(1) We take two cards (without replacement) from a well-shuffled standard deck of 52 cards. Let X denote the number of these two cards that are aces and let Y denote the number that are hearts.
(a) Tabulate the joint PMF for X and Y.
(b) Compute the PMF for Y both directly and as a marginal of the above (this provides a check on your computations).
(c) What is the covariance of X and Y?

(2) Suppose X_1 and X_2 are independent and Geometric(p) distributed. What is the probability that $X_1 \geq X_2$?

(3) Consider events A_i for each natural number $1 \leq i \leq n$ and let X_i denote the associated indicator random variable.
(a) Show that

$$\mathbb{P}\big(\text{none of } A_i \text{ occur}\big) = \mathbb{E}\bigg\{\prod_{i=1}^{n}[1 - X_i]\bigg\} = \sum_{k=0}^{n}(-1)^k \sum_{\#S=k} \mathbb{E}\bigg\{\prod_{i\in S} X_i\bigg\}$$

where the S sum is over all k-element subsets of $\{1, \ldots, n\}$.
(b) What is the probability that a randomly chosen permutation of $\{1, \ldots, n\}$ has no fixed points?
(c) Show that this probability converges to $\exp(-1)$ as $n \to \infty$.

(4) Each cereal box comes with a toy and we are encouraged to "collect all n." How many boxes will we need to buy? We assume the toys have equal frequency and appear independently. Let Y_r denote the (random) number of boxes purchased when we first have r distinct toys. Let $X_r = Y_r - Y_{r-1}$ with $Y_0 = 0$.
(a) What is the PMF for X_r?
(b) What is $\mathbb{E}(X_r)$ for each $1 \leq r \leq n$?
(c) What is $\mathbb{E}(Y_r)$ for each $1 \leq r \leq n$?

(5) Show that the CDF $F_X(x)$ of any random variable has the following properties:
(a) F_X is non-decreasing: $x \leq y \quad \Longrightarrow \quad F_X(x) \leq F_X(y)$.
(b) $\lim\limits_{x\to-\infty} F_X(x) = 0$ and $\lim\limits_{x\to\infty} F_X(x) = 1$.
(c) For any $x \in \mathbb{R}$,

$$\lim_{y\uparrow x} F_X(y) = \mathbb{P}(X < x) \quad \text{and} \quad \lim_{y\downarrow x} F_X(y) = F_X(x)$$

I.e. F_X is càdlàg — continuous from the right and limits from the left exist.