170A Killip Homework 8 Due Fri, Mar 1

(1) Let $\Theta \sim \text{Uniform}(0, 2\pi)$. What are the PDF and CDF of $X = \cos(\Theta)$.

(2) Let X and Y be real-valued random variables on some probability space and let $g : \mathbb{R}^2 \to \mathbb{R}$ be continuous. Show that $g(X, Y)$ is also a random variable — it is measurable!

Remark: A similar argument shows that if $f_X(x)$ is a PDF and so measurable, then $xf_X(x)$ and $|x|f_X(x)$ are also measurable.

(3) Let $X \sim \text{Gamma}(k, \lambda)$.
(a) Determine $\mathbb{E}(X^n)$ for each $n \in \mathbb{N}$.
(b) Determine $\mathbb{E}(e^{tX})$ for all $t \in \mathbb{R}$ for which it is defined.
(c) Show that if $Y \sim N(0, \sigma^2)$, then Y^2 is Gamma distributed and determine the parameters.

(4) Suppose $Y \sim N(0, \sigma^2)$.
(a) Suppose $h : \mathbb{R} \to \mathbb{R}$ is smooth and satisfies

$$|h'(x)| \leq C(1 + x^{2N})$$

for some constants $C > 0$ and $N \in \mathbb{N}$. Show that

$$\mathbb{E}\big[Yh(Y)\big] = \sigma^2\, \mathbb{E}\big[h'(Y)\big]$$

(b) Use the above to evaluate $\mathbb{E}(Y^n)$ for all $n \in \mathbb{N}$.
(c) Verify that when n is even, your answer to (c) agrees with what you obtain from your answers to the previous problem.
(d) Show that when $\sigma = 1$ and n is even, $\mathbb{E}(Y^n)$ is equal to the number of ways of forming n objects into (unordered) pairs.

(5) Let X be a random variable taking positive values and suppose $\ln(X) \sim N(0, 1)$.
(a) What is the probability that $X \geq 2$?
(b) What is the probability that $\frac{1}{2} \leq X \leq 2$?
(c) For which value of λ is $\mathbb{P}(X > \lambda) = \frac{1}{10}$?
Base your answers on the following tabulated values for the normal distribution:

`https://www.itl.nist.gov/div898/handbook/eda/section3/eda3671.htm`