(1) Let X_1 and X_2 be independent and uniformly distributed on the interval $[0,1]$. Given $0 < \alpha < 1$, we generate a random variable Y as follows: If $X_1 > \alpha$ then $Y = X_1$, otherwise $Y = X_2$.
(a) What are the CDF and PDF of Y.
(b) What is $\mathbb{E}(Y)$.
(c) For which value of α is $\mathbb{E}(Y)$ largest.
(d) For comparison, find the CDF, PDF, and mean of $Z = \max\{X_1, X_2\}$.

Remark: Imagine we are offered salary X_1 at job 1 and need to accept or decline before finding out the salary X_2 at job 2.

(2) Suppose U and V are independent and follow a Uniform(0,1) law. Show that

$$X = \sqrt{-2\ln(U)}\cos(2\pi V) \quad \text{and} \quad Y = \sqrt{-2\ln(U)}\sin(2\pi V)$$

define independent random variables each with a $N(0,1)$ law. This is known as the Box–Muller transform.

(3) Recall (class/book) the definition of the χ^2_ν random variable: $X \sim \chi^2_\nu$ if and only if

$$f_X(x) = \begin{cases} \frac{1}{2^{\nu/2}\Gamma(\nu/2)} x^{\frac{\nu}{2}-1} e^{-x/2} & : x \geq 0 \\ 0 & : \text{otherwise} \end{cases}$$

where $\nu > 0$ and $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x}\, dx$ for all $\alpha > 0$.

Suppose $X \sim \chi^2_\nu$ and $Y \sim \chi^2_k$ are independent. Show that $X + Y \sim \chi^2_{\nu+k}$. In this way, discover the value of Euler's Beta Integral

$$\int_0^1 u^{\frac{\nu}{2}-1}(1-u)^{\frac{k}{2}-1}\, du$$

as a ratio of Gamma functions.

(4) Suppose $X \sim \chi^2_\nu$ and $Z \sim N(0,1)$ are independent. Find the pdf of

$$T = Z\sqrt{\tfrac{\nu}{X}}.$$

Solving this problem will lead you to the discovery (also of Euler) that

$$\int_{-\infty}^{\infty} (1 + \tfrac{1}{\nu}t^2)^{-\frac{\nu+1}{2}}\, dt = \frac{\sqrt{\nu\pi}\,\Gamma(\frac{\nu}{2})}{\Gamma(\frac{\nu+1}{2})}.$$

We say that T follows a Student t-distribution with ν degrees of freedom. 'Student' was the pseudonym used by W. S. Gosset on the paper introducing this distribution.