

## Problem Set 2

Due: Wednesday, January 25 at the beginning of class

We will assume rings have identity and ring homomorphisms are unital (send 1 to 1) unless stated otherwise. Do Problem A but do not turn it in. Turn in Problems 1–8.

**Problem A.** Decide which of the following are ring homomorphisms from  $M_2(\mathbb{Z})$  to  $\mathbb{Z}$ :

- (a)  $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto a$  (projection onto the 1,1 entry)
- (b)  $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto a + d$  (the *trace* of the matrix)
- (c)  $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto ad - bc$  (the *determinant* of the matrix).

**Problem 1.** For the following two rings, give an example of a prime ideal that is not maximal (and prove that your answer is correct):

- (a)  $\mathbb{Z}[x]$
- (b)  $F[x, y]$  for a field  $F$ .

**Problem 2.** Prove that  $R$  is a division ring if and only if its only left ideals are  $(0)$  and  $R$ . (The analogous result holds when “left” is replaced by “right”.)

**Problem 3.** Let  $R$  be a commutative ring. Prove that the principal ideal generated by  $x$  in the polynomial ring  $R[x]$  is a prime ideal if and only if  $R$  is an integral domain. Prove that  $(x)$  is a maximal ideal if and only if  $R$  is a field.

**Problem 4.** Prove that  $\text{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z}) \cong \mathbb{Z}/(n, m)\mathbb{Z}$ .

**Problem 5.** Let  $R$  be a commutative ring and let  $M$  be a left  $R$ -module. Prove that  $\text{Hom}_R(R, M)$  and  $M$  are isomorphic as left  $R$ -modules.

**Problem 6.** Let  $R$  be commutative ring. Show that an  $R$ -module  $M$  is irreducible if and only if  $M$  is isomorphic (as an  $R$ -module) to  $R/I$  where  $I$  is a maximal ideal of  $R$ .

**Problem 7.**

- (a) Let  $R = M_n(\mathbb{C})$ . Let  $V = \mathbb{C}^n$  considered as a left  $R$ -module in the natural way, i.e., the action of a matrix  $A \in M_n(\mathbb{C})$  on a column vector  $\mathbf{x}$  of length  $n$  is equal to the product  $A\mathbf{x}$ . Determine the submodules of  $V$ .
- (b) Now consider  $V = \mathbb{C}^n$  as a left  $\mathbb{C}\mathcal{S}_n$ -module, where the action is given by  $\pi e_i = e_{\pi(i)}$  for  $\pi \in \mathcal{S}_n$ , and where  $e_1, e_2, \dots, e_n$  denotes the standard basis of  $\mathbb{C}^n$ . Determine the submodules of  $V$ .

**Problem 8.** Determine all 2-dimensional  $\mathbb{C}$ -algebras. This means (1) give a list of nonisomorphic 2-dimensional  $\mathbb{C}$ -algebras, and (2) show that any 2-dimensional  $\mathbb{C}$ -algebra is isomorphic to one on the list.