Math 110BH Homework 9

Topics: modules over PIDs, matrix forms

Due: Thursday, March 16th

Problem 1*

Use the fundamental theorem of finitely generated modules over PIDs to show that a finitely generated module over a PID is free if and only if it is torsion free.

Problem 2

Let R be a PID and suppose M is a finitely generated torsion module. Prove that M is a simple module (i.e. $M \neq 0$ and the only submodules of M are 0 and M) if and only if $M = \langle x \rangle$ for some $x \in M$ with $\text{Ann}(x) = (p)$ where (p) is a nonzero prime ideal.

Problem 3

Determine how many abelian groups there are of order 400.

Problem 4

Prove that two 3×3 matrices with entries from a field F are similar if and only if they have the same minimal polynomial and the same characteristic polynomial.

Problem 5

Give an explicit example to show the above statement is false for 4×4 matrices.

Problem 6*

Find all similarity classes of 6×6 matrices with entries from $\mathbb{Q}$ that have the minimal polynomial equal to $(x+2)^2(x-1)$.

Problem 7

Find all Jordan canonical forms for 2×2, 3×3, and 4×4 matrices over $\mathbb{C}$.

Problem 8

Let A be a matrix over a field F. Prove that if $A^2 = A$ then A is similar to a diagonal matrix where each diagonal entry is either 1 or 0.

Problem 9*

Let V be a finite dimensional vector space over $\mathbb{Q}$ and suppose $T : V \to V$ is a linear transformation with characteristic polynomial $(x^4 - 1)^2 = 0$. Prove that T is invertible.

Problem 10

(Problem 9, continued) Determine how many similarity classes of such transformations T there are.