HOMEWORK 5

1. Show that over any field there exist infinitely many non-associate irreducible polynomials.

2. Prove that the factor ring $\mathbb{Z}[i]/(1+i)\mathbb{Z}[i]$ is a field of two elements.

3. Let $f, g \in \mathbb{Q}[X]$ with $fg \in \mathbb{Z}[X]$. Prove that there is $a \in \mathbb{Q}^\times$ such that $af \in \mathbb{Z}[X]$ and $a^{-1}g \in \mathbb{Z}[X]$.

4. Let F be a field. Prove that the set R of all polynomials in $F[X]$ whose X-coefficient is equal to 0 is a subring of $F[X]$ and that R is not a UFD. (Hint: Use $X^6 = (X^2)^3 = (X^3)^2$.)

5. Find all irreducible polynomials of degree ≤ 4 in $(\mathbb{Z}/2\mathbb{Z})[X]$.

6. Let $f \in \mathbb{Z}[X]$, $a, b \in \mathbb{Z}$, $a \neq b$. Prove that $a - b$ divides $f(a) - f(b)$. (Hint: $a - b$ divides $a^n - b^n$.)

7. Prove that $X^n + Y^n - 1$ is irreducible in $\mathbb{Z}[X, Y]$ for every $n > 0$. (Hint: Use Eisenstein's Criterion.)

8. Let f be a monic polynomial in $\mathbb{Z}[X]$. Prove that if $a \in \mathbb{Q}$ is a root of f then $a \in \mathbb{Z}$.

9. Find all roots of $f = X^p - X$ in $(\mathbb{Z}/p\mathbb{Z})[X]$ (p prime) and factor f into a product of irreducible polynomials. (Hint: Use Fermat's Little Theorem.)

10. Determine whether $X^4 + 4$ is irreducible in $\mathbb{Z}[X]$.