Math 131A - Analysis
Final exam
Monday Jun. 12 15:00 PST

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You have **3 hours** to answer the questions in this exam. Write your first name, last name, and UID at the top of each page.

Section 1 contains the questions for this exam. There are 10 questions to this exam. Each question is worth 20 marks. **Write your solutions to the corresponding questions in ??.** You are not permitted the use of calculators, phones, or other electronics. You are not allowed to have notes/cheat sheets. By signing this page, you adhere to UCLA's policy on Academic Integrity.

Turn off or set your electronics to silent mode.

Unless you are asked to prove them, you may use results from the lectures/problem sheets provided you explicitly state them clearly.

1 Questions

Q1 Lecture notes In this question, we denote by S a subset of $\mathbb{R}$.

(a) Define what it means for S to be bounded above, bounded below, and bounded.

(b) If S is bounded, define the supremum and infimum of S.

(c) State the Completeness Axiom for the reals.

(d) Let $S \subset \mathbb{R}$ be bounded above and non-empty. Prove that $M = \sup S$ satisfies the following two statements

S1) $\forall s \in S, s \leq M$.

S2) $\forall L < M, \exists s \in S$ such that $L < s$.

(e) Let $S \subset \mathbb{R}$ be bounded above and non-empty. Prove that if $M \in \mathbb{R}$ is a number which satisfies **S1)** and **S2)**, then $M = \sup S$.

Q2 Lecture notes

(a) **Example 2.1.3** Prove that $\lim_{n\to\infty} \frac{3n+1}{7n-4} = \frac{3}{7}$.

(b) **Example 2.1.4** Prove that the sequence $a_n = (-1)^n$ does not converge to any real number $a \in \mathbb{R}$.

Q3 Induction

(a) **Example from lecture** Using mathematical induction, prove that

$$\sum_{j=1}^{n} j = \frac{1}{2}n(n+1), \quad \forall n \in \mathbb{N}.$$

(b) **PS1 Q3** Using mathematical induction, prove that the number $7^n - 6n - 1$ is divisible by 36 for every $n \in \mathbb{N}$.

Q4 Lecture notes In this question, let $(a_n)_{n\in\mathbb{N}} \subset \mathbb{R}$ denote a sequence of real numbers and $a \in \mathbb{R}$.

(a) Define what it means for $a_n \to a$ as $n \to \infty$.

(b) Define what it means for $(a_n)_{n\in\mathbb{N}}$ to be a Cauchy sequence.

(c) Define the limits superior and inferior of $(a_n)_{n\in\mathbb{N}}$.

(d) Define what it means for $\sum a_n$ to converge to $a \in \mathbb{R}$.

(e) Prove that the sequence $a_n = \frac{1}{n}$ converges to 0 as $n \to \infty$.

Q5 Lecture notes In this question, we fix three numbers $a < c < b$; two functions $f : (a,b) \to \mathbb{R}$ and $g : (a,b) \setminus \{c\} \to \mathbb{R}$; and $L \in \mathbb{R}$.

(a) Define what it means for f to be continuous at c (in the $\epsilon - \delta$ sense).

(b) Define what it means for f to be sequentially continuous at c.

(c) Define what it means for g to approach L as x converges to c (in the $\epsilon - \delta$ sense).

(d) Define what it means for f to be differentiable at c with $f'(c) = L$ (in the $\epsilon - \delta$ sense).

Q6 Lecture notes Let $(a_n)_{n\in\mathbb{N}} \subset \mathbb{R}$ be a sequence of real numbers and $a \in \mathbb{R}$.

(a) Define what it means for the sequence (a_n) to be bounded.

(b) **Lemma 2.2.1** Suppose $a_n \to a$ as $n \to \infty$. Prove that (a_n) is a bounded sequence.

Q7 Lecture notes

(a) Let $(a_n)_{n\in\mathbb{N}} \subset \mathbb{R}$ denote a sequence of real numbers and $a \in \mathbb{R}$. Define what it means for $\sum a_n$ to converge to $a \in \mathbb{R}$.

(b) **Theorem 2.6.2** State and prove the Comparison test for series.

Q8 PS4 Q2 Let $(a_n)_{n\in\mathbb{N}} \subset \mathbb{R}$ be a sequence which converges to $a \in \mathbb{R}$.

(a) Show that if $a_n \geq 0$ for every $n \in \mathbb{N}$, then $a \geq 0$.

(b) Is the above result true with strict inequalities? In other words, if $a_n > 0$ for every $n \in \mathbb{N}$, can we say that $a > 0$?

(c) Let now $(b_n)_{n\in\mathbb{N}}$ be a sequence which converges to $b \in \mathbb{R}$. Assume that $a_n \geq b_n$ for every $n \in \mathbb{N}$. Show that $a \geq b$.

Q9 PS4 Q4 Let $(a_n)_{n\in\mathbb{N}} \subset \mathbb{R}$ be a sequence with $a \in \mathbb{R}$.

(a) Prove that $a_n \to a \implies |a_n| \to |a|$. *Hint:* Use the reverse triangle inequality.

(b) Is the converse true? In other words, does $|a_n| \to |a|$ imply $a_n \to a$? Why or why not?

Q10 Theorem 2.4.3 State and prove the Bolzano-Weierstrass Theorem. Your proof may use any result from the course provided you clearly and explicitly state it.

Q11 PS2 Q10 Let $a, b \in \mathbb{R}$. Show that the following statements are equivalent.

(a) $a \leq b$.

(b) For any $\varepsilon > 0. a \leq b + \varepsilon$.

Q12 PS3 Q9 and PS4 Q11 Let $a_0 = 1$ and define the recursive sequence $a_{n+1} = \sqrt{a_n + 1}$ for $n \in \mathbb{N}$.

(a) Show that (a_n) is a bounded sequence. *Hint:* Induction.

(b) Show that (a_n) is monotonically increasing. *Hint:* Induction.

(c) Hence, show that (a_n) is convergent and find the limit.

Q13 Example 2.6.4 Prove that the harmonic series $\sum \frac{1}{n}$ diverges. *Hint: Compare this series with a cleverly chosen divergent series. You do not need to explicitly write the formula for the terms of the series, but you should clearly describe the construction.*

Q14 **Lecture notes Theorem 4.2.1** State and prove Rolle's Theorem. Your proof may use any result from the course provided you clearly and explicitly state it.

Q15 Consider the following functions defined on $\mathbb{R}$

$$f(x) = \begin{cases} \sin\frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}, \quad g(x) = \begin{cases} x\sin\frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}.$$

(a) **PS7 Q4** Prove that f is discontinuous at $x = 0$. *Hint:* Use sequential continuity.

(b) **PS7 Q4** Prove that g is continuous at $x = 0$. *Hint:* Use the fact that $|\sin y| \leq 1$ for all $y \in \mathbb{R}$.

Q16 **PS8 Q3** Let $a, b \in \mathbb{R}$.

(a) Prove that $\min(a, b) = \frac{1}{2}(a + b) - \frac{1}{2}|a - b|$.

(b) Prove a similar formula for $\max(a, b)$.

(c) Fix two continuous functions $f(x)$ and $g(x)$ on $\mathbb{R}$. Prove that $h(x) = \min(f(x), g(x))$ is continuous on $\mathbb{R}$. Prove that $i(x) = \max(f(x), g(x))$ is continuous on $\mathbb{R}$.

Q17 **Lecture notes** Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Let $P = \{a = t_0 < t_1 < \cdots < t_n = b\}$ be a partition of $[a, b]$ for some $n \in \mathbb{N}$.

(a) Define the upper and lower Darboux sums $U(f, P)$ and $L(f, P)$ of f with respect to P.

(b) Define the upper and lower Darboux integrals $U(f)$ and $L(f)$ over $[a, b]$.

(c) Define what it means for f to be integrable on $[a, b]$.

Q18 **Theorem 5.1.1** State and prove the Cauchy criterion for integrability of a bounded function $f : [a, b] \to \mathbb{R}$.

Q19 **PS8 Q6** Let $a < b$ be real numbers and $f : (a, b) \to \mathbb{R}$ a function. We say that f is *Lipschitz on* (a, b) if and only if the following statement is true

$$\exists L \geq 0, \text{ such that } |f(x) - f(y)| \leq L|x - y|, \quad \forall x, y \in (a, b).$$

(a) Prove that if $f : (a, b) \to \mathbb{R}$ is Lipschitz on (a, b), then f is uniformly continuous on (a, b).

(b) Suppose $f : (a, b) \to \mathbb{R}$ is differentiable. Assume moreover that its derivative f' is bounded on (a, b). i.e. there is some $M \geq 0$ such that $|f'(x)| \leq M$ for every $x \in (a, b)$. Prove that f is Lipschitz on (a, b). *Hint: Mean Value Theorem.*

(c) Find a Lipschitz function which is not differentiable. More specifically, you should explicitly write down a formula for a function, prove it is Lipschitz, and prove that there is (at least) one point at which the function is not differentiable.

Q20 **PS7 Q10** Consider the following functions

$$f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}, \quad g(x) = \begin{cases} x, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}.$$

(a) Prove that f is discontinuous on all of $\mathbb{R}$.

(b) Prove that g is continuous at $x = 0$ and at no other point in $\mathbb{R}$.

Q21 **Proposition 5.2.2** Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Prove that f is integrable. Your proof may use any result from the course provided you clearly and explicitly state it.

Q22 **Lecture notes**

(a) **Theorem 3.2.2** State and prove the Intermediate Value Theorem. Your proof may use any result from the course provided you clearly and explicitly state it.

(b) **Example 3.2.2** Let $f : [0, 1] \to [0, 1]$ be a continuous function. In other words, $f(x) \in [0, 1]$ for every $x \in [0, 1]$. Prove that there exists a point $x_0 \in [0, 1]$ such that $f(x_0) = x_0$. *Hint: Consider the function* $g(x) = f(x) - x$.

(c) **PS8 Q4** In this question, you may take for granted that sine is a continuous function. Show that there exists a point $x_0 \in [0, \pi/2]$ such that

$$\sin x_0 = \frac{3}{10}.$$

Q23 In this question, you may take for granted that

$$\frac{d}{dx} e^x = e^x, \quad \forall x \in \mathbb{R}$$

and

$$\frac{d}{dx} \log x = \frac{1}{x}, \quad \forall x > 0,$$

where $\log = \ln$.

(a) **Lecture notes** State the Chain Rule (you do not need to prove it).

(b) **Lecture notes** State the first version of the Fundamental Theorem of Calculus (you do not need to prove it). This is a statement about the integral of a derivative.

(c) **PS9 Q11, Q13** Using the Chain Rule and Product Rule, calculate

$$\frac{d}{dx} e^{-x^2}, \quad \text{and} \quad \frac{d}{dx}(x \log x).$$

(d) **PS9 Q11, Q13** Calculate

$$\int_0^1 x e^{-x^2} dx, \quad \text{and} \quad \int_0^1 \log x dx.$$

Hint: You may use the fact that $x \log x$ evaluated at $x = 0$ is 0.

Q24 **PS6 Q5** Suppose we have two convergent series $\sum_{n=0}^{\infty} a_n = A$ and $\sum_{n=0}^{\infty} b_n = B$ where $A, B \in \mathbb{R}$.

(a) Prove that $\sum_{n=0}^{\infty}(a_n + b_n) = A + B$

(b) Prove that whenever $k \in \mathbb{R}$, we have $\sum_{n=0}^{\infty} k a_n = kA$.

(c) Is it the case that $\sum_{n=0}^{\infty} a_n b_n = AB$? Why or why not?