Math 110BH Homework 2

Topics: prime and maximal ideals, Zorn's Lemma, localization

Due: Wednesday, January 25th at 11:59pm

Problem 1

Let R be a ring.

a) Prove there is a unique ring homomorphism $\phi : \mathbb{Z} \to R$.

b) Note the kernel of the unique homomorphism ϕ will be an ideal of $\mathbb{Z}$ and thus of the form $n\mathbb{Z}$ for some nonnegative integer n. We call n the *characteristic* of the ring R and write $\text{char}(R) = n$. For each of the following, give an example of a ring with this property or prove no such example exists:

   i) A ring R with infinitely many elements such that $\text{char}(R) = 3$

   ii) A noncommutative ring R with $\text{char}(R) = 6$

   iii) A ring R with finitely many elements whose characteristic is 0

Problem 2*

Let R be an integral domain with finitely many elements. Prove that $\text{char}(R)$ is a prime number and that R must be a field.

Problem 3

Suppose R is a commutative ring of characteristic $p > 0$. Prove that the function $f : R \to R$ given by $f(x) = x^p$ is a ring homomorphism. This is called the *Frobenius homomorphism*.

Problem 4*

Let R be a commutative ring. Use Zorn's Lemma to prove that the set of prime ideals of R has a minimal element with respect to inclusion (this could be the zero ideal).

Problem 5

In class we learned about the *ring of quotients*. This exercise generalizes this notion. Let R be a commutative ring and S be a *multiplicative subset* of R. That is, $1 \in S$ and for all $a, b \in S$, $ab \in S$.

a) Define $\mathcal{F} = \{(r, s) \mid r \in R, s \in S\}$. Define a relation on $\mathcal{F}$ via $(a, b) \sim (c, d)$ if $tad = tbc$ for some $t \in S$. Prove that $\sim$ is an equivalence relation.

b) Let $S^{-1}R$ denote the set of equivalence classes. Write $\frac{r}{s}$ for the equivalence class $[(r, s)]$. Prove that the following operations make $S^{-1}R$ into a ring:

$$\frac{a}{b} + \frac{b}{d} = \frac{ad + bc}{bd}, \qquad \frac{a}{b} \cdot \frac{c}{d} = \frac{ab}{cd}.$$

Problem 6

This continues the previous problem.

a) Suppose S contains a nilpotent element. What is $S^{-1}R$?

b) We have a ring homomorphism $\phi : R \to S^{-1}R$ given by $\phi(r) = \frac{r}{1}$ (you should check this is a ring homomorphism, but you don't need to write up the details). Suppose $0 \notin S$. What is $\ker(\phi)$?

Problem 7

Let R be a commutative ring and consider the polynomial ring $R[x]$. Prove that the principal ideal generated by x is a prime ideal if and only if R is an integral domain. Prove the principal ideal generated by x is a maximal ideal if and only if R is a field.

Problem 8

Let I and J be two ideals of a commutative ring R. We define IJ to be the ideal consisting of all finite sums $\sum_{i=1}^{n} a_i b_i$ where $a_i \in I$ and $b_i \in J$. Assume P is a prime ideal of R that contains IJ. Prove that $I \subseteq P$ or $J \subseteq P$.

Problem 9

Let R, S be commutative rings and let $\phi : R \to S$ be a ring homomorphism.

a) Prove that if P is a prime ideal of S then $\phi^{-1}(P)$ is a prime ideal of R.

b) Prove that if ϕ is surjective, then if M is a maximal ideal of S, then $\phi^{-1}(M)$ is a maximal ideal of R. Then find a counterexample to show this statement is false if we don't assume ϕ is surjective.

Problem 10*

A *local ring* is a commutative ring with a unique maximal ideal. Let R be a commutative ring.

a) Prove that if R is a local ring, then all elements in $R - M$ are units.

b) Prove that if the set M of non-units forms an ideal, then R is a local ring with unique maximal ideal M.