(1) (a) Suppose $X \sim \text{Poisson}(\lambda)$. By repeatedly differentiating the power series for $\exp(\lambda)$, compute
$$\mathbb{E}\big\{X(X-1)(X-2)\cdots(X+1-\ell)\big\}$$
for each $\ell = 1, 2, 3, \ldots$.
(b) By a parallel method evaluate such expectations when $X \sim \text{Binomial}(n, p)$.
(c) If $n \to \infty$ with $p = \lambda/n$ show that your (b) answer converges to that for (a).

(2) (a) Determine the mean and variance of the following random variables:
$$\text{Poisson}(\lambda), \quad \text{Binomial}(n,p), \quad \text{and} \quad \text{Geometric}(p).$$
(b) Tabulate then memorize these results.

(3) (a) Let $X : \Omega \to \mathbb{Z}$ be a *non-negative* random variable. Show that
$$\mathbb{E}(X) = \sum_{n=1}^{\infty} \mathbb{P}(X \geq n),$$
including the assertion that $\mathbb{E}(X)$ exists if and only if this sum converges.
(b) More generally, what relation between functions G and g ensures
$$\mathbb{E}\big(G(X)\big) = \sum_{n=0}^{\infty} g(n)\,\mathbb{P}(X \geq n)$$
To simplify questions of convergence, let us just assume $g(k) \geq 0$ for all k.

(4) We throw a die independently four times and let X denote the minimal value rolled.
(a) For each $n \in \mathbb{N}$, find probability that $X \geq n$.
(b) Compute the PMF of X.
(c) Determine the mean and variance of X.

(5) I play the following game using a coin that lands heads with probability p. I start with $X_0 = \$1$ and at each stage I gamble all I have on the toss of the coin. If it lands heads I end up with twice what I started with; if it lands tails I lose everything. All coin tosses are statistically independent.
(a) With X_n denoting how much money I have after the nth toss, find
$$\mathbb{E}(X_{n+1} | X_n = k)$$
in terms of k.
(b) Find $\mathbb{E}(X_n)$ for all n.

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(6) I carry out an infinite sequence of independent Bernoulli(p) trials, $0 < p < 1$. For each $r \in \mathbb{N}$ we let Y_r denote the position of my r^{th} success. In a previous homework, we found the PMF of X — it follows the negative Binomial law

$$\mathbb{P}(Y_r = k) = \begin{cases} \binom{k-1}{r-1} p^r (1-p)^{k-r} & : \ k \in \mathbb{N} \quad \text{and} \quad k \geq r \\ 0 & : \ \text{otherwise} \end{cases}$$

See also page 27 in the text.

(a) Given a number $k \geq r$, verify that the event $Y_r = k$ has positive probability.

(b) Compute

$$\mathbb{E}\big(Y_{r+1}\big|Y_r = k\big) \quad \text{and} \quad \text{var}\big(Y_{r+1}\big|Y_r = k\big).$$

The latter denotes the variance of Y_{r+1} under the probability law $\mathbb{P}(\,\cdot\,|Y_r = k)$.

(c) Use (b) to find a recurrence between the means/variances of Y_r as r varies.

(d) Find the mean and variance of Y_r.