AAP PLF Analysis Problem Set

March 2024

1 Warmup

1. Write down the formal definition of the supremum.
2. Write down the definition of Cauchy convergence and convergence.
3. Write down the definition of limsup and liminf.

2 Problems with Fields

1. Let $(F, +, 0, \cdot, 1, \leq)$ be an ordered field. Prove that
2. (a) Write down the definition of the triangle inequality and prove it.
   (b) Prove the following inequality:

   $$||a| - |b|| \leq |a - b|$$

   This is also called **the reverse triangle inequality.**
3. Prove the following:

   $$\forall a, b \in F : |a \cdot b| = |a| \cdot |b|$$

   Proof. □

3 Supremum and Completeness Problems

3.1 Definitions and Important Proofs

1. Write down the Archimedean Property of the reals.
2. (a) State the Completeness Axiom for Real Numbers.
   (b) Assuming the completeness axiom, prove the analogous statement for the infimum.
3. Let $S = [a, b)$, where $a < b$. Find $\inf(S)$ and $\sup(S)$.
4. Let $S = \left\{\frac{n}{2^n} : n \in \mathbb{N}, n \neq 0\right\}$. Find $\inf(S)$ and $\sup(S)$.
5. Let $A \subseteq \mathbb{R}$ such that $A \neq \emptyset$ and bounded above. Let $b > 0$. Define

   $$bA = \{ba : a \in A\}.$$

   Show that $\sup(bA) = b \sup(A)$. What is $\sup(bA)$ if $b < 0$ and A is bounded below?
6. Prove that for any non-empty $A, B \subseteq \mathbb{Q}$ admitting suprema, we have

   $$A \subseteq B \implies \sup(A) \leq \sup(B)$$

   Then, show that under these conditions $A \cup B$ admits a supremum and show

$$\sup(A \cup B) = \max(\sup(A), \sup(B))$$

7. Given two sets $A, B \subseteq \mathbb{Q}$, denote

$$A + B := \{a + b : a \in A \wedge b \in B\}$$

Assuming that both A, B are non-empty and admit suprema, prove that

$$\sup(A + B) = \sup(A) + \sup(B)$$

8. We say that B **is dense in** A if $\forall a \in A : \forall \epsilon > 0 : \exists b \in B : |a - b| < \epsilon$. Prove that if A is dense in B, B is dense in A and if both sets don't contain their supremums, then $\sup A = \sup B$.
9. Let A be bounded above and B bounded below. Assume $\sup A = \inf B$. Prove that $\forall \epsilon > 0 : \exists a \in A : \exists b \in B : |a - b| < \epsilon$.
10. (Challenge Question) Let $A \sim B := A$ is dense in B. Is this an equivalence relation on $P(\mathbb{R})$?

4 Sequences

4.1 Give an example of each of the following or prove that it is impossible.

1. A sequence that is convergent but not monotonic.

4.2 Important Proofs

1. Prove the multiplication law for limits.
2. Prove that every convergent sequence is bounded. Then, prove that every Cauchy sequence is bounded. Give an example of a bounded sequence that is not convergent (with a proof).
3. Prove that every monotone bounded sequence converges.

4.3 Textbook Problems

Ross, Section 10: 6, 8, 9, 10

4.4 Problems

1. Prove that for every real number x, there exists a sequence of **rationals** q_n such that q_n converges to x.
2. Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence that diverges to $-\infty$. Prove that $\{a_n^2\}_{n\in\mathbb{N}}$ diverges to $+\infty$.
3. Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers. If $L \in$ is not a subsequential limit of $\{a_n\}_{n\in\mathbb{N}}$, there exists some $\epsilon > 0$ such that only finitely many terms of the sequence $\{a_n\}_{n\in\mathbb{N}}$ lie in the interval $(L-\epsilon, L+\epsilon)$.
4. Let $a_1 = 1$ and $a_{n+1} = 3 - \frac{1}{a_n}$. Prove that a_n converges. Find what the limit is.

5 Subsequences

5.1 Give an example of each of the following or prove that it is impossible.

1. A monotonic sequence with no convergent subsequences.
2. A sequence with 3 subsequential limits.
3. A sequence that has every integer as a subsequential limit.
4. A sequence that has every rational as a subsequential limit.
5. A sequence that has every real number as a subsequential limit.
6. An unbounded sequence with a convergent subsequence.

6 Liminf and Limsup

6.1 Give an example of each of the following or prove that it is impossible.

6.2 Problems

1. If $\limsup_{n\to\infty} a_n = +\infty$, $\limsup_{n\to\infty} k \cdot a_n = +\infty$ for any $k > 0$.
2. Prove that there's a subsequence that converges to the limsup.
3. Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers and let a be a subsequential limit of a_n. Prove that

$$\liminf_{n\to\infty} a_n \leq a \leq \limsup_{n\to\infty} a_n$$

.

4. Given bounded sequences of real numbers $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$, prove that

$$\limsup_{n\to\infty}(a_n + b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$$

Give an example where the inequality is strict.

5. Given bounded non-negative sequences of real numbers $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$, prove that

$$\limsup_{n\to\infty}(a_n + b_n) \leq (\limsup_{n\to\infty} a_n)(\limsup_{n\to\infty} b_n)$$

Give an example where the inequality is strict.

7 Infinite Series

7.1 Give an example of each of the following or prove that it is impossible.

1. A convergent infinite series where the root test applies but the ratio test doesn't apply.
2. A convergent infinite series where neither the root or the ratio test apply.
3. A divergent infinite series where neither the root or the ratio test apply.
4. An alternating series whose terms go to 0 but doesn't converge.

7.2 Problems

1. Suppose that $\{a_n\}_{n\in\mathbb{N}}$ is a sequence taking on finitely many values in the open interval $(-1,1)$. Show that $\sum_{n=0}^{\infty}(a_n)^n$ converges. Hint: It's crucial that it's the open interval.
2. Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers such that $\liminf_{n\to\infty} a_n > 0$. Prove that there's no subsequence $\{a_{n_k}\}$ such that $\sum_{n=0}^{\infty} a_{n_k}$ converges.
3. Let $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ be sequences of real numbers such that $a_n \geq 0$ and $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ converge. Prove that $\sum_{n=0}^{\infty} a_n b_n^2$ also converges.
4. Given sequences of real numbers $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ such that $\sum_{n=0}^{\infty}|a_n|$ converges and $\{b_n\}$ is bounded, prove that $\sum_{n=0}^{\infty} a_n b_n$ converges.

Challenge Problem Given sequences of real numbers $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ such that

$$\sum_{n=0}^{\infty} a_n \text{ convergent } \wedge b_n \text{ monotone bounded}$$

, prove that

$$\sum_{n=0}^{\infty} a_n b_n \text{ convergent}$$

8 Continuity

8.1 Give an example of each of the following or prove that it is impossible.

1. A function that satisfies IVP but is not continuous.
2.

8.2 Definitions and Important Proofs

1. Show the equivalence of the ϵ-δ definition of continuity and the sequential definition of continuity.
2. Prove EVT and IVT.

8.3 Textbook Problems

Ross, Section 17: 7,8,9,12,13

Ross, Section 18: 5,6,7,8,10

8.4 Problems

1. Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by $f(x) = x(1-x)$. Prove that f is continuous using the ϵ-δ definition.
2. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function and assume C is a closed subset of $Im(f)$. Prove that $f^{-1}(C)$ is also closed. Repeat this with open sets.
3. Let $f : \mathbb{R} \to \mathbb{R}$ be the floor function. Prove that f is continuous at x if and only if $x \notin \mathbb{Z}$.

9 Uniform Continuity

9.1 Give an example of each of the following or prove that it is impossible.

9.2 Problems

1. Prove that $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$ is not uniformly continuous.
2. A function is said to be **Lipschitz continuous** if $\exists M > 0 : \forall x, y \in Dom(f)$:

$$|f(x) - f(y)| \leq |x - y|$$

   Prove that Lipschitz continuity implies uniform continuity.
3. (a) Prove that $|\sin x - \sin y| \leq |x - y|$ for all $x, y \in \mathbb{R}$.
   (b) Show that $\sin x$ is uniformly continuous on $\mathbb{R}$.
4. Prove that $f(x) = \sqrt{x}$ is uniformly continuous on $(0, 1]$ and $[1, \infty)$. Conclude that $f(x)$ is uniformly continuous on $(0, \infty)$.

10 Differentiation

10.1 Definition and Basic Properties

Definition 1. A function $f : I \to \mathbb{R}$ is **differentiable** at $c \in I$ if the following limit exists.

$$\lim_{z \to x} \frac{f(z) - f(x)}{z - x}$$

If the limit it exists, it's denoted by f'. We can treat f' as a function defined when this limit exists.

An equivalent way to express this limit is the following

$$\lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

Notice that in both cases, the function isn't even defined at the limit point. This is no loss as the value at x is irrelevant for the limit at x.

Lemma 10.1. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable at $x \in \mathrm{int}(\mathrm{Dom}(f))$. Then, f is continuous at x.

Lemma 10.2 (Differentiation Laws)**.** Let $f, g : \mathbb{R} \to \mathbb{R}$ with $x \in \mathrm{int}(\mathrm{Dom}(f))$. If f and g are both differentiable at x, then so are $(f + g)(x)$ and $(f \cdot g)(x)$ and

Finish writing this out.

Lemma 10.3. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable at $x \in \mathrm{int}(\mathrm{Dom}(f))$. If f has a local extremum at x, then $f'(x) = 0$.

Proof. We'll consider the left and right limit of the derivative. **Finish this proof.** □

A function might have a local extremum at a point where the derivative doesn't exist. One such example is $f(x) = |x|$, which has a minimum at $x = 0$ but isn't differentiable at 0. However, the one-sided derivatives exist and obey the corresponding inequalities.

10.2 The Mean Value Theorem

Theorem 10.4 (Rolle's MVT)**.** Let $a < b$ reals and $f : [a, b] \to \mathbb{R}$ a function (with $\mathrm{Dom}(f) = [a, b]$) that is continous on $[a, b]$ and differentiable on (a, b). Then,

$$f(a) = f(b) \implies \exists x \in (a, b) : f'(x) = 0$$

Proof. □

Theorem 10.5 (Lagrange's MVT)**.** Let $a < b$ reals and $f : [a, b] \to \mathbb{R}$ a function (with $\mathrm{Dom}(f) = [a, b]$) that is continous on $[a, b]$ and differentiable on (a, b). Then,

$$\exists x \in (a, b) : f'(x) = \frac{f(b) - f(a)}{b - a}$$

Proof. □

Theorem 10.6 (The derivative satisfies IVP)**.**

Proof. The idea of this proof is to construct some function such that at the extremum points of the function, we get $f'(x) = t$, where t is the value we'd like to achieve. □

10.3 Miscallenous

Lemma 10.7. Let $f : \mathbb{R} \to \mathbb{R}$ be an everywhere differentiable function. f is Lipschitz continuous if and only if f' is bounded.

10.4 Warmup Exercises

1. Use the definition of the derivative to compute $f'(2)$, where $f(x) = x^3$.

10.5 Give an example of each of the following or prove that it is impossible.

1. A continuous but not differentiable function.

10.6 Problems

1. Ross 28.8

2. Ross 29.2

Ross 29.3 Assume f is differentiable on $\mathbb{R}$ and $f(0) = 0, f(1) = 1, f(2) = 1$. Show that $f'(x) = \frac{1}{2}$ for some $x \in (0, 2)$ and $f'(x) = \frac{1}{7}$ for some $x \in (0, 2)$.

3. Ross 29.4

Ross 29.5 Let f be defined on $\mathbb{R}$ and assume

$$\forall x, y \in \mathbb{R} : |f(x) - f(y)| \leq (x - y)^2$$

Prove that f is the constant function.

4. Ross 29.7

5. Ross 29.9

Ross 29.10

Ross 29.11 Show that $\forall x > 0 : \sin(x) \leq x$.

Ross 29.14 Let f, g be differentiable on $\mathbb{R}$, $f(0) = g(0)$ and $\forall x \geq 0 : f'(x) \leq g'(x)$. Prove that $\forall x \geq 0 : f(x) \leq g(x)$.

6. Assume f is differentiable on $\mathbb{R}$ and $\forall x \in \mathbb{R} : 1 \leq f'(x) \leq 2$. Prove that $\forall x \geq 0 : x \leq f(x) \leq 2x$.

Ross 29.18

7. Show that there's no function whose derivative is the Dirichlet function.

10.6.1 A Discontinuous Derivative

Let $f = x^2 \sin(\frac{1}{x})$ when $x \neq 0$ and $f(0) = 0$. f is clearly differentiable at $a \neq 0$ with

$$f'(x) = 2x \sin(\frac{1}{x}) - \cos(\frac{1}{x})$$

f is also differentiable at 0 using the limit definition of the derivative, and $f'(0) = 0$.

However, we can show that f' is discontinuous using the sequential definition of continuity. Let

$$a_n = \frac{1}{2\pi n}$$

$$b_n = \frac{1}{2\pi n + \pi}$$

Both of these sequences converge to 0, but $f'(a_n)$ converges to 0 whereas $f'(b_n)$ does not.

Follow-up Question: Why did we use x^2? What happens if we try to set $f = x \sin(\frac{1}{x})$? Is this function continuous/differentiable at 0?

10.7 Solutions to Problems

Ross 28.8 Do what you gotta do.

Ross 29.2 Turn this expression into the derivative and use the fact that the derivative of cos is sin.

Ross 29.3 The first x comes from applying MVT to $f(0)$ and $f(1)$. The second x comes from the IVP property of the derivative coupled with the fact that $f'(x) = 0$ for some $x \in (1, 2)$.

Ross 29.4 Use the hint, and it's immediate.

Ross 29.5 Take the limit and show that the derivative is 0 everywhere.

Ross 29.7 Solution in the back of the book.

Ross 29.9 Solution in the back of the book.

Ross 29.10 Do what you gotta do. The point is that the function is not increasing even though the derivative is positive.

Ross 29.11 Take the derivative of $f(x) = x - \sin(x)$.

Ross 29.14 Use contradiction and MVT.

Ross 29.18 Immediate.

1. The derivative should satisfy the IVP property.

11 Taylor's Theorem

11.1 Problems

1. Find the Taylor series for $\cos(x)$, $\sin(x)$ and e^x. Prove that the radius of convergence is ∞.
2.

12 Integration

12.1 Problems

1. Oscillation lemma
2. Ross 32.7, 32.8, 33.7, 33.8 (all in Notability)
3. Ross 33.4 – 1 at rationals and -1 at irrationals
4. Ross 33.13 – IVT for integrals
5. MATH131BH Homework 5 Problem 3
6. MATH131BH Homework 5 Problem 5

13 Fundamental Theorem of Calculus

1. Lipschitz continuity of the antiderivative
2. Ross 34.5
3.

14 Challenge Problems

1. Let $\{s_n\}_{n\in\mathbb{N}}$ a sequence of real numbers with $s_1 := \sqrt{2}$ and $s_{n+1} := \sqrt{2+\sqrt{s_n}}$. Prove that s_n converges.

2. Let $\{x_n\}_{n\in\mathbb{N}} \subset (0,1)$ be a real-valued sequence.

   (a) Show that $\{x_n\}_{n\in\mathbb{N}}$ admits a convergent subsequence.

   (b) If there is a convergent subsequence $\{x_{n_k}\}_{n\in\mathbb{N}}$, show with a counter example that it doesn't need to converge to a point within $(0,1)$.

3. Given two reals a_0, b_0, define $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ recursively so that

$$\forall n \in \mathbb{N} : a_{n+1} = \frac{a_n + b_n}{2} \wedge b_{n+1} = \sqrt{a_n + b_n}$$

   Prove that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$.