Math 110BH Homework 4

Topic(s): polynomial rings

Due: Wednesday, February 8th at 11:59pm

For all of this assignment R denotes a commutative ring.

Problem 1

Consider the following ideals in the polynomial ring $\mathbb{Z}[x, y]$. For each ideal determine if it is prime and then determine if it is maximal:

$$I_1 = (x, y), \quad I_2 = (5, x, y), \quad I_3 = (6, x).$$

Problem 2*

Prove that a polynomial ring in more than one variable over R is not a principal ideal domain.

Problem 3*

Prove that the rings $F[x, y]/(y^2 - x)$ and $F[x, y]/(y^2 - x^2)$ are not isomorphic for any field F.

Problem 4

Let $f(t) \in R[t]$ be a polynomial of degree $n \geq 1$ whose leading term is a unit in R. For a polynomial $g(t) \in R[t]$ let $\overline{g(t)}$ denote the image of $g(t)$ in the quotient ring $R[t]/(f(t))$. Prove that for any $g(t)$ there exists a polynomial $g_0(t)$ of degree less than or equal to $n - 1$ such that $\overline{g(t)} = \overline{g_0(t)}$.

Problem 5*

Let F be a finite field of order q and $f(t)$ be a polynomial of degree $n \geq 1$. Prove that $F[t]/(f(t))$ is a ring with q^n elements.