Math 110BH Homework 7

Topic(s): Direct sums, free modules, Noetherian modules

February 24, 2023

For all of this assignment R denotes an arbitrary ring unless stated otherwise.

Problem 1*

Let M and $M_1, \dots, M_n$ be R-modules. Prove that $M \cong \bigoplus_{j=1}^n M_j$ if and only if there exist R-linear maps $\iota_j : M_j \to M$ and $p_j : M \to M_j$ such that

i) $p_j \circ \iota_j$ is the identity map on M_j for all j,

ii) $p_j \circ \iota_k = 0$ for all $j \neq k$, and

iii) $\iota_1 \circ p_1 + \cdots + \iota_n \circ p_n$ is the identity on M.

Problem 2

Let M, N be free R-modules with bases α, β, respectively. Use the universal property of free R-modules to prove that if there exists a bijection $g : \alpha \to \beta$, then there exists an R-module isomorphism $f : M \to N$ such that $f(a) = g(a)$ for all $a \in \alpha$.

Problem 3

Let M be a R-module. Prove that all of the following conditions are equivalent:

i) All submodules of M are finitely generated.

ii) M satisfies the ascending chain condition (ACC) for submodules. That is, whenever

$$M_1 \subseteq M_2 \subseteq M_3 \subseteq \dots$$

is an increasing chain of submodules, then there is a positive integer N such that for all $k \geq N$, $M_k = M_N$.

iii) Every nonempty set of submodules of M contains a maximal element under inclusion.

Note: If M satisfies any (so then all) of the above, then we say M is a **Noetherian module**.

Problem 4*

Prove that if M and N are Noetherian modules, then the direct sum $M \oplus N$ is Noetherian.

Problem 5*

Let M be a Noetherian R-module and $f : M \to M$ be a surjective module homomorphism.

a) Prove that there exists an n such that $\ker(f^n) = \ker(f^{n+k})$ for all $k \geq 0$. Then prove for such n that $\ker(f^n) \cap \operatorname{im}(f^n) = \{0\}$.

b) Prove that f is an R-module isomorphism.