# Problem Set 3

Due: Wednesday, February 1 at the beginning of class

We will assume rings have identity and ring homomorphisms are unital (send 1 to 1) unless stated otherwise. Read 10.3 Problem 27. Turn in Problems 1–8.

**Problem 1.** An element $e \in R$ is called a *central idempotent* if $e^2 = e$ and $er = re$ for all $r \in R$. If $e$ is a central idempotent in $R$, prove that $M = eM \oplus (1-e)M$.

**Problem 2.** An element $m$ of the $R$-module $M$ is called a *torsion element* if $rm = 0$ for some nonzero element $r \in R$. The set of torsion elements is denoted

$$\text{Tor}(M) = \{m \in M \mid rm = 0 \text{ for some nonzero } r \in R\}.$$

(a) Prove that if $R$ is an integral domain then $\text{Tor}(M)$ is a submodule of $M$ (called the *torsion* submodule of $M$).
(b) Give an example of a ring $R$ and an $R$-module $M$ such that $\text{Tor}(M)$ is not a submodule.
(c) If $R$ has zero divisors show that every nonzero $R$-module has nonzero torsion elements.

**Problem 3.** Let $\phi : M \to N$ be an $R$-module homomorphism. Prove that $\phi(\text{Tor}(M)) \subseteq \text{Tor}(N)$.

**Problem 4.** Let $R = \mathbb{Z}[x]$ and let $M = (2, x)$ be the ideal generated by 2 and $x$, considered as a submodule of $R$. Show that $\{2, x\}$ is not a basis of $M$. Show that the rank of $M$ is 1 but that $M$ is not free of rank 1.

**Problem 5.** Let $F$ be a field. Give a simple description of the set of zero divisors of $M_n(F)$ in terms of concepts from linear algebra.

**Problem 6.** Show that if $M_1$ and $M_2$ are irreducible $R$-modules, then any nonzero $R$-module homomorphism from $M_1$ to $M_2$ is an isomorphism. Deduce that if $M$ is irreducible then $End_R(M)$ is a division ring (this result is called Schur's Lemma).

**Problem 7.** Show that if $R = \mathbb{Z}$, $I = \mathbb{Z}_{>0}$, and $M_i = \mathbb{Z}/i\mathbb{Z}$ for each $i \in I$, then $\bigoplus_{i \in I} M_i$ is not isomorphic to $\prod_{i \in I} M_i$.

**Problem 8.** Let $R$ be a commutative ring. Prove that there is no isomorphism of $R$-modules between $R^2$ and $R^3$. Note: your proof must use that $R$ is commutative since the statement is false without it (see 10.3 Problem 27).