HW

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Exercise 1

The forward direction was proved on last week's homework.

Suppose M is torsion free. By the fundamental theorem of finitely generated modules over PIDs, we have $M \cong R^s \oplus R/(d_1) \oplus \cdots \oplus R/(d_k)$ for some $d_1, \ldots, d_k \in R$. We claim that the decomposition of M cannot contain components of the form $R/(d_i)$. Indeed, if this were the case, then let x be an element in M corresponding to some element in $R/(d_i)$. Then $d_i x = 0$, contradicting the fact that M is torsion free. We conclude that $M \cong R^s$. But R^s is clearly free, so M is free. This completes the proof.

Exercise 2

Suppose M is simple. Take $x \neq 0$. Then $\langle x \rangle \neq \{0\}$, so we must have $\langle x \rangle = M$. By problem 1 of homework 6, we have $M \cong R/Ann(x)$. Now $Ann(x) \neq \{0\}$, for if this were the case, then $M \cong R$. But any integral domain is a torsion free module over itself, contradicting the fact that M is torsion. To show that $Ann(x)$ is prime, it suffices to show that $Ann(x)$ is maximal. Suppose that I is an ideal such that $Ann(x) \subset I$. We know that $I/Ann(x)$ is a submodule of $R/Ann(x)$. But since $M \cong R/Ann(x)$ is simple, we must have $I/Ann(x) = \{0 + Ann(x)\}$ or $I/Ann(x) = R/Ann(x)$. In the first case $I = Ann(x)$ and in the second case $I = R$. We conclude that $Ann(x)$ is maximal.

Suppose $M = \langle x \rangle$ and $Ann(x)$ is a nonzero prime ideal. Again, we know $M \cong R/Ann(x)$. Suppose that $N \subset M$ is a submodule. By the correspondence theorem, we can identify N with a submodule (ideal) I of R containing $Ann(x)$. Since R is a PID and $Ann(x)$ is prime, $Ann(x)$ is in fact maximal, hence $I = Ann(x)$ or $I = R$. If $I = Ann(x)$, then the previous correspondence gives $N = \{0\}$, and if $I = R$, then $N = R/Ann(x)$. We conclude that M is simple. This completes the proof.

Exercise 3

Observe that $400 = 2^4 \times 5^2$. Let M be an abelian group (equivalently a $\mathbb{Z}$-module) of order 400. Note that M must be finitely generated, for example $M = \langle M \rangle$. Thus by the elementary divisor form of the fundamental theorem of finitely generated modules over PIDs, we have $M \cong \mathbb{Z}^s \oplus \mathbb{Z}/p_1^{e_1} \oplus \cdots \oplus \mathbb{Z}/p_r^{e_r}$. We must have $s = 0$, or else $|M| = \infty$. Now $|\mathbb{Z}/p_1^{e_1} \oplus \cdots \oplus \mathbb{Z}/p_r^{e_r}| = p_1^{e_1} \dots p_r^{e_r} = 400$, so by the fundamental theorem of arithmetic $p_i \in \{2, 5\}$ for all i, and it suffices to find all possible assignments of $e_1, \dots, e_r$ To make this equality hold. One can check that the unique assignments are

a. $\mathbb{Z}/25 \oplus \mathbb{Z}/16$

b. $\mathbb{Z}/25 \oplus \mathbb{Z}/8 \oplus \mathbb{Z}/2$

c. $\mathbb{Z}/25 \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/4$

d. $\mathbb{Z}/25 \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2$

e. $\mathbb{Z}/25 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2$

f. $\mathbb{Z}/5 \oplus \mathbb{Z}/5 \oplus \mathbb{Z}/16$

g. $\mathbb{Z}/5 \oplus \mathbb{Z}/5 \oplus \mathbb{Z}/8 \oplus \mathbb{Z}/2$

h. $\mathbb{Z}/5 \oplus \mathbb{Z}/5 \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/4$

i. $\mathbb{Z}/5 \oplus \mathbb{Z}/5 \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2$

j. $\mathbb{Z}/5 \oplus \mathbb{Z}/5 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2$

Exercise 4

If suffices to consider matrices in rational canonical form. Suppose $M, N \in F^{3\times 3}$. Let $p_1, p_2, p_3 \in F[t]$ s.t. $q_1 \mid q_2 \mid q_3$ and q_1, q_2, q_3 are either 1 or the invariant factors of M. Define q_1, q_2, q_3 for N in a similar fashion. The fact that M and N have the same minimal polynomial implies $p_3 = q_3$, and the fact that they have the same characteristic polynomial implies $p_1 p_2 p_3 = q_1 q_2 q_3$. We will show that $p_i = q_i$ for $i = 1, 2, 3$. We proceed by casework.

Case 1: $\deg(p_3) = \deg(q_3) = 3$: since the degree of the characteristic polynomial is 3, we must have $p_1 = p_2 = q_1 = q_2 = 1$.

Case 2: $\deg(p_3) = \deg(q_3) = 2$: since the degree of the characteristic polynomial is 3 and $\deg(p_1) \leq \deg(p_2), \deg(q_1) \leq \deg(q_2)$, we must have $\deg(p_1) = \deg(q_1) = 0$ and $\deg(p_2) = \deg(q_2) = 1$. In particular $p_1 = q_1 = 1$. Thus the condition on the characteristic polynomials reduces to $p_2 p_3 = q_2 q_3$. But since $p_3 = q_3$, this implies $p_2 = q_2$.

Case 3: $\deg(p_3) = \deg(q_3) = 1$: since $p_1 \mid p_2 \mid p_3$, we have $\deg(p_1) \leq \deg(p_2) \leq \deg(p_3) = 1$. But the degree of the characteristic polynomial is $\deg(p_1 p_2 p_3) = \deg(p_1) + \deg(p_2) + \deg(p_3) = 3$, so we must have $\deg(p_i) = 1$ for $i = 1, 2, 3$. Combined with the divisibility requirement, we get $p_1 = p_2 = p_3$. Similarly $q_1 = q_2 = q_3$. Thus $p_i = p_3 = q_3 = q_i$ for all i.

This completes the proof.

Exercise 5

Consider the matrices

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

It is easy to check that both matrices have minimal polynomial t^2 and characteristic polynomial t^4. However, by the uniqueness of rational canonical form, A and B are not similar.

Exercise 6

By the existence and uniqueness of rational canonical form, it suffices to construct the invariant factors of the matrix. One of the invariant factors is the minimal polynomial $(x+2)^2(x-1)$. The other remaining factors must divide the minimal polynomial, hence have the form $(x+2)^a(x-1)^b$ for $a \in \{0, 1, 2\}, b \in \{0, 1\}$ with a and b not both zero. In addition, the characteristic polynomial has degree 6, so the degree product of the other invariant factors is $6 - 3 = 3$. Under these constraints, it is not hard to list out all possibilities for the invariant factors:

a. $x - 1, x - 1, x - 1, (x - 1)(x + 2)^2$

b. $x + 2, x + 2, x + 2, (x - 1)(x + 2)^2$

c. $x - 1, (x - 1)(x + 2), (x - 1)(x + 2)^2$

d. $x + 2, (x - 1)(x + 2), (x - 1)(x + 2)^2$

e. $x + 2, (x + 2)^2, (x - 1)(x + 2)^2$

f. $(x - 1)(x + 2)^2, (x - 1)(x + 2)^2$

The corresponding matrices can be constructed using the companion matrices of each invariant factor:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -3 \end{pmatrix}, \begin{pmatrix} -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -3 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -3 \end{pmatrix}$$

$$\begin{pmatrix} -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -3 \end{pmatrix}, \begin{pmatrix} -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -4 & 0 & 0 & 0 \\ 0 & 1 & -4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -3 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 4 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -3 \end{pmatrix}$$

Exercise 7

Let $a, b, c, d \in F$. We can simply list out all possibilities; to avoid construction matrices that are similar, we place the largest Jordan blocks first.

The 2×2 Jordan forms are

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \begin{pmatrix} a & 0 \\ 1 & a \end{pmatrix}$$

The 3×3 Jordan forms are

$$\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}, \begin{pmatrix} a & 0 & 0 \\ 1 & a & 0 \\ 0 & 0 & b \end{pmatrix}, \begin{pmatrix} a & 0 & 0 \\ 1 & a & 0 \\ 0 & 1 & a \end{pmatrix}$$

The 4×4 Jordan forms are

$$\begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{pmatrix}, \begin{pmatrix} a & 0 & 0 & 0 \\ 1 & a & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & c \end{pmatrix}, \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 1 & b \end{pmatrix}, \begin{pmatrix} a & 0 & 0 & 0 \\ 1 & a & 0 & 0 \\ 0 & 1 & a & 0 \\ 0 & 0 & 0 & b \end{pmatrix}, \begin{pmatrix} a & 0 & 0 & 0 \\ 1 & a & 0 & 0 \\ 0 & 1 & a & 0 \\ 0 & 0 & 1 & a \end{pmatrix}$$

Exercise 8

Let $A : V \to V$. We claim that $\mathrm{im}(A) \cap \ker(A) = \{0\}$. Indeed, if $x \in im(A) \cap \ker(A)$, then there exists $y \in V$ such that $Ay = x$, and also $Ax = 0$. But then

$$x = Ay = A^2 y = A(Ay) = Ax = 0,$$

as desired.

Thus we have the (inner) direct sum $\mathrm{im}\, A \oplus \ker A$ as a subspace of V. By the first isomorphism theorem (rank-nullity), we have $\dim \mathrm{im}\, A + \dim \ker A = \dim V$, hence $V = \mathrm{im}\, A \oplus \ker A$.

By definition, $Ax = 0$ for all $x \in ker(A)$. On the other hand, suppose $x \in \mathrm{im}(A)$ with $Ay = x$. Then $x = Ay = A^2 y = A(Ay) = Ax$, so A maps any element $x \in \mathrm{im}(A)$ to itself. With this observation, let $\mathcal{B}$ be a basis for $\ker A$ and $\mathcal{C}$ a basis for $\mathrm{im}\, A$. Then $\mathcal{B} \cup \mathcal{C}$ is a basis for V, and by construction the representation of A in this basis is diagonal with zeros on the diagonal for vectors from $\mathcal{B}$ and ones on the diagonal for vectors from $\mathcal{C}$.

Exercise 9

By Cayley-Hamilton, we have $(T^4 - I)^2 = T^8 - 2T^4 + I = 0$. Rearranging gives $I = T(-T^7 + 2T^3)$, which shows that T is invertible.

Exercise 10

We claim that there are 8 similarity classes. Indeed, the factorization of the characteristic polynomial into irreducibles if $(x^2+1)^2(x-1)^2(x+1)^2$. By Cayley-Hamilton, the minimal polynomial must contain these same factors, hence the minimal polynomial is $(x^2+1)^a(x-1)^b(x+1)^c$ for $\{a,b,c\} \in \{1,2\}$. In fact, the choice of a, b, c uniquely determines the invariant factors, as the second invariant factor must containing the remaining factors (observe that we cannot have 3 or more invariant factors because that would imply that some irreducible factor appears with multiplicity at least 3). We conclude that there are $2^3 = 8$ similarity classes, as desired.