$.

4. Show that a submodule of a cyclic module over a PID is also cyclic.

5. Let a and b be nonzero elements of a PID R. Prove that $R/aR \oplus R/bR \simeq R/cR \oplus R/dR$, where c is a least common multiple and d is a greatest common divisor of a and b.

6. Let M be a finitely generated torsion module over a PID R and let $n = |IF(M)|$. Prove that M can be generated by n elements and cannot be generated by less than n elements.

7. A module is called *indecomposable* if it is not equal to the direct sum of its nonzero submodules. Prove that a finitely generated module M over a PID R is indecomposable if and only if $M \simeq R$ or $M \simeq R/P^n$, where P is a prime ideal of R and $n \geq 0$.

8. Let n be an integer. Prove that every abelian group A with $nA = 0$ has the structure of a $\mathbb{Z}/n\mathbb{Z}$-module.

9. Classify all finite $\mathbb{Z}/n\mathbb{Z}$-modules up to isomorphism. (Hint: Use the classification of finite abelian groups.)

10. Let M be a subgroup of a free abelian group F of finite rank. Suppose that $M \cap pF = pM$ for all prime integers p. Prove that the quotient group F/M is free.