Homework 8

Graded

Student

NATHAN LEUNG

Total Points

25 / 25 pts

Question 1

1 — **5** / 5 pts

- ✔ − 0 pts Correct
- − 1 pt Minor issue(s)
- − 3 pts Major issue(s)
- − 5 pts Completely incorrect

Question 2

2 — **5** / 5 pts

- ✔ − 0 pts Correct
- − 1 pt Minor issue(s)
- − 3 pts Major issue(s)
- − 5 pts Completely incorrect

Question 3

3 — **5** / 5 pts

- ✔ − 0 pts Correct
- − 1 pt Minor issue(s)
- − 3 pts Major issue(s)
- − 5 pts Completely incorrect

Question 4

4 — **5** / 5 pts

- ✔ − 0 pts Correct
- − 1 pt Minor issue(s)
- − 3 pts Major issue(s)
- − 5 pts Completely incorrect

Question 5

5	**5** / 5 pts

- **− 0 pts** Correct
- **− 1 pt** Minor issue(s)
- **− 3 pts** Major issue(s)
- **− 5 pts** Completely incorrect

Question assigned to the following page: 1

NATHAN LEUNG
MATH 170A
1 MARCH 2024

HOMEWORK 8

1 $\Theta \sim \text{Uniform}(0, 2\pi)$ [redacted].

PDF $f_\Theta(x) = \begin{cases} \frac{1}{2\pi} & \text{if } 0 \le x \le 2\pi \\ 0 & \text{otherwise} \end{cases}$

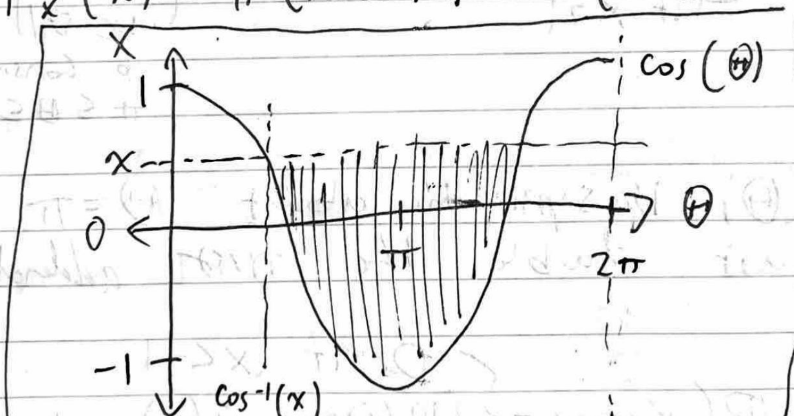
CDF $F_\Theta(x) = \mathbb{P}(\Theta \le x) = \int_{-\infty}^{x} f(u)\,du = \int^{\min(x,2\pi)} \frac{1}{2\pi}\,du$

$$= \begin{cases} \frac{x}{2\pi} & \text{if } 0 \le x \le 2\pi \\ 0 & \text{if } x < 0 \\ 1 & \text{if } x > 2\pi \end{cases}$$

$X = \cos(\Theta)$

First, we find the CDF of X, $F_X(x)$. We can differentiate to find the PDF.

$F_X(x) = \mathbb{P}(X \le x) = \mathbb{P}(\cos(\Theta) \le x)$ $\quad \theta \in [0, 2\pi]$



$\cos(\Theta) \le x$ in shaded area

Recall $\cos^{-1}(x)$ is defined on $x \in [-1, 1]$ and has range $[0, \pi]$. Note also that $\cos(\Theta)$ is symmetric about $\Theta = \pi$ and is strictly decreasing on $\Theta \in [0, \pi]$.

With that, note that for a given $x \in [-1, 1]$, we can find the corresponding left-hand side θ by

HOMEWORK 8

1 evaluating $\cos^{-1}(x)$. This gives us a cont value in $[0, \pi]$. Since $\cos(\Theta)$ is decreasing on $[0, \pi)$, we have

$$\mathbb{P}(X < x) = \mathbb{P}(\cos(\Theta) < x)$$

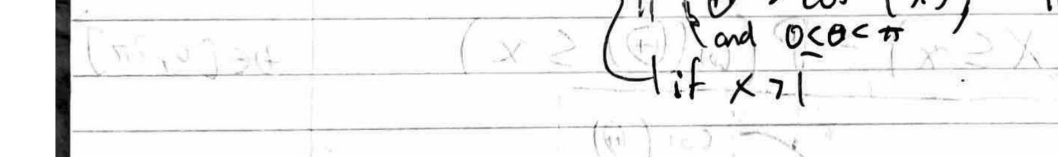
$\cos^{-1}(x)$ reverses order on $\theta \in [0, \pi]$ since cos is decreasing on that interval

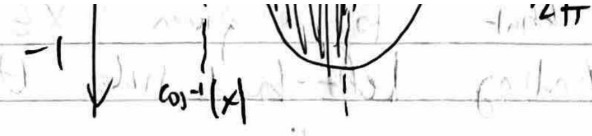
$$= \begin{cases} 0 & \text{if } x < -1 \\ \mathbb{P}(\Theta > \cos^{-1}(x) \text{ and } 0 \le \Theta < \pi) + \mathbb{P}(\cos(\Theta) < x \text{ and } \pi \le \Theta \le 2\pi) & \text{when } -1 \le x \le 1 \\ 1 & \text{if } x > 1 \end{cases}$$

we still need to consider when $\pi \le \Theta \le 2\pi$

But $\cos(\Theta)$ is symmetric about $\Theta = \pi$. So we can just double the first addend above i.e.

$$F_X(x) = \mathbb{P}(X \le x) = \begin{cases} 0 & \text{if } x < -1 \\ 2\,\mathbb{P}(\Theta > \cos^{-1}(x) \text{ and } \Theta \le \pi) & \text{if } -1 \le x \le 1 \\ 1 & \text{if } x > 1 \end{cases}$$





-1
cos^{-1}(x)
2π

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1 cont. That is, our CDF for X is

$$F_X(x) = \begin{cases} 0 & \text{if } x < -1 \\ 2\mathbb{P}(\cos^{-1}(x) < \Theta \leq \pi) & \text{if } -1 \leq x \leq 1 \\ 1 & \text{if } x > 1 \end{cases}$$

$\cos^{-1}(x) \in [0, \pi]$ on this interval

We can find a closed form for the $-1 \leq x \leq 1$ case by evaluating $F_\Theta(x)$. We get

$$2\mathbb{P}(\cos^{-1}(x) < \Theta \leq \pi) = 2\int_{\cos^{-1}(x)}^{\pi} f_\Theta(u)\, du$$

$$= 2\int_{\cos^{-1}(x)}^{\pi} \frac{1}{2\pi}\, du = 2 \cdot \frac{1}{2\pi} \int_{\cos^{-1} x}^{\pi} du = \frac{1}{\pi}\left[\pi - \cos^{-1}(x)\right]$$

$$= \frac{\pi - \cos^{-1}(x)}{\pi}$$

Our final CDF for X is thus

$$\boxed{F_X(x) = \begin{cases} 0 & \text{if } x \leq -1 \\ \frac{\pi - \cos^{-1}(x)}{\pi} & \text{if } -1 < x < 1 \\ 1 & \text{if } x \geq 1 \end{cases}} = 1 - \frac{1}{\pi}\cos^{-1}(x)$$

To find the PDF $f_X(x)$, we take the derivative:

$$\boxed{f_X(x) = \frac{d}{dx} F_X(x) = \begin{cases} 0 & \text{if } x \leq -1 \\ \frac{1}{\pi\sqrt{1-x^2}} & \text{if } -1 < x < 1 \\ 0 & \text{if } x \geq 1 \end{cases}}$$

strict inequality to avoid division by 0

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2 $X, Y: \Omega \to \mathbb{R}$ $\qquad g: \mathbb{R}^2 \to \mathbb{R}$ cts

WTS $Z = g(X, Y)$ is a random variable,
i.e. it is a mapping $\Omega \to \mathbb{R}$ s.t.
$\{\omega \in \Omega : Z(\omega) \leq x\} = \{\omega \in \Omega : g(X(\omega), Y(\omega)) \leq x\} \in \mathcal{F}$
i.e. is measurable

Clearly, $Z(\omega) = g(X(\omega), Y(\omega))$ is a mapping $\Omega \to \mathbb{R}$ by the definition of g.
Thus, we just need to show that

$$\{\omega \in \Omega : g(X(\omega), Y(\omega)) \leq x\} \in \mathcal{F} \quad \text{for all } x \in \mathbb{R},$$

i.e. each such set is measurable.

Fix $x \in \mathbb{R}$. We will show that

$$\{\omega \in \Omega : g(X(\omega), Y(\omega)) \leq x\} \in \mathcal{F}$$

Since σ-algebras are closed under complements, it suffices to show that

$$\{\omega \in \Omega : g(X(\omega), Y(\omega)) \leq x\}^c$$

$$= \{\omega \in \Omega : g(X(\omega), Y(\omega)) > x\} \in \mathcal{F}.$$

We can express this complement set equivalently as

$$\{\omega \in \Omega : (X(\omega), Y(\omega)) \in g^{-1}[(z, \infty)]\}$$

i.e. all $\omega \in \Omega$ such that g evaluated at the point $(X(\omega), Y(\omega))$ is in (z, ∞). This

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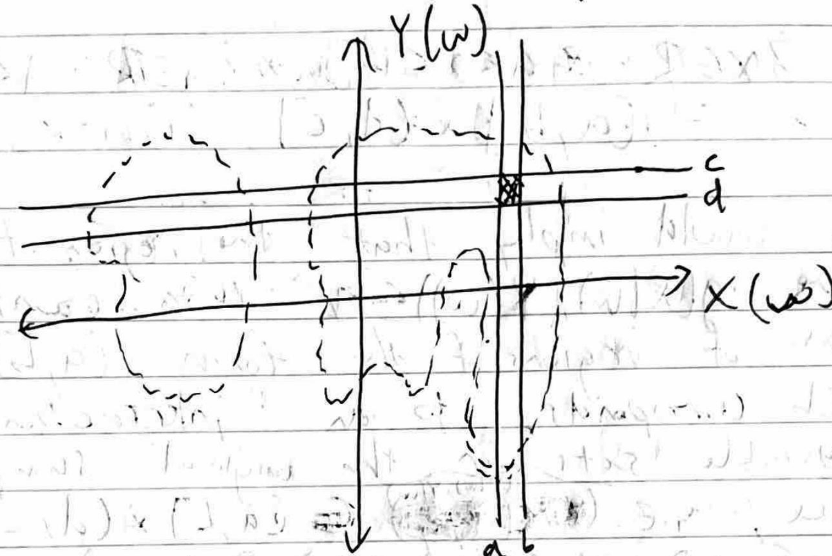
2 follows from the definition of 7. Note that cont (z, ∞) is an open set in $\mathbb{R}$. Since g is continuous, the preimage $g^{-1}[(z, \infty)]$ must also be an open set too (but in $\mathbb{R}^2$).

We have the following picture:



The dotted lines indicate the regions in $\mathbb{R}^2$ where $g(X(\omega), Y(\omega)) \leq x$ for our fixed $x \in \mathbb{R}$.

Since X and Y are random variables, we know that the sets $\{\omega \in \Omega : a < X(\omega) \leq b\}$ and $\{\omega \in \Omega : d < Y(\omega) \leq c\}$ are both measurable. Hence, the countable intersection (of two sets)

$$\{\omega \in \Omega : a < X(\omega) \leq b\} \cap \{\omega \in \Omega : d < Y(\omega) \leq c\}$$
$$= \{\omega \in \Omega : a < X(\omega) \leq b \text{ and } d < Y(\omega) \leq c\}$$

must also be measurable (we can turn a countable intersection into a countable union by taking complements and applying De Morgan)

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2 cont. In the picture, the values of X and Y corresponding to this intersection are represented by the shaded box.

We claim that any open set in $\mathbb{R}^2$ can be constructed as a countable union of such half-open boxes

$$\{x \in \mathbb{R} : a < x \leq b\} \times \{y \in \mathbb{R} : d < y \leq c\}$$
$$= (a, b] \times (d, c]$$

This would imply that the region of $\mathbb{R}^2$ where $g(X(\omega), Y(\omega)) \leq x$ is a countable union of regions of the form $(a, b] \times (d, c]$, which corresponds to an intersection of two measurable sets in the original sample space, i.e. $(X(\omega), Y(\omega)) \in (a, b] \times (d, c] \iff \omega \in \{\omega \in \Omega : a < X(\omega) \leq b\} \cap \{\omega \in \Omega : d < Y(\omega) \leq c\}$.

Since σ-algebras are closed under countable unions (and hence countable intersections, via complementation), showing the claim consequently implies that $\{\omega \in \Omega : (X(\omega), Y(\omega)) \in g^{-1}((z, \infty))\} = \{\omega \in \Omega : g(X(\omega), Y(\omega)) > x\}$, as an open set (and hence countable union of such half-open box sets) is in the σ-algebra and hence measurable, which is what we want to show.

Indeed, fix $O \subseteq \mathbb{R}^2$ an arbitrary open set. We claim

[redacted]

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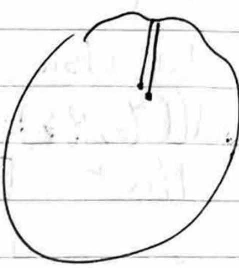

2 cont. $$\mathcal{O} = \bigcup_{(q_n,q_m)\in\mathbb{Q}^2\cap\mathcal{O}} \left[q_n - \frac{d_{nm}}{2}, q_n+\frac{d_{nm}}{2}\right] \times \left[q_m - \frac{d_{nm}}{2}, q_m + \frac{d_{nm}}{2}\right]$$

(i.e. over every rational point in $\mathcal{O}$)

where d_{nm} = the distance from (q_n, q_m) to $\mathcal{O}^c$ (i.e. the minimum distance needed to go in any direction away from (q_n,q_m) such that one leaves $\mathcal{O}$). Note $\mathbb{Q}^2$ is countable, so this is indeed a countable union.

To show equality, we need to show the inclusion both ways. Let $(x,y)\in\mathcal{O}$, and let d = distance from (x,y) to $\mathcal{O}^c$ and the openness of $\mathcal{O}$. By the density of $\mathbb{Q}$ in $\mathbb{R}$, we can pick $(q_x,q_y)\in\mathbb{Q}^2\cap\mathcal{O}$ such that the distance from (q_x,q_y) to (x,y) is less than $\frac{d}{100}$.

[redacted]

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2 cont. We claim that $(x,y) \in (q_x - \frac{d_{xy}}{2}, q_x + \frac{d_{xy}}{2}] \times (q_y - \frac{d_{xy}}{2}, q_y + \frac{d_{xy}}{2}]$,

i.e. this arbitrary point (x,y) is in one of the half-open boxes in the union, where d_{xy} = distance from (q_x, q_y) to $\mathcal{O}^c$.

$B_\varepsilon((x,y))$ = ball of radius ε centred at (x,y)

d = distance fn

We chose $(q_x, q_y) \in \mathbb{Q}^2 \cap \mathcal{O}$ such that $d((q_x, q_y), (x,y)) < \frac{d}{100}$. We chose d such that $B_d((x,y)) \subseteq \mathcal{O}$, i.e. $B_d((x,y)) \cap \mathcal{O}^c = \emptyset$.

By the fact that $d((q_x, q_y), (x,y)) < \frac{d}{100}$, we know that $(q_x, q_y) \in B_{\frac{d}{100}}((x,y))$ and $(x,y) \in B_{d/100}((q_x, q_y))$.

We claim that $d_{xy} = d((q_x, q_y), \mathcal{O}^c)$ = min distance from (q_x, q_y) to $\mathcal{O}^c \geq \frac{d}{10}$. Suppose for contradiction that $d_{xy} < d/10$, i.e. the distance from (q_x, q_y) to $\mathcal{O}^c$ was less than $d/10$. Then $B_{d/10}((q_x, q_y)) \cap \mathcal{O}^c \neq \emptyset$; i.e. there must exist some point in $\mathcal{O}^c$ at most $d/10$ distance away from (q_x, q_y); otherwise, the minimum distance would be $\geq d/10$.

We claim $B_{d/10}((q_x, q_y)) \subseteq B_d((x,y))$. Indeed, pick $(a,b) \in B_{d/10}((q_x, q_y))$. Then by the triangle inequality, we get

$$d((a,b),(x,y)) \leq d((a,b),(q_x,q_y)) + d((q_x,q_y),(x,y))$$

$$< \underbrace{\frac{d}{10}}_{\text{by choice of } (a,b)} + \underbrace{\frac{d}{100}}_{\text{by choice of } (q_x, q_y)}$$

$$= \frac{11d}{100} < \frac{20d}{100} = \frac{d}{5}$$

100 100 5

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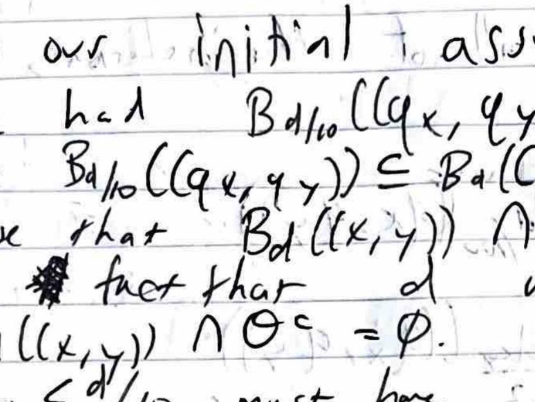
2. cont. Since $d((a,b),(x,y)) \leq \frac{d}{5} < d$, we thus have $(a,b) \in B_d((x,y))$. Since (a,b) was arbitrary, we get the relation $B_{d/10}((q_x, q_y)) \subseteq B_d((x,y))$.

By our initial assumption (for contradiction), we had $B_{d/10}((q_x,q_y)) \cap \Theta^c \neq \emptyset$. But then, since $B_{d/10}((q_x,q_y)) \subseteq B_d((x,y))$, it must also be the case that $B_d((x,y)) \cap \Theta^c \neq \emptyset$, which contradicts the fact that d was chosen such that $B_d((x,y)) \cap \Theta^c = \emptyset$. Thus, our assumption that $d_{xy} < d/10$ must have been false, and in fact, $d_{xy} \geq d/10$.

To finish, we need to show that $(x,y) \in [q_x - \frac{d_{xy}}{2}, q_x + \frac{d_{xy}}{2}] \times [q_y - \frac{d_{xy}}{2}, q_y + \frac{d_{xy}}{2}]$. We have the picture:



Note that the circle of radius $\frac{d_{xy}}{2}$ is contained within the square w/ side lengths d_{xy}.

Thus we have $B_{\frac{d_{xy}}{2}}((q_x, q_y)) \subseteq [q_x - \frac{d_{xy}}{2}, q_x + \frac{d_{xy}}{2}] \times [q_y - \frac{d_{xy}}{2}, q_y + \frac{d_{xy}}{2}]$

we showed this earlier

But $\frac{d_{xy}}{2} \geq \frac{1}{2}\left(\frac{d}{10}\right) = \frac{d}{20}$, so

$$B_{d/20}((q_x, q_y)) \subseteq B_{\frac{d_{xy}}{2}}((q_x, q_y)).$$

And $d > d$, so we also have

And $\frac{d}{20} > \frac{d}{100}$, so we always have

$$B_{d/100}((q_x, q_y)) \subseteq B_{d/20}((q_x, q_y)).$$

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2 cont. Finally, we recall that we chose (q_x, q_y) such that $d((q_x, q_y), (x,y)) < \frac{1}{100}$, so $(x,y) \in B_{d/100}((q_x, q_y))$.

Putting together all of the inclusions, we find that

$$(x,y) \in B_{d/100}((q_x, q_y))$$

$$\subseteq B_{\frac{d_{xy}}{2}}((q_x, q_y))$$

$$\subseteq (q_x - \tfrac{d_{xy}}{2}, q_x + \tfrac{d_{xy}}{2}] \times (q_y - \tfrac{d_{xy}}{2}, q_y + \tfrac{d_{xy}}{2}]$$

$$(\subseteq \mathcal{O} \text{ since } d_{xy} \text{ is min. dist to } \mathcal{O}^c)$$

Since any $(x,y) \in \mathcal{O}$ is in one of the half-open boxes centered at a rational point included in the union, we get the relation

$$\mathcal{O} \subseteq \bigcup_{(q_n, q_m) \in \mathbb{Q}^2 \cap \mathcal{O}} (q_n - \tfrac{d_{nm}}{2}, q_n + \tfrac{d_{nm}}{2}] \times (q_m - \tfrac{d_{nm}}{2}, q_n + \tfrac{d_{nm}}{2}]$$

For the reverse direction, note that every point in the infinite union is in one (or more) of the half-open boxes, each which is constructed specifically so that it does not extend past the boundary of $\mathcal{O}$ (since we only go $d_{nm}/2$ distance away from the point). Since every box in the union is contained in $\mathcal{O}$, every point in the union must be contained in $\mathcal{O}$ also. Thus

$$\mathcal{O} = \bigcup_{(q_n, q_m) \in \mathbb{Q}^2 \cap \mathcal{O}} \left(q_n - \frac{d_{nm}}{2}, q_n + \frac{d_{nm}}{2}\right] \times \left(q_m - \frac{d_{nm}}{2}, q_m + \frac{d_{nm}}{2}\right]$$

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2 cont. Hence, we can conclude every open set $O \subseteq \mathbb{R}^2$ is the countable union of ~~[illegible]~~ the half-open boxes as constructed in our expression above.

Specifically, $g^{-1}[(z,\infty)] \subseteq \mathbb{R}^2$, an open set, is the countable union of half-open boxes. This corresponds exactly to a countable union of intersected subsets of Ω (that is, each half-open box ~~corresponds to~~ $(a,b] \times (c,d]$ corresponds to $\{\omega \in \Omega : a < X(\omega) \leq b\} \cap \{\omega \in \Omega : c < Y(\omega) \leq d\}$) that are in the σ-algebra since X and Y are measurable.

Since σ-algebras are closed under countable unions and complements, this means that the subset of Ω corresponding to $g^{-1}[(z,\infty)]$, i.e. $\{\omega \in \Omega : g(X(\omega), Y(\omega)) > x\} \subseteq \Omega$, is in the σ-algebra also. Hence, its complement $\{\omega \in \Omega : g(X(\omega), Y(\omega)) \leq x\}$ is in the σ-algebra also. Since $x \in \mathbb{R}$ was arbitrary, this means that $g(X, Y)$ ~~is a well-defined random variable~~ satisfies the definition of a random variable (i.e. $F_{g(X,Y)}(x)$ exists for all x) which is what we wanted to show.

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3 $X \sim \text{Gamma}(k, \lambda)$ $\quad k > 0,\ \lambda > 0$

a $\mathbb{E}[X^n]$ for each $n \in \mathbb{N}$

In the case of $n=1$, we have

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x f(x)\, dx \qquad f(x) = \begin{cases} \frac{1}{\Gamma(k)} \lambda^k x^{k-1} e^{-\lambda x} & \text{if } x > 0 \\ 0 & \text{if } x \le 0 \end{cases}$$

$$= \int_0^{\infty} x \cdot \frac{1}{\Gamma(k)} \lambda^k x^{k-1} e^{-\lambda x}\, dx$$

$$= \int_0^{\infty} \frac{k}{k} \cdot \frac{1}{\Gamma(k)} \cdot \frac{\lambda}{\lambda} \cdot \lambda^k \cdot x^k \cdot e^{-\lambda x}\, dx$$

$$= \int_0^{\infty} \frac{k}{\Gamma(k+1)} \cdot \frac{\lambda^{k+1}}{\lambda} \cdot x^k e^{-\lambda x}\, dx$$

$$= \frac{k}{\lambda} \underbrace{\int_0^{\infty} \frac{1}{\Gamma(k+1)} \lambda^{k+1} x^k e^{-\lambda x}\, dx}_{\text{PDF of Gamma}(k+1,\ \lambda)}$$

$$= \frac{k}{\lambda}(1) \qquad \text{integral of PDF over support is 1}$$

$$= \frac{k}{\lambda}$$

In general, for $n \in \mathbb{N}$, we have

$$\mathbb{E}[X^n] = \int^{\infty} x^n \cdot \frac{1}{\ldots} \lambda^k x^{k-1} e^{-\lambda x} \ldots$$

$$\mathbb{E}[X^{\cdot}] = \int_0 x^{\cdot} \cdot \frac{}{\Gamma(k)} \lambda \; x \; e \quad dx$$

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3a cont.

$$= \int_0^\infty \frac{1}{\Gamma(k)} \lambda^k x^{k+n-1} e^{-\lambda x}\, dx$$

$$= \int_0^\infty \frac{k(k+1)\cdots(k+n-2)(k+n-1)}{k(k+1)\cdots(k+n-2)(k+n-1)} \cdot \frac{1}{\Gamma(k)} \cdot \frac{\lambda^n}{\lambda^n} \cdot \lambda^k x^{k+n-1} e^{-\lambda x}\, dx$$

$$= \int_0^\infty \frac{k(k+1)\cdots(k+n-2)(k+n-1)}{\Gamma(k+n)} \cdot \frac{\lambda^{k+n}}{\lambda^n} x^{k+n-1} e^{-\lambda x}\, dx$$

$$= \frac{k(k+1)\cdots(k+n-2)(k+n-1)}{\lambda^n} \underbrace{\int_0^\infty \frac{1}{\Gamma(k+n)} \lambda^{k+n} x^{k+n-1} e^{-\lambda x}\, dx}_{\text{PDF of Gamma}(k+n,\, \lambda)}$$

$$= \boxed{\frac{k(k+1)\cdots(k+n-2)(k+n-1)}{\lambda^n}}$$

3b $\mathbb{E}[e^{tx}]$ for $t \in \mathbb{R}$

$$= \int_0^\infty e^{tx} \frac{1}{\Gamma(k)} \lambda^k x^{k-1} e^{-\lambda x}\, dx$$

$$= \int_0^\infty \frac{1}{\Gamma(k)} \lambda^k x^{k-1} e^{(-\lambda + t)x}\, dx$$

$$= \int_0^\infty \frac{1}{\Gamma(k)} \lambda^k x^{k-1} e^{-(\lambda - t)x}\, dx$$

Note that this integral diverges when $\lambda - t \leq 0$, i.e. $\lambda \leq t$ because then the exponent of e is positive.

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3b cont To proceed, assume that $\lambda > t$. Then we can write

$$\mathbb{E}[e^{tX}] = \int_0^\infty \frac{1}{\Gamma(k)} \lambda^k x^{k-1} e^{-(\lambda - t)x} dx$$

$$= \lambda^k \int_0^\infty \frac{1}{\Gamma(k)} \frac{(\lambda-t)^k}{(\lambda-t)^k} x^{k-1} e^{-(\lambda-t)x} dx$$

$$= \frac{\lambda^k}{(\lambda-t)^k} \underbrace{\int_0^\infty \frac{1}{\Gamma(k)} (\lambda-t)^k x^{k-1} e^{-(\lambda-t)x} dx}_{\text{PDF of Gamma}(k,\, \lambda - t)}$$

$\lambda - t > 0$ by choice

$$= \boxed{\left(\frac{\lambda}{\lambda - t}\right)^k} \text{ when } \lambda > t$$

3c $Y \sim N(0, \sigma^2)$

WTS Y^2 is Gamma distributed

By the definition of a normal distribution,

$$f_Y(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2} \frac{(x-\mu)^2}{\sigma^2}\right)$$

It suffices to show that the PDF of Y^2 is equal to that of a Gamma-distributed random variable for some parameters k and λ.

First, we compute the CDF

$F_{\cdot}(\cdot) = \mathbb{P}(Y^2 \leq \cdot)$

$F_{Y^2}(y) = \mathbb{P}(Y^2 \leq y)$

$= \mathbb{P}(-\sqrt{y} \leq Y \leq \sqrt{y})$

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3c cont

$F_{Y^2}(y) = \mathbb{P}(-\sqrt{y} < Y \leq \sqrt{y})$

$= \mathbb{P}(Y \leq \sqrt{y}) - \mathbb{P}(Y \leq -\sqrt{y})$

CDF of Y is cts, so $\mathbb{P}(Y = -\sqrt{y}) = 0$

for $y \geq 0$, since $\sqrt{\ }$ is not defined on $y \leq 0$, $y \in \mathbb{R}$

Thus $f_{Y^2}(y) = \frac{d}{dy} F_{Y^2}(y)$

$= \frac{d}{dy}\left(\mathbb{P}(Y \leq \sqrt{y}) - \mathbb{P}(Y \leq -\sqrt{y})\right)$ for $y \geq 0$

$= \frac{d}{dy}\left(F_Y(\sqrt{y}) - F_Y(-\sqrt{y})\right)$

$= \frac{d}{dy} F_Y(\sqrt{y}) - \frac{d}{dy} F_Y(-\sqrt{y})$

$= \frac{1}{2\sqrt{y}} f_Y(\sqrt{y}) + \frac{1}{2\sqrt{y}} F_Y(-\sqrt{y})$ Chain Rule

$= \frac{1}{2\sqrt{y}}\left(f_Y(\sqrt{y}) + f_Y(-\sqrt{y})\right)$

$= \frac{1}{2\sqrt{y}} \cdot \frac{1}{\sqrt{2\pi\sigma^2}}\left(\exp\left(-\frac{1}{2}\frac{(\sqrt{y}-\mu)^2}{\sigma^2}\right) + \exp\left(-\frac{1}{2}\frac{(-\sqrt{y}-\mu)^2}{\sigma^2}\right)\right)$

$= \frac{1}{2\sqrt{y}} \cdot \frac{1}{\sqrt{2\pi\sigma^2}}\left(\exp\left(-\frac{1}{2\sigma^2}y\right) + \exp\left(-\frac{1}{2\sigma^2}y\right)\right)$ since $\mu = 0$

$= \frac{1}{\cancel{2}\sqrt{y}} \cdot \frac{1}{\sqrt{2\pi\sigma^2}} \cdot \cancel{2}\exp\left(-\frac{y}{2\sigma^2}\right)$

$= \frac{1}{\sqrt{\pi}} \cdot \frac{1}{\sqrt{2\sigma^2}} \cdot y^{-1/2} \cdot e^{-\frac{1}{2\sigma^2}y}$

$= \frac{1}{\sqrt{\pi}} \cdot \left(\frac{1}{2\sigma^2}\right)^{1/2} \cdot y^{-1/2} \cdot e^{-\frac{1}{2\sigma^2}y}$

Note that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ (conversion to spherical coordinates)

Questions assigned to the following page: 3 and 4

HOMEWORK 8

3c cont. Thus we have

$$f_{Y^2}(y) = \frac{1}{\Gamma(\frac{1}{2})} \cdot \left(\frac{1}{2\sigma^2}\right)^{1/2} \cdot y^{-1/2} \cdot e^{\frac{-1}{2\sigma^2} y} \quad \text{for } y \geq 0$$

which is exactly the PDF for a $\text{Gamma}(\frac{1}{2}, \frac{1}{2\sigma^2})$ random variable (the pdf is 0 for $y < 0$ also, since when $y < 0$ there is no real number whose ~~[illegible]~~ square is less).

That is, $\boxed{Y^2 \sim \text{Gamma}\left(\frac{1}{2}, \frac{1}{2\sigma^2}\right)}$

4 We have $Y \sim N(0, \sigma^2)$

a We have $h: \mathbb{R} \to \mathbb{R}$ smooth (i.e. infinitely differentiable) and satisfying

$$|h'(x)| \leq C(1 + x^{2N}) \quad \text{for some } C > 0 \text{ and } N \in \mathbb{N}$$

i.e. $h'(x)/h(x)$ grows at most polynomially. We want to show that

$$\mathbb{E}[Yh(Y)] = \sigma^2 \mathbb{E}[h'(Y)]$$

Note that both expectations exist since $f_Y(y)$ is exponential decay, but h is polynomial

absolutely

By the definition of expectation, we have

$$\mathbb{E}[Yh(Y)] = \int_{-\infty}^{\infty} y h(y) f_Y(y)\, dy$$

Proceed by integration by parts.

$$u = h(y) \qquad v = \int y f_Y(y)\, dy$$

h is smooth $\rightarrow$ $du = h'(y)dy \qquad dv = y f_Y(y)\, dy$

VI is smooth

Question assigned to the following page: 4

HOMEWORK 8

4a cont For v, we have

$$v = \int y f_Y(y)\,dy$$

$$= \int y \cdot \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2}\frac{(y-\mu)^2}{\sigma^2}\right) dy$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \int y e^{\frac{-y^2}{2\sigma^2}}\,dy \qquad (\mu = 0)$$

Let $w = -\frac{1}{2\sigma^2} y^2$. Then $dw = \frac{-1}{\sigma^2} y\,dy$ and so $y\,dy = -\sigma^2 dw$. Making the change of variables yields

$$v = \frac{1}{\sqrt{2\pi\sigma^2}} \int e^w \cdot (-\sigma^2\,dw)$$

$$= \frac{-\sigma^2}{\sqrt{2\pi\sigma^2}} \int e^w\,dw$$

$$= \frac{-\sigma^2}{\sqrt{2\pi\sigma^2}} e^w$$

$$= \frac{-\sigma^2}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2} y^2}$$

Then proceeding w/ integration by parts, we have

$$\int_{-\infty}^{\infty} y h(y) f_Y(y)\,dy = \left[h(y) \cdot \frac{-\sigma^2}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2} y^2}\right]_{-\infty}^{\infty}$$

$$- \int \frac{-\sigma^2 [illegible]}{\sqrt{2\pi\sigma^2}} e^{[illegible]} h'(y)\,dy$$

Question assigned to the following page: 4

HOMEWORK 8

4a cont $\int_{-\infty}^{\infty} y h(y) f_Y(y) dy =$

$$\frac{-\sigma^2}{\sqrt{2\pi\sigma^2}} \left[h(y) e^{-\frac{1}{2\sigma^2} y^2} \right]_{-\infty}^{\infty}$$

$$+ \sigma^2 \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2} y^2} h'(y)\, dy$$

Note that the first term above goes to zero since the negative exponential dominates the polynomial growth of $h(y)$. Thus we are left with

$$\int_{-\infty}^{\infty} y h(y) f_Y(y) dy = \sigma^2 \int_{-\infty}^{\infty} h'(y) \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2} y^2} dy$$

$$= \sigma^2 \mathbb{E}[h'(y)]$$

which is what we wanted to show.

4b We want to use the relation we found above to evaluate $\mathbb{E}[Y^n]$ for all $n \in \mathbb{N}$.

We claim the following: for $n = 2m$ even,

$$\mathbb{E}[Y^n] = \mathbb{E}[Y^{2m}] = \left[\prod_{i=1}^{m} (2i-1)\right] \sigma^{2m}$$

For $n = 2m-1$ odd,

$$\mathbb{E}[Y^n] = \mathbb{E}[Y^{2m-1}] = \vec{0}$$

Question assigned to the following page: 4

Homework 8

4b We proceed by induction on $m \in \mathbb{N}$ (in
cont. effect proving the odd case and even case at the same time)

In the base case, $m=1$, we have $n_{odd} = 2m-1 = 1$ and $n_{even} = 2m = 2$.

$$\mathbb{E}[Y^{n_{odd}}] = \mathbb{E}[Y] = \sigma^2 \mathbb{E}[0] = 0$$

(where $h(y) = 1$ and we apply part (a))

So the ~~base case~~ claim holds in the odd base case.

$$\mathbb{E}[Y^{n_{even}}] = \mathbb{E}[Y^2] = \sigma^2 \mathbb{E}[1] = \sigma^2 = \sigma^{2m} \cdot \prod_{i=1}^{m} (2i-1)$$

where $h(y) = y$, $h'(y) = 1$

only has 1 term, 1

and so the ~~base case~~ claim holds in the even base case.

Now assume that the claim holds for all natural numbers $\mathbb{N}$ up to some m. We want to show that this implies that it holds for $m+1$. We have (in the odd case)

$$\mathbb{E}[Y^{2(m+1)-1}] = \mathbb{E}[Y^{2m+2-1}] = \mathbb{E}[Y^{2m+1}]$$

$$= \mathbb{E}[Y^{2m} \cdot Y] = \sigma^2 \mathbb{E}[2m Y^{2m-1}]$$

$$= \sigma^2 (2m) \cdot \mathbb{E}[Y^{2m-1}]$$

$$= \sigma^2 (2m)(0)$$ by inductive hypothesis

$= 0$

Question assigned to the following page: 4

HOMEWORK 8

4b cont. In the even case, we have

$$\mathbb{E}[Y^{2(m+1)}] = \mathbb{E}[Y^{2m+2}]$$

$$= \mathbb{E}[Y \cdot Y^{2m+1}]$$

$$= \sigma^2 \mathbb{E}[(2m+1) Y^{2m}]$$

$$= \sigma^2 (2m+1) \mathbb{E}[Y^{2m}]$$

$$= \sigma^2 (2m+1) \left(\prod_{i=1}^{m} (2i-1)\right) \sigma^{2m}$$ inductive hypothesis

$$= \sigma^{2m+2} \left[\prod_{i=1}^{m+1} (2i-1)\right]$$

$$= \sigma^{2(m+1)} \left[\prod_{i=1}^{m+1} (2i-1)\right]$$

and so the claim holds in the even case as well.

4c We verify that our answer to Part (b) above matches the result found in Problem 3, Part (a), specifically for n even.

That is, when $n = 2m$ is even, we need to show

$$\mathbb{E}[Y^n] = \mathbb{E}[Y^{2m}] = \mathbb{E}[(Y^2)^m]$$

$$= \mathbb{E}[X^m] = \frac{k(k+1)\cdots(k+m-1)}{\lambda^m}$$ Problem 3 Part (a)

where $X \sim \text{Gamma}\left(\frac{1}{2}, \frac{1}{2\sigma^2}\right)$ by Problem 3, Part (c).

Question assigned to the following page: 4

HOMEWORK 8

4c We have
cont.

$$\mathbb{E}[Y^n] = \mathbb{E}[Y^{2m}] = \left[\prod_{i=1}^{m}(2i-1)\right]\sigma^{2m}$$

$$\mathbb{E}[X^m] = \frac{\frac{1}{2}\left(\frac{1}{2}+1\right)\cdots\left(\frac{1}{2}+m-1\right)}{\left(\frac{1}{2\sigma^2}\right)^m}$$

$$= \frac{\frac{1}{2}\left(\frac{3}{2}\right)\cdots\left(\frac{1+2m-2}{2}\right)}{\left(\frac{1}{2\sigma^2}\right)^m}$$

$$= \frac{\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\cdots\left(\frac{2m-1}{2}\right)}{\frac{1}{(2\sigma^2)^m}}$$

$$= \left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\cdots\left(\frac{2m-1}{2}\right)\cdot(2\sigma^2)^m$$

$$= \prod_{i=1}^{m}\left(\frac{2i-1}{2}\right)\cdot 2^m\cdot\sigma^{2m}$$

$$= \frac{\prod_{i=1}^{m}(2i-1)}{2^m}\cdot 2^m\cdot\sigma^{2m}$$

$$= \left[\prod_{i=1}^{m}(2i-1)\right]\sigma^{2m}$$

$$= \mathbb{E}[Y^n]$$

which is what we wanted to show,

Question assigned to the following page: 4

HOMEWORK 8

4d When $\sigma = 1$ and n is even we want to show that $E(Y^n)$ is the number of ways of forming n objects into unordered pairs.

When $\sigma = 1$, and $n = 2m$ is even, we get

$$E[Y^n] = E[Y^{2m}] = \left[\prod_{i=1}^{m} (2i-1)\right]\sigma^{2m}$$

~~A the~~

$$= \left[\prod_{i=1}^{m} (2i-1)\right] \quad \text{since } \sigma = 1$$

= product of all positive odd integers up to $n = 2m$

At the same time, the number of ways to form $n = 2m$ objects into m unordered pairs is given by

$$\binom{n}{\underbrace{2,2,\ldots,2,2}_{m \text{ times}}} \cdot \frac{1}{m!}$$

since the multinomial coefficient gives the number of ways of splitting n objects into m distinct 2-element subsets and we divide by a factor of $m!$ because the m different possible pairs should not be considered [illegible]

should not be considered distinct for
our purposes and they can
be ordered in m! different ways.

Question assigned to the following page: 4

Homework 8

4d cont.

$$\underbrace{\binom{n}{2,2,\ldots,2}}_{m \text{ times}} \cdot \frac{1}{m!} = \overbrace{\binom{n}{2}\binom{n-2}{2}\cdots\binom{4}{2}\binom{2}{2}}^{m \text{ factors}} \cdot \frac{1}{m!}$$

$$= \frac{n!}{\cancel{(n-2)!}\,2!} \cdot \frac{\cancel{(n-2)!}}{\cancel{(n-4)!}\,2!} \cdot \cdots \cdot \frac{4!}{\cancel{2!}\cdot 2!}\,\frac{\cancel{2!}}{2!}\,\frac{1}{m!}$$

Note that the product telescopes, in that every factor cancels except for the initial $n!$ and m instances of $2!$ in the denominator. We are left with

$$\binom{n}{2,2,2,\ldots 2} \cdot \frac{1}{m!} = \frac{n!}{2^m} \cdot \frac{1}{m!}$$

$$= \frac{n\cdot(n-1)\cdot(n-2)\cdots(3)(2)(1)}{2^m} \cdot \frac{1}{m!}$$

$$= \underbrace{\left(\frac{n}{2}\right)(n-1)\left(\frac{n-2}{2}\right)\cdots(3)\left(\frac{2}{2}\right)}_{\text{there are } m \text{ even factors in } n!} \cdot 1 \cdot \frac{1}{m!}$$

$$= \left[(n-1)(n-3)\cdots(3)(1)\right]\left[\left(\frac{n}{2}\right)\left(\frac{n-2}{2}\right)\cdots\left(\frac{2}{2}\right)\right] \cdot \frac{1}{m!}$$

$$= \left[\prod_{i=1}^{m}(2i-1)\right]\left[\left(\frac{n}{2}\right)\left(\frac{n}{2}-1\right)\cdots(1)\right] \cdot \frac{1}{m!}$$

$$= \left[\prod_{i=1}^{m}(2i-1)\right]\left[m(m-1)\cdots(1)\right] \cdot \frac{1}{m!}$$

$$= \left[\prod_{i=1}^{m}(2i-1)\right]\left[m!\right] \cdot \frac{1}{m!} = \boxed{\prod_{i=1}^{m}(2i-1)}$$

which is exactly $\mathbb{E}[Y^n]$ when $\sigma = 1$

Question assigned to the following page: 5

HOMEWORK 8

5a $\ln(x) \sim N(0,1)$. Let $Y = \ln(x)$

a $\mathbb{P}(X \geq 2) = \mathbb{P}(\ln(x) \geq \ln(2))$ since ln is increasing

$= \mathbb{P}(Y \geq \ln(2))$

$= 1 - \mathbb{P}(Y < \ln(2))$

$= 1 - \mathbb{P}(Y < 0.693147)$ used calculator

$= 1 - (\mathbb{P}(0 < Y < 0.693147) + 0.5)$ N(0,1) is symmetric about 0, both halves are $\frac{1}{2}$ prob.

$\approx 1 - (0.25490 + 0.5)$ read from table

$\approx \boxed{0.245}$

b $\mathbb{P}(\frac{1}{2} \leq X \leq 2) = \mathbb{P}(\ln(\frac{1}{2}) \leq \ln(x) \leq \ln(2))$ ln is increasing

$= \mathbb{P}(-0.693147 \leq \ln(x) \leq 0.693147)$

$= 2\mathbb{P}(0 \leq \ln(x) \leq 0.693147)$ N(0,1) is symmetric

$= 2(0.25490) = \boxed{0.50980}$ read from table

c For which value of λ is $\mathbb{P}(X > \lambda) = \frac{1}{10}$?

$\cancel{\mathbb{P}(X > \lambda)}$ $\mathbb{P}(X > \lambda) = \frac{1}{10} \iff \mathbb{P}(\ln(x) > \ln(\lambda)) = \frac{1}{10}$

i.e. $\iff \mathbb{P}(Y > \ln(\lambda)) = \frac{1}{10}$

$\iff 1 - \mathbb{P}(Y > \ln(\lambda)) = \frac{9}{10}$

$$\Leftrightarrow \mathbb{P}(Y \leq \ln(\lambda)) = \frac{9}{10}$$

Question assigned to the following page: 5

HOMEWORK 8

5c Cont According to the table, $P(Y \leq y) = \frac{9}{10}$ when $y = 1.28$. Thus we need to solve

$$\ln(\lambda) = 1.28$$

$$e^{\ln(\lambda)} = e^{1.28}$$

$$\boxed{\lambda = 3.5966}$$