23W-MATH-131A-LEC-2 Final

STEPHANIE WU

TOTAL POINTS

29 / 34

QUESTION 1

1 Problem 1 6 / 6

✓ **- 0 pts** *Correct*

- 4 pts Limited progress on upper bound

- 1 pts Attempted induction, but mistake in lower bound

- 4.5 pts Limited progress (e.g. not much past a base case.)

- 1 pts Missing step

- 3 pts Some good progress

- 0.5 pts Minor issues

- 2 pts Fairly important issue

- 1 pts Some algebra issues

- 5 pts Largely incorrect

QUESTION 2

2 Problem 2 7 / 8

- 0 pts Correct

- 1 pts (c) Incorrect justification

- 0.5 pts n^{1/n} converges to 1 (need to at least mention something along these lines, or for example bound it by sqrt(n) and fold it into the numerator).

- 2 pts (c) Incorrect

- 2 pts (d) Incorrect

- 2 pts (a) Incorrect

- 2 pts (b) Incorrect

- 1 pts (a) What is the limit?

- 1 pts (c) Need a little more explanation here.

+ 1 pts Partial credit for incorrect parts

- 1 pts (c) Claimed a limit of +infty or -infty

- 0 pts (c) Didn't rule out convergence to +/- infty.

- 1 pts (d) Incorrect justification

- 0.5 pts (d) Need to say a little more -- e.g. that it's a Dirichlet series with p=1.

- 0.5 pts (c) Need a little more explanation

✓ **- 1 pts** *(b) Missing/incorrect justification*

- 1 pts (b) Missing limit

QUESTION 3

3 Problem 3 5 / 5

✓ **- 0 pts** *Correct*

- 1 pts Missing step

- 0 pts Minor issue with implications going the wrong way. (Though steps were reversible.)

- 4 pts Said something correct, but the proof technique doesn't work.

- 0.5 pts Small missing step

- 3 pts Tried to show decreasing/bounded but little additional progress.

- 2.5 pts Limited progress showing decreasing/bounded. Got limit correct.

- 2 pts Limit correct, some progress towards showing decreasing/bounded

- 1.5 pts Moderately important issue

- 3 pts Demonstrating some reasonable understanding, but a lot of details missing.

- 1 pts Only proved decreasing conditional on bounded below by 1 (but didn't prove this.)

- 1.5 pts Limit missing

- 1 pts Incorrect limit

- 0.5 pts small missing step

QUESTION 4

4 Problem 4 **1 / 5**

- 0 pts Correct

- 4 *Point adjustment*

1 The ratio test is a **sufficient** condition for convergence, **not** a necessary one; that is, if $$\limsup \frac{|a_{n+1}|}{|a_n|}<1$$ then $$\sum a_n$$ is convergent but not the other way around. So we cannot state that $$\limsup \frac{|a_{n+1}|}{|a_n|}<1$$.

2 Same here.

QUESTION 5

5 Problem 5 **5 / 5**

✓ **- 0 pts** *Correct*

3 $$53$$

QUESTION 6

6 Problem 6 **5 / 5**

✓ **- 0 pts** *Correct*

Math 131A-2, Winter 2022
Analysis

Final

Instructions: You have **3 hours** to complete this exam. There are **6 questions** worth a total of **34 points**. This test is a closed book and closed notes. Please, put away all electronic devices and make sure they are in a silent mode or turned off not to disturb the exam.

For full credit show all of your work legibly and justify your answers. Please write your solutions in the space below the questions; **INDICATE** if you go over the page and/or use scrap paper. **DO NOT FOLD** the pages so we can scan the tests.

Important: Do not forget to write your **NAME** and **UID** in the space below.

Name: Stephanie Wu
Student ID number: 905912292

Question	Points	Score
1	6	
2	8	
3	5	
4	5	
5	5	
6	5	
Total:	34	

Problem 1. *6pts.*
Prove that

$$\frac{1}{2}-\frac{1}{2(n+1)^2}<\frac{1}{1^3}+\frac{1}{2^3}+\frac{1}{3^3}+\cdots+\frac{1}{n^3}\leq\frac{3}{2}-\frac{1}{2n^2},\quad \forall n\in\mathbb{N}.$$

Base case: $\frac{1}{2}-\frac{1}{2(1+1)^2}<\frac{1}{1^3}\leq\frac{3}{2}-\frac{1}{2(1)^2}$.

Inductive step: Suppose for some $k-1\in\mathbb{N}$, $\frac{1}{2}-\frac{1}{2k^2}<\frac{1}{1^3}+\cdots+\frac{1}{(k-1)^3}\leq\frac{3}{2}-\frac{1}{2(k-1)^2}$

It suffices to show $-\frac{1}{2(k+1)^2}+\frac{1}{2k^2}<\frac{1}{k^3}\leq-\frac{1}{2k^2}+\frac{1}{2(k-1)^2}$ for inductive step.

Let $f(x)=-\frac{1}{2x^2}$ Then by MVT, there exists some $c\in(x,x+1)$ s.t.

$\Downarrow$

$f'(x)=\frac{1}{x^3}$

$$-\frac{1}{2(x+1)^2}+\frac{1}{2x^2}=(x+1-x)f'(c)=\frac{1}{c^3}.$$

$\frac{1}{c^3}<\frac{1}{k^3}$

Letting $k\in(x,c)$ we have the left inequality. Simililarly, $\exists\ d\in(x-1,x)$ we have by MVT

$$-\frac{1}{2x^2}+\frac{1}{2(x-1)^2}=(x-(x-1))f'(d)=\frac{1}{d^3}.$$

Since $d<k$ clearly $\frac{1}{d^3}>\frac{1}{k^3}$ so we have the right inequality, and our induction is complete.

Problem 2. *8pts.*

For each sequence below, identify whether it converges or diverges. If it converges, give its limit. For full credit, provide *brief justifications* for your answers. Complete proofs are not required.

(a) $\frac{3^n - n\cdot 2^n + \cos n}{n!}$

(b) $n^{\frac{1}{n}} \cdot \frac{n^2+4n-12}{(-1)^n n^3+n+4}$

(c) $\sin \frac{\pi n}{4} - n \cos \frac{\pi n}{4}$

(d) $\sum_{k=1}^{n} \frac{1}{k}$

a. The sequence converges, $\lim_{n\to\infty} \frac{3^n - n\cdot 2^n + \cos n}{n!} = 0$. as $n!$ grows much faster than $3^n - n\cdot 2^n + \cos n$. (or by ratio test for the series which means sequential limit is 0).

b. The sequence converges, $\lim_{n\to\infty} b_n = 0$ as the function behaves similarly to $\frac{1}{n^p}$ for $|p|<1$, which converges to 0.

c. The sequence diverges as the subsequential limit of $c_{8k+2} = 1$ and $c_{8k+6} = -1$, and subsequential limits must be the same for all subsequences for convergence.

d. The sequence diverges as $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{k}$ is a Direchlet series of degree ≤ 1, which diverges.

Problem 3. *5pts.*
Let $a_1 = 2$, and

$$a_{n+1} = \frac{a_n}{2\sqrt{a_n} - 1}, \quad \forall n \geq 1.$$

Prove that (a_n) is convergent and find rigorously its limit.

I claim that $1 \leq a_{n+1} < a_n \leq 2$. Proceed with induction:

Base case: For $a_2 = \frac{2}{2\sqrt{2}-1}$, $a_1 = 2$, $1 \leq \frac{2}{2\sqrt{2}-1} < 2 \leq 2$.

Inductive step: suppose for some $k \in \mathbb{N}$, $1 \leq a_{k+1} < a_k \leq 2$. Then we have $a_{k+1} \leq 2$,
$1 \leq a_{k+1}$
and by def'n $a_{k+1} - a_{k+2} =$ $\quad 1 = \sqrt{1} \leq \sqrt{a_{k+1}}$

$$a_{k+1} - \frac{a_{k+1}}{2\sqrt{a_{k+1}} - 1} = \frac{2a_{k+1}\sqrt{a_{k+1}} - 2a_{k+1}}{2\sqrt{a_{k+1}} - 1} = \frac{2a_{k+1}(\sqrt{a_{k+1}} - 1)}{2\sqrt{a_{k+1}} - 1} \geq 0$$

so $a_{k+2} < a_{k+1}$, and finally

$$a_{k+2} - 1 = \frac{a_{k+1}}{2\sqrt{a_{k+1}} - 1} - 1 = \frac{a_{k+1} - 2\sqrt{a_{k+1}} + 1}{2\sqrt{a_{k+1}} - 1} = \frac{(\sqrt{a_{k+1}} - 1)^2}{2\sqrt{a_{k+1}} - 1} \geq 0$$

so $1 \leq a_{k+2} < a_{k+1} \leq 2$ as desired.

Let $S = \lim_{n\to\infty} a_n$. Then

$S = \lim_{n\to\infty} a_{n+1} = \frac{S}{2\sqrt{S} - 1}$, or $2S^{3/2} - S = S$. $S = 1$ is thus the limit of a_n.

Problem 4. *5pts.*

Assume that $a_n \geq 0$, and both $\sum a_n$ and $\sum b_n$ are convergent. Prove that $\sum a_n b_n^2$ is also convergent.

Since $\sum a_n$, $\sum b_n$ are convergent, by the ratio test, for some $N_1 \in \mathbb{N}$, $n > N_1$ implies

$\left|\frac{a_{n+1}}{a_n}\right| < 1$ and for $N_2 \in \mathbb{N}$, $n > N_2$ implies $\left|\frac{b_{n+1}}{b_n}\right| < 1$.

Then $\left|\frac{a_{n+1} b_{n+1}^2}{a_n b_n^2}\right| = \left|\frac{a_{n+1}}{a_n}\right| \cdot \left|\frac{b_{n+1}}{b_n}\right|^2 < 1$ for all $n > \max(N_1, N_2)$

gives $\sum |a_n b_n^2|$ is convergent,

and $\sum |a_n b_n^2| = \sum a_n b_n^2$ (since $a_n \geq 0$ and $b_n^2 \geq 0$). $\square$

Problem 5. *5pts.*

Assume that an ant moves on a plane continuously; that is, its coordinates $x(t)$ and $y(t)$ are continuous functions of time t. The ant starts moving from the point $(-2, 5)$ and arrives at $(7, -4)$ five minutes later. Prove that the ant has at least once crossed

(a) the parabola $y = x^2$,

(b) the curve $x = \sin y$. $x = (-1, 1)$

a. Let $r(t) = x^2(t) - y(t)$. Then the ant is on the parabola if $r(t) = 0$.

Since $r(t)$ is a combination of the product and difference of continuous functions $x(t)$ and $y(t)$, $r(t)$ is also continuous. Furthermore, at $t=0$, $r(0) = (-2)^2 - 5 = -1$ and at $t=5$ minutes, $r(5) = 7^2 - (-4)$ 3 Since $r(t)$ is cont., we can apply IVT, and since $r(0) = -1 < 0 < 65 = r(5)$, for some $c \in (0,5)$, $r(c) = 0$, or the ant crossed the parabola.

b. Let $r(t) = \sin(y(t)) - x(t)$. ant crosses if $r(t) = 0$. Since $\sin(y(t))$ is continuous as its the composition of cont. functions, $r(t)$ is continuous. At $t=0$, $r(0) = \sin(5) - (-2)$, so $r(0) \in (-1, 3)$, and at $t=5$ $r(5) = \sin(-4) - (7)$ so $r(5) \in (-8, -6)$. By IVT, we are guaranteed some $t \in (r(0), r(5))$ since $r(5) < 0 < r(0)$, so the ant crosses the curve.

Problem 6. *5pts.*
Prove that

$$\left|\frac{1}{(1-x)^2} - 1 + 2(x-2) - 3(x-2)^2 + 4(x-2)^3 - 5(x-2)^4\right| \le 6(x-2)^5, \quad \forall x > 2.$$

Let $f(x) = \dfrac{1}{(1-x)^2}$, so

$f'(x) = \dfrac{2}{(1-x)^3}$, $f''(x) = \dfrac{6}{(1-x)^4}$, $f^{(3)}(x) = \dfrac{24}{(1-x)^5}$, $f^{(4)}(x) = \dfrac{120}{(1-x)^6}$, $f^{(5)}(x) = \dfrac{720}{(1-x)^7}$.

Let $x_0 = 2$. Then Taylor's Theorem gives $f(x) - \sum_{k=0}^{4} (x-x_0)^k = \dfrac{f^{(5)}(c)}{5!}(x-2)^5$ for some $c \in (2, x)$.

$$\frac{1}{(1-x)^2} - \left(1 + \frac{2}{2!(1-2)^3}(x-2) + \frac{6}{3!(1-2)^4}(x-2)^2 + \frac{24}{4!(1-2)^5}(x-2)^3 + \frac{120}{5!(1-2)^6}(x-2)^4\right) = \frac{720}{5!(1-c)^7}(x-2)^5.$$

$$\rightarrow \frac{1}{(1-x)^2} - 1 + 2(x-2) - 3(x-2)^2 + 4(x-2)^3 - 5(x-2)^4 = \frac{6}{(1-c)^7}(x-2)^5.$$

Since $c \in (2, x)$ and $x > 2$, we take $c = 2$ to bound the above equality at $-6(x-2)^5$, giving the desired absolute value inequality.