Homework 7

● Graded

Student

NATHAN LEUNG

Total Points

24 / 25 pts

Question 1

1 **5** / 5 pts

✔ **− 0 pts** Correct

− 1 pt Minor issue(s)

− 3 pts Major issue(s)

− 5 pts Completely incorrect

Question 2

2 **5** / 5 pts

✔ **− 0 pts** Correct

− 1 pt Minor issue(s)

− 3 pts Major issue(s)

− 5 pts Completely incorrect

Question 3

3 **5** / 5 pts

✔ **− 0 pts** Correct

− 1 pt Minor issue(s)

− 3 pts Major issue(s)

− 5 pts Completely incorrect

Question 4

4 **5** / 5 pts

✔ **− 0 pts** Correct

− 1 pt Minor issue(s)

− 3 pts Major issue(s)

− 5 pts Completely incorrect

Question 5

5

4 / 5 pts

– 0 pts Correct

✔ **– 1 pt** Minor issue(s)

– 3 pts Major issue(s)

– 5 pts Completely incorrect

1 You've shown the limit along a particular sequence, but how do you know that this is actually the full limit?

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MATH 170A
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HOMEWORK 7

1 $X = \#$ of aces $\quad X \in \{0,1,2\}$
$Y = \#$ of hearts $\quad Y \in \{0,1,2\}$

$$\Omega = (\underbrace{\{A,2,3,4,\dots,9,10,J,Q,K\}}_{\text{rank}} \times \underbrace{\{\heartsuit,\clubsuit,\spadesuit,\diamondsuit\}}_{\text{suit}})^2$$

1a Joint PMF for X and Y

Hearts \ Aces	$X=0$	$X=1$	$X=2$	Total
$Y=0$	$\frac{\binom{36}{2}}{1,326} = \frac{630}{1,326}$	$\frac{3\cdot 36}{1326} = \frac{108}{1326}$	$\frac{3}{1326}$	$\frac{741}{1326}$
$Y=1$	$\frac{12\cdot 36}{1326} = \frac{432}{1326}$	$\frac{72}{1326}$	$\frac{3}{1326}$	$\frac{507}{1326}$
$Y=2$	$\frac{\binom{12}{2}}{1326} = \frac{66}{1326}$	$\frac{12}{1326}$	$\frac{0}{1326}$	$\frac{78}{1326}$
Total	$\frac{1128}{1326}$	$\frac{192}{1326}$	$\frac{6}{1326}$	

Total # of two-card selections in Ω is $\binom{52}{2} = 1,326$

$\boxed{\text{Justification } (X,Y)}$

$(0,0) = 0$ aces, 0 hearts $\qquad \binom{36}{2}$

There are 4 aces, 13 hearts, and 1 ace of hearts. Thus, a total of $4+13-1 = 17-1 = 16$ cards that are either an ace or heart. There are $52-16 = 36$ cards that are not an ace or a heart. We choose 2.

$(1,0) = 1$ ace, 0 hearts $\qquad \binom{3}{1}\binom{36}{1} = 3\cdot 36$

Of the three aces that are not hearts we pick 1. Of the 36 remaining non-ace/non-heart cards, we pick 1.

HOMEWORK 7

1a cont.

<u>$(2,0) = 2$ aces, 0 hearts</u> $\quad \binom{3}{2}$

There are 3 aces which are not hearts. We pick 2 of them.

<u>$(0,1) = 0$ aces, 1 heart</u> $\quad \binom{12}{1}\binom{36}{1} = 12 \cdot 36$

There are 12 heart cards which are not aces, and 36 non-ace/non-heart cards. We pick one of each.

<u>$(1,1) = 1$ ace, 1 heart</u> $\quad \binom{36}{1} + \binom{3}{1}\binom{12}{1} = 36 + 36$

There are two cases; we either pick the ace of hearts or we do not. If we pick the ace of hearts, then we pick one of the 36 remaining non-ace/non-heart cards. If we don't pick the ace of hearts, we pick one of the 3 non-heart aces then one of the 12 non-ace hearts.

<u>$(2,1) = 2$ aces, 1 heart</u> $\quad \binom{3}{1} = 3$

In this case, we must pick the ace of hearts as one of our cards, then we pick one of the remaining 3 aces.

<u>$(0,2) = 0$ aces, 2 hearts</u> $\quad \binom{12}{2}$

We pick 2 of the 12 non-ace heart cards.

$(1,2) = 1$ ace, 2 hearts $\quad \binom{12}{1} = 12$

$\underline{(1,2) = 1 \text{ ace}, 2 \text{ hearts}.}$ $(1,1) = 12.$

First, we need the ace of hearts. Then we choose one of the 12 remaining non-ace hearts.

Question assigned to the following page: 1

HOMEWORK 7

1a cont $(2,2)$ = 2 aces, 2 hearts

There is only one ace of hearts, so this case is impossible.

1b We can compute the PMF for Y as a marginal of the joint PMF of X and Y.

$$p_Y(0) = \sum_{k=0}^{2} p_{X,Y}(k,0)$$

$$= p_{X,Y}(0,0) + p_{X,Y}(1,0) + p_{X,Y}(2,0)$$

$$= \frac{630}{1326} + \frac{108}{1326} + \frac{3}{1326} = \frac{741}{1326}$$

Similarly

$$p_Y(1) = \frac{432}{1326} + \frac{72}{1326} + \frac{3}{1326} = \frac{507}{1326}$$ (from table in 1(a))

$$p_Y(2) = \frac{66}{1326} + \frac{12}{1326} + \frac{0}{1326} = \frac{78}{1326}$$

Directly, we have

39 non-heart cards

$$\mathbb{P}(\{Y=0\}) = \mathbb{P}(\{0 \text{ hearts}\}) = \frac{\binom{39}{2}}{\binom{52}{2}} = \frac{741}{1326}$$

$$\mathbb{P}(\{Y=1\}) = \mathbb{P}(\{1 \text{ heart}\}) = \frac{\binom{13}{1}\binom{39}{1}}{1326} = \frac{507}{1326}$$

$$\mathbb{P}(\{Y=2\}) = \mathbb{P}(\{2 \text{ hearts}\}) = \frac{\binom{13}{2}}{1326} = \frac{78}{1326}$$

which matches what we derived as a marginal from the original joint distribution.

HOMEWORK 7

1c $\mathrm{Cov}(X, Y) = \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])]$ def'n

$= \mathbb{E}[XY - \mathbb{E}[X]Y - \mathbb{E}[Y]X + \mathbb{E}[X]\mathbb{E}[Y]]$ expand

$= \mathbb{E}[XY] - \mathbb{E}[\mathbb{E}[X]Y] - \mathbb{E}[\mathbb{E}[Y]X] + \mathbb{E}[\mathbb{E}[X]\mathbb{E}[Y]]$ linearity

$= \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] - \mathbb{E}[Y]\mathbb{E}[X] + \mathbb{E}[X]\mathbb{E}[Y]$

$= \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$

By the definition of expectation, we have

$$\mathbb{E}[X] = \sum_{k=0}^{2} k\,\mathbb{P}(X=k)$$

$$= \mathbb{P}(X=1) + 2\,\mathbb{P}(X=2)$$

$$= \frac{192}{1326} + 2\left(\frac{6}{1326}\right) \quad \text{(from table in 1(a))}$$

$$= \frac{204}{1326}$$

$$\mathbb{E}[Y] = \sum_{k=0}^{2} k\,\mathbb{P}(Y=k)$$

$$= \mathbb{P}(\cancel{X}Y=1) + 2\,\mathbb{P}(Y=2)$$

$$= \frac{507}{1326} + 2\left(\frac{78}{1326}\right) = \frac{663}{1326}$$

$$\mathbb{E}[XY] = \sum_{\substack{0 \le j \le 2 \\ 0 \le k \le 2}} jk\,\mathbb{P}(X=j, Y=k)$$

possible products jk are 0, 1, 2, 4

$$= \underbrace{1 \cdot \frac{72}{1326}}_{(1,1)} + \underbrace{2 \cdot \frac{3}{1326}}_{(2,1)} + \underbrace{2 \cdot \frac{12}{1326}}_{(1,2)} + 4 \cdot \frac{0}{}$$

(2,2)

$$= \frac{102}{1326}$$

Questions assigned to the following page: 1 and 2

HOMEWORK 7

1c cont.

$$\text{Cov}(X,Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$$

$$= \frac{102}{1326} - \left(\frac{204}{1326}\right)\left(\frac{663}{1326}\right)$$

$$= \frac{102(1326)}{1326^2} - \frac{204(663)}{1326^2}$$

$$= \frac{135252}{1326^2} - \frac{135252}{1326^2}$$

$$= \boxed{0}$$

2 X_1, X_2 are independent $X_1, X_2 \sim \text{Geometric}(p)$

$$\mathbb{P}(X_1 \geq X_2) = \sum_{i=1}^{\infty} \mathbb{P}(X_1 \geq X_2 \mid X_2 = i)\,\mathbb{P}(X_2 = i)$$ partition theorem

$$= \sum_{i=1}^{\infty} \mathbb{P}(X_1 \geq i) \cdot \mathbb{P}(X_2 = i)$$

Note that $\mathbb{P}(X_1 \geq i) = \sum_{k=i}^{\infty} \mathbb{P}(X_i = k)$

$$= \sum_{k=i}^{\infty} (1-p)^{k-1} p$$ Def of Geometric(p)

$$= p \sum_{k=i}^{\infty} (1-p)^{k-1}$$

$$= p\left(\frac{(1-p)^{i-1}}{1-(1-p)}\right)$$ Geometric series sum

$$= p\left(\frac{(1-p)^{i-1}}{p}\right)$$

$$= (1-p)^{i-1}$$

Question assigned to the following page: 2

Homework 7

2 cont. Making the substitution

$$\mathbb{P}(X_1 \geq i) = (1-p)^{i-1}$$

we get

$$\mathbb{P}(X_1 \geq X_2) = \sum_{i=1}^{\infty} \mathbb{P}(X_1 \geq i) \cdot \mathbb{P}(X_2 = i)$$

$$= \sum_{i=1}^{\infty} (1-p)^{i-1} \cdot \boxed{\mathbb{P}(X_2 = i)}$$

$$= \sum_{i=1}^{\infty} (1-p)^{i-1} (1-p)^{i-1} p$$

$$= \sum_{i=1}^{\infty} \left((1-p)^2\right)^{i-1} p$$

$$= p \sum_{i=1}^{\infty} \left((1-p)^2\right)^{i-1}$$

$$= p \left(\frac{1}{1-(1-p)^2} \right)$$ Geometric series sum

$$= p \left(\frac{1}{1-(1-2p+p^2)} \right)$$

$$= p \left(\frac{1}{2p-p^2} \right)$$

$$= p \left(\frac{1}{p(2-p)} \right)$$

=

$$= \boxed{\frac{1}{2-p}}$$

HOMEWORK 7

3a $\mathbb{P}(\{\text{none of } A_i \text{ occur}\}) = \mathbb{P}(X=1)$ where X is the indicator random variable associated with the event $\{\text{none of } A_i \text{ occur}\}$

By the definition of expectation, $\mathbb{P}(X=1) = \mathbb{E}[X]$ since X is an indicator r.v.

Note that if X_i is the indicator random variable for A_i, $(1-X_i)$ is the indicator r.v. for A_i^c, i.e. A_i does not occur.

Similarly, $(1-X_i)(1-X_j)$ is the indicator r.v. for $A_i^c \cap A_j^c$, i.e. A_i and A_j both do not occur. Hence, we can write

$$X = (1-X_1)(1-X_2)\cdots(1-X_n)$$

$$= \prod_{i=1}^{n} [1-X_i]$$

So $\mathbb{P}(\{\text{none of } A_i \text{ occur}\}) = \mathbb{P}(X=1) = \mathbb{E}[X]$

$$= \mathbb{E}\left[\prod_{i=1}^{n} [1-X_i]\right]$$

Now, we need to show that

$$\mathbb{E}\left(\prod_{i=1}^{n} [1-X_i]\right) = \sum_{k=0}^{n} (-1)^k \sum_{\#S=k} \mathbb{E}\left[\prod_{i\in S} X_i\right]$$

First, we expand the product

$$\prod_{i=1}^{n} [1-X_i] = (1-X_1)(1-X_2)\cdots(1-X_n)$$

HOMEWORK 7

3a cont. By the definition of multiplication, to expand this product, we need to choose 1 term from each binomial factor, either 1 or $-X_i$. The sum over all possible choices of the (product of the chosen terms) is the expanded product.

That is,

$$(1-X_1)(1-X_2)\cdots(1-X_n) = \sum_{\substack{\text{binary} \\ \text{strings } b \\ \text{of length } n}} (-X_1)^{b[1]}(-X_2)^{b[2]}\cdots(-X_n)^{b[n]}$$

where $b[1], b[2], \ldots, b[n]$ are either 0 or 1, depending on whether the bit in position i of $b[i]$ is a 0 or 1.

If the bit is a 0, that indicates the 1 term was chosen; if the bit is a 1, that indicates the $-X_i$ term was chosen.

Note that we can equivalently sum over all subsets of $\{1, \ldots, n\}$, since binary strings of length n are in bijection w/ $\mathcal{P}(\{1, \ldots, n\})$ the power set. (We showed this in Homework 1, Problem 1).

If we split the sum over subsets by the cardinality of each subset, we get exactly the sum stated in the problem, after factoring out -1 from the k $(-x_i)$ terms.

$$\sum_{\text{binary strings}} (-x_1)^{b(1)} \cdots (-x_n)^{b(n)} = \sum_{k=0}^{n} (-1)^k \sum_{|S|=k} \left[\prod_{i \in S} x_i\right]$$

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HOMEWORK 7

3a cont. Applying $\mathbb{E}$ to both sides and then linearity of expectation, we obtain the second equality

$$\mathbb{E}\left[\prod_{i=1}^{n}(1-X_i)\right] = \sum_{k=0}^{n}(-1)^k \sum_{\#S=k} \mathbb{E}\left[\prod_{i\in S} X_i\right]$$

The transitive property of equality gives the entire three-expression equality

3b Let X_i be the indicator random variable for $\{i \text{ is a fixed point of the permutation}\}$

Note that
$$\mathbb{E}[X_i] = \mathbb{P}(X_i = 1)$$
$$= \mathbb{P}(\{i \text{ is a fixed point}\})$$
$$= \frac{(n-1)!}{n!} = \frac{1}{n}$$

since out of the $n!$ total permutations, there are $(n-1)!$ w/ i as a fixed point (once we fix i, there are $(n-1)!$ ways to order/assign the remaining $n-1$ elements.

We can apply Part (a) to determine the probability that none of $i=1, i=2, \ldots, i=n$ are fixed points.

We have $\mathbb{P}(\{\text{none of } X_i = 1\}) = \mathbb{E}\left[\prod_{i=1}^{n}(1-X_i)\right]$

$$= \sum_{k=0}^{n}(-1)^k \sum_{\#S=k} \mathbb{E}\left[\prod_{i\in S} X_i\right]$$

We know that $\mathbb{E}[\prod X_i] = \frac{(n-k)!}{\;}$ for $|S|=k$

We claim that $\mathbb{E}\left[\prod_{i \in S} 1_{i} \right] = \frac{\ldots}{n!}$ for $|S| = k$.

HOMEWORK 7

3b cont. Indeed, $\mathbb{E}[\prod_{i\in S} X_i]$ is exactly $\mathbb{P}(X_i = 1 \text{ for } i \in S)$, i.e., i is a fixed point of the permutation for $i \in S$.

Since $|S| = k$, this means we are fixing k elements and letting the remaining $n-k$ be assigned arbitrarily. There are $(n-k)!$ ways of doing this assignment out of $n!$ total permutations.

Thus, $\mathbb{E}[\prod_{i\in S} X_i] = \frac{(n-k)!}{n!}$. Our original expression becomes

$$\mathbb{P}(\{\text{none of } X_i = 1\}) = \sum_{k=0}^{n} (-1)^k \sum_{\#S=k} \mathbb{E}\left[\prod_{i\in S} X_i\right]$$

$$= \sum_{k=0}^{n} (-1)^k \sum_{\#S=k} \left(\frac{(n-k)!}{n!}\right)$$

$$= \sum_{k=0}^{n} (-1)^k \binom{n}{k} \left(\frac{(n-k)!}{n!}\right)$$

There are $\binom{n}{k}$ ways to pick a k-element subset of n elements

$$= \sum_{k=0}^{n} (-1)^k \left(\frac{n!}{(n-k)!\,k!}\right)\left(\frac{(n-k)!}{n!}\right)$$

$$= \sum_{k=0}^{n} (-1)^k \frac{1}{k!}$$

$$\boxed{= \sum_{k=0}^{n} \frac{(-1)^k}{k!}}$$

3c Note that $e^x = \sum \frac{x^k}{k!}$ by definition

Hence $e^{-1} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!}$. As $n \to \infty$ in our

Questions assigned to the following page: 3 and 4

Homework 7

3b cont. original answer to Part (b), we get

$$\lim_{n\to\infty} \sum_{k=0}^{n} \frac{(-1)^k}{k!} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} = e^{-1}$$

definition of infinite sum

which is what we wanted to show.

4a We claim that $X_r \sim \text{Geometric}\left(\frac{n-(r-1)}{n}\right)$.

Indeed, fix an arbitrary elementary outcome $\omega \in \Omega$. Then $X_r(\omega) = Y_r(\omega) - Y_{r-1}(\omega)$, i.e. the number of boxes purchased when we have r toys minus the number of boxes purchased when we have $r-1$ toys. This is exactly the number of boxes we needed to purchase to get from $r-1$ distinct toys to r distinct toys.

Since toys have equal frequency and appear independently, when we have $\frac{r-1}{n}$ distinct toys, every time we buy a new box, we have a $1-\frac{r-1}{n} = \frac{n-(r-1)}{n}$ chance of getting a new toy distinct from the $r-1$ toys we already have. The number of boxes we need to buy to successfully get that rth toy is thus modeled exactly like a $\text{Geometric}\left(\frac{n-(r-1)}{n}\right)$ variable. By the definition of a Geometric distribution, then, we have

$$\left(\frac{r-1}{n}\right)^{k-1}\left(\frac{n-(r-1)}{n}\right) \quad \text{if } k \geq 1$$

$$\boxed{P_{X_n}(k) = \begin{cases} \ldots \\ 0 & \text{otherwise} \end{cases}}$$

Homework 7

4b $\mathbb{E}[X_r] = \dfrac{1}{\left(\dfrac{(n-(r-1))}{n}\right)} = \dfrac{n}{n-(r-1)} = \boxed{\dfrac{n}{n-r+1}}$

since the $\mathbb{E}$ of a Geometric(p) r.v. is $1/p$.

4c $\mathbb{E}[Y_r]$? We know that $X_r = Y_r - Y_{r-1}$ so

$$
\begin{aligned}
Y_r &= X_r + Y_{r-1} \\
&= X_r + [X_{r-1} + Y_{r-2}] \\
&= X_r + [X_{r-1} + [X_{r-2} + Y_{r-3}]] \\
&\quad \cdots
\end{aligned}
$$

inductively expand recurrence...

$$= \sum_{k=1}^{r} X_k + Y_0$$

$$= \sum_{k=1}^{r} X_k \qquad \text{since } Y_r = 0$$

Thus
$$
\begin{aligned}
\mathbb{E}[Y_r] &= \mathbb{E}\left[\sum_{k=1}^{r} X_k\right] \\
&= \sum_{k=1}^{r} \mathbb{E}[X_k] \qquad \text{linearity of expectation} \\
&= \sum_{k=1}^{r} \frac{n}{n-k+1} \\
&= n \sum_{k=1}^{r} \frac{1}{n-k+1} \\
&= n \sum^{r-1} \frac{1}{n-k} \qquad \text{reindex}
\end{aligned}
$$

$k=0$

$$= n\left(H_n - H_{n-r}\right)$$

Homework 7

4c cont. where $H_n = \sum_{k=1}^{n} \frac{1}{n}$ the n^{th} sum of the harmonic series.

5a We want to show that the CDF $F_X(x)$ of an arbitrary random variable X is non-decreasing, i.e. $x \leq y \Rightarrow F_X(x) \leq F_X(y)$.

To start, suppose $x \leq y$. By definition, we have $F_X(x) = \mathbb{P}(X \leq x)$ and $F_X(y) = \mathbb{P}(X \leq y)$.

Note that $\{X \leq x\}$ and $\{x < X \leq y\}$ are disjoint sets. More precisely, let $\omega \in \Omega$. If $X(\omega) \leq x$, then ω cannot be an event in the other set, by the ordering of the real/rational/integers. Similarly, if $x < X(\omega) \leq y$, then ω cannot be an event in the first set.

Moreover, $\{X \leq y\} = \{X \leq x\} \cup \{x < X \leq y\}$ since $x \leq y$. The inclusions are clear from the ordering of the real line. Since we have a disjoint union, we can apply the axioms of probability to write

$$\begin{aligned} F_X(y) &= \mathbb{P}(\{X \leq y\}) \\ &= \mathbb{P}(\{X \leq x\} \cup \{x < X \leq y\}) \\ &= \mathbb{P}(\{X \leq x\}) + \mathbb{P}(\{x < X \leq y\}) && \text{disjoint union} \\ &= F_X(x) + \mathbb{P}(\{x < X \leq y\}) \\ &\geq F_X(x) && \text{since } \mathbb{P} \geq 0 \text{ always} \end{aligned}$$

which is what we wanted to show.

which is what the [illegible]

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HOMEWORK 7

5b We want to show that

$$\lim_{x \to -\infty} F_X(x) = 0, \quad \lim_{x \to \infty} F_X(x) = 1$$

We have

$$\lim_{x \to -\infty} F_X(x) = \lim_{x \to -\infty} \mathbb{P}(\{X \le x\})$$

$$= \lim_{n \to \infty} \mathbb{P}(\{X \le -n\})$$

if the limit exists over $\mathbb{R}$, it exists and is equal to the limit over $\mathbb{N}$

Note this works for any decreasing sequence $\to -\infty$

Let $A_n = \{X \le -n\}$.
Note that $A_n \supseteq A_{n+1} \supseteq A_{n+2} \supseteq \cdots$.
Moreover, $\bigcap_{n=1}^{\infty} A_n = \emptyset$ since for any real number, $\exists n \in \mathbb{N}$ w/ n less than that number.

Thus, by continuity of probability, we have

$$\lim_{x \to -\infty} F_X(x) = \lim_{n \to \infty} (\mathbb{P}(A_n))$$

$$= \mathbb{P}\left(\bigcap_{n=1}^{\infty} A_n\right)$$

$$= \mathbb{P}(\emptyset)$$

$$= 0$$

which is what we wanted to show.

For the other limit, we have

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Question assigned to the following page: 5

HOMEWORK 7

5b cont.
$$\lim_{x\to\infty} F_X(x) = \lim_{x\to\infty} \mathbb{P}(X \le x)$$
$$= \lim_{n\to\infty} \mathbb{P}(X \le n)$$

If limit over $\mathbb{R}$ exists, it is equal to limit over $\mathbb{N} \subseteq \mathbb{R}$

Let $B_n = \{X \le n\}$. Then $B_n^c = \{X > n\}$. We have

$$B = \bigcup_{n=1}^{\infty} B_n \quad \text{and} \quad B^c = \left(\bigcup_{n=1}^{\infty} B_n\right)^c$$
$$= \bigcap_{n=1}^{\infty} B_n^c \quad \text{DeMorgan}$$
$$= \bigcap_{n=1}^{\infty} \{X > n\}$$

Note that $B^c = \emptyset$ since any real number has a natural number greater than it. Thus we have

$$0 = \mathbb{P}(\emptyset) = \mathbb{P}(B^c)$$

which means $\mathbb{P}(B) = 1 - \mathbb{P}(B^c) = 1 - 0 = 1$ and hence

$$\lim_{x\to\infty} F_X(x) = \lim_{n\to\infty} \mathbb{P}(X \le n) = \lim_{n\to\infty} \mathbb{P}(B_n)$$
$$= \mathbb{P}\left(\bigcup_{n=1}^{\infty} B_n\right) = \mathbb{P}(B) = 1$$

$= P\left(\bigcup_{n\geq 1} B_n\right) = P(B)$

continuity of probability

which is what we wanted to show

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HOMEWORK 7

5c We want to show for any $x \in \mathbb{R}$,

$$\lim_{y \to x^-} F_X(y) = \mathbb{P}(X < x) \quad \text{and}$$

$$\lim_{y \to x^+} F_X(y) = F_X(x).$$

For the first limit, we have

works for any increasing sequence of sets

$$\lim_{y \to x^-} F_X(y) = \lim_{n \to \infty} F_X\left(x - \tfrac{1}{n}\right)$$

$$= \lim_{n \to \infty} \mathbb{P}\left(\left\{X \le x - \tfrac{1}{n}\right\}\right)$$

if the LHS limit exists, it must equal the RHS limit since $x - \frac{1}{n} \to x$ as $n \to \infty$

Let $A_n = \{X \le x - \frac{1}{n}\}$. Let $A = \bigcup_{n=1}^{\infty} \{X \le x - \frac{1}{n}\}$. Then $A = \{X < x\}$. Continuing, we ~~have~~

$$\lim_{y \to x^-} F_X(y) = \lim_{n \to \infty} (\mathbb{P}(A_n))$$

$$= \mathbb{P}(A) \qquad \text{continuity of } \mathbb{P},\ A_1 \subset A_2 \subset A_3 \subset \dots$$

$$= \mathbb{P}(\{X < x\})$$

which is what we wanted to show.

For the second limit we have

$$\lim_{y \to x^+} F_X(y) = \lim_{n \to \infty} F_X\left(x + \tfrac{1}{n}\right)$$

works for any decreasing sequence of sets

$$= \lim_{n \to \infty} \mathbb{P}\left(\left\{X \le x + \tfrac{1}{n}\right\}\right)$$

$$= \mathbb{P}\left(\bigcap \{X \le x + \tfrac{1}{n}\}\right)$$ continuity, each set is superset

$$= \mathbb{P}(X \le x) = F_X(x)$$ by def