170A Killip Homework 1 Due Fri, Jan 19

(1) Fix a sample space Ω.
(a) Show the intersection of any collection of σ-algebras is a σ-algebra.
(b) What is the intersection of all σ-algebras on Ω.
(c) Show that for any collection $\mathcal{A}$ of subsets of Ω, there is a smallest σ-algebra containing $\mathcal{A}$. This called the *σ-algebra generated by $\mathcal{A}$* and denoted $\sigma(\mathcal{A})$.

(2) Fix the sample space $\Omega = [0, 1]$, the unit interval in $\mathbb{R}$.
(a) Show that any σ-algebra on Ω that contains every closed interval $[a, b]$ must also contain every open set and every closed set.
(b) Deduce that the σ-algebra generated by the closed intervals is the same as that generated by all open sets.

Remark: When Ω has a metric and/or topology (and so a notion of an open set), we call the σ-algebra generated by the collection of open sets the *Borel σ-algebra*.

(3) A bag contains either a blue or yellow counter with either possibility equally likely. A yellow counter is added and the bag well shaken. A randomly chosen counter is removed from the bag and turns out to be yellow. What is the probability that the remaining counter is yellow? Invent a sample space and events to explain your answer.

(4) A magnetic tape storing information in binary form has been corrupted, so it can no longer be read reliably. The probability that you correctly detect a 0 is 0.9, while the probability that you correctly detect a 1 is 0.85. Each digit is a 1 or a 0 with equal probability. Given that you read a 1, what is the probability that this is a correct reading?

(5) A crime is committed by a single person on an island with population 50 000. The implement used is only owned by 100 people on the island; we know one of these people is the criminal.
(a) Given that someone is not the criminal, what is the probability that they own the implement?
(b) Given that someone owns the implement, what is the probability that they are the criminal?
(c) (For private consideration) Which probability should be presented to a jury?

(6) Consider a university with Arts and Engineering schools (A and E). We wish to model whether Assistant Professors are promoted to tenure (which we label T) or not (written N) and whether this depends on their gender (M or F). Using the letters A, E, T, N, M, F to represent these events, we find the following proportions:

M	A	E
T	1/16	7/32
N	3/32	1/8

F	A	E
T	1/8	1/8
N	3/16	1/16

This data is made up, but exemplifies certain observed phenomena.
(a) Write the chances that a male/female is promoted, across the university as a whole, as conditional probabilities. Compute them!
(b) Do the same for the probabilities that males/females in A are promoted, as well as, in E.
(c) What aspect of your answers constitutes Simpson's Paradox?
(d) Determine $\mathbb{P}(A|F)$ and $\mathbb{P}(F|A)$.
(e) How does one combine the in-school probabilities that a female is promoted to find the university probability? Provide a general formula!
(f) Explain, via a simple* example, how the following is possible in a single survey: In my respondents, Group A reported a higher average happiness score than Group B. Moreover, the same was found among your respondents. Nevertheless, when we pooled our data, Group B had the higher average score.

Moral: You have to be careful about averaging averages!

*I claim five total respondents are enough, but maybe it is easier to explain the idea with six. You don't need to make an minimal example, but do try to make it simple!