Homework 5

Graded

Student

NATHAN LEUNG

Total Points

30 / 30 pts

Question 1

1 — **5** / 5 pts

✔ **− 0 pts** Correct

− 1 pt Minor issue(s)

− 3 pts Major issue(s)

− 5 pts Completely incorrect

Question 2

2 — **5** / 5 pts

✔ **− 0 pts** Correct

− 1 pt Minor issue(s)

− 3 pts Major issue(s)

− 5 pts Completely incorrect

Question 3

3 — **5** / 5 pts

✔ **− 0 pts** Correct

− 1 pt Minor issue(s)

− 3 pts Major issue(s)

− 5 pts Completely incorrect

1 This is much more detail than is needed. You can just rearrange the sums and adjust the bounds of summation accordingly.

Question 4

4

5 / 5 pts

✔ **– 0 pts** Correct

– 1 pt Minor issue(s)

– 3 pts Major issue(s)

– 5 pts Completely incorrect

Question 5

5

5 / 5 pts

✔ **– 0 pts** Correct

– 1 pt Minor issue(s)

– 3 pts Major issue(s)

– 5 pts Completely incorrect

Question 6

6

5 / 5 pts

✔ **– 0 pts** Correct

– 1 pt Minor issue(s)

– 3 pts Major issue(s)

– 5 pts Completely incorrect

Question assigned to the following page: 1

MATH 170A Homework 5

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9 February 2024

Problem 1

Part (a)

Suppose $X \sim \text{Poisson}(\lambda)$. By repeatedly differentiating the power series for $\exp(\lambda)$, compute

$$\mathbb{E}\{X(X-1)(X-2)\cdots(X+1-\ell)\}$$

for each $\ell = 1, 2, 3, ...$

We can write $\exp(\lambda) = 1 + \lambda + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \cdots$. By the definition of the Poisson distribution, we also have

$$p_X(k) = \begin{cases} \frac{\lambda^k}{k!}e^{-\lambda} & \text{when } k \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

We want to calculate

$$\begin{aligned} \mathbb{E}\{X(X-1)(X-2)\cdots(X+1-\ell)\} &= \sum_{k=0}^{\infty} k(k-1)(k-2)\cdots(k+1-\ell)p_X(k) \\ &= \sum_{k=0}^{\infty} k(k-1)(k-2)\cdots(k+1-\ell)\left(\frac{\lambda^k}{k!}e^{-\lambda}\right) \end{aligned}$$

Note the following pattern:

$$\begin{aligned} \exp(\lambda) &= 1 + \lambda + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \cdots \\ &= \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \\ \frac{d}{d\lambda}\exp(\lambda) &= \sum_{k=0}^{\infty} k \cdot \frac{\lambda^{k-1}}{k!} && \text{Power rule} \\ \frac{d^2}{d\lambda^2}\exp(\lambda) &= \sum_{k=0}^{\infty} k(k-1) \cdot \frac{\lambda^{k-2}}{k!} \end{aligned}$$

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$$\vdots$$
$$\frac{d^\ell}{d\lambda^\ell}\exp(\lambda) = \sum_{k=0}^{\infty} k(k-1)\cdots(k+1-\ell)\cdot\frac{\lambda^{k-\ell}}{k!}$$

Thus we can write

$$\begin{aligned}
\lambda^\ell \cdot \left(\frac{d^\ell}{d\lambda^\ell}\exp(\lambda)\right) &= \lambda^\ell \cdot \left(\sum_{k=0}^{\infty} k(k-1)\cdots(k+1-\ell)\cdot\frac{\lambda^{k-\ell}}{k!}\right) \\
\lambda^\ell \exp(\lambda) &= \sum_{k=0}^{\infty} k(k-1)\cdots(k+1-\ell)\cdot\frac{\lambda^{k}}{k!} && \text{Since } \frac{d}{d\lambda}\exp(\lambda) = \exp(\lambda)
\end{aligned}$$

We can substitute the above into the expectation we are trying to calculate:

$$\begin{aligned}
&\mathbb{E}\{X(X-1)(X-2)\cdots(X+1-\ell)\} \\
&\qquad = \sum_{k=0}^{\infty} k(k-1)(k-2)\cdots(k+1-\ell)\left(\frac{\lambda^k}{k!}e^{-\lambda}\right) \\
&\qquad = e^{-\lambda}\sum_{k=0}^{\infty} k(k-1)(k-2)\cdots(k+1-\ell)\cdot\frac{\lambda^k}{k!} \\
&\qquad = e^{-\lambda}\left(\lambda^\ell \exp(\lambda)\right) \\
&\qquad = e^{-\lambda}e^{\lambda}\lambda^\ell && \text{Definition of exp} \\
&\qquad = \boxed{\lambda^\ell}
\end{aligned}$$

Part (b)

By a parallel method evaluate such expectations when $X \sim \text{Binomial}(n, p)$.

By the definition of the Binomial distribution, we have

$$p_X(k) = \begin{cases} \binom{n}{k} p^k (1-p)^{n-k} & \text{when } 0 \le k \le n \\ 0 & \text{otherwise} \end{cases}$$

We want to calculate

$$\begin{aligned}
\mathbb{E}\{X(X-1)(X-2)\cdots(X+1-\ell)\} &= \sum_{k=0}^{n} k(k-1)(k-2)\cdots(k+1-\ell)p_X(k) \\
&= \sum_{k=0}^{n} k(k-1)(k-2)\cdots(k+1-\ell)\binom{n}{k}p^k(1-p)^{n-k}
\end{aligned}$$

Consider the following polynomial equalities:

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$$
\begin{aligned}
(x+y)^n &= \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} && \text{Binomial theorem} \\
n(x+y)^{n-1} &= \sum_{k=0}^{n} \binom{n}{k} k x^{k-1} y^{n-k} && \frac{d}{dx} \text{ on both sides} \\
n(n-1)(x+y)^{n-2} &= \sum_{k=0}^{n} \binom{n}{k} k(k-1) x^{k-2} y^{n-k} && \frac{d}{dx} \text{ on both sides} \\
&\vdots \\
n(n-1)\cdots(n+1-\ell)(x+y)^{n-\ell} &= \sum_{k=0}^{n} \binom{n}{k} k(k-1)\cdots(k+1-\ell) x^{k-\ell} y^{n-k}
\end{aligned}
$$

Multiplying both sides of the last equation above by x^ℓ gives us the equality

$$
x^\ell \cdot n(n-1)\cdots(n+1-\ell)(x+y)^{n-\ell} = \sum_{k=0}^{n} \binom{n}{k} k(k-1)\cdots(k+1-\ell) x^k y^{n-k}
$$

Plugging in $x = p$ and $y = 1-p$ yields

$$
\begin{aligned}
p^\ell \cdot n(n-1)\cdots(n+1-\ell)(p+(1-p))^{n-\ell} &= \sum_{k=0}^{n} \binom{n}{k} k(k-1)\cdots(k+1-\ell) p^k (1-p)^{n-k} \\
p^\ell \cdot n(n-1)\cdots(n+1-\ell)1^{n-\ell} &= \sum_{k=0}^{n} \binom{n}{k} k(k-1)\cdots(k+1-\ell) p^k (1-p)^{n-k} \\
p^\ell \cdot n(n-1)\cdots(n+1-\ell) &= \sum_{k=0}^{n} \binom{n}{k} k(k-1)\cdots(k+1-\ell) p^k (1-p)^{n-k}
\end{aligned}
$$

This yields something we can substitute into our original calculation for the expectation:

$$
\begin{aligned}
\mathbb{E}\{X(X-1)(X-2)\cdots(X+1-\ell)\} &= \sum_{k=0}^{n} k(k-1)(k-2)\cdots(k+1-\ell)\binom{n}{k} p^k (1-p)^{n-k} \\
&= p^\ell \cdot n(n-1)\cdots(n+1-\ell) \\
&= \boxed{p^\ell \cdot \frac{n!}{(n-\ell)!}}
\end{aligned}
$$

Note that the above expression is only valid for $0 \le \ell \le n$; if $\ell > n$, then one of factors k, $k-1$, $k-2$, ..., $k+1-\ell$ in the summands will always be zero for $0 \le k \le n$. Thus, each summand in the sum for the expectation will be 0 and hence the overall expectation will be 0 also.

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Part (c)

If $n \to \infty$ with $p = \frac{\lambda}{n}$, show that your **Part (b)** answer converges to that for **Part (a)**.

In **Part (b)**, we saw that $\mathbb{E}\{X(X-1)(X-2)\cdots(X+1-\ell)\} = p^\ell \cdot \frac{n!}{(n-\ell)!}$. Computing the limit, we have

$$\begin{aligned}
&\lim_{n\to\infty} \left(\mathbb{E}\left\{X(X-1)(X-2)\cdots(X+1-\ell)\right\}\right) \\
&\quad= \lim_{n\to\infty}\left(p^\ell \cdot \frac{n!}{(n-\ell)!}\right) \\
&\quad= \lim_{n\to\infty}\left(\left(\frac{\lambda}{n}\right)^\ell \cdot \frac{n!}{(n-\ell)!}\right) && \text{Definition of } \lambda \\
&\quad= \lim_{n\to\infty}\left(\frac{\lambda^\ell}{n^\ell} \cdot \frac{n!}{(n-\ell)!}\right) \\
&\quad= \lim_{n\to\infty}\left(\frac{\lambda^\ell}{n^\ell} \cdot \frac{n(n-1)\cdots(2)(1)}{(n-\ell)(n-\ell-1)\cdots(2)(1)}\right) \\
&\quad= \lim_{n\to\infty}\left(\frac{\lambda^\ell}{n^\ell} \cdot n(n-1)\cdots(n-\ell+2)(n-\ell+1)\right) \\
&\quad= \lim_{n\to\infty}\left(\lambda^\ell \cdot \frac{n(n-1)\cdots(n-\ell+2)(n-\ell+1)}{n^\ell}\right) \\
&\quad= \lim_{n\to\infty}\left(\lambda^\ell \cdot \frac{\Pi_{k=0}^{\ell-1}(n-k)}{n^\ell}\right) \\
&\quad= \left(\lim_{n\to\infty}\lambda^\ell\right) \cdot \lim_{n\to\infty}\left(\frac{\Pi_{k=0}^{\ell-1}(n-k)}{n^\ell}\right) && \text{Algebraic limit theorem} \\
&\quad= \left(\lim_{n\to\infty}\lambda^\ell\right) \cdot 1 && \text{Limit of quotient of equal-degree polynomials} \\
& && \text{is limit of ratio of leading coefficients} \\
&\quad= \left(\lim_{n\to\infty}\lambda^\ell\right) \\
&\quad= \lambda^\ell && \text{Assuming } \lambda \text{ is independent of } n \text{ after} \\
& && \text{making the substitution } p = \frac{\lambda}{n}
\end{aligned}$$

which is what we wanted to show, since λ^ℓ is the answer we got in **Part (a)**.

Problem 2

Part (a)

Determine the mean and variance of the following random variables:

Poisson(λ)   Binomial(n, p)   Geometric(p)

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Poisson(λ)

Let $X \sim \text{Poisson}(\lambda)$. Then for expectation, we have

$$\begin{aligned}
\mathbb{E}[X] &= \sum_{k=0}^{\infty} k\mathbb{P}(X=k) && \text{Definition of } \mathbb{E} \\
&= \sum_{k=0}^{\infty} k \cdot \frac{\lambda^k}{k!} \cdot e^{-\lambda} && \text{Definition of Poisson}(\lambda) \\
&= \sum_{k=1}^{\infty} k \cdot \frac{\lambda^k}{k!} \cdot e^{-\lambda} && \text{First summand is 0, so we can index from 1} \\
&= \sum_{k=1}^{\infty} \frac{\lambda^k}{(k-1)!} \cdot e^{-\lambda} \\
&= \lambda e^{-\lambda} \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!} \\
&= \lambda e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} && \text{Reindex sum} \\
&= \lambda e^{-\lambda} e^{\lambda} && \text{Definition of } e^{\lambda} \\
&= \lambda
\end{aligned}$$

For variance, we have

$$\text{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2$$

We can compute $\mathbb{E}[X^2]$ separately:

$$\begin{aligned}
\mathbb{E}[X^2] &= \sum_{k=0}^{\infty} k^2 \cdot \mathbb{P}(X=k) && \text{Definition of } \mathbb{E} \\
&= \sum_{k=0}^{\infty} k^2 \cdot \frac{\lambda^k}{k!} \cdot e^{-\lambda} && \text{Definition of Poisson}(\lambda) \\
&= \sum_{k=1}^{\infty} k^2 \cdot \frac{\lambda^k}{k!} \cdot e^{-\lambda} && \text{First summand is 0, so we can index from 1} \\
&= \sum_{k=1}^{\infty} k \cdot \frac{\lambda^k}{(k-1)!} \cdot e^{-\lambda} \\
&= \lambda \cdot \sum_{k=1}^{\infty} k \cdot \frac{\lambda^{k-1}}{(k-1)!} \cdot e^{-\lambda} \\
&= \lambda \cdot \sum_{k=0}^{\infty} (k+1) \cdot \frac{\lambda^k}{k!} \cdot e^{-\lambda} && \text{Reindex}
\end{aligned}$$

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$$
\begin{aligned}
&= \lambda \cdot \sum_{k=0}^{\infty} (k+1) \cdot \mathbb{P}(X = k) \\
&= \lambda \cdot \mathbb{E}[X+1] && \text{Definition of } \mathbb{E} \\
&= \lambda \cdot (\mathbb{E}[X] + \mathbb{E}[1]) && \text{Linearity of expectation} \\
&= \lambda \cdot (\lambda + 1) && \text{From previous part} \\
&= \lambda^2 + \lambda
\end{aligned}
$$

Thus, we get

$$
\begin{aligned}
\text{Var}(X) &= \mathbb{E}[X^2] - \mathbb{E}[X]^2 \\
&= (\lambda^2 + \lambda) - \lambda^2 && \text{From above} \\
&= \lambda
\end{aligned}
$$

Binomial(n, p)

Let $X \sim \text{Binomial}(n, p)$. Then, for expectation, we have

$$
\begin{aligned}
\mathbb{E}[X] &= \sum_{k=0}^{n} k \mathbb{P}(X = k) && \text{Definition of } \mathbb{E} \\
&= \sum_{k=0}^{n} k \cdot \binom{n}{k} p^k (1-p)^{n-k} && \text{Definition of Binomial}(n, p)
\end{aligned}
$$

In **Problem 1, Part (b)**, we saw (by differentiating both sides of the Binomial Theorem) that

$$
n(x+y)^{n-1} = \sum_{k=0}^{n} \binom{n}{k} k x^{k-1} y^{n-k}
$$

If we let $x = p$ and $y = 1 - p$, we get the equality

$$
\begin{aligned}
n(p + (1-p))^{n-1} &= \sum_{k=0}^{n} \binom{n}{k} k p^{k-1} (1-p)^{n-k} \\
n(1)^{n-1} &= \sum_{k=0}^{n} \binom{n}{k} k p^{k-1} (1-p)^{n-k} \\
n &= \sum_{k=0}^{n} \binom{n}{k} k p^{k-1} (1-p)^{n-k} \\
np &= \sum_{k=0}^{n} \binom{n}{k} k p^{k} (1-p)^{n-k} && \text{Multiply both sides by p}
\end{aligned}
$$

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The last sum above is exactly $\mathbb{E}[X]$, i.e. we can substitute to conclude that $\mathbb{E}[X] = np$.
For variance, we have the formula

$$\mathrm{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2$$

We can compute $\mathbb{E}[X^2]$ separately:

$$\begin{aligned}
\mathbb{E}[X^2] &= \sum_{k=0}^{n} k^2 \cdot \mathbb{P}(X = k) && \text{Definition of } \mathbb{E} \\
&= \sum_{k=0}^{n} k^2 \cdot \binom{n}{k} p^k (1-p)^{n-k} && \text{Definition of Binomial}(n, p)
\end{aligned}$$

As we did before, we observe that in **Problem 1, Part (b)**, we saw (by repeatedly differentiating both sides of the Binomial Theorem) that

$$\begin{aligned}
n(n-1)(x+y)^{n-2} &= \sum_{k=0}^{n} \binom{n}{k} k(k-1) x^{k-2} y^{n-k} \\
&= \sum_{k=0}^{n} \binom{n}{k} (k^2 - k) x^{k-2} y^{n-k}
\end{aligned}$$

If we multiply both sides of the above by x^2, we get

$$\begin{aligned}
x^2 n(n-1)(x+y)^{n-2} &= \sum_{k=0}^{n} \binom{n}{k} (k^2 - k) x^k y^{n-k} \\
&= \left(\sum_{k=0}^{n} \binom{n}{k} k^2 x^k y^{n-k} \right) - \left(\sum_{k=0}^{n} \binom{n}{k} k x^k y^{n-k} \right)
\end{aligned}$$

If we let $x = p$ and $y = 1 - p$, we get the equality

$$\begin{aligned}
p^2 n(n-1)(p + (1-p))^{n-2} &= \left(\sum_{k=0}^{n} \binom{n}{k} k^2 p^k (1-p)^{n-k} \right) - \left(\sum_{k=0}^{n} \binom{n}{k} k p^k (1-p)^{n-k} \right) \\
p^2 n(n-1)(1)^{n-2} &= \left(\sum_{k=0}^{n} \binom{n}{k} k^2 p^k (1-p)^{n-k} \right) - \left(\sum_{k=0}^{n} \binom{n}{k} k p^k (1-p)^{n-k} \right) \\
p^2 n(n-1) &= \left(\sum_{k=0}^{n} \binom{n}{k} k^2 p^k (1-p)^{n-k} \right) - \left(\sum_{k=0}^{n} \binom{n}{k} k p^k (1-p)^{n-k} \right)
\end{aligned}$$

Rearranging the last equality yields

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$$p^2 n(n-1) + \sum_{k=0}^{n} \binom{n}{k} k p^k (1-p)^{n-k} = \sum_{k=0}^{n} \binom{n}{k} k^2 p^k (1-p)^{n-k}$$

Observe that

$$\mathbb{E}[X] = \sum_{k=0}^{n} k \binom{n}{k} p^k (1-p)^{n-k} = np$$

We can substitute this into our equation to get

$$\begin{aligned} p^2 n(n-1) + \mathbb{E}[X] &= \sum_{k=0}^{n} \binom{n}{k} k^2 p^k (1-p)^{n-k} \\ p^2 n(n-1) + np &= \sum_{k=0}^{n} \binom{n}{k} k^2 p^k (1-p)^{n-k} \end{aligned}$$

which is exactly the expression we have for $\mathbb{E}[X^2]$, i.e. $\mathbb{E}[X^2] = p^2 n(n-1) + np = n^2p^2 - np^2 + np$. Thus, we get

$$\begin{aligned} \mathrm{Var}(X) &= \mathbb{E}[X^2] - \mathbb{E}[X]^2 \\ &= (n^2p^2 - np^2 + np) - (np)^2 && \text{From above} \\ &= n^2p^2 - np^2 + np - n^2p^2 \\ &= -np^2 + np \\ &= np(-p+1) \\ &= np(1-p) \end{aligned}$$

Geometric(p)

Let $X \sim \mathrm{Geometric}(p)$. Then, for expectation, we have

$$\begin{aligned} \mathbb{E}[X] &= \sum_{k=0}^{\infty} k \mathbb{P}(X = k) && \text{Definition of } \mathbb{E} \\ &= \sum_{k=0}^{\infty} k \cdot (1-p)^{k-1} p && \text{Definition of Geometric}(p) \\ &= p \sum_{k=0}^{\infty} k \cdot (1-p)^{k-1} \end{aligned}$$

Consider the power series equality $\frac{1}{1-x} = \sum_{i=0}^{\infty} x^i$ which holds for $0 < x < 1$. If we differentiate both sides, we get

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$$\frac{1}{1-x} = \sum_{i=0}^{\infty} x^i$$
$$\frac{1}{(1-x)^2} = \sum_{i=0}^{\infty} ix^{i-1}$$

If we let $x = 1 - p$ (which is valid since $0 < p < 1$, so $-1 < -p < 0$ and hence $0 < 1 - p < 1$), we get the equality

$$\frac{1}{(1-(1-p))^2} = \sum_{i=0}^{\infty} i(1-p)^{i-1}$$
$$\frac{1}{p^2} = \sum_{i=0}^{\infty} i(1-p)^{i-1}$$

We can substitute this back into our expression for $\mathbb{E}[X]$ to get

$$\begin{aligned}\mathbb{E}[X] &= p\sum_{k=0}^{\infty} k \cdot (1-p)^{k-1} \\ &= p\left(\frac{1}{p^2}\right) \\ &= \frac{1}{p}\end{aligned}$$

For variance, we have the formula

$$\text{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2$$

We can compute $\mathbb{E}[X^2]$ separately:

$$\begin{aligned}\mathbb{E}[X^2] &= \sum_{k=0}^{\infty} k^2 \cdot \mathbb{P}(X = k) && \text{Definition of } \mathbb{E} \\ &= \sum_{k=0}^{\infty} k^2 \cdot (1-p)^{k-1}p && \text{Definition of Geometric}(p)\end{aligned}$$

Similar to what we did for expectation, we can take derivatives of the formal geometric power series to determine a closed form for the above sum. We had

$$\frac{1}{(1-x)^2} = \sum_{i=0}^{\infty} ix^{i-1}$$

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$$\begin{aligned}
\frac{2}{(1-x)^3} &= \sum_{i=0}^{\infty} i(i-1)x^{i-2} && \frac{d}{dx} \text{ on both sides} \\
\frac{2x}{(1-x)^3} &= \sum_{i=0}^{\infty} i(i-1)x^{i-1} && \text{Multiply both sides by } x \\
\frac{2x}{(1-x)^3} &= \sum_{i=0}^{\infty} (i^2-i)x^{i-1} \\
\frac{2x}{(1-x)^3} &= \left(\sum_{i=0}^{\infty} i^2 x^{i-1}\right) - \left(\sum_{i=0}^{\infty} i x^{i-1}\right) \\
\frac{2x}{(1-x)^3} + \left(\sum_{i=0}^{\infty} i x^{i-1}\right) &= \sum_{i=0}^{\infty} i^2 x^{i-1}
\end{aligned}$$

Note that we already know that $\sum_{i=0}^{\infty} ix^{i-1} = \frac{1}{(1-x)^2}$ from our calculations for $\mathbb{E}[X]$. Making that substitution yields

$$\frac{2x}{(1-x)^3} + \frac{1}{(1-x)^2} = \sum_{i=0}^{\infty} i^2 x^{i-1}$$

If we let $x = 1-p$, we get

$$\begin{aligned}
\frac{2(1-p)}{(1-(1-p))^3} + \frac{1}{(1-(1-p)^2} &= \sum_{i=0}^{\infty} i^2 (1-p)^{i-1} \\
\frac{2-2p}{p^3} + \frac{1}{p^2} &= \sum_{i=0}^{\infty} i^2 (1-p)^{i-1} \\
\frac{2-2p}{p^3} + \frac{p}{p^3} &= \sum_{i=0}^{\infty} i^2 (1-p)^{i-1} \\
\frac{2-p}{p^3} &= \sum_{i=0}^{\infty} i^2 (1-p)^{i-1}
\end{aligned}$$

Substituting this into our expression for $\mathbb{E}[X^2]$ gives

$$\begin{aligned}
\mathbb{E}[X^2] &= \sum_{k=0}^{\infty} k^2 \cdot (1-p)^{k-1} p \\
&= p\left(\sum_{k=0}^{\infty} k^2 \cdot (1-p)^{k-1}\right)
\end{aligned}$$

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$$= p\left(\frac{2-p}{p^3}\right)$$
$$= \frac{2-p}{p^2}$$

Thus, we get

$$\begin{aligned}
\mathrm{Var}(X) &= \mathbb{E}[X^2] - \mathbb{E}[X]^2 \\
&= \left(\frac{2-p}{p^2}\right) - \left(\frac{1}{p}\right)^2 && \text{From above} \\
&= \frac{2-p}{p^2} - \frac{1}{p^2} \\
&= \frac{1-p}{p^2}
\end{aligned}$$

Part (b)

Tabulate then memorize these results.

	Poisson(λ)	**Binomial(n, p)**	**Geometric(p)**
Mean	λ	np	$\frac{1}{p}$
Variance	λ	$np(1-p)$	$\frac{1-p}{p^2}$

Problem 3

Part (a)

Let $X : \Omega \to \mathbb{Z}$ be a *non-negative* random variable. Show that

$$\mathbb{E}(X) = \sum_{n=1}^{\infty} \mathbb{P}(X \geq n)$$

including the assertion that $\mathbb{E}(X)$ exists if and only if this sum converges.

First, we show equality regardless of convergence (i.e. in a formal sense, in that the series for expectation on the left-hand side and the series given on the right-hand side have the same summands). By the definition of $\geq$, we can write

$$\begin{aligned}
\mathbb{P}(X \geq n) &= \mathbb{P}(X = n) + \mathbb{P}(X = n+1) + ... \\
&= \sum_{i=0}^{\infty} \mathbb{P}(X = n + i)
\end{aligned}$$

Plugging this into the right-hand side of the equality we want to show above, we get

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$$\sum_{n=1}^{\infty}\left(\sum_{i=0}^{\infty}\mathbb{P}(X=n+i)\right)$$

Ignoring questions of convergence for now, we may re-order the sum by considering the following table:

	$n=1$	$n=2$	$n=3$	$n=4$
$i=0$	$\mathbb{P}(X=1)$	$\mathbb{P}(X=2)$	$\mathbb{P}(X=3)$	$\mathbb{P}(X=4)$
$i=1$	$\mathbb{P}(X=2)$	$\mathbb{P}(X=3)$	$\mathbb{P}(X=4)$	$\mathbb{P}(X=5)$
$i=2$	$\mathbb{P}(X=3)$	$\mathbb{P}(X=4)$	$\mathbb{P}(X=5)$	$\mathbb{P}(X=6)$
$i=3$	$\mathbb{P}(X=4)$	$\mathbb{P}(X=5)$	$\mathbb{P}(X=6)$	$\mathbb{P}(X=7)$

In this table, each column corresponds to the outer index n in the original sum $\sum_{n=1}^{\infty}\left(\sum_{i=0}^{\infty}\mathbb{P}(X=n+i)\right)$. For each n, we sum all probabilities corresponding to $X \geq n$ using the inner index i.

We reindex this sum from pairs (n,i) into single indices $m=n+i$ where we multiply the summand of index m by the number of ways we can solve $m=n+i$ subject to $i \geq 0$ and $n \geq 1$. Visually, we index according to the color scheme in the following table, where each color corresponds to one summand in the reindexed sum:

	$n=1$	$n=2$	$n=3$	$n=4$
$i=0$	$\mathbb{P}(X=1)$	$\mathbb{P}(X=2)$	$\mathbb{P}(X=3)$	$\mathbb{P}(X=4)$
$i=1$	$\mathbb{P}(X=2)$	$\mathbb{P}(X=3)$	$\mathbb{P}(X=4)$	$\mathbb{P}(X=5)$
$i=2$	$\mathbb{P}(X=3)$	$\mathbb{P}(X=4)$	$\mathbb{P}(X=5)$	$\mathbb{P}(X=6)$
$i=3$	$\mathbb{P}(X=4)$	$\mathbb{P}(X=5)$	$\mathbb{P}(X=6)$	$\mathbb{P}(X=7)$

More precisely, we claim that there are m solutions $(n,i)\in\mathbb{Z}^2$ to the equation $m=n+i$ subject to the constraints $i \geq 0$ and $n \geq 1$ (in the table, for instance, we see that $\mathbb{P}(X=3)$ appears 3 times). In general, we can enumerate. The solutions to the equation subject to the constraints will be exactly the m pairs of the form

$$\begin{aligned}(m,0)\\(m-1,1)\\\vdots\\(1,m-1)\end{aligned}$$

with $m \geq 1$ since it is bounded below by $n \geq 1$ and $i \geq 0$. Thus, under this indexing, we can write the sum as

$$\sum_{n\geq 1,k\geq 0}\mathbb{P}(X=n+k)=\sum_{m=0}^{\infty}m\mathbb{P}(X=m)$$

Question assigned to the following page: 3

By the definition of $\mathbb{E}$, we know that

$$\mathbb{E}(X) = \sum_{n=0}^{\infty} n\mathbb{P}(X = n)$$

where our summation begins at $n = 0$ since X is non-negative. This is exactly the expression we got for $\sum_{n=1}^{\infty} \mathbb{P}(X \geq n)$ above, i.e. we can conclude that

$$\mathbb{E}(X) = \sum_{n=1}^{\infty} \mathbb{P}(X \geq n)$$

in a formal sense, in that the series has the same summands as that given by the definition of expectation.

Next, we make this notion of equality more precise by considering questions of convergence. We claim that $\mathbb{E}(X)$ exists (and is equal to $\sum_{n=1}^{\infty} \mathbb{P}(X \geq n)$) if and only if $\sum_{n=1}^{\infty} \mathbb{P}(X \geq n)$ converges. Moreover, we claim that $E(X)$ is infinite if and only if $\sum_{n=1}^{\infty} \mathbb{P}(X \geq n)$ is infinite (i.e. if we expand the definition of equality to also include the infinite case, equality between the two expressions always holds).

First, suppose that $\mathbb{E}(X)$ exists. By definition, that means that the sum

$$\mathbb{E}(X) = \sum_{n=0}^{\infty} n\mathbb{P}(X = n)$$

converges absolutely. But then that means we can re-index it safely into the double sum

$$\sum_{n=1}^{\infty} \left(\sum_{i=0}^{\infty} \mathbb{P}(X = n + i) \right)$$

which we showed was equal to $\sum_{n=1}^{\infty} \mathbb{P}(X \geq n)$. Similarly, if $\sum_{n=1}^{\infty} \mathbb{P}(X \geq n)$ converged, it must also converge absolutely (since probabilities are always nonnegative), and hence we can re-index in the reverse direction to get the definition of expectation $\sum_{n=0}^{\infty} n\mathbb{P}(X = n)$.

On the other hand, if $\mathbb{E}(X) = \sum_{n=0}^{\infty} n\mathbb{P}(X = n)$ did not converge, since X is a non-negative random variable, it must diverge to positive infinity. Note that $\sum_{n=0}^{\infty} n\mathbb{P}(X = n)$ is an infinite sum of nonnegative summands. Tonelli's Theorem tells us that any rearrangement of this sum — such as $\sum_{n=1}^{\infty} \mathbb{P}(X \geq n)$, which we showed above — preserves convergence/divergence behavior. Thus, $\sum_{n=1}^{\infty} \mathbb{P}(X \geq n)$ must also diverge to positive infinity as well.

Part (b)

More generally, what relation between functions G and g ensures

$$\mathbb{E}(G(X)) = \sum_{n=0}^{\infty} g(n)\mathbb{P}(X \geq n)$$

To simplify questions of convergence, let us just assume $g(k) \geq 0$ for all k.

We consider each side of the equation separately. First, on the left-hand side, by the definition of expectation, we have

Questions assigned to the following page: 3 and 4

$$\mathbb{E}(G(X)) = \sum_{n=0}^{\infty} G(n)\mathbb{P}(X = n)$$

That is, for a fixed $k \in \mathbb{Z}_{\geq 0}$, the coefficient of $\mathbb{P}(X = k)$ in the overall sum is exactly $G(k)$.

On the right-hand side, from our work in **Part (a)** (assuming absolute convergence/validity of rearrangement), we have

$$\begin{aligned}
\sum_{n=0}^{\infty} g(n)\mathbb{P}(X \geq n) &= \sum_{n=0}^{\infty} g(n)\left(\sum_{i=0}^{\infty} \mathbb{P}(X = n+i)\right) \\
&= \sum_{n=0}^{\infty}\left(\sum_{i=0}^{\infty} g(n)\mathbb{P}(X = n+i)\right) \\
&= \sum_{\substack{k=n+i \\ n\geq 0 \\ k \geq 0}} g(n)\mathbb{P}(X = k)
\end{aligned}$$

In this sum, for a fixed $k \in \mathbb{Z}_{\geq 0}$, there is a $\mathbb{P}(X = k)$ term for all integer solutions (n, i) to $k = n + i$ subject to $n \geq 0$ and $k \geq 0$, with coefficient $g(n)$ (note that n starts from 0 here, unlike in **Part (a)**). For any given n, there is exactly corresponding i which satisfies the equation. Thus, the coefficient of $\mathbb{P}(X = k)$ in the overall sum is exactly the sum of all the $g(n)$s for each $0 \leq n \leq k$, i.e. the coefficient is $\sum_{n=0}^{k} g(k)$. In order for the two sides of the original equation to be equal, the corresponding coefficients of $\mathbb{P}(X = k)$ in the overall sum must also be equal, i.e. for all $k \in \mathbb{Z}_{\geq 0}$, G and g must be related by

$$\boxed{G(k) = \sum_{n=0}^{k} g(n)}$$

Problem 4

We throw a die independently four times and let X denote the minimal value rolled.

Part (a)

For each $n \in \mathbb{N}$, find the probability that $X \geq n$.

We have a sample space $\Omega = \{1, 2, 3, 4, 5, 6\}^4$ where the first component corresponds to the first roll, the second component to the second roll, etc. The event $X \geq n$ (i.e. the minimal value rolled is greater than or equal to n) corresponds to the subset

$$E = \{(x_1, x_2, x_3, x_4) : n \leq x_1 \leq 6, n \leq x_2 \leq 6, n \leq x_3 \leq 6, n \leq x_4 \leq 6\}$$

Question assigned to the following page: 4

i.e. every roll is a number greater than or equal to n. Assuming the die is fair, since the rolls are independent, each elementary event $(x_1, x_2, x_3, x_4) \in \Omega$ is equiprobable with probability $\frac{1}{6^4}$ (there are $|\Omega| = 6^4$ total possible elementary events).

Thus, to calculate the probability that $X \geq n$, it suffices to count the number of elements in E and divide by the total number of elements in Ω. Precisely, for each roll x_i, if $x_i \geq n$ (with $1 \leq n \leq 6$), there are $6 - n + 1 = 7 - n$ possibilities which satisfy the constraint: the die roll could be any of $n, n+1, ..., 6$. Thus, there are $(7-n)^4$ total possibilities among the four rolls. Hence, the overall probability for $1 \leq n \leq 6$ is

$$\mathbb{P}(X \geq n) = \mathbb{P}(E) = \frac{(7-n)^4}{6^4} = \left(\frac{7-n}{6}\right)^4$$

For the remaining $n \in \mathbb{N}$ not satisfying $1 \leq n \leq 6$ we can say $\mathbb{P}(X \geq n) = 1$ if $n < 1$ and $\mathbb{P}(X \geq n) = 0$ if $n > 6$. Altogether, we have

$$\boxed{\mathbb{P}(X \geq n) = \begin{cases} 1 & \text{when } n < 1 \\ \left(\frac{7-n}{6}\right)^4 & \text{when } 1 \leq n \leq 6 \\ 0 & \text{when } n > 6 \end{cases}}$$

Part (b)

Compute the PMF of X.

First, we note that $\mathbb{P}(X \geq n) = \sum_{i=n}^{6} \mathbb{P}(X = i)$. But then we have

$$\begin{aligned}
\mathbb{P}(X \geq n) - \mathbb{P}(X \geq n+1) &= \sum_{i=n}^{6} \mathbb{P}(X = i) - \sum_{i=n+1}^{6} \mathbb{P}(X = i) \\
&= \mathbb{P}(n) + \sum_{i=n+1}^{6} \mathbb{P}(X = i) - \sum_{i=n+1}^{6} \mathbb{P}(X = i) \\
&= \mathbb{P}(n)
\end{aligned}$$

Thus our pmf p_X becomes (for $1 \leq k \leq 6$)

$$\begin{aligned}
p_X(n) = \mathbb{P}(X = n) &= \mathbb{P}(X \geq n) - \mathbb{P}(X \geq n+1) \\
&= \left(\frac{7-n}{6}\right)^4 - \left(\frac{7-(n+1)}{6}\right)^4 && \text{From } \textbf{Part (b)} \\
&= \left(\frac{7-n}{6}\right)^4 - \left(\frac{6-n}{6}\right)^4
\end{aligned}$$

In full, we have

$$\boxed{p_X(n) = \begin{cases} \left(\frac{7-n}{6}\right)^4 - \left(\frac{6-n}{6}\right)^4 & \text{when } 1 \leq n \leq 6 \\ 0 & \text{otherwise} \end{cases}}$$

Question assigned to the following page: 4

Part (c)

Determine the mean and variance of X.

For the mean, note that X is a non-negative random variable. By **Problem 3, Part (a)**, we have

$$\begin{aligned}
\mathbb{E}[X] &= \sum_{k=1}^{\infty} \mathbb{P}(X \geq n) \\
&= \sum_{k=1}^{6} \mathbb{P}(X \geq n) + \sum_{k=7}^{\infty} \mathbb{P}(X \geq n) \\
&= \sum_{k=1}^{\infty} \left(\frac{7-n}{6}\right)^4 + 0 && \text{From } \textbf{Part (a)} \\
&= 1 + \left(\frac{5}{6}\right)^4 + \left(\frac{2}{3}\right)^4 + \left(\frac{1}{2}\right)^4 + \left(\frac{1}{3}\right)^4 + \left(\frac{1}{6}\right)^4
\end{aligned}$$

For the variance, recall the formula

$$\mathrm{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2$$

We will compute $\mathbb{E}[X^2]$ separately. Let $G(k) = k^2$. By **Problem 3, Part (b)**, we can alternatively find some $g(k)$ such that $k^2 = G(k) = \sum_{n=0}^{k} g(n)$. Indeed, consider

$$g(k) = \begin{cases} 2k-1 & \text{when } k \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

We verify that the relation is satisfied for $k \geq 0$. Indeed, we have

$$\begin{aligned}
G(0) &= 0 & \sum_{n=0}^{0} g(n) &= g(0) = 0 \\
G(1) &= 1 & \sum_{n=0}^{1} g(n) &= g(0) + g(1) = 0 + 1 = 1 \\
G(2) &= 4 & \sum_{n=0}^{2} g(n) &= g(0) + g(1) + g(2) = 0 + 1 + 3 = 4
\end{aligned}$$

and in general we can do a rough inductive argument to show that

$$\begin{aligned}
G(k+1) &= (k+1)^2 \\
&= k^2 + 2k + 1 \\
&= G(k) + 2k + 1
\end{aligned}$$

Question assigned to the following page: 4

$$
\begin{aligned}
&= \sum_{n=0}^{k} g(n) + (2k+1) && \text{Inductive hypothesis} \\
&= \sum_{n=0}^{k} g(n) + (2k+2-1) \\
&= \sum_{n=0}^{k} g(n) + (2(k+1)-1) \\
&= \sum_{n=0}^{k} g(n) + g(k+1) \\
&= \sum_{n=0}^{k+1} g(n)
\end{aligned}
$$

i.e. this relation holds for all $k \in \mathbb{N}$. Thus, applying the result of **Problem 3, Part (b)** with $G(k) = k^2$, we get

$$
\begin{aligned}
\mathbb{E}(G(X)) &= \sum_{n=0}^{\infty} g(n)\mathbb{P}(X \geq n) \\
\mathbb{E}(X^2) &= \sum_{n=1}^{\infty} g(n)\mathbb{P}(X \geq n) && \text{Since } g(0) = 0 \\
&= \sum_{n=1}^{6} g(n)\mathbb{P}(X \geq n) && \text{Since } \mathbb{P}(X \geq n) = 0 \text{ for } n > 6 \\
&= \sum_{n=1}^{6} (2n-1)\mathbb{P}(X \geq n)
\end{aligned}
$$

Expanding the final sum above gives us

$$
\begin{aligned}
\mathbb{E}(X^2) &= \mathbb{P}(X \geq 1) + 3\mathbb{P}(X \geq 2) + 5\mathbb{P}(X \geq 3) + 7\mathbb{P}(X \geq 4) + 9\mathbb{P}(X \geq 5) + 11\mathbb{P}(X \geq 6) \\
&= 1 + 3\left(\frac{5}{6}\right)^4 + 5\left(\frac{2}{3}\right)^4 + 7\left(\frac{1}{2}\right)^4 + 9\left(\frac{1}{3}\right)^4 + 11\left(\frac{1}{6}\right)^4
\end{aligned}
$$

Plugging this back into the formula for variance yields

Questions assigned to the following page: 4 and 5

$$
\begin{aligned}
&\mathrm{Var}(X) \\
&\quad = \mathbb{E}[X^2] - \mathbb{E}[X]^2 \\
&\quad = \left(1 + 3\left(\frac{5}{6}\right)^4 + 5\left(\frac{2}{3}\right)^4 + 7\left(\frac{1}{2}\right)^4 + 9\left(\frac{1}{3}\right)^4 + 11\left(\frac{1}{6}\right)^4\right) \\
&\quad - \left(1 + \left(\frac{5}{6}\right)^4 + \left(\frac{2}{3}\right)^4 + \left(\frac{1}{2}\right)^4 + \left(\frac{1}{3}\right)^4 + \left(\frac{1}{6}\right)^4\right)^2
\end{aligned}
$$

Problem 5

I play the following game using a coin that lands heads with probability p. I start with $X_0 = \$1$ and at each stage I gamble all I have on the toss of the coin. If it lands heads I end up with twice what I started with; if it lands tails I lose everything. All coin tosses are statistically independent.

Part (a)

With X_n denoting how much money I have after the nth toss, find

$$\mathbb{E}(X_{n+1} \mid X_n = k)$$

in terms of k.

Our sample space is $\Omega = \{H, T\}^{\mathbb{N}}$. Based on the rules of the game as given in the problem, we can write explicit definitions for each X_n with respect to an elementary event $\omega \in \Omega$. More precisely, we have

$$
\begin{aligned}
X_0(\omega) &= 1 \\
X_1(\omega) &= \begin{cases} 2 & \text{if } \omega \in \{H\} \times \Omega \\ 0 & \text{otherwise} \end{cases} \\
X_2(\omega) &= \begin{cases} 4 & \text{if } \omega \in \{H\}^2 \times \Omega \\ 0 & \text{otherwise} \end{cases} \\
&\vdots \\
X_n(\omega) &= \begin{cases} 2^n & \text{if } \omega \in \{H\}^n \times \Omega \\ 0 & \text{otherwise} \end{cases}
\end{aligned}
$$

Since all tosses are independent, the probability of the elementary event $\omega = \{H\}^n \times \Omega$ is simply p^n. With that, we can determine the pmf for each X_n:

$$p_{X_n}(k) = \begin{cases} p^n & \text{if } k = 2^n \\ 0 & \text{otherwise} \end{cases}$$

Question assigned to the following page: 5

From here, we can calculate the expectation

$$\begin{aligned}
\mathbb{E}(X_{n+1} \mid X_n = k) &= \sum_{j=0}^{\infty} j \cdot \mathbb{P}(X_{n+1} = j \mid X_n = k) \\
&= \sum_{j=0}^{\infty} j \cdot \frac{\mathbb{P}(X_{n+1} = j \cap X_n = k)}{\mathbb{P}(X_n = k)}
\end{aligned}$$

Based on the definition of $\mathbb{P}(X_{n+1})$ we have above, there are only two possibilities: $\mathbb{P}(X_{n+1}) = 2^{n+1}$ or $\mathbb{P}(X_{n+1}) = 0$. Since only $j = 2^{n+1}$ is nonzero, we can simplify the sum to

$$\mathbb{E}(X_{n+1} \mid X_n = k) = 2^{n+1} \cdot \frac{\mathbb{P}(X_{n+1} = 2^{n+1} \cap X_n = k)}{\mathbb{P}(X_n = k)}$$

We have two cases. If $k \neq 2^n$, then $\mathbb{P}(X_n = k) = 0$ and this expectation is not defined. Thus, we assume that $k = 2^n$. In this case, we have

$$\mathbb{E}(X_{n+1} \mid X_n = k) = 2^{n+1} \cdot \frac{\mathbb{P}(X_{n+1} = 2^{n+1} \cap X_n = 2^n)}{\mathbb{P}(X_n = 2^n)}$$

Note that if $X_{n+1} = 2^{n+1}$, we must necessarily have had $X_n = 2^n$ — the only other possibility for X_n is if $X_n = 0$, but then the game would've been over. That is, $X_{n+1} = 2^{n+1} \cap X_n = 2^n = X_{n+1} = 2^{n+1}$, so we can write

$$\begin{aligned}
\mathbb{E}(X_{n+1} \mid X_n = k) &= 2^{n+1} \cdot \frac{\mathbb{P}(X_{n+1} = 2^{n+1})}{\mathbb{P}(X_n = 2^n)} \\
&= 2^{n+1} \cdot \frac{p_{X_{n+1}}(2^{n+1})}{p_{X_n}(2^n)} \\
&= 2^{n+1} \cdot \frac{p^{n+1}}{p^n} \\
&= 2^{n+1} p \\
&= 2(2^n) p \\
&= \boxed{2kp} && \text{Since } k = 2^n
\end{aligned}$$

Part (b)

Find $\mathbb{E}(X_n)$ for all n.

By the definition of expectation, we have

Questions assigned to the following page: 5 and 6

$$
\begin{aligned}
\mathbb{E}(X_n) &= \sum_{k=0}^{\infty} k\mathbb{P}(X_n = k) \\
&= \sum_{k=0}^{\infty} k p_{X_n}(k) \\
&= 2^n p_{X_n}(2^n) && \text{Only one nonzero summand} \\
&= 2^n p^n \\
&= \boxed{(2p)^n}
\end{aligned}
$$

Problem 6

I carry out an infinite sequence of independent Bernoulli(p) trials, $0 < p < 1$. For each $r \in \mathbb{N}$ we let Y_r denote the position of my rth success. In a previous homework, we found the PMF of X — it follows the negative Binomial law

$$
\mathbb{P}(Y_r = k) = \begin{cases} \binom{k-1}{r-1} p^r (1-p)^{k-r} & k \in \mathbb{N} \text{ and } k \geq r \\ 0 & \text{otherwise} \end{cases}
$$

See also page 27 in the text.

Part (a)

Given a number $k \geq r$, verify that the event $Y_r = k$ has positive probability.

It suffices to show that $\mathbb{P}(Y_r = k) > 0$. By the PMF given in the problem statement above, we have

$$
\mathbb{P}(Y_r = k) = \binom{k-1}{r-1} p^r (1-p)^{k-r} \qquad k \geq r \text{ and } k \in \mathbb{N} \text{ by hypothesis}
$$

Since $0 < p < 1$, we know that $p^r(1-p)^{k-r} > 0$ since the product of positive (nonzero) numbers must be positive and nonzero. Similarly, since $k \geq r$, the binomial coefficient $\binom{k-1}{r-1}$ is a well-defined product of positive, nonzero numbers. Hence, the complete expression must be positive and nonzero as well, i.e. $\mathbb{P}(Y_r = k) > 0$, which is what we wanted to show.

Since the events $Y_r = k$ have positive probability for $k \geq r$, it is well-defined to condition on them in the subsequent parts of this problem.

Part (b)

Compute

$$
\mathbb{E}(Y_{r+1} \mid Y_r = k) \qquad \text{and} \qquad \operatorname{Var}(Y_{r+1} \mid Y_r = k)
$$

Question assigned to the following page: 6

By the definition of expectation, we have

$$\mathbb{E}(Y_{r+1} \mid Y_r = k) = \sum_{i=1}^{\infty} i \cdot \mathbb{P}(Y_{r+1} = i \mid Y_r = k)$$

But note that $\mathbb{P}(Y_{r+1} = i \mid Y_r = k)$ is simply the probability of the $(r+1)$st success on the ith trial, given that the rth success was on the kth trial. For $i \leq k$, this probability is zero, and for $i > k$, since the trials are independent, this reduces to the probability of getting the next success on the $(i-k)$th trial after the kth trial (e.g. if $i = k+2$, it would be the probability of getting the next success on the $(k+2-k) = 2$nd trial after the kth trial. Ignoring the first k trials, this is exactly what it means to be geometrically distributed. Hence, we can conclude that $\mathbb{P}(Y_{r+1} = i \mid Y_r = k) = (1-p)^{i-k-1}p$. Incorporating this logic into our expression for the expectation, we have

$$\begin{aligned}
\mathbb{E}(Y_{r+1} \mid Y_r = k) &= \sum_{i=1}^{\infty} i \cdot \mathbb{P}(Y_{r+1} = i \mid Y_r = k) \\
&= \sum_{i=k+1}^{\infty} i \cdot \mathbb{P}(Y_{r+1} = i \mid Y_r = k) && \text{When } i < k\text{, probability is } 0 \\
&= \sum_{i=1}^{\infty} (k+i) \cdot \mathbb{P}(Y_{r+1} = k+i \mid Y_r = k) && \text{Reindex} \\
&= \sum_{i=1}^{\infty} (k+i) \cdot (1-p)^{(k+i)-k-1}p \\
&= \sum_{i=1}^{\infty} (k+i) \cdot (1-p)^{i-1}p \\
&= \left(\sum_{i=1}^{\infty} k \cdot (1-p)^{i-1}p\right) + \left(\sum_{i=1}^{\infty} i \cdot (1-p)^{i-1}p\right)
\end{aligned}$$

From **Problem 2, Part (a)**, note that $\sum_{i=1}^{\infty} i \cdot (1-p)^{i-1}p$ is exactly the expression for the expectation of a random variable distributed according to a Geometric(p) distribution. Thus, we can write

$$\begin{aligned}
\mathbb{E}(Y_{r+1} \mid Y_r = k) &= \left(\sum_{i=1}^{\infty} k \cdot (1-p)^{i-1}p\right) + \left(\sum_{i=1}^{\infty} i \cdot (1-p)^{i-1}p\right) \\
&= \left(\sum_{i=1}^{\infty} k \cdot (1-p)^{i-1}p\right) + \frac{1}{p} \\
&= k \cdot \left(\sum_{i=1}^{\infty} (1-p)^{i-1}p\right) + \frac{1}{p} \\
&= k \cdot 1 + \frac{1}{p} && \text{Sum over pmf of Geometric}(p)
\end{aligned}$$

Question assigned to the following page: 6

$$= k + \frac{1}{p}$$

To calculate the variance, we first note the definition of the variance of an arbitrary random variable X under a generic probability law $\mathbb{P}(\cdot)$.

$$\begin{aligned}
\mathrm{Var}(X) &= \mathbb{E}[(X-\mu)^2] && \text{Definition of variance} \\
&= \mathbb{E}[g(X)] && \text{Let } g(x) = (x-\mu)^2 \\
&= \sum_{i=0}^{\infty} g(i)\mathbb{P}(X=i) && \text{Law of the subconscious statistician} \\
&= \sum_{i=0}^{\infty} (i-\mu)^2\mathbb{P}(X=i)
\end{aligned}$$

Recall that $\mu = \mathbb{E}(X) = \sum_{j=0}^{\infty} j \cdot \mathbb{P}(X=j)$. Making this substitution, we have

$$\mathrm{Var}(X) = \sum_{i=0}^{\infty}\left(\left(i - \left(\sum_{j=0}^{\infty} j \cdot \mathbb{P}(X=j)\right)\right)^2 \cdot \mathbb{P}(X=i)\right)$$

We are told that $\mathrm{Var}(\cdot \mid Y_r = k)$ denotes the variance under the probability law $\mathbb{Q}(\cdot) = \mathbb{P}(\cdot \mid Y_r = k)$. Thus, for a random variable X, we have

$$\begin{aligned}
\mathrm{Var}(X \mid Y_r = k) &= \sum_{i=0}^{\infty}\left(\left(i - \left(\sum_{j=0}^{\infty} j \cdot \mathbb{Q}(X=j)\right)\right)^2 \cdot \mathbb{Q}(X=i)\right) \\
&= \sum_{i=0}^{\infty}\left(\left(i - \left(\sum_{j=0}^{\infty} j \cdot \mathbb{P}(X=j \mid Y_r = k)\right)\right)^2 \cdot \mathbb{P}(X=i \mid Y_r = k)\right)
\end{aligned}$$

In our case, we care about Y_{r+1}, not X. Substituting Y_{r+1} in place of X tells us that

$$\mathrm{Var}(Y_{r+1} \mid Y_r = k) = \sum_{i=0}^{\infty}\left(\left(i - \left(\sum_{j=0}^{\infty} j \cdot \mathbb{P}(Y_{r+1}=j \mid Y_r = k)\right)\right)^2 \cdot \mathbb{P}(Y_{r+1}=i \mid Y_r = k)\right)$$

But in **Part (a)**, we essentially argued that the probability law $\mathbb{P}(Y_{r+1} = i \mid Y_r = k)$ was given by the pmf

$$p_{Y_{r+1}|Y_r=k}(i) = \begin{cases} (1-p)^{i-k-1}p & \text{when } i > k \\ 0 & \text{otherwise} \end{cases}$$

Question assigned to the following page: 6

since the trials are independent. This means that we can start summing up from $i = k + 1$ in our calculation for the expectation, since all summands for indices less than or equal to k will be multiplied by a factor of 0 from the pmf above. We have

$$\mathrm{Var}(Y_{r+1} \mid Y_r = k) = \sum_{i=k+1}^{\infty} \left(\left(i - \left(\sum_{j=0}^{\infty} j \cdot \mathbb{P}(Y_{r+1} = j \mid Y_r = k) \right) \right)^2 \cdot \mathbb{P}(Y_{r+1} = i \mid Y_r = k) \right)$$

If we reindex the outer sum from 1 to ∞ instead of $k+1$ to ∞, we get

$$\mathrm{Var}(Y_{r+1} \mid Y_r = k) = \sum_{i=1}^{\infty} \left(\left((k+i) - \left(\sum_{j=0}^{\infty} j \cdot \mathbb{P}(Y_{r+1} = j \mid Y_r = k) \right) \right)^2 \cdot \mathbb{P}(Y_{r+1} = k+i \mid Y_r = k) \right)$$

Similarly, the index of the inner sum can start from $k+1$ instead of 0 since all terms corresponding to indices k or less will have a factor of 0 also.

$$\mathrm{Var}(Y_{r+1} \mid Y_r = k) = \sum_{i=1}^{\infty} \left(\left((k+i) - \left(\sum_{j=k+1}^{\infty} j \cdot \mathbb{P}(Y_{r+1} = j \mid Y_r = k) \right) \right)^2 \cdot \mathbb{P}(Y_{r+1} = k+i \mid Y_r = k) \right)$$

Reindexing the inner sum to start from 1 also yields

$$\mathrm{Var}(Y_{r+1} \mid Y_r = k) = \sum_{i=1}^{\infty} \left(\left((k+i) - \left(\sum_{j=1}^{\infty} (k+j) \cdot \mathbb{P}(Y_{r+1} = k+j \mid Y_r = k) \right) \right)^2 \cdot \mathbb{P}(Y_{r+1} = k+i \mid Y_r = k) \right)$$

But then note that the reindexed probability laws $\mathbb{P}(Y_{r+1} = k+i \mid Y_r = k)$ and $\mathbb{P}(Y_{r+1} = k+j \mid Y_r = k)$ (treating k as fixed and i and j as parameters) then become defined by the pmf given by

$$p_{\text{reindexed}}(i) = \begin{cases} (1-p)^{i-1}p & \text{when } i > 0 \\ 0 & \text{otherwise} \end{cases}$$

which is exactly the pmf of the Geometric(p) distribution (or in our case, the random variable Y_1). That is, for each $i \geq 1$, we have the equality

$$\mathbb{P}(Y_{r+1} = k+i \mid Y_r = k) = \mathbb{P}(Y_1 = i)$$

Substituting this into our expression for the variance above gives us

$$\begin{aligned} &\mathrm{Var}(Y_{r+1} \mid Y_r = k) \\ &\quad = \sum_{i=1}^{\infty} \left(\left((k+i) - \left(\sum_{j=1}^{\infty} (k+j) \cdot \mathbb{P}(Y_1 = j) \right) \right)^2 \cdot \mathbb{P}(Y_1 = i) \right) \end{aligned}$$

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$$
\begin{aligned}
&= \sum_{i=1}^{\infty} \left(\left((k+i) - \left(\sum_{j=1}^{\infty} k \cdot \mathbb{P}(Y_1 = j) + \sum_{j=1}^{\infty} j \cdot \mathbb{P}(Y_1 = j) \right) \right)^2 \cdot \mathbb{P}(Y_1 = i) \right) \\
&= \sum_{i=1}^{\infty} \left(\left((k+i) - \left(k \cdot \sum_{j=1}^{\infty} \mathbb{P}(Y_1 = j) + \mathbb{E}[Y_1] \right) \right)^2 \cdot \mathbb{P}(Y_1 = i) \right) \\
&= \sum_{i=1}^{\infty} ((k+i) - (k \cdot 1 + \mathbb{E}[Y_1]))^2 \cdot \mathbb{P}(Y_1 = i) \\
&= \sum_{i=1}^{\infty} ((k+i) - (k + \mathbb{E}[Y_1]))^2 \cdot \mathbb{P}(Y_1 = i) \\
&= \sum_{i=1}^{\infty} \left((k + i - k - \mathbb{E}[Y_1])^2 \cdot \mathbb{P}(Y_1 = i) \right) \\
&= \sum_{i=1}^{\infty} (i - \mathbb{E}[Y_1])^2 \cdot \mathbb{P}(Y_1 = i) \\
&= \sum_{i=1}^{\infty} g(i) \cdot \mathbb{P}(Y_1 = i) && \text{Where } g(i) = (i - \mathbb{E}[Y_1])^2 \\
&= \mathbb{E}[g(Y_1)] && \text{Subconscious statistician} \\
&= \mathbb{E}[(Y_1 - \mathbb{E}[Y_1])^2] \\
&= \mathrm{Var}(Y_1) \\
&= \frac{1-p}{p^2} && Y_1 \text{ is Geometric}(p)
\end{aligned}
$$

Part (c)

Use **Part (b)** to find a recurrence between the means/variances of Y_r as r varies.

Note that the events corresponding to $Y_r = r$, $Y_r = r+1$, ... partition the sample space Ω, since if we carry out an infinite sequence of trials, the rth success must occur at either position r, $r+1$, ... More precisely, we partition the sample space Ω into the sets E_i for $i \geq r$ defined by

$$
\begin{aligned}
E_r &= \{\omega \in \Omega : Y_r(\omega) = r\} \\
E_{r+1} &= \{\omega \in \Omega : Y_r(\omega) = r+1\} \\
E_{r+2} &= \{\omega \in \Omega : Y_r(\omega) = r+2\} \\
&\vdots \\
E_i &= \{\omega \in \Omega : Y_r(\omega) = i\} \\
&\vdots
\end{aligned}
$$

By the partition theorem for expectation, we can thus write

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$$\begin{aligned}
\mathbb{E}[Y_{r+1}] &= \sum_{k=r}^{\infty} \mathbb{E}[Y_{r+1} \mid E_i] \cdot \mathbb{P}(E_k) \\
&= \sum_{k=r}^{\infty} \mathbb{E}[Y_{r+1} \mid Y_r = k] \cdot \mathbb{P}(Y_r = k) && \text{Definition of } E_k \\
&= \sum_{k=r}^{\infty} \left(k + \frac{1}{p}\right) \cdot \mathbb{P}(Y_r = k) && \text{From } \textbf{Part (b)} \\
&= \sum_{k=r}^{\infty} k \cdot \mathbb{P}(Y_r = k) + \sum_{k=r}^{\infty} \frac{1}{p} \cdot \mathbb{P}(Y_r = k) \\
&= \mathbb{E}[Y_r] + \sum_{k=r}^{\infty} \frac{1}{p} \cdot \mathbb{P}(Y_r = k) && \text{Definition of } \mathbb{E} \text{ and} \\
& && \mathbb{P}(Y_r = k) = 0 \text{ for } k < r \\
&= \mathbb{E}[Y_r] + \frac{1}{p} \cdot \sum_{k=r}^{\infty} \mathbb{P}(Y_r = k) \\
&= \mathbb{E}[Y_r] + \frac{1}{p} \cdot 1 && \mathbb{P}(Y_r = k) = 0 \text{ for } k < r \text{ so} \\
& && \text{we sum over entire pmf} \\
&= \mathbb{E}[Y_r] + \frac{1}{p}
\end{aligned}$$

For variance, we have

$$\begin{aligned}
\mathrm{Var}(Y_{r+1}) &= \mathbb{E}[(Y_{r+1} - \mu)^2] && \text{Where } \mu = \mathbb{E}[Y_{r+1}] \\
&= \mathbb{E}[Y_{r+1}^2 - 2Y_{r+1}\mu + \mu^2] \\
&= \mathbb{E}[Y_{r+1}^2] - \mathbb{E}[2Y_{r+1}\mu] + \mathbb{E}[\mu^2] && \text{Linearity of } \mathbb{E} \\
&= \mathbb{E}[Y_{r+1}^2] - 2\mu\mathbb{E}[Y_{r+1}] + \mu^2 \\
&= \mathbb{E}[Y_{r+1}^2] - 2\mu\left(\frac{r+1}{p}\right) + \mu^2 && \text{From } \textbf{Part (d)} \\
&= \left(\sum_{k=0}^{\infty} \mathbb{E}[Y_{r+1}^2 \mid Y_r = k] \cdot \mathbb{P}(Y_r = k)\right) - 2\mu\left(\frac{r+1}{p}\right) + \mu^2 && \text{Partition Theorem}
\end{aligned}$$

Note that we have the following relation of conditional variance and expectation

$$\mathrm{Var}(Y_{r+1} \mid Y_r = k) = \mathbb{E}[Y_{r+1}^2 \mid Y_r = k] - (\mathbb{E}[Y_{r+1} \mid Y_r = k])^2$$

We can rearrange this to get

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$$\begin{aligned}
\mathbb{E}[Y_{r+1}^2 \mid Y_r = k] &= \mathrm{Var}(Y_{r+1} \mid Y_r = k) + \left(\mathbb{E}[Y_{r+1} \mid Y_r = k]\right)^2 \\
&= \frac{1-p}{p^2} + \left(k + \frac{1}{p}\right)^2 && \text{From } \textbf{Part (b)} \\
&= \frac{1-p}{p^2} + \left(k^2 + \frac{2k}{p} + \frac{1}{p^2}\right) \\
&= \frac{2-p}{p^2} + \frac{2k}{p} + k^2
\end{aligned}$$

Plugging this back into the expression for variance we have above, we get

$$\begin{aligned}
\mathrm{Var}(Y_{r+1}) &= \left(\sum_{k=0}^{\infty} \mathbb{E}[Y_{r+1}^2 \mid Y_r = k] \cdot \mathbb{P}(Y_r = k)\right) - 2\mu\left(\frac{r+1}{p}\right) + \mu^2 \\
&= \left(\sum_{k=0}^{\infty} \left(\frac{2-p}{p^2} + \frac{2k}{p} + k^2\right) \cdot \mathbb{P}(Y_r = k)\right) - 2\mu\left(\frac{r+1}{p}\right) + \mu^2 \\
&= \left(\left(\sum_{k=0}^{\infty} \frac{2-p}{p^2} \cdot \mathbb{P}(Y_r = k)\right) + \left(\sum_{k=0}^{\infty} \frac{2k}{p} \cdot \mathbb{P}(Y_r = k)\right) + \left(\sum_{k=0}^{\infty} k^2 \cdot \mathbb{P}(Y_r = k)\right)\right) - 2\mu\left(\frac{r+1}{p}\right) + \mu^2 \\
&= \left(\left(\frac{2-p}{p^2} \cdot \sum_{k=0}^{\infty} \mathbb{P}(Y_r = k)\right) + \left(\frac{2}{p} \cdot \sum_{k=0}^{\infty} k \cdot \mathbb{P}(Y_r = k)\right) + \left(\sum_{k=0}^{\infty} k^2 \cdot \mathbb{P}(Y_r = k)\right)\right) - 2\mu\left(\frac{r+1}{p}\right) + \mu^2 \\
&= \left(\left(\frac{2-p}{p^2} \cdot 1\right) + \left(\frac{2}{p} \cdot \mathbb{E}[Y_r]\right) + \mathbb{E}[Y_r^2]\right) - 2\mu\left(\frac{r+1}{p}\right) + \mu^2 \\
&= \left(\frac{2-p}{p^2} + \left(\frac{2}{p} \cdot \frac{r}{p}\right) + \mathbb{E}[Y_r^2]\right) - 2\mu\left(\frac{r+1}{p}\right) + \mu^2 \\
&= \frac{2-p}{p^2} + \frac{2r}{p^2} + \mathbb{E}[Y_r^2] - 2\mu\left(\frac{r+1}{p}\right) + \mu^2 \\
&= \frac{2-p+2r}{p^2} + \mathbb{E}[Y_r^2] - 2\mu\left(\frac{r+1}{p}\right) + \mu^2 \\
&= \frac{2-p+2r}{p^2} + \mathbb{E}[Y_r^2] - 2\left(\frac{r+1}{p}\right)\left(\frac{r+1}{p}\right) - \left(\frac{r+1}{p}\right)^2 \\
&= \frac{2-p+2r}{p^2} + \mathbb{E}[Y_r^2] - \left(\frac{r+1}{p}\right)^2 \\
&= \frac{2-p+2r}{p^2} + \mathbb{E}[Y_r^2] - \frac{r^2+2r+1}{p^2} \\
&= \frac{-r^2+1-p}{p^2} + \mathbb{E}[Y_r^2]
\end{aligned}$$

We can express Y_r^2 in terms of the variance of Y_r using the formula

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$$\begin{aligned}
\text{Var}(Y_r) &= \mathbb{E}[Y_r^2] - \mathbb{E}[Y_r]^2 \\
\mathbb{E}[Y_r^2] &= \text{Var}(Y_r) + \mathbb{E}[Y_r]^2 \\
\mathbb{E}[Y_r^2] &= \text{Var}(Y_r) + \left(\frac{r}{p}\right)^2 && \text{From } \textbf{Part (d)} \\
\mathbb{E}[Y_r^2] &= \text{Var}(Y_r) + \frac{r^2}{p^2}
\end{aligned}$$

This yields the recurrence

$$\begin{aligned}
\text{Var}(Y_{r+1}) &= \frac{-r^2 + 1 - p}{p^2} + \mathbb{E}[Y_r^2] \\
&= \frac{-r^2 + 1 - p}{p^2} + \text{Var}(Y_r) + \frac{r^2}{p^2} \\
&= \text{Var}(Y_r) + \frac{1-p}{p^2}
\end{aligned}$$

Part (d)

Find the mean and variance of Y_r.

For the mean (expectation), we have the recurrence

$$\begin{aligned}
E[Y_{r+1}] &= E[Y_r] + \frac{1}{p} \\
&= \left(E[Y_{r-1}] + \frac{1}{p}\right) + \frac{1}{p} \\
&\vdots \\
&= E[Y_{r-(r-1)}] + r \cdot \frac{1}{p} \\
&= E[Y_1] + r \cdot \frac{1}{p} \\
&= \frac{1}{p} + r \cdot \frac{1}{p} && Y_1 \text{ is Geometric}(p) \\
&= \frac{1}{p} + \frac{r}{p} \\
&= \frac{r+1}{p}
\end{aligned}$$

For the variance, we ahve the recurrence

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$$\begin{aligned}
\mathrm{Var}(Y_{r+1}) &= \mathrm{Var}(Y_r) + \frac{1-p}{p^2} \\
&= \left(\mathrm{Var}(Y_{r-1}) + \frac{1-p}{p^2}\right) + \frac{1-p}{p^2} \\
&= \qquad \vdots \\
&= \mathbb{E}[Y_{r-(r-1)}] + r \cdot \frac{1-p}{p^2} \\
&= \mathbb{E}[Y_1] + r \cdot \frac{1-p}{p^2} \\
&= \frac{1-p}{p^2} + r\left(\frac{1-p}{p^2}\right) && Y_1 \text{ is Geometric}(p) \\
&= (r+1)\left(\frac{1-p}{p^2}\right)
\end{aligned}$$

With respect to Y_r, then, our final answers are

$$\boxed{\mathbb{E}[Y_r] = \frac{r}{p}} \qquad \boxed{\mathrm{Var}(Y_r) = r\left(\frac{1-p}{p^2}\right)}$$