HOMEWORK 7

1. Prove that the intersection of two principal ideals in a UFD is a principal ideal.

2. Find an example of a non-free submodule $N \subset M$ of a free module M over some domain R.

3. Show that a submodule N of a module M generated by n elements over a PID also can be generated by n elements.

4. Prove that the group $\mathbb{Z}^n$ cannot be generated by $n-1$ elements.

5. Find two non-free modules M and N over $\mathbb{Z}/6\mathbb{Z}$ such that $M \oplus N$ is free.

6. Let R be a PID and let M be a torsion finitely generated R-module with the invariant factors $d_1|d_2|\dots|d_k$. Set

$$I = \{a \in R \text{ such that } aM = 0\}.$$

Prove that $I = d_k R$.

7. Classify all abelian groups of order 300.

8. Find the rank of the subgroup in $\mathbb{Z}^3$ generated by $(2,-2,0)$, $(0,4,-4)$ and $(5,0,-5)$.

9. Determine the invariant factors of the factor group $\mathbb{Z}^3/N$, where N is generated by $(3,-3,3)$, $(0,6,-12)$ and $(9,0,-9)$.

10. Let M be a finitely generated torsion module over a PID R. Prove that M is cyclic if and only if every two elementary divisors of M are relatively prime.