Problem Set 1

Due: Wednesday, January 18

We will assume rings have identity and ring homomorphisms are unital (send 1 to 1) unless stated otherwise. Do Problems A–C but do not turn these in. Turn in Problems 1–9.

Problem A. Let R be a ring. Show that $(-1)^2 = 1$ in R.

Problem B. Decide which of the following are subrings of the ring of all functions from the closed interval $[0, 1]$ to $\mathbb{R}$:

(a) the set of all functions $f(x)$ such that $f(q) = 0$ for all $q \in \mathbb{Q} \cap [0, 1]$
(b) the set of all polynomial functions
(c) the set of all functions which only have a finite number of zeros, together with the zero function
(d) the set of all functions which have an infinite number of zeros
(e) the set of all functions f such that $\lim_{x\to 1^-} f(x) = 0$.
(f) the set of all rational linear combinations of the functions $\sin(nx)$ and $\cos(mx)$, where $m, n \in \{0, 1, 2, \ldots\}$.

Problem C. Decide which of the following are ideals of the ring $\mathbb{Z} \times \mathbb{Z}$:

(a) $\{(a, a)|a \in \mathbb{Z}\}$
(b) $\{(2a, 2b)|a, b \in \mathbb{Z}\}$
(c) $\{(2a, 0)|a \in \mathbb{Z}\}$
(d) $\{(a, -a)|a \in \mathbb{Z}\}$.

Problem 1. An element x in a ring R is called *nilpotent* if $x^m = 0$ for some $m \in \mathbb{Z}^+$. Let x be a nilpotent element of the commutative ring R.

(a) Prove that x is either zero or a zero divisor.
(b) Prove that rx is nilpotent for all $r \in R$.
(c) Prove that $1 + x$ is a unit in R.
(d) Deduce that the sum of a nilpotent element and a unit is a unit.

Problem 2. Let R be a ring with $1 \neq 0$. A nonzero element a is called a *left zero divisor* in R if there is a nonzero element $x \in R$ such that $ax = 0$. Symmetrically, $b \neq 0$ is a *right zero divisor* if there is a nonzero $y \in R$ such that $yb = 0$ (so a zero divisor is an element which is either a left or a right zero divisor). An element $u \in R$ has a *left inverse* in R if there is some $s \in R$ such that $su = 1$. Symmetrically, v has a *right inverse* if $vt = 1$ for some $t \in R$.

Let $\mathbb{F}$ be a field. An $\mathbb{F}$-*algebra* is a ring A together with a (unital) ring homomorphism $f : \mathbb{F} \to A$ such that the image $f(\mathbb{F})$ is contained in the center of A, where the *center* of a ring A is the set $\{a \in A : ab = ba \text{ for every } b \in A\}$. It follows from this definition that an $\mathbb{F}$-algebra is also an $\mathbb{F}$-vector space. So a finite-dimensional $\mathbb{F}$-algebra is an $\mathbb{F}$-algebra that is also finite-dimensional as an $\mathbb{F}$-vector space.

(a) Prove that u is a unit if and only if it has both a right and a left inverse (i.e. u must have a two-sided inverse).
(b) Prove that if u has a right inverse then u is not a right zero divisor.
(c) Prove that if u has more than one right inverse then u is a left zero divisor.
(d) Prove that if R is a finite-dimensional algebra over a field then every element that has a right inverse is a unit (i.e., has a two-sided inverse).

Problem 3. Let $\mathcal{K} = \{k_1, \ldots, k_m\}$ be a conjugacy class in the finite group G.

(a) Prove that the element $K = k_1 + \cdots + k_m$ is in the center of the group ring RG.

(b) Let $\mathcal{K}_1, \ldots, \mathcal{K}_r$ be the conjugacy classes of G and for each $\mathcal{K}_i$ let K_i be the element of RG that is the sum of the members of $\mathcal{K}_i$. Prove that an element $\alpha \in RG$ is in the center of RG if and only if $\alpha = a_1K_1 + a_2K_2 + \cdots + a_rK_r$ for some $a_1, a_2, \ldots, a_r \in R$.

Problem 4. Prove that the rings $\mathbb{Z}[x]$ and $\mathbb{Q}[x]$ are not isomorphic.

Problem 5. Decide which of the following are ideals of the ring $\mathbb{Z}[x]$ (and justify your answer):

(a) the set of all polynomials whose constant term is a multiple of 3

(b) the set of all polynomials whose coefficient of x^2 is a multiple of 3

(c) the set of all polynomials whose constant term, coefficient of x, and coefficient of x^2 are zero

(d) $\mathbb{Z}[x^2]$ (i.e., the polynomials in which only even powers of x appear)

(e) the set of polynomials whose coefficients sum to zero

(f) the set of polynomials $p(x)$ such that $p'(0) = 0$, where $p'(x)$ is the usual first derivative of $p(x)$ with respect to x.

Problem 6. Prove that every (two-sided) ideal of $M_n(R)$ is equal to $M_n(J)$ for some (two-sided) ideal J of R.

Problem 7. Let I and J be ideals of R.

(a) Prove that $I + J$ is the smallest ideal of R containing both I and J.

(b) Prove that IJ is an ideal contained in $I \cap J$.

(c) Give an example where $IJ \neq I \cap J$.

(d) Prove that if R is commutative and if $I + J = R$ then $IJ = I \cap J$.

Problem 8. Let R be the ring of all continuous functions from the closed interval $[0, 1]$ to $\mathbb{R}$ and for each $c \in [0, 1]$ let $M_c = \{f \in R | f(c) = 0\}$ (recall that M_c was shown to be a maximal ideal of R).

(a) Prove that if M is *any* maximal ideal of R then there is a real number $c \in [0, 1]$ such that $M = M_c$.

(b) Prove that if b and c are distinct points in $[0, 1]$ then $M_b \neq M_c$.

(c) Prove that M_c is not equal to the principal ideal generated by $x - c$.

(d) Prove that M_c is not a finitely generated ideal.

Problem 9. (Bonus) Let $\mathcal{S}_3$ denote the symmetric group on three letters. Determine all nonzero minimal two-sided ideals of $\mathbb{C}\mathcal{S}_3$ (a nonzero two-sided ideal of a ring is *minimal* if the only two-sided ideals it contains are 0 and itself).