Math 110BH Homework 1

Topics: rings, subrings, ring homomorphisms, ideals

Due: Wednesday, January 18th at 11:59pm

Problem 1*

The *center* of a ring R is the subset $Z(R) = \{z \in R \mid zr = rz \text{ for all } r \in R\}$.

a) Prove the center of a ring is a subring.

b) Let $\phi : R \to S$ be a surjective ring homomorphism. Prove the image of the center of R is contained in the center of S.

Problem 2

Let R be a commutative ring. What is the center of the ring $M_n(R)$? Prove your answer is correct.

Problem 3

Let R be a commutative ring with $0 \neq 1$. Prove that R is an integral domain if and only if the *cancellation law* holds. That is, R is an integral domain if and only if whenever $a, b, c \in R$ with a nonzero, $ab = ac$ implies $b = c$.

Problem 4

Let R be a commutative ring. Prove that all ideals of $M_n(R)$ are of the form $M_n(I)$ where I is an ideal in R.

Problem 5*

Suppose R is a ring with the property that $a^2 = a$ for all $a \in R$. Prove that R is commutative.

Problem 6

Let R be a commutative ring. Prove the binomial theorem holds in R. That is, for all $a, b \in R$ and $n \in \mathbb{Z}^+$

$$(a+b)^n = \sum_{k=1}^{n} \binom{n}{k} a^k b^{n-k}.$$

Problem 7

An element x of a ring R is called *nilpotent* if $x^m = 0$ for some $m \in \mathbb{Z}^+$. Describe all nilpotent elements in $\mathbb{Z}/n\mathbb{Z}$ where $n \in \mathbb{Z}^+$.

Problem 8*

Assume R is a commutative ring.

a) Prove that if x is nilpotent, then $1 + x$ is a unit.

b) Prove that the set of nilpotent elements in R forms an ideal. (This is called the *nilradical* of R.)

Problem 9

Let $\phi : R \to S$ be a ring homomorphism.

a) Prove that if J is an ideal of S, then $\phi^{-1}(J)$ is an ideal of R.

b) Prove that if ϕ is surjective and I is an ideal of R then $\phi(I)$ is an ideal of S. Is this still true if we remove the hypothesis that ϕ is surjective? If so, prove it. If not, provide a counterexample.

Problem 10

Let R be a commutative ring. Define the set $R[[x]]$ of formal power series in the indeterminate x with coefficients from R to be the collection of infinite sums

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$$

Define addition and multiplication of power series in the expected way as if they were "polynomials of infinite degree":

$$\left(\sum_{n=0}^{\infty} a_n x^n\right) + \left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n=0}^{\infty} (a_n + b_n) x^n,$$

$$\left(\sum_{n=0}^{\infty} a_n x^n\right) \cdot \left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k b_{n-k}\right) x^n.$$

One can check this forms a ring (you should check this, but you don't need to write it up). Prove that $\sum_{n=0}^{\infty} a_n x^n$ is a unit in $R[[x]]$ if and only if a_0 is a unit in R.