(1) Suppose $f(x)$ is a continuous function as well as a pdf. We also assume that there is some number $M > 0$ so that

$$0 \le f(x) \le M \text{ for all } x \in \mathbb{R} \quad \text{ and } f(x) = 0 \text{ for all } |x| \ge M.$$

Let $U_n \sim \text{Uniform}([-M, M])$ and $V_n \sim \text{Uniform}([0, M])$, $n \in \mathbb{N}$, all be independent. Setting $n = 1$ initially, we perform the following: If $V_n \le f(U_n)$ we return the value U_n and stop; if not we increase n by one and repeat.

(a) What is the distribution of the number of iterations n that we perform? With what parameter?

(b) What is the pdf of the returned value U_n?

Remark: This is a simple example of a rejection sampling algorithm.

(2) Suppose X_1, X_2, and X_3 are independent and uniformly distributed on $[0, 1]$.

(a) What is the probability of the event $X_1 \le X_2 \le X_3$?

(b) What is the joint pdf of X_1 and X_2 conditioned on the event $X_1 \le X_2$?

(c) What is the joint pdf of X_1, X_2, and X_3 conditioned on $X_1 \le X_2 \le X_3$?

(d) Find the PDF for the median of these three random variables.

(3) Suppose $X \sim \text{Exponential}(\lambda = 1)$ and the law of Y is then defined via

$$f_{Y|X}(y|x) = xe^{-xy} 1_{[0,\infty)}(y).$$

We may say that $Y \sim \text{Exponential}(x)$, conditioned on $X = x$.

(a) What is the joint pdf of X and Y?

(b) What is the marginal law of Y?

(c) Find $\mathbb{E}(\frac{1}{1+Y})$ in two ways: via part (b) and via the law of total expectation.

(d) What is the law of X conditioned on $Y = y$?

(e) What is $\mathbb{E}(X|Y = y)$?

(4) Let $T_1, T_2, T_3 \sim \text{Exponential}(\lambda)$ be independent and let

$$Y_1 = T_1, \quad Y_2 = T_1 + T_2, \quad \text{and} \quad Y_3 = T_1 + T_2 + T_3.$$

(a) Determine the pdf of Y_2 given $Y_1 = y_1$.

(b) Determine the pdf of Y_2 given $Y_1 = y_1$ and $Y_2 = y_2$, written $f_{Y_3|Y_2,Y_1}(y_3|y_2, y_1)$.

(c) Deduce the joint pdf of Y_1, Y_2, and Y_3.

(d) Determine $f_{Y_1,Y_2|Y_3}(y_1, y_2|y_3)$.

(5) Given a vector $\vec{\mu} \in \mathbb{R}^2$ and a 2×2 positive definite matrix Σ, we say that $(X_1, X_2) \sim N(\vec{\mu}, \Sigma)$ if they have joint pdf

$$f_{X_1,X_2}(\vec{x}) = [\det(2\pi\Sigma)]^{-1/2} \exp\bigl\{-\tfrac{1}{2}(\vec{x} - \vec{\mu}) \cdot \Sigma^{-1}(\vec{x} - \vec{\mu})\bigr\}.$$

(a) For a fixed vector $\vec{a}$, find the mean and variance of the random variable

$$Z = \vec{a} \cdot \vec{X} = a_1X_1 + a_2X_2.$$

(b) Show that Z is normally distributed