Complex Analysis

Boran Erol

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1 Introduction

The Prime Number Theorem (PNT) is one of the most celebrated theorems in mathematics. In this writeup, we look at some of the history of the Prime Number Theorem and review Newman's proof with a little bit more detail and references to our class this quarter. At the end, I will give a brief overview of Alan Turing's relationship to the Riemann Hypothesis and point out an interesting connection to a remark made in class this quarter.

Let $\pi(x)$ be the function that counts the primes up to x, i.e.

$$\pi(x) := \text{number of primes less than or equal to } x.$$

The Prime Number Theorem states that $\pi(x) \sim \frac{x}{\log x}$, where log denotes the natural logarithm. Here, $\sim$ means that

$$\lim_{x\to\infty} \frac{\pi(x)}{x/\log x} = 1$$

In other words, the Prime Number Theorem states that the nth prime will have size roughly $n \log n$. In fact, Barkley Rosser proved in 1939 that the nth prime number is strictly greater than $n \log n$.

The Prime Number Theorem was first conjectured by Gauss when he was around 15 (in fact, he conjectured that $\pi(x) \sim Li(x)$, where $Li(x)$ is the logarithmic integral function , which turns out to be a better approximation) using prime number tables compiled by Johann Heinrich Lambert, who is known for proving that π is irrational. Legendre, seemingly independently from Gauss, guessed in 1808 that

$$\pi(x) \sim \frac{x}{\log x - A(x)}$$

for some $A(x)$ that tends to a constant as x goes to infinity. In fact, some sources state as

$$\pi(x) \sim \frac{x}{\log x - 1.08366}$$

instead, but no explanation for this constant has been found in Legendre's notes. Following Gauss and Legendre, Chebyshev came along and proved that $\pi(x) = \Theta(x)$, and provided extremely good constants. More specifically, Chebyshev proved that for sufficiently large x we have that

$$0.92\frac{x}{\log x} \leq \pi(x) \leq 1.10\frac{x}{\log x}$$

In fact, Newman's short proof still uses Chebyshev's idea to first prove the upper bound for $\pi(x)$, so we'll see the upper bound in a little bit.

In 1859, Riemann came along and published the paper that introduces the Riemann Hypothesis in an 8-page paper. In the paper, Riemann also mentions the conjecture that $\pi(x) < Li(x)$ (though it's not certain whether he believed it), which would later be disproved by Littlewood and capture Alan Turing's interest. More importantly, Riemann considers the Zeta function first introduced by Euler in 1737 with complex inputs and proves that proving the Prime Number Theorem reduces to showing that the Zeta function doesn't have any zeros with $Re(s) = 1$.

In 1896, approximately a 100 year after Gauss's conjecture, Hadamard and de la Vallée Poussin independently proved the Prime Number Theorem using Riemann's proposed method. De la Vallée Poussin's initial solution was apparently incredibly messy, and he also admits that his solution was worse than Hadamard's. Over the next 100 years, the proofs would be improved with new tricks and observations, and in 1980 Newman discovered the short proof that we present in this writeup.

In 1949, Selberg and Erdös (there's a priority dispute, and Richard Borcherds thinks that Selberg wins this dispute since Erdös used her identity to prove the result, which is the key ingredient) proved the Prime Number Theorem using non-analytic methods.

The famous Riemann Hypothesis is also related to the Prime Number Theorem in that the positions of the zeros of the zeta function would allow us to control the error term for $\pi(x)$ better, but we'll get to this after proving the Prime Number Theorem.

Let's now introduce some notation. Throughout, s will denote complex numbers with $s = \sigma + it$ and $\sigma, t \in \mathbb{R}$. Moreover, p will denote a prime and ρ will denote a zero of $\zeta(s)$. When we use the notation

$$\sum_p, \sum_\rho$$

this means summing (or multiplying) over all primes or the zeros of ζ, respectively.

There are many resources online that try to explain the Prime Number Theorem and many posts on MSE that clarify the proof. As I was exploring these resources myself to try to understand the Prime Number Theorem, I realized that (like me) most beginners had trouble understanding the so-called 'Analytic Theorem' in Newman's paper.

Thus, in order to make this writeup somewhat unique and more useful to novice mathematicians like me, I will start by focusing solely on this part of Newman's proof and later move on to the actual proof.

2 The Only Hard Part in Newman's Proof

Well, the hardness stated in the title is clearly subjective. However, I don't understand this proof intuively and [Korevaar] states that this is essentially a poor-man's Ikehara-Wiener Theorem and was the major simplification in Newman's proof, so that's at least some evidence of hardness. Here is the statement:

Lemma 2.1. Let $f : [0, \infty)$ be a bounded and locally integrable function. Define

$$g(z) := \int_0^\infty f(t)e^{-tz}dt, Re(z) > 0$$

Assume $g(z)$ extends holomorphically to $Re(z) \geq 0$. Then, $\int_0^\infty f(t)dt$ exists and equals to $g(0)$.

Proof. $\square$

3 Chebsyhev's Upper Bound

In this section, we present Chebyshev's upper bound and introduce Chebyshev's function.

4 Newman's Proof

5 Alan Turing and the Riemann Zeta Function

I come from a computer science background, and therefore I was fascinated to learn that Alan Turing also did some work on the Riemann Zeta function. In fact, one of his contributions involved developing the so-called Turing method to aid in the numerical verification that obviates the use of the Argument Principle sketched in the class.

Recall that during class we argued that we can use the argument principle and the fact that the zeros of the Zeta function are symmetric across the critical line to verify the Riemann hypothesis numerically.

The argument went as follows: use numerical methods to estimate the location of a zero, use the Argument Principle to integrate the ?? around the purported zero, and check whether the numerical integral gives a value close to 1. Even though numerical integration is imprecise, since the winding number is an integer, we can be certain that the zero is on the critical line, since otherwise we would have two zeros.

Instead, Alan Turing proposed using the formula (that I don't really understand)

$$N(t) = 1 + \frac{1}{\pi}\theta(t) + S(t)$$

where $N(t)$ counts the number of zeros in the critical strip with imaginary part in $[0, t]$ and $\theta(t)$ is a smooth function called the Riemann-Siegel theta function. Since $N(t)$ isn't smooth and $1 + \frac{1}{\pi}\theta(t)$ is, $S(t)$ should also have jump discontinuities to catch up with $N(t)$. Moreover, if $S(t)$ jumps by two at any point (unless the Zeta function has non-simple zeros, which, as stated in lecture, would be super surprising for reasons beyond me), we would gather that the zero was not on the critical line. I found this super interesting since it both pertained to something we talked about in lecture and Alan Turing!

Another gem I found while looking into this is the following story: Alan Turing apparently got a 40 pound grant from the Royal Society during his time in Cambridge to run computer simulations for

Here's a quote from his notes:

So not only did Alan Turing not believe in the Riemann Hypothesis, he also believed that he could find a counterexample with a pretty small imaginary value! Alan Turing's belief is confirmed by his 1953 paper, where he states