Analysis Notes

March 2024

Can you have 3 disjoint dense sets in a set?

1 Metric Spaces

Lemma 1.1. Let (X, ρ) be a metric space and $A \subseteq X$. $g : X \to \mathbb{R}$ defined by

$$g(x) = \rho(x, A)$$

is a continuous function.

Proof. We'll prove a stronger condition, namely, we'll prove that $\forall x, y \in X : |g(x) - g(y)| \leq \rho(x, y)$. Let $x, y \in X$. Observe that for any $p \in A$,

$$g(x) = \inf_{a \in A} \rho(x, a) \leq \rho(x, y) + \rho(y, p)$$

Then, $g(x) - \rho(x, y)$ is a lower bound on $A = \{\rho(y, a) : a \in A\}$ and thus $g(x) - \rho(x, y) \leq g(y)$ by the *greatest* lower bound property of the infimum. Rearranging produces the desired inequality. $\square$

In fact, notice that it is uniformly continuous.

Lemma 1.2. Let (X, ρ) be a metric space and $A \subseteq B \subseteq X$. Then, $diam A \leq diam B$.

2 Metric Space Compactness

Lemma 2.1. Let (X, ρ) be a complete metric space. Let E_n be a sequence of non-empty, closed and bounded subsets such that $\forall n \in \mathbb{N} : E_{n+1} \subseteq E_n$. If $\lim_{n\to\infty} diam(E_n) = 0$, $\bigcap_{i=1}^{\infty} E_n$ has exactly one point.

In other words, each nested sequence of shrinking closed sets in a complete space has exactly one point in common.

Proof. Before we prove the (harder) existence, let's first prove uniqueness. Assume by contradiction that E contains two distinct points p and q. Let $d := \frac{1}{2}\rho(p, q)$. Then, $diam(E) > d$. However, p and q are also in every E_n so we also just got a lower bound for $diam(E_n)$ for all $n \in \mathbb{N}$, which contradicts the fact that the diameters are going to 0.

Let's now prove that the intersection is not empty. Since E is an intersection of closed sets, E is closed. Thus, it suffices to find a limit point of E.

By invoking the Axiom of Choice, pick some $x_n \in E_n$ for all $n \in \mathbb{N}$. We'll prove that this sequence is Cauchy.

Let $\epsilon > 0$. Since $\lim_{n\to\infty} diam(E_n) = 0$, there's some $N \in \mathbb{N} : \forall n \geq N : diam(E_n) < \epsilon$. Then, by using the fact that the sequences are nested, we immediately get that x_n is $\square$

3 Infinite Series

3.1 Cauchy Condensation Test

Theorem 3.1. Let a_n be a non-negative and decreasing sequence. Then,

$$\sum_{n=1}^{\infty} a_n \leq 2^n \sum_{n=1}^{\infty} a_{2^n} \leq 2 \sum_{n=1}^{\infty} a_n$$

3.2 Rearrangement Theorems

Theorem 3.2. Any rearrangement of an absolutely convergent series converges to the same sum.

The idea behind the proof is to use the Cauchy Criterion and cancel out the first N terms by taking m to be large enough. The rest of the proof is just fun and games with the triangle inequality.

3.3 Dirichlet and Abel Summation Tests

4 Continuity

Lemma 4.1. The inverse of a monotonic function is monotonic in the same way.

Proof. Without loss of generality, assume f is monically increasing. Then, $x < y \implies f(x) < f(y)$ so $f^{-1}(x) < f^{-1}(y) \implies x < y$. Considering the contrapositive gives the required result. $\square$

5 Intermediate Value Theorem

Theorem 5.1 (Intermediate Value Theorem)**.**

Corollary 5.1.1. Every odd-degree polynomial has a root in $\mathbb{R}$.

Corollary 5.1.2. Let $I \subseteq \mathbb{R}$ be a compact interval and $f : I \to I$ be a continuous function with $\mathrm{Dom}(f) = I$. Then, $\exists x \in I : f(x) = x$.

Proof. □

6 Differentiation

6.0.1 Inverse Function Rule

The following lemma is from Exercise 5.2 in Rudin:

Lemma 6.1. Let $f : [a, b] \to \mathbb{R}$ and assume $f'(x) > 0$ on (a, b). Let g be the inverse of f. Then, g is differentiable and

$$g'(f(x)) = \frac{1}{f'(x)}$$

Proof. □

Here's a homework exercise from MATH131BH. Notice that we need f to be injective for the inverse to be defined. Also notice that no regularity assumptions are made about f apart from the behavior at x. In particular, f might be discontinuous and monotone on $(x - \delta, x + \delta)$ for any $\delta > 0$.

Lemma 6.2. Let $f : \mathbb{R} \to \mathbb{R}$ be an injective function on $\mathrm{Dom}(f)$. Let $x \in \mathrm{int}(\mathrm{Dom}(f))$ such that $f(x) \in \mathrm{int}(\mathrm{Ran}(f))$. Assume that the inverse function is continuous at $f(x)$. If $f'(x) \neq 0$, f^{-1} is differentiable at $f(x)$ with

$$(f^{-1})'(x) = \frac{1}{f'(x)}$$

Proof. □

The requirement that f^{-1} is continous can be replaced by different assumptions. For example, f can be continously differentiable.

Ross has another version of this lemma.

7 Applications of Uniform Convergence:

8 Fixed Point Theorems

A good resource is Raymond Chu's notes for MATH204 at UCLA.

The first of the fixed point theorems was a corollary to IVT: see Corollary 5.1.2.

Theorem 8.1 (Banach Contraction Mapping Theorem)**.** Let (X, d) be a non-empty complete metric space. Let $f : X \to X$ be c-Lipschitz for some $c \in (0, 1)$. Then, f has a unique fixed point.

Proof. The uniqueness is immediate by the Lipschitz condition.

We'll consider the sequence given by continuously applying f. Since X is complete, it suffices to show that this sequence is Cauchy. $\square$

We can't take $c = 1$ since for example $f(x) = x + 1$ satisfies it but has no fixed points.

9 Banach Spaces

Definition 1. Let X be an F vector space and d be a metric induced by a norm on X. Assume (X, d) is a complete metric space. This is called a **Banach space**.