Linear Algebra

Boran Erol

March 2024

1 Basic Definitions, Examples and Theorems

Here are some examples of vector spaces:

Example 1.1. $V = F[x]$ is a vector space over F with basis $1, x, x^2, ...$

Example 1.2. Let V be the space of real-valued functions on $[a, b]$ where $a < b$.

Example 1.3. Let V be the space of continuous real-valued functions on $[a, b]$ where $a < b$.

Example 1.4. Let F be a field and $f(x) = x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0 \in F[x]$. Consider $F[x]/(f(x))$. This is an n-dimensional vector spaces over F.

Expound on this using the Euclidean algorithm.

Theorem 1.5. Let V be an n dimensional vector space over F. Then, $V \cong F^n$.

Theorem 1.6. Let V be a vector space over F and W be a subspace of V. Then, V/W is a vector space with $\dim(V) = \dim(W) + \dim(V/W)$.

Corollary 1.6.1. Let $\phi : U \to V$ be a linear transformation of vector spaces over F. Then,

$$\dim V = \dim(\ker \phi) + \dim(\phi(V))$$

.

1.1 Exercises

Lemma 1.7. Let V be a finite dimensional vector space and $\phi \in L(V)$. Then,

$$\exists m \in \mathbb{Z} : Im(\phi^m) \cap \ker(\phi^m) = \{0\}$$

.

Proof. □

Lemma 1.8. Let V be an n dimensional vector space and $\phi \in L(V)$ such that $\phi^2 = 0$. Then, $Im(\phi) \subseteq \ker(\phi)$. Therefore, $\text{rank}(\phi) \leq n/2$.

Proof. $Im(\phi) \subseteq \ker(\phi)$ is easy: if x in the image doesn't map to 0, $\phi^2 \neq 0$. $\text{rank}(\phi) \leq n/2$ is an immediate consequence of Rank-Nullity. □

2 Diagonalization

Lemma 2.1. Every linear operator on a finite-dimensional complex vector space has an eigenvalue and a corresponding eigenvector.

Proof. In Sheldon Axler's Linear Algebra Done Right. Basically uses the fact that $\mathbb{C}$ is algebraically closed. □

Definition 1. Let $A \in M_n(F)$. The **characteristic polynomial of A** is

$$\det(xI_n - A) = x^n + ... + (-1)^n \cdot \det A$$

Notice that this is a monic polynomial and the degree of the polynomial is the dimension of the vector space V.

3 Dual Spaces

Lemma 3.1. Let V be a finite dimensional vector spaces and $f_1, ..., f_k$ be elements of V^* with kernels $N_1, ..., N_k$. Prove that $f \in V^*$ is a linear combination of $f_1, ..., f_k$ if and only if $N := ker(f)$ contains $\bigcap_{i=1}^k N_i$.

Proof. The forward implication is immediate. For the reverse implication, let $f \in V^*$ such that $\square$

4 Determinants

Lemma 4.1. Let A be an $n \times n$ matrix with column vectors $a_1, ..., a_n$. Find the determinant of the matrix with column vectors $a_1 + a_2, a_2 + a_3, ..., a_n + a_1$.

5 Positive Definite Matrices

Lemma 5.1. Let A, B be positive-definite matrices such that $A \leq B$, i.e. $B - A$ is positive-definite. Prove then that $B^{-1} \leq A_{-1}$.

Proof. Recall that inverses of positive definite matrices are positive definite.

$\square$

6 Similar Matrices and Canonical Forms

Definition 2. Let $A, B \in M_n(F)$. A and B are **similar** if there's an invertible $M \in M_n(F)$ such that $A = MBM^{-1}$.

Similarity is an equivalence relation.

Similar matrices have the same eigenvalues. The converse is not true.

7 Inner Products

Orthonormal bases are useful in the sense that they allow us to compute the coordinates of vectors and inner products of vectors in an efficient fashion.

For example, let $b_1, ..., b_n$ be an orthonormal basis for $\mathbb{R}^n$.

Let $c = \sum_{i=1}^{n} \alpha_i b_i$ and $c' = \sum_{i=1}^{n} \alpha_i' b_i$.

Then, $\langle c, c' \rangle = \sum_{i=1}^{n} \alpha_i \alpha_i'$.

7.1 Exercises

Let V be the vector space of real-valued polynomials over $\mathbb{R}$ of degree at most 2. Define the usual inner product:

$$\langle p, q \rangle = \int_0^1 p(t)q(t)dt$$

Find an orthogonal basis containing $p(x) = x$.

Use Gram-Schmidt. One element can be $1 - \frac{3}{2}x$.

Lemma 7.1. Let V be a finite-dimensional complex inner product space. Then, there's no invertible linear operator T on V such that $\forall v \in V : \langle Tv, v \rangle = 0$.

Proof. Linear operators on V have at least one eigenvalue: use this to reach a contradiction. $\square$

However, this is not true for $\mathbb{R}$. Just consider the 90 degree rotation on $\mathbb{R}^2$ with the Euclidean norm.

8 Various Exercises

Lemma 8.1. Let f be a linear operator in a vector space V over $\mathbb{R}$ such that $\forall v \in V : f(f(v)) = -v$. V has the structure of a vector space over $\mathbb{C}$ such that $\forall v \in V : f(v) = iv$.

Proof. $\square$