170A Killip Homework 6 Due Fri, Feb 16

(1) Each of n people (whom we label 1, 2, $\ldots$, n) are randomly and independently assigned a number from the set $\{1, 2, 3, \ldots, 365\}$ according to the uniform distribution. We will call this number their birthday. Let j and k be distinct labels (between 1 and n) and let A_{jk} denote the event that the corresponding people share a birthday. Let X_{jk} denote the indicator random variable associated to A_{jk}.
(a) Tabulate the joint PMF for X_{12} and X_{13}. Compute the PMF for the product $X_{12}X_{13}$.
(b) Tabulate the joint PMF for X_{12} and X_{34}. Compute the PMF for the product $X_{12}X_{34}$.
(c) Are A_{12} and A_{34} independent? Are they independent conditioned on A_{13}?
(d) Are A_{12} and A_{13} independent? Are they independent conditioned on A_{23}?
(e) Compute the expected number of pairs of people who share a birthday (hint: write this the number as a sum of X_{jk}s).
(f) Compute the second moment and variance of the number of pairs of people who share a birthday.

(2) An infant repeatedly attempts to build a stack of three blocks. Her probability of balancing any particular block is p (including the first block of the stack), independent of any other attempt. Failure to balance any particular block results in the whole stack collapsing.
(a) What is the probability she successfully builds the stack on any one attempt?
(b) What is the PMF for the number of attempts needed to first successfully complete the stack.
(c) What is the (conditional) PMF for the number of blocks successfully balanced in any particular attempt to build the stack, given that the attempt fails?
(d) What is the expected number of blocks balanced succesfully in a failed attempt to build the stack?
(e) Fix $\ell \in \{1, 2, 3, \ldots\}$. Given that the first successful completion of the stack happend on the ℓth attempt, what is the expected value of the total number of blocks balanced successfully over the course of these ℓ attempts?
(f) What is the expected total number of blocks balanced successfully up to and including the first successful completion of the whole stack?

(3) Suppose $X \sim \text{Poisson}(\lambda)$ and $Y \sim \text{Poisson}(\mu)$ are statistically independent. Show that $X + Y \sim \text{Poisson}(\mu + \lambda)$. Hint: Use the binomial theorem.

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(4) We have a physical process that produces independent identically distributed random variables X_j whose distribution is unknown to us. We wish to find the true mean μ via experimentation. We may assume that $\mathbb{E}(X_i^2) < \infty$.
(a) Whose inequality shows $\mathbb{E}(|X_i|) < \infty$? Elaborate.
(b) Show that the sample mean

$$\bar{X} = \tfrac{1}{n}\big(X_1 + \cdots + X_n\big)$$

is an unbiased estimate of the true mean.
(c) Determine $\mathrm{var}(\bar{X})$ in terms of $\mathrm{var}(X_i)$.
(d) Show that if the mystery distribution is Poisson(λ), albeit with λ unknown, then $\bar{X}$ is the maximum likelihood estimator for μ.

(5) (Continued from previous problem.) We may also wish to know the (common) variance of our random variables X_i, which we may estimate using

$$Z = \tfrac{1}{n}\big([X_1 - \bar{X}]^2 + \cdots + [X_n - \bar{X}]^2\big),$$

or, if by some miracle we knew the true mean μ of the distribution

$$\tilde{Z} = \tfrac{1}{n}\big([X_1 - \mu]^2 + \cdots + [X_n - \mu]^2\big).$$

(a) Prove that $Z \le \tilde{Z}$ for every elementary outcome.
(b) Prove that

$$Z = \left(\tfrac{1}{n}\sum_{i=1}^{n} X_i^2\right) - \bar{X}^2.$$

(b) Find $\mathbb{E}(Z)$ and $\mathbb{E}(\tilde{Z})$ in terms of $\mathrm{var}(X_i)$.
(c) The idea of using $\frac{n}{n-1}Z$ as an estimator for the true variance is known as the Bessel correction. Explain how this leads to bias (systematic underestimation) in determining the standard deviation.