Homework 2

Graded

Student

NATHAN LEUNG

Total Points

30 / 30 pts

Question 1

1 **5** / 5 pts

✔ **– 0 pts** Correct

– 1 pt Minor issue(s)

– 3 pts Major issue(s)

– 5 pts Completely incorrect

1 typically this would just be A^c in LaTeX - no need for any fancy C

2 Need to make sure that S is nonempty

Question 2

2 **5** / 5 pts

✔ **– 0 pts** Correct

– 1 pt Minor issue(s)

– 3 pts Major issue(s)

– 5 pts Completely incorrect

Question 3

3 **5** / 5 pts

✔ **– 0 pts** Correct

– 1 pt Minor issue(s)

– 3 pts Major issue(s)

– 5 pts Completely incorrect

Question 4

4 **5** / 5 pts

✔ **– 0 pts** Correct

– 1 pt Minor issue(s)

– 3 pts Major issue(s)

– 5 pts Completely incorrect

Question 5

5 **5** / 5 pts

✔ **– 0 pts** Correct

– 1 pt Minor issue(s)

– 3 pts Major issue(s)

– 5 pts Completely incorrect

Question 6

6 **5** / 5 pts

✔ **– 0 pts** Correct

– 1 pt Minor issue(s)

– 3 pts Major issue(s)

– 5 pts Completely incorrect

Question assigned to the following page: 1

MATH 170A Homework 2

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Problem 1

Fix a sample space Ω.

Part (a)

Show the intersection of any collection of σ-algebras is a σ-algebra.

Let $S = \{\mathcal{F}_i \mid i \in I\}$ be a collection of σ-algebras, where I is an indexing set. We want to show that

$$\mathcal{F}_{\text{all}} = \bigcap_{i \in I} \mathcal{F}_i$$

is also a σ-algebra. It suffices to show that $\mathcal{F}_{\text{all}}$ satisfies the axioms of a σ-algebra, namely

1. $\Omega, \varnothing \in \mathcal{F}_{\text{all}}$
2. $A \in \mathcal{F}_{\text{all}} \implies A^{\complement} \in \mathcal{F}_{\text{all}}$
3. $A_n \in \mathcal{F}_{\text{all}}, n \in \mathbb{N} \implies \bigcup_{n \in \mathbb{N}} A_n \in \mathcal{F}_{\text{all}}$

For the first axiom, fix $i \in I$ and $\mathcal{F}_i \in S$, some σ-algebra in the collection S. Then, $\Omega \in \mathcal{F}_i$ since $\mathcal{F}_i$ is a σ-algebra itself. Since $i \in I$ was arbitrary, we can conclude that $\Omega \in \mathcal{F}_i$ for all $i \in I$. Hence, by the definition of set intersection, $\Omega \in \bigcap_{i \in I} \mathcal{F}_i = \mathcal{F}_{\text{all}}$. Similar logic tells us that $\varnothing \in \mathcal{F}_{\text{all}}$ also (since $\varnothing$ is in each σ-algebra $\mathcal{F}_i$).

For the second axiom, fix $A \in \mathcal{F}_{\text{all}}$. We want to show that $A^{\complement} \in \mathcal{F}_{\text{all}}$. By the definition of set intersection, $A \in \mathcal{F}_i$ for all $i \in I$. Then, fix $i \in I$ and $\mathcal{F}_i \in S$, some σ-algebra in the collection S. Since $A \in \mathcal{F}_i$, a σ-algebra, $A^{\complement} \in \mathcal{F}_i$ also. Since $i \in I$ was arbitrary, we can conclude that $A^{\complement} \in \mathcal{F}_i$ for all $i \in I$. Then, by the definition of set intersection, $A^{\complement} \in \bigcap_{i \in I} \mathcal{F}_i = \mathcal{F}_{\text{all}}$, which is what we needed to show.

Finally, for the third axiom, let $T = \{A_n \mid n \in \mathbb{N}\} \subset \mathcal{F}_{\text{all}}$ be an arbitrary countable collection of subsets in $\mathcal{F}_{\text{all}}$. We want to show that $\bigcup_{n \in \mathbb{N}} A_n \in \mathcal{F}_{\text{all}}$. Fix an arbitrary $i \in I$. Also, fix an arbitrary $n \in \mathbb{N}$ and $A_n \in T$. Since $A_n \in T \subset \mathcal{F}_{\text{all}}$, by the definition of set intersection, $A_n \in \mathcal{F}_i$. Since $n \in \mathbb{N}$ was arbitrary, $A_n \in \mathcal{F}_i$ for all $n \in \mathbb{N}$. Thus, since $\mathcal{F}_i$ is a σ-algebra, $\bigcup_{n \in \mathbb{N}} A_n \in \mathcal{F}_i$. Since $i \in I$ was arbitrary, $\bigcup_{n \in \mathbb{N}} A_n \in \mathcal{F}_i$ for all $i \in I$. Hence, by the definition of set intersection, $\bigcup_{n \in \mathbb{N}} A_n \in \bigcap_{i \in I} \mathcal{F}_i = \mathcal{F}_{\text{all}}$, which is what we wanted to show.

Since $\mathcal{F}_{\text{all}}$ satisfies the axioms of a σ-algebra, it must be a σ-algebra. Since S was an arbitrary collection of σ-algebras, we can conclude that the intersection of any collection of σ-algebras is a σ-algebra also.

Questions assigned to the following page: 2 and 1

Part (b)

What is the intersection of all σ-algebras on Ω?

Let $S = \{\mathcal{F} \mid \mathcal{F} \text{ is a } \sigma\text{-algebra on } \Omega\}$ and $\mathcal{F}_{\text{min}} = \bigcap_{\mathcal{F} \in S} \mathcal{F}$, the intersection of all σ-algebras on Ω. We claim that $\mathcal{F}_{\text{min}} = \{\varnothing, \Omega\}$.

To show set equality, it suffices to show that $\{\varnothing, \Omega\} \subset \mathcal{F}_{\text{min}}$ and $\mathcal{F}_{\text{min}} \subset \{\varnothing, \Omega\}$.

For the first inclusion, note that for all $\mathcal{F} \in S$, $\varnothing \in \mathcal{F}$ and $\Omega \in \mathcal{F}$ since $\mathcal{F}$ is a σ-algebra. Thus, $\{\varnothing, \Omega\} \subset \mathcal{F}$ for all $\mathcal{F} \in S$. Then, by the definition of set intersection, $\{\varnothing, \Omega\} \subset \mathcal{F}_{\text{min}}$ too.

For the second inclusion, note that $\{\varnothing, \Omega\}$ is a σ-algebra on Ω. It trivially satisfies the first axiom, $\Omega^{\complement} = \varnothing$ and $\varnothing^{\complement} = \Omega$ so it satisfies the second axiom, and the only union is $\Omega \cup \varnothing = \Omega$ which is in the σ-algebra. Since $\{\varnothing, \Omega\}$ is a σ-algebra on Ω, it is included in the intersection $\mathcal{F}_{\text{min}} = \bigcap_{\mathcal{F} \in S} \mathcal{F}$. By the definition of set intersection, then, $\mathcal{F}_{\text{min}} \subset \{\varnothing, \Omega\}$, which is what we wanted to show.

Since $\{\varnothing, \Omega\} \subset \mathcal{F}_{\text{min}}$ and $\mathcal{F}_{\text{min}} \subset \{\varnothing, \Omega\}$, we can conclude that $\bigcap_{\mathcal{F} \in S} \mathcal{F} = \mathcal{F}_{\text{min}} = \{\varnothing, \Omega\}$, i.e. the intersection of all σ-algebras on Ω is $\{\varnothing, \Omega\}$.

Part (c)

Show that for any collection $\mathcal{A}$ of subsets of Ω, there is a smallest σ-algebra containing $\mathcal{A}$. This is called the *σ-algebra generated by* $\mathcal{A}$ and denoted $\sigma(\mathcal{A})$.

Fix $\mathcal{A} \subset \mathcal{P}(\Omega)$, a collection of subsets of Ω.

Let $S = \{\mathcal{F} \subset \mathcal{P}(\Omega) \mid \mathcal{F} \text{ is a } \sigma\text{-algebra on } \Omega \text{ containing } \mathcal{A}\}$, the set of all σ-algebras on Ω containing $\mathcal{A}$. We claim that

$$\sigma(\mathcal{A}) = \bigcap_{\mathcal{F} \in S} \mathcal{F}$$

That is, the smallest σ-algebra containing $\mathcal{A}$ is the intersection of all σ-algebras containing $\mathcal{A}$.

Since $\mathcal{A} \subset \mathcal{F}$ for all $\mathcal{F} \in S$, by the definition of set intersection, $\mathcal{A} \subset \bigcap_{\mathcal{F} \in S} \mathcal{F}$. By **Part (a)**, we know that $\bigcap_{\mathcal{F} \in S} \mathcal{F}$ is a σ-algebra also. Thus, to finish, we only need to show that $\bigcap_{\mathcal{F} \in S} \mathcal{F}$ is the *smallest* σ-algebra containing $\mathcal{A}$. But indeed, suppose for contradiction that there existed a strictly smaller σ-algebra $\mathcal{F}_{\text{smaller}}$ containing $\mathcal{A}$, i.e. $\mathcal{F}_{\text{smaller}} \subsetneq \bigcap_{\mathcal{F} \in S} \mathcal{F}$. Then $\mathcal{F}_{\text{smaller}} \in S$ and hence would be included in the intersection $\bigcap_{\mathcal{F} \in S} \mathcal{F}$. By the definition of set intersection, though, we also have $\bigcap_{\mathcal{F} \in S} \mathcal{F} \subset \mathcal{F}_{\text{smaller}}$. Putting together all the inclusions, we get $\mathcal{F}_{\text{smaller}} \subsetneq \bigcap_{\mathcal{F} \in S} \mathcal{F} \subset \mathcal{F}_{\text{smaller}}$, that is, $\mathcal{F}_{\text{smaller}} \subsetneq \mathcal{F}_{\text{smaller}}$, a contradiction since $\mathcal{F}_{\text{smaller}}$ must equal itself. Thus, our assumption that there existed a strictly smaller σ-algebra $\mathcal{F}_{\text{smaller}}$ containing $\mathcal{A}$ must have been incorrect, and hence $\bigcap_{\mathcal{F} \in S} \mathcal{F}$ is indeed the smallest σ-algebra on Ω containing $\mathcal{A}$.

Problem 2

Fix the sample space $\Omega = [0, 1]$, the unit interval in $\mathbb{R}$.

Part (a)

Show that any σ-algebra on Ω that contains every closed interval $[a, b]$ must also contain every open set and every closed set.

Let $\mathcal{F}_{\text{closed}}$ be an σ-algebra on Ω that contains every closed interval $[a, b]$. We want to show it contains every open set and every closed set. Note that without loss of generality, we can just

Question assigned to the following page: 2

show that $\mathcal{F}_{\text{closed}}$ contains every open set. Every closed set is the complement of an open set, and σ-algebras are closed under complementation, so if $\mathcal{F}_{\text{closed}}$ contains every open set, it must also contain every closed set (or contrapositively, if it didn't contain every closed set, then under closure by complementation $\mathcal{F}_{\text{closed}}$ would also not contain a certain open set, a contradiction).

Thus, to proceed, we show that $\mathcal{F}_{\text{closed}}$ contains every open set. Let $A \subset [0,1]$ be an arbitrary open set. It suffices to show that A is the union of a countable number of sets in $\mathcal{F}_{\text{closed}}$; σ-additivity then would tell us that $A \in \mathcal{F}_{\text{closed}}$.

Specifically, we claim that A is the union of a countable number of open intervals in $\mathcal{F}_{\text{closed}}$. First, note that $\mathcal{F}_{\text{closed}}$ contains every open interval as well. Indeed, let $(a,b) \subset [0,1]$ be an arbitrary open interval. But $[0,a] \cup [b,1] \in \mathcal{F}_{\text{closed}}$ (since $\mathcal{F}_{\text{closed}}$ contains every closed interval and is closed under finite unions) and hence $([0,a] \cup [b,1])^{\complement} = (a,b) \in \mathcal{F}_{\text{closed}}$ also. To be precise, we note that intervals of the form $[0,a)$ and $(b,1]$ are open in the subspace topology on $[0,1]$. But similar logic applies; $[0,a)$ is the complement of the closed interval $[a,1]$ and $(b,1]$ is the complement of the closed interval $[0,b]$. So open intervals of this form are also in $\mathcal{F}_{\text{closed}}$.

Now, let $\mathcal{B} = \{(p,q) \mid p,q \in A \cap \mathbb{Q}, p < q, (p,q) \subset A\} \cup \{[0,q) \mid q \in A \cap \mathbb{Q}, [0,q) \subset A\} \cup \{(p,1] \mid p \in A \cap \mathbb{Q}, (p,1] \subset A\}$; that is, $\mathcal{B}$ is the set of all open (in the subspace topology relative to $[0,1]$) intervals with rational endpoints contained in A. Note that $\mathcal{B}$ is a countable set; it is a subset of all ordered pairs of rationals $(p,q) \in \mathbb{Q} \times \mathbb{Q}$, which is countable. Since $\mathcal{F}_{\text{closed}}$ contains every open interval, $\mathcal{B} \subset \mathcal{F}_{\text{closed}}$.

We claim that $A = \bigcup_{I \in \mathcal{B}} I$, i.e. A is the union of all intervals I in $\mathcal{B}$. To show set equality, we need to show the inclusion in both directions.

First, let $a \in A$. Then, since A is open, there exists $\epsilon > 0$ such that $(a-\epsilon, a+\epsilon) \subset A$, i.e. the ϵ-ball around a is contained in the set A. By the density of the rationals in the reals, there exists $q_1, q_2 \in \mathbb{Q}$ satisfying $a - \epsilon < q_1 < a < q_2 < a + \epsilon$, so $a \in (q_1, q_2) \in \mathcal{B}$. Since $a \in A$ was arbitrary, we can conclude that $A \subset \bigcup_{I \in \mathcal{B}} I$.

Next, let $b \in \bigcup_{I \in \mathcal{B}} I$. Then $b \in (p,q)$ for some rational open interval (i.e. $p,q \in \mathbb{Q}$), with the interval contained in A, by the construction of $\mathcal{B}$. That is, $b \in (p,q) \subset A$. Since $b \in \bigcup_{I \in \mathcal{B}} I$ was arbitrary, we can conclude that $\bigcup_{I \in \mathcal{B}} I \subset A$.

Since the inclusion goes both ways, we have $A = \bigcup_{I \in \mathcal{B}} I$. Since $\mathcal{B} \subset \mathcal{F}_{\text{closed}}$ is countable, the σ-additivity axiom tells us that $A = \bigcup_{I \in \mathcal{B}} I \in \mathcal{F}_{\text{closed}}$. Since A was an arbitrary open set, we can conclude that $\mathcal{F}_{\text{closed}}$ contains every open set, which as we showed above suffices to show that $\mathcal{F}_{\text{closed}}$ contains every closed set as well.

Part (b)

Deduce that the σ-algebra generated by the closed intervals is the same as that generated by all open sets.

Denote the σ-algebra generated by the closed intervals $\mathcal{F}_{\text{closed}}$, and denote the σ-algebra generated by the open sets $\mathcal{F}_{\text{open}}$.

By **Problem 1, Part (c)**, the σ-algebra generated by the closed intervals $\mathcal{F}_{\text{closed}}$ is the smallest σ-algebra containing the closed intervals. Similarly, the σ-algebra generated by all open sets $\mathcal{F}_{\text{open}}$ is the smallest σ-algebra containing all open sets.

To show the equality of these σ-algebras we need to show inclusion in both directions. First, by **Part (a)**, we know that $\mathcal{F}_{\text{closed}}$ contains all open sets. Since $\mathcal{F}_{\text{open}}$ is the smallest σ-algebra containing all open sets, we know that $\mathcal{F}_{\text{open}} \subset \mathcal{F}_{\text{closed}}$.

Questions assigned to the following page: 2 and 3

At the same time, note that the σ-algebra generated by all open sets $\mathcal{F}_{\text{open}}$ contains all closed intervals: any closed interval $[a, b] \subset [0, 1]$ is the complement of the union of the open intervals $[0, a)$ and $(b, 1]$. Since $\mathcal{F}_{\text{closed}}$ is the smallest σ-algebra containing all closed intervals, $\mathcal{F}_{\text{closed}} \subset \mathcal{F}_{\text{open}}$.

Since the inclusion goes both ways, we have $\mathcal{F}_{\text{open}} = \mathcal{F}_{\text{closed}}$, which is what we wanted to show.

Problem 3

A bag contains either a blue or yellow counter with either possibility equally likely. A yellow counter is added and the bag well shaken. A randomly chosen counter is removed from the bag and turns out to be yellow. What is the probability that the remaining counter is yellow? Invent a sample space and events to explain your answer.

First, we define our sample space:

$$\begin{aligned}\Omega &= \underbrace{\{B, Y\}}_{\text{initial counter in bag}} \times \underbrace{\{1, 2\}}_{\substack{\text{whether initial counter 1} \\ \text{or added (yellow) counter 2} \\ \text{was drawn first}}} \\ &= \{(B, 1), (B, 2), (Y, 1), (Y, 2)\}\end{aligned}$$

In this case, $(B, 1)$, for instance, would mean that the blue counter was originally in the bag, and that original blue counter was drawn first (as opposed to the added yellow counter). On the other hand, $(B, 2)$ indicates the event that the blue counter was originally in the bag but the added yellow counter was drawn first.

Note that the events represented by each coordinate of the Cartesian product are independent — the initial counter in the bag being blue or yellow does not have an effect on which counter ends up getting drawn first (the initial or the added counter). Thus, all four elementary events in our sample space are equiprobable with probability $\frac{1}{4}$.

We are asked to calculate the probability that the remaining, second counter is yellow, given that the first counter is removed from the bag and turns out to be yellow. To proceed, we first formalize the events. We define:

$$\begin{aligned}E_{\text{yellow first}} &= \{(B, 2), (Y, 1), (Y, 2)\} \\ E_{\text{yellow second}} &= \{(B, 1), (Y, 1), (Y, 2)\}\end{aligned}$$

Then, we can apply the definition of conditional probability:

$$\begin{aligned}\mathbb{P}(E_{\text{yellow second}} \mid E_{\text{yellow first}}) &= \frac{\mathbb{P}(E_{\text{yellow second}} \cap E_{\text{yellow first}})}{\mathbb{P}(E_{\text{yellow first}})} \\ &= \frac{\mathbb{P}(\{(Y, 1), (Y, 2)\})}{\frac{3}{4}} \\ &= \frac{\frac{2}{4}}{\frac{3}{4}}\end{aligned}$$

Questions assigned to the following page: 3 and 4

$$= \frac{2}{3}$$

Given that the first counter drawn from the bag is yellow, the probability that the remaining counter is yellow is $\boxed{\frac{2}{3}}$.

Problem 4

A magnetic tape storing information in binary form has been corrupted, so it can no longer be read reliably. The probability that you correctly detect a 0 is 0.9, while the probability that you correctly detect a 1 is 0.85. Each digit is a 1 or a 0 with equal probability. Given that you read a 1, what is the probability that this is a correct reading?

We define our sample space as follows:

$$\begin{aligned} \Omega &= \underbrace{\{0,1\}}_{\text{detected digit}} \times \underbrace{\{0,1\}}_{\text{true digit}} \\ &= \{(0,0),(0,1),(1,0),(1,1)\} \end{aligned}$$

Next, we formalize some events:

$$\begin{aligned} E_0 &= \{0,1\} \times \{0\} && \text{true digit is 0} \\ E_1 &= \{0,1\} \times \{1\} && \text{true digit is 1} \\ E_{\text{correct}} &= \{(0,0),(1,1)\} && \text{correct reading} \\ E_{\text{correct}}^{\complement} = E_{\text{incorrect}} &= \{(0,1),(1,0)\} && \text{incorrect reading} \\ E_{\text{read 1}} &= \{1\} \times \{0,1\} = \{(1,0),(1,1)\} && \text{read 1} \end{aligned}$$

We can now express the probabilities given in the problem in terms of the events:

$$\begin{aligned} \mathbb{P}(E_{\text{correct}} \mid E_0) &= 0.9 \\ \mathbb{P}(E_{\text{correct}} \mid E_1) &= 0.85 \\ \mathbb{P}(E_0) = \mathbb{P}(E_1) &= 0.5 \end{aligned}$$

We can derive the probabilities of some elementary events by applying the definition of conditional probability to the expressions above and substituting known probabilities:

$$\begin{aligned} 0.9 = \mathbb{P}(E_{\text{correct}} \mid E_0) &= \frac{\mathbb{P}(E_{\text{correct}} \cap E_0)}{\mathbb{P}(E_0)} = \frac{\mathbb{P}(\{(0,0)\})}{0.5} \\ \mathbb{P}(\{(0,0)\}) &= 0.9 \cdot 0.5 = 0.45 \end{aligned}$$

Question assigned to the following page: 4

$$0.85 = \mathbb{P}(E_{\text{correct}} \mid E_1) = \frac{\mathbb{P}(E_{\text{correct}} \cap E_1)}{\mathbb{P}(E_1)} = \frac{\mathbb{P}(\{(1,1)\})}{0.5}$$
$$\mathbb{P}(\{(1,1)\}) = 0.85 \cdot 0.5 = 0.425$$

We want to calculate $\mathbb{P}(E_{\text{correct}} \mid E_{\text{read 1}})$. We can apply the definition of conditional probability to get

$$\begin{aligned}
\mathbb{P}(E_{\text{correct}} \mid E_{\text{read 1}}) &= \frac{\mathbb{P}(E_{\text{correct}} \cap E_{\text{read 1}})}{\mathbb{P}(E_{\text{read 1}})} \\
&= \frac{\mathbb{P}(\{(0,0),(1,1)\} \cap \{(1,0),(1,1)\})}{\mathbb{P}(\{(1,0),(1,1)\})} \\
&= \frac{\mathbb{P}(\{(1,1)\})}{\mathbb{P}(\{(1,0),(1,1)\})} \\
&= \frac{\mathbb{P}(\{(1,1)\})}{\mathbb{P}(\{(1,0)\}) + \mathbb{P}(\{(1,1)\})} && \text{Events are disjoint}
\end{aligned}$$

We know $\mathbb{P}(\{(1,1)\}$ from our work above, so we just need to find $\mathbb{P}(\{(1,0)\})$. Note that this corresponds exactly to the event where the detected digit is 1 but the true digit is 0. We can figure out this probability by taking the complement of some known probabilities.

We know from the problem that the probability that the true digit is 0 when the detected digit is 0 (i.e. a correct reading of 0) is $\mathbb{P}(E_{\text{correct}} \mid E_0) = 0.9$. Recall from lecture that $\mathbb{Q} = \mathbb{P}(\cdot \mid E_0)$ also defines a probability measure. Since $E_{\text{correct}}^{\complement} = E_{\text{incorrect}}$, by the axioms of a probability measure we know that $\mathbb{P}(E_{\text{incorrect}} \mid E_0) = \mathbb{Q}(E_{\text{incorrect}}) = 1 - \mathbb{Q}(E_{\text{correct}}) = 1 - \mathbb{P}(E_{\text{correct}} \mid E_0) = 1 - 0.9 = 0.1$. Thus, we have

$$\begin{aligned}
0.1 = \mathbb{P}(E_{\text{incorrect}} \mid E_0) &= \frac{\mathbb{P}(E_{\text{incorrect}} \cap E_0)}{\mathbb{P}(E_0)} \\
&= \frac{\mathbb{P}(\{(0,1),(1,0)\} \cap E_0)}{\mathbb{P}(E_0)} \\
&= \frac{\mathbb{P}(\{(1,0)\}}{0.5} \\
\mathbb{P}(\{(1,0)\} &= 0.5 \cdot 0.1 = 0.05
\end{aligned}$$

Plugging this into the equation we have above, we get

$$\begin{aligned}
\mathbb{P}(E_{\text{correct}} \mid E_{\text{read 1}}) &= \frac{\mathbb{P}(\{(1,1)\})}{\mathbb{P}(\{(1,0)\}) + \mathbb{P}(\{(1,1)\})} \\
&= \frac{0.425}{0.05 + 0.425} \\
&= \frac{0.425}{0.475} \\
&\approx 0.895
\end{aligned}$$

Questions assigned to the following page: 4 and 5

In other words, given that we read a 1, the probability that this is actually a correct reading is approximately $\boxed{0.895}$.

Problem 5

A crime is committed by a single person on an island with population 50,000. The implement used is only owned by 100 people on the island; we know one of these people is the criminal.

Part (a)

Given that someone is not the criminal, what is the probability that they own the implement?

First, we define our sample space:

$$\Omega = \{P_1, P_2, ..., P_{50,000}\} \text{ where each } P_i \text{ for } 1 \leq i \leq 50,000 \text{ denotes a unique person}$$

Denote the person who is the criminal $P_c \in \Omega$. Now, we define our events:

$$\begin{aligned} E_{\text{criminal}} &= \{P_c\} \\ E_{\text{owns implement}} &= \{P_i \mid 1 \leq i \leq 50,000, P_i \text{ owns implement}\} \\ E_{\text{not criminal}} &= E^{\complement}_{\text{criminal}} = \omega \setminus E_{\text{criminal}} = \omega \setminus \{P_c\} \end{aligned}$$

The problem tells us that $E_{\text{criminal}} \subset E_{\text{owns implement}}$ and $|E_{\text{owns implement}}| = 100$.

We want to calculate the probability that someone owns the implement, given that they are not the criminal. Symbolically, we want to calculate $\mathbb{P}(E_{\text{owns implement}} \mid E_{\text{not criminal}})$. By the definition of conditional probability, we have

$$\begin{aligned} \mathbb{P}(E_{\text{owns implement}} \mid E_{\text{not criminal}}) &= \frac{\mathbb{P}(E_{\text{owns implement}} \cap E_{\text{not criminal}})}{\mathbb{P}(E_{\text{not criminal}})} \\ &= \frac{\frac{99}{50000}}{\frac{49999}{50000}} && \text{Of the 100 people who own the implement,} \\ & && \text{99 are not the criminal} \\ &= \frac{99}{49999} \end{aligned}$$

Part (b)

Given that someone owns the implement, what is the probability that they are the criminal?

Essentially, we want to calculate $\mathbb{P}(E_{\text{criminal}} \mid E_{\text{owns implement}})$. Applying the definition of conditional probability, we have

$$\mathbb{P}(E_{\text{criminal}} \mid E_{\text{owns implement}}) = \frac{\mathbb{P}(E_{\text{criminal}} \cap E_{\text{owns implement}})}{\mathbb{P}(E_{\text{owns implement}})}$$

Questions assigned to the following page: 5 and 6

$$
\begin{aligned}
&= \frac{\mathbb{P}(E_{\text{criminal}})}{\mathbb{P}(E_{\text{owns implement}})} && \text{Since } E_{\text{criminal}} \subset E_{\text{owns implement}} \\
&= \frac{\frac{1}{50000}}{\frac{100}{50000}} \\
&= \frac{1}{100}
\end{aligned}
$$

Part (c)

Which probability should be presented to a jury?

It depends if I'm the prosecution or the defense.

Problem 6

Consider a university with Arts and Engineering schools (A and E). We wish to model whether Assistant Professors are promoted to tenure (which we label T) or not (written N) and whether this depends on their gender (M or F). Using the letters A, E, T, N, M, F to represent these events, we find the following proportions:

M	A	E
T	1/16	7/32
N	3/32	1/8

F	A	E
T	1/8	1/8
N	3/16	1/16

The data is made up, but exemplifies certain observed phenomena.

Part (a)

Write the chances that a male/female is promoted, across the university as a whole, as conditional probabilities. Compute them!

$$
\begin{aligned}
\mathbb{P}(T \mid M) &= \frac{\mathbb{P}(T \cap M)}{\mathbb{P}(M)} \\
&= \frac{\frac{1}{16} + \frac{7}{32}}{\frac{1}{16} + \frac{7}{32} + \frac{3}{32} + \frac{1}{8}} \\
&= \frac{\frac{9}{32}}{\frac{16}{32}} \\
&= \frac{9}{16} && \text{Chance male is promoted} \\
\mathbb{P}(T \mid F) &= \frac{\mathbb{P}(T \cap F)}{\mathbb{P}(F)} \\
&= \frac{\frac{1}{8} + \frac{1}{8}}{\frac{1}{8} + \frac{1}{8} + \frac{3}{16} \frac{1}{16}}
\end{aligned}
$$

Question assigned to the following page: 6

$$= \frac{\frac{8}{32}}{\frac{16}{32}}$$
$$= \frac{8}{16}$$
$$= \frac{1}{2} \qquad \text{Chance female is promoted}$$

Part (b)

Do the same for the probabilities that males/females in A are promoted, as well as in E.

$$\begin{aligned}
\mathbb{P}(T \mid M \cap A) &= \frac{\mathbb{P}(T \cap (M \cap A))}{\mathbb{P}(M \cap A)} \\
&= \frac{\frac{1}{16}}{\frac{1}{16} + \frac{3}{32}} \\
&= \frac{\frac{2}{32}}{\frac{5}{32}} \\
&= \frac{2}{5} && \text{Probability of promotion, given male in } A \\
\mathbb{P}(T \mid F \cap A) &= \frac{\mathbb{P}(T \cap (F \cap A))}{\mathbb{P}(F \cap A)} \\
&= \frac{\frac{1}{8}}{\frac{1}{8} + \frac{3}{16}} \\
&= \frac{\frac{2}{16}}{\frac{5}{16}} \\
&= \frac{2}{5} && \text{Probability of promotion, given female in } A \\
\mathbb{P}(T \mid M \cap E) &= \frac{\mathbb{P}(T \cap (M \cap E))}{\mathbb{P}(M \cap E)} \\
&= \frac{\frac{7}{32}}{\frac{7}{32} + \frac{1}{8}} \\
&= \frac{\frac{7}{32}}{\frac{11}{32}} \\
&= \frac{7}{11} && \text{Probability of promotion, given male in } E \\
\mathbb{P}(T \mid F \cap E) &= \frac{\mathbb{P}(T \cap (F \cap E))}{\mathbb{P}(F \cap E)} \\
&= \frac{\frac{1}{8}}{\frac{1}{8} + \frac{1}{16}} \\
&= \frac{\frac{2}{16}}{\frac{3}{16}}
\end{aligned}$$

Question assigned to the following page: 6

$$= \frac{2}{3} \qquad \text{Probability of promotion, given female in } E$$

Part (c)

What aspect of your answers constitutes Simpson's Paradox?

In **Part (a)**, we see that across the university, male assistant professors are slightly more likely to be promoted than female assistant professors ($\frac{9}{16} > \frac{1}{2}$).

However, in **Part (b)**, we see that when we look at probability of promotion within each school (Arts and Engineering), female assistant professors are just as likely or slightly more likely to be promoted than male assistant professors ($\frac{2}{5} \geq \frac{2}{5}$ and $\frac{2}{3} \geq \frac{7}{11}$).

This is Simpson's paradox: a trend which appears in two different groups of data (Arts and Engineering) reverses when we look at the combined data.

Part (d)

Determine $\mathbb{P}(A \mid F)$ and $\mathbb{P}(F \mid A)$.

We apply the definition of conditional probability:

$$\begin{aligned}
\mathbb{P}(A \mid F) &= \frac{\mathbb{P}(A \cap F)}{\mathbb{P}(F)} \\
&= \frac{\frac{1}{8} + \frac{3}{16}}{\frac{1}{8} + \frac{3}{16} + \frac{1}{8} + \frac{1}{16}} \\
&= \frac{\frac{5}{16}}{\frac{8}{16}} \\
&= \frac{5}{8} \\
\mathbb{P}(F \mid A) &= \frac{\mathbb{P}(F \cap A)}{\mathbb{P}(A)} \\
&= \frac{\frac{1}{8} + \frac{3}{16}}{\frac{1}{8} + \frac{3}{16} + \frac{1}{16} + \frac{3}{32}} \\
&= \frac{\frac{5}{16}}{\frac{10}{16}} \\
&= \frac{5}{10} \\
&= \frac{1}{2}
\end{aligned}$$

Part (e)

How does one combine the in-school probabilities that a female is promoted to find the university probability?

Question assigned to the following page: 6

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MATH 170A Homework 2
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Let $\mathbb{Q}(\cdot) = \mathbb{P}(\cdot \mid F)$. Recall from lecture that this defines a probability measure. Then, $\mathbb{Q}(T)$ represents the probability that a female is promoted. We can apply the **Partition Theorem (1.48)** on this measure $\mathbb{Q}$ to determine the university probability from the in-school probability.

First, we note that A and E are disjoint (for simplicity, we assume no joint appointments). Moreover, since $A \cup E = \Omega$, A and E in fact form a partition of F. Applying the Partition Theorem, we have

$$
\begin{aligned}
\mathbb{Q}(T) &= \mathbb{Q}(T \mid E)\mathbb{Q}(E) + \mathbb{Q}(T \mid A)\mathbb{Q}(A) \\
&= \frac{\mathbb{Q}(T \cap E)}{\mathbb{Q}(E)} \cdot \mathbb{Q}(E) + \frac{\mathbb{Q}(T \cap A)}{\mathbb{Q}(A)} \cdot \mathbb{Q}(A) && \text{Definition of conditional probability} \\
&= \frac{\mathbb{P}(T \cap E \mid F)}{\mathbb{P}(E \mid F)} \cdot \mathbb{P}(E \mid F) + \frac{\mathbb{P}(T \cap A \mid F)}{\mathbb{P}(A \mid F)} \cdot \mathbb{P}(A \mid F) && \text{Definition of } \mathbb{Q} \\
&= \frac{\frac{\mathbb{P}((T\cap E)\cap F)}{\mathbb{P}(F)}}{\frac{\mathbb{P}(E\cap F)}{\mathbb{P}(F)}} \cdot \mathbb{P}(E \mid F) + \frac{\frac{\mathbb{P}((T\cap A)\cap F)}{\mathbb{P}(F)}}{\frac{\mathbb{P}(A\cap F)}{\mathbb{P}(F)}} \cdot \mathbb{P}(A \mid F) \\
&= \frac{\frac{\mathbb{P}(T\cap (E\cap F))}{\mathbb{P}(F)}}{\frac{\mathbb{P}(E\cap F)}{\mathbb{P}(F)}} \cdot \mathbb{P}(E \mid F) + \frac{\frac{\mathbb{P}(T\cap (A\cap F))}{\mathbb{P}(F)}}{\frac{\mathbb{P}(A\cap F)}{\mathbb{P}(F)}} \cdot \mathbb{P}(A \mid F) && \text{Set intersection is associative} \\
&= \frac{\mathbb{P}(T \cap (E \cap F))}{\mathbb{P}(E \cap F)} \cdot \mathbb{P}(E \mid F) + \frac{\mathbb{P}(T \cap (A \cap F))}{\mathbb{P}(A \cap F)} \cdot \mathbb{P}(A \mid F) && \text{Cancel common denominator} \\
&= \mathbb{P}(T \mid E \cap F) \cdot \mathbb{P}(E \mid F) + \mathbb{P}(T \mid A \cap F) \cdot \mathbb{P}(A \mid F) && \text{Definition of conditional probability} \\
&= \frac{2}{3} \cdot \mathbb{P}(E \mid F) + \frac{2}{5} \cdot \mathbb{P}(A \mid F) && \text{From \textbf{Part (b)}}
\end{aligned}
$$

In **Part (d)**, we calculated $\mathbb{P}(A \mid F) = \frac{5}{8}$. In a similar way, we can calculate $\mathbb{P}(E \mid F)$:

$$
\begin{aligned}
\mathbb{P}(E \mid F) &= \frac{\mathbb{P}(E \cap F)}{\mathbb{P}(F)} \\
&= \frac{\frac{1}{8} + \frac{1}{16}}{\frac{1}{8} + \frac{3}{16} + \frac{1}{8} + \frac{1}{16}} \\
&= \frac{\frac{3}{16}}{\frac{8}{16}} \\
&= \frac{3}{8}
\end{aligned}
$$

Thus, we get the school-wide probability

$$
\begin{aligned}
\mathbb{P}(T \mid F) &= \mathbb{Q}(T) \\
&= \frac{2}{3} \cdot \frac{3}{8} + \frac{2}{5} \cdot \frac{5}{8}
\end{aligned}
$$

Question assigned to the following page: 6

$$
\begin{aligned}
&= \frac{2}{8} + \frac{2}{8} \\
&= \frac{4}{8} \\
&= \frac{1}{2}
\end{aligned}
$$

which is the same as what we got in **Part (a)**. In general, we can use the Partition Theorem to determine a combined probability from group-specific probabilities.

Part (f)

Explain, via a simple example, how the following is possible in a single survey: In my respondents, Group A reported a higher average happiness score than Group B. Moreover, the same was found among your respondents. Nevertheless, when we pooled our data, Group B had the higher average score.

Consider the following survey results (average happiness measured from 1 to 10):

A	B
10	9
	9

My respondents

A	B
2	1
2	

Your respondents

Note that the average happiness of Group A is higher than Group B in both surveys: $10 > 9$ and $2 > 1$. However, consider the combined data:

A	B
10	9
2	9
2	1

Combined respondents

In the combined data, the average happiness of Group A is $(10 + 2 + 2)/3 = 14/3 \approx 4.667$. On the other hand, the average happiness of Group B is $(9 + 9 + 1) = 19/3 \approx 6.333$, and $6.333 > 4.667$.

The reason this is possible (if we ignore the fact that this is completely artificial data generated to prove a point) is because there are probably two underlying types of people in the population — naturally happier people and naturally sadder people. For some reason, my survey only surveyed the naturally happier people, and your survey only surveyed the naturally sadder people. Moreover, there are more of Group B that are naturally happy than there are of Group A (could be a characteristic of the group, or could just be an artifact of the survey). In any case, if we weight the averages equally (what we might intuitively do), we underweight the happy Group B people in my survey. Once we weight the happy Group B people proportionately (in the combined data), their contribution to Group B's happiness is accounted for and we see the higher average score among Group B.