Complex Analysis

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1 Resources

- Stein and Shakarchi
- Serge Lang (I didn't like how he didn't introduce topology. It's lame.)
- BrightSide of Math Youtube Series

2 Introduction to Complex Numbers

Example 2.0.1. Consider $f : \mathbb{C} \to \mathbb{C}$ defined by $f(z) = \frac{1}{z}$. This function maps the unit circle outside of the unit circle and maps the outside of the unit circle to the unit circle. It's a bijection. It's called an **inversion** through the unit circle.

Example 2.0.2. Consider $f : \mathbb{C} \to \mathbb{C}$ defined by $f(z) = \frac{1}{\bar{z}}$. This function maps the unit circle outside of the unit circle and maps the outside of the unit circle to the unit circle. It's a bijection. It's called an **reflection** through the unit circle.

Lemma 2.1. Let z_n be a sequence of complex numbers with $z_n = x_n + iy_n$. z_n is Cauchy if and only if x_n and y_n are Cauchy.

Proof. The forward direction is immediate since $|x_n - x_m| \leq |z_n - z_m|$ and similarly for y_n.

Notice that $|z_n - z_m| = \sqrt{(x_n - x_m)^2 + (y_n - y_m)^2}$, which proves the converse. □

Definition 1. Let $f : U \to \mathbb{C}$. f is **(complex) differentiable** at $z_0 \in U$ there's $f'(z_0) \in \mathbb{C}$ and $\phi : \mathbb{C} xrightarrow \mathbb{C}$ with

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \phi(z)$$

where $\lim_{z \to z_0} \frac{\phi(z)}{z - z_0} = 0$.

Definition 2. Let $f : U \to \mathbb{C}$. f is **holomorphic** if f is complex differentiable on U.

Definition 3. Let $f_R : \mathbb{R}^2 \to \mathbb{R}^2$. f is **totally differentiable**