(1) Here again are the probabilities from the last problem on HW2:

M	A	E
T	1/16	7/32
N	3/32	1/8

F	A	E
T	1/8	1/8
N	3/16	1/16

Show that F and T are not independent, but are independent conditioned on A. Does this information tend to suggest or to refute discrimination?

(2) I toss a fair coin many many times. Each trial is independent of the others. For each pair of integers $1 \leq k \leq n$, we define

$$H_k = \{ \text{ the } k^{\text{th}} \text{ throw lands heads } \}$$
$$Y_{0,n} = \{ \text{ the first } n \text{ trials contain no tails } \}$$
$$Y_{k,n} = \{ \text{ the } k^{\text{th}} \text{ throw is the last (largest } k\text{) tail among the first } n \text{ trials } \}$$
$$R_n = \{ \text{ the first } n \text{ trials contain a run of 4 consecutive heads } \}.$$

(a) Show that all sets $Y_{\ell,n}$ are in the σ-algebra generated by $H_1, H_2, \ldots$.
(b) What are

$$\mathbb{P}(H_k), \quad \mathbb{P}(Y_{0,n}), \quad \text{and} \quad \mathbb{P}(Y_{k,n}), \quad \text{for each } 1 \leq k \leq n?$$

(c) Show that H_k and $Y_{\ell,n}$ are independent if $k < \ell \leq n$, but not if $\ell \leq k$.
(d) Provide a formula for $\mathbb{P}(R_n^c)$ by using a partition built from the Y-sets and the Partition Theorem (that helps with what follows).
(e) Deduce the recursion formula

$$\mathbb{P}(R_n^c) = \tfrac{1}{2}\mathbb{P}(R_{n-1}^c) + \tfrac{1}{4}\mathbb{P}(R_{n-2}^c) + \tfrac{1}{8}\mathbb{P}(R_{n-3}^c) + \tfrac{1}{16}\mathbb{P}(R_{n-4}^c)$$

at least if $n \geq 5$. (Cf. Book Problem 1.11.)
(f) Tabulate enough such probabilities to determine the smallest n for which $\mathbb{P}(R_n^c) \leq \frac{1}{2}$. (You may use a calculator/computer.)

(3) Suppose my knowledge/ignorance of the number of branches of a certain store (in my city) is given by the following probability law:

$$\mathbb{P}(k \text{ branches}) = (1-p)p^k \quad \text{where } 0 < p < 1 \text{ is fixed and } k = 0, 1, 2, 3, \ldots.$$

(a) If I subsequently discover that they have at least 7 branches (e.g. I walk into store and it says 'Branch #7') what new probability law describes my revised knowledge.
(b) What is the probability of this observation if there are k stores? What number of stores makes the observation most likely?
(c) What number of stores has greatest posterior probability?
(d) Which method for estimating the total number of stores is influenced by the choice of prior distribution: maximum likelihood (b) or maximum a posteriori probability (c)?

(4) Each of n people are randomly and independently assigned a number from the set $\{1, 2, 3, \ldots, 365\}$ according to the uniform distribution. We will call this number their birthday.
(a) What is the probability that no two people share a birthday?
(b) Use a computer or calculator to evaluate your answer as a decimal for $n = 22$ and $n = 23$.

(5) I repeatedly attempt the same task. My probability of success, on the k^{th} attempt is $p \in (0, 1)$, independent of the outcome of all previous attempts.
(a) Explain why each trial is also independent of all future attempts.
(b) What is the probability that my first success occurs on the k^{th} trial?
(c) Given that my first seven attempts failed, what is the probability that I will need to make k more attempts before succeeding?
(d) Given that my second success occurred on the n^{th} trial ($n \geq 2$), what is the probability that my first success occurred on the k^{th} trial?
(e) What is the probability that my first success occurs on an odd numbered attempt.

(6) Consider the roll of a single fair die. Define three events A, B, and C with the all following properties, which you must verify:
(a) A and B are independent.
(b) A and C are independent.
(c) A and $B \cup C$ are *not* independent.
(d) A and $B \cap C$ are *not* independent.