Homework 9

Graded

Student

NATHAN LEUNG

Total Points

20 / 20 pts

Question 1

1 — 5 / 5 pts

- ✔ **– 0 pts** Correct
- **– 1 pt** Minor issue(s)
- **– 3 pts** Major issue(s)
- **– 5 pts** Completely incorrect

Question 2

2 — 5 / 5 pts

- ✔ **– 0 pts** Correct
- **– 1 pt** Minor issue(s)
- **– 3 pts** Major issue(s)
- **– 5 pts** Completely incorrect

Question 3

3 — 5 / 5 pts

- ✔ **– 0 pts** Correct
- **– 1 pt** Minor issue(s)
- **– 3 pts** Major issue(s)
- **– 5 pts** Completely incorrect

Question 4

4 — 5 / 5 pts

- ✔ **– 0 pts** Correct
- **– 1 pt** Minor issue(s)
- **– 3 pts** Major issue(s)
- **– 5 pts** Completely incorrect

Question assigned to the following page: 1

NATHAN LEUNG
MATH 170A
8 MARCH 2023

HOMEWORK 9

1 $X_1, X_2 \sim \text{Uniform}(0,1)$ and independent

$\alpha \in (0,1)$ fixed

$$Y(\omega) = \begin{cases} X_1(\omega) & \text{if } X_1(\omega) > \alpha \\ X_2(\omega) & \text{otherwise} \end{cases}$$

a CDF of Y

$$F_Y(y) = \mathbb{P}(Y \le y)$$

$$= \mathbb{P}(Y \le y \cap X_1 > \alpha) + \mathbb{P}(Y \le y \cap X_1 \le \alpha)$$

$$= \mathbb{P}(X_1 \le y \cap X_1 > \alpha) + \mathbb{P}(X_2 \le y \cap X_1 \le \alpha)$$

def of Y

independence

$$= \mathbb{P}(\alpha < X_1 \le y) + \mathbb{P}(X_2 \le y)\,\mathbb{P}(X_1 \le \alpha)$$

$$= \begin{cases} 0 & \text{if } y \le 0 \\ \alpha y & \text{if } 0 < y < \alpha \\ (y-\alpha) + y\alpha & \text{if } \alpha \le y < 1 \\ 1 & \text{if } y \ge 1 \end{cases} \qquad \alpha \in (0,1)$$

by def. of uniform distribution on [0, 1]

Question assigned to the following page: 1

HOMEWORK 9

1a cont. We can differentiate to get the PDF

$$F_Y(y) = \begin{cases} 0 & \text{if } y \le 0 \\ \alpha y & \text{if } 0 < y \le \alpha \\ (y-\alpha) + y\alpha & \text{if } \alpha \le y < 1 \\ 1 & \text{if } y \ge 1 \end{cases}$$

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \begin{cases} 0 & \text{if } y \le 0 \\ \alpha & \text{if } 0 < y < \alpha \\ 1+\alpha & \text{if } \alpha \le y < 1 \\ 0 & \text{if } y \ge 1 \end{cases}$$

1b $$E[Y] = \int_{-\infty}^{\infty} x p_Y(x)\,dx$$

$$= \int_0^1 x p_Y(x)\,dx \qquad p_Y(y) = 0 \text{ when } y \notin [0,1]$$

$$= \int_0^\alpha x\alpha\,dx + \int_\alpha^1 x(1+\alpha)\,dx$$

$$= \alpha \int_0^\alpha x\,dx + (1+\alpha)\int_\alpha^1 x\,dx$$

$$= \alpha\left[\frac{x^2}{2}\right]_{x=0}^{x=\alpha} + (1+\alpha)\left[\frac{x^2}{2}\right]_{x=\alpha}^{x=1}$$

$$= \alpha\left[\frac{\alpha^2}{2}\right] + (1+\alpha)\left[\frac{1}{2} - \frac{\alpha^2}{2}\right]$$

$$= \frac{\alpha^3}{2} + (1+\alpha)\left[\frac{1-\alpha^2}{2}\right]$$

$$= \frac{\alpha^3}{2} + \frac{(1+\alpha)(1-\alpha^2)}{2}$$

Question assigned to the following page: 1

Homework 9

1b. cont. $\mathbb{E}[Y] = \frac{\alpha^3}{2} + \frac{1-\alpha^2+\alpha-\alpha^3}{2}$

$$= \boxed{\frac{1+\alpha-\alpha^2}{2}}$$

1c $\mathbb{E}[Y] = \frac{1+\alpha-\alpha^2}{2}$

To maximize, we take the derivative.

$$f(\alpha) = \frac{1+\alpha-\alpha^2}{2}$$

$$f'(\alpha) = \frac{d}{d\alpha}\left(\frac{1+\alpha-\alpha^2}{2}\right)$$

$$= \frac{1}{2}\frac{d}{d\alpha}\left(1+\alpha-\alpha^2\right)$$

$$= \frac{1}{2}\left(1-2\alpha\right)$$

Then we solve for points where $\frac{d}{d\alpha}$ is 0:

$$\frac{1}{2}(1-2\alpha) = 0$$

$$1-2\alpha = 0$$

$$1 = 2\alpha$$

$\alpha = \frac{1}{2}$ a critical point

$$f''(\alpha) = \frac{d}{d\alpha}\frac{1}{2}(1-2\alpha)$$

$$= \frac{1}{2}(-2) = -1$$

So by the 2nd derivative test, $\alpha = \frac{1}{2}$

is a local max.

Question assigned to the following page: 1

HOMEWORK 9

1c cont. Since $\alpha \in (0,1)$ and

$$f(0) = \frac{1}{2} \qquad f(1) = \frac{1}{2}$$

$$f\left(\frac{1}{2}\right) = \frac{1 + \frac{1}{2} - \frac{1}{4}}{2} = \frac{\frac{5}{4}}{2} = \frac{5}{8}$$

$$\frac{5}{8} > \frac{1}{2}$$

The point $\boxed{\alpha = \frac{1}{2}}$ maximizes $\mathbb{E}[Y]$.

1d $Z = \max(X_1, X_2)$

$$F_Z(z) = \mathbb{P}(Z \leq z)$$

$$= \mathbb{P}(X_1 \leq z \cap X_2 \leq z) \quad \text{def of max}$$

$$= \mathbb{P}(X_1 \leq z)\, \mathbb{P}(X_2 \leq z) \quad \text{independence}$$

$$= \begin{cases} 0 & \text{if } x \leq 0 \\ z^2 & \text{if } 0 < x < 1 \\ 1 & \text{if } x \geq 1 \end{cases} \quad \text{def of uniform}$$

$$f_Z(z) = \begin{cases} 0 & \text{if } x \leq 0 \\ 2z & \text{if } 0 < x < 1 \\ 0 & \text{if } x \geq 1 \end{cases} \quad \text{take derivative}$$

$$\mathbb{E}[Z] = \int_{-\infty}^{\infty} z f_Z(z)\, dz = \int_0^1 z \cdot 2z\, dz = \int_0^1 2z^2\, dz$$

$$= 2\left[\frac{z^3}{3}\right]_{z=0}^{z=1} = 2\left(\frac{1}{3}\right) = \boxed{\frac{2}{3}}$$

Question assigned to the following page: 2

HOMEWORK 9

2 $X = \sqrt{-2\ln(U)}\cos(2\pi V)$ $\quad U, V \sim \text{Uniform}(0,1)$ independent
$Y = \sqrt{-2\ln(U)}\sin(2\pi V)$

Define the region

$A(x,y) = \{(s,t) \in \mathbb{R}^2 : s \le x \text{ and } t \le y\}$.

Then we can write

$$F_{X,Y}(x,y) = \mathbb{P}(X \le x \text{ and } Y \le y)$$
$$= \mathbb{P}((X,Y) \in A(x,y))$$

Also, define the function

$$g_A(x,y) = \begin{cases} 1 & \text{if } (x,y) \in A \\ 0 & \text{otherwise} \end{cases}$$ any region A

Since $X = \sqrt{-2\ln(U)}\cos(2\pi V)$ and $Y = \sqrt{-2\ln(U)}$ $\sin(2\pi V)$, we can write (for some region A)

Question assigned to the following page: 2

HOMEWORK 9

2 cont. $\mathbb{P}((X,Y)\in A)$

$$= \iint_{\mathbb{R}^2} g_A\left(\sqrt{-2\ln(u)}\cos(2\pi v), \sqrt{-2\ln(u)}\sin(2\pi v)\right) f_{u,v}(u,v)\,du\,dv$$

$$= \iint_{[0,1]^2} g_A\left(\sqrt{-2\ln(u)}\cos(2\pi v), \sqrt{-2\ln(u)}\sin(2\pi v)\right) du\,dv$$ since $U,V \sim$ Uniform$[0,1]$

We proceed by changing variables. Let

$$r = \sqrt{-2\ln(u)} \qquad \theta = 2\pi v$$

We have the change-of-variables transform

$$T((u,v)) = \left(\sqrt{-2\ln(u)}, 2\pi v\right)$$

and Jacobian

$$J_T(r,\theta) = \begin{vmatrix} \frac{du}{dr} & \frac{du}{d\theta} \\ \frac{dv}{dr} & \frac{dv}{d\theta} \end{vmatrix}$$

Solving for the inverse transform, we have

$$r = \sqrt{-2\ln(u)}$$

$$r^2 = -2\ln(u)$$ $r \geq 0$ always since we take principal $\sqrt{\ }$

$$-\frac{1}{2}r^2 = \ln(u)$$

$$u = e^{-\frac{1}{2}r^2}$$

Question assigned to the following page: 2

HOMEWORK 9

2 cont. $\theta = 2\pi v$, so $v = \frac{\theta}{2\pi}$.

Thus, $\frac{du}{dr} = -re^{-\frac{1}{2}r^2}$ $\qquad \frac{du}{d\theta} = 0$

$\frac{dv}{dr} = 0$ $\qquad \frac{dv}{d\theta} = \frac{1}{2\pi}$

and $J_p(r, \theta) = \begin{vmatrix} -re^{-\frac{1}{2}r^2} & 0 \\ 0 & \frac{1}{2\pi} \end{vmatrix}$

$$= -\frac{1}{2\pi} re^{-\frac{1}{2}r^2}$$

The transformed region is

Question assigned to the following page: 2

HOMEWORK 9

2 cont. $\{(u,v) : 0 \le u \le 1$ and $0 \le v \le 1\}$

$= \{(r,\theta) : 0 \le e^{-\frac{1}{2}r^2} \le 1$ and $0 \le \frac{\theta}{2\pi} \le 1\}$

$= \{(r,\theta) : -\infty \le -\frac{1}{2}r^2 \le 0$ and $0 \le \theta \le 2\pi\}$ ln is order preserving

$= \{(r,\theta) : -\infty \le -r^2 \le 0$ and $0 \le \theta \le 2\pi\}$

$= \{(r,\theta) : -\infty \le r \le \infty$ and $0 \le \theta \le 2\pi\}$ squares are always positive, so inequality always holds

$= \mathbb{R} \times [0, 2\pi]$

Thus, our integral after changing variables from (u,v) to (r,θ) becomes

$$\iint_{\mathbb{R}\times[0,2\pi]} g_A(r\cos\theta, r\sin\theta) \left| -\frac{1}{2\pi} r e^{-\frac{1}{2}r^2} \right| dr\, d\theta$$

$$= \iint_{\mathbb{R}\times[0,2\pi]} g_A(r\cos\theta, r\sin\theta) \cdot \frac{1}{2\pi} e^{-\frac{1}{2}r^2} r\, dr\, d\theta$$

We change variables once more. Let

$x = r\cos\theta$ and $y = r\sin\theta$

so $r = \sqrt{x^2 + y^2}$ and $\theta = \operatorname{atan2}(y, x)$ (two-argument arctangent)

$\tan^{-1}\left(\frac{y}{x}\right)$ ...

$$= \begin{cases} \tan^{-1}\left(\frac{y}{x}\right) & \text{if } x \geq 0 \\ \tan^{-1}\left(\frac{y}{x}\right) + \pi & \text{if } x < 0 \end{cases}$$

Question assigned to the following page: 2

HOMEWORK 9

2 cont. This gives us a change-of-variable transform

$$S((r,\theta)) = (r\cos\theta, r\sin\theta)$$

with Jacobian

$$J_S = \begin{vmatrix} \frac{dr}{dx} & \frac{dr}{dy} \\ \frac{d\theta}{dx} & \frac{d\theta}{dy} \end{vmatrix}$$

$$\frac{dr}{dx} = \frac{1}{2}(x^2+y^2)^{-\frac{1}{2}} \cdot 2x = \frac{x}{\sqrt{x^2+y^2}}$$

$$\frac{dr}{dy} = \frac{1}{2}(x^2+y^2)^{-\frac{1}{2}} \cdot 2y = \frac{y}{\sqrt{x^2+y^2}}$$

$$\frac{d\theta}{dx} = \frac{1}{1+\frac{x^2}{y^2}} \cdot \frac{1}{y} = \frac{1}{y^2+\frac{x^2}{y}} = \frac{y}{y^2+x^2}$$

$$\frac{d\theta}{dy} = \frac{1}{1+\frac{x^2}{y^2}} \cdot \frac{-x}{y^2} = \frac{-x}{y^2+x^2}$$

Question assigned to the following page: 2

HOMEWORK 9

2 cont.

$$J_s = \begin{vmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} \\ \frac{y}{x^2+y^2} & \frac{-x}{x^2+y^2} \end{vmatrix}$$

$$= \frac{-x^2}{(x^2+y^2)\sqrt{x^2+y^2}} - \frac{y^2}{(x^2+y^2)\sqrt{x^2+y^2}} = \frac{-(x^2+y^2)}{(x^2+y^2)(\sqrt{x^2+y^2})}$$

$$= \frac{-1}{\sqrt{x^2+y^2}}$$

~~Proo~~ The transformed region is

$\{(r, \theta) : r \in \mathbb{R},\ 0 \leq \theta \leq 2\pi\}$

$= \{(x,y) : \sqrt{x^2+y^2} \in \mathbb{R},\ 0 \leq \operatorname{atan2}(y,x) \leq 2\pi\}$

$= \mathbb{R}^2$ since both condition are always true and atan2 is defined on all $\mathbb{R}^2$

Thus our integral becomes

$$\iint_{\mathbb{R}^2} g_A(x,y) \cdot \frac{1}{2\pi} e^{-\frac{1}{2}(x^2+y^2)} \sqrt{x^2+y^2} \left| \frac{-1}{\sqrt{x^2+y^2}} \right| dx\,dy$$

$$= \iint_{\mathbb{R}^2} g_A(x,y) \cdot \frac{1}{2\pi} e^{-\frac{1}{2}(x^2+y^2)}\, dx\,dy$$

Question assigned to the following page: 2

Homework 9

2 cont. That is, we have the equality

$$\mathbb{P}((X,Y)\in A) = \iint_{\mathbb{R}^2} g_A(x,y)\cdot \frac{1}{2\pi} e^{-\frac{1}{2}(x^2+y^2)}\,dx\,dy$$

Note that $F_{X,Y}(x,y)$

$$= \mathbb{P}(X\le x \text{ and } Y \le y)$$

$$= \mathbb{P}((X,Y)\in A(x,y)) \quad \text{as defined earlier}$$

$$= \iint_{\mathbb{R}^2} g_{A(x,y)}(s,t)\cdot \frac{1}{2\pi} e^{-\frac{1}{2}(s^2+t^2)}\,ds\,dt$$

$$= \int_{-\infty}^{x}\int_{-\infty}^{y} \frac{1}{2\pi} e^{-\frac{1}{2}(s^2+t^2)}\,ds\,dt \quad \text{by definition of } A(x,y) \text{ and } g_{A(x,y)}$$

We want to show that X and Y are both $N(0,1)$ distributed and independent.

We have ← compute marginal

$$F_X(x) = \mathbb{P}(X\le x) = \int_{-\infty}^{x}\int_{-\infty}^{\infty} \frac{1}{2\pi} e^{-\frac{1}{2}(s^2+t^2)}\,ds\,dt$$

$$= \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}s^2}\left(\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2}\,dt\right) ds$$

(s and t are symmetric above
so we can switch)

Question assigned to the following page: 2

HOMEWORK 19

2 cont.

$$F_X(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}s^2} (1)\, ds$$

$$= \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}s^2} ds$$

So $f_X(x) = \frac{d}{dx} F_X(x) = \frac{d}{dx} \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}s^2} ds$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$$

which is exactly the PDF for a $N(0,1)$ distributed random variable. Hence $X \sim N(0,1)$. A symmetric argument shows $Y \sim N(0,1)$.

Finally, to show independence, note that

$$F_{X,Y}(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} \frac{1}{2\pi} e^{-\frac{1}{2}(s^2+t^2)} ds\, dt$$

So their joint density function is

$$\frac{\partial}{\partial x \partial y} F_{X,Y}(x,y) = \frac{1}{2\pi} e^{-\frac{1}{2}(s^2+t^2)}$$

$$= \left(\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}\right)\left(\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2}\right)$$

Since the joint density function can be expressed as a product of a function of x and a function of y, X and Y must be independent.

Question assigned to the following page: 3

HOMEWORK 19

3 $X \sim \chi_v^2$, $Y \sim \chi_k^2$ independent

WTS $X+Y \sim \chi_{k+v}^2$

We let $Z = X+Y$ and proceed using the convolution formula.

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z-x)\, dx$$

$$= \int_0^{\infty} f_X(x) f_Y(z-x)\, dx$$

since pdf of χ_k^2 random variable is 0 for negative values

Note if $z \le 0$, then $f_Z(z) = 0$. Assume $z > 0$

$$\rightarrow = \int_0^{z} f_X(x) f_Y(z-x)\, dx$$

when $x > z$, $z-x < 0$ so $f_Y(z-x) = 0$

$$= \int_0^z \left(\frac{1}{2^{v/2}\Gamma(v/2)} x^{\frac{v}{2}-1} e^{-\frac{x}{2}}\right)\left(\frac{1}{2^{k/2}\Gamma(k/2)}(z-x)^{\frac{k}{2}-1} e^{\frac{-(z-x)}{2}}\right) dx$$

$$= \frac{1}{2^{(k+v)/2}\Gamma(\frac{v}{2})\Gamma(\frac{k}{2})} \int_0^z x^{\frac{v}{2}-1}(z-x)^{\frac{k}{2}-1} e^{-\frac{z}{2}}\, dx$$

Let $A = \frac{1}{2^{(k+v)/2}\Gamma(\frac{k}{2})\Gamma(\frac{k}{2})}$. Then we can write

$$= A e^{-z/2} \int^z x^{\frac{v}{2}-1}(z-x)^{\frac{k}{2}-1}\, dx$$

Question assigned to the following page: 3

HOMEWORK 7

3 We make the substitution
cont.

$$u = \frac{x}{z} \qquad x = uz$$

$$du = \frac{1}{z}dx$$

Note $z > 0$ means this substitution is ok

$$z\,du = dx$$

and we get

$$f_Z(z) = Ae^{-z/2}\int_0^1 (uz)^{\frac{v}{2}-1}(z-uz)^{\frac{k}{2}-1}\cdot z\,du$$

$$= Ae^{-z/2}\int_0^1 z^{\frac{v}{2}-1}u^{\frac{v}{2}-1}z^{\frac{k}{2}-1}(1-u)^{\frac{k}{2}-1}z\,du$$

$$= Ae^{-z/2}z^{\frac{v+k}{2}-1}\int_0^1 u^{\frac{v}{2}-1}(1-u)^{\frac{k}{2}-1}du$$

Let $C = \int_0^1 u^{\frac{v}{2}-1}(1-u)^{\frac{k}{2}-1}du$

Then we have $f_Z(z) = Ae^{-z/2}z^{\frac{v+k}{2}-1}\cdot C$

Let $W \sim \text{Gamma}\left(\frac{v+k}{2}, \frac{1}{2}\right) = \chi^2_{v+k}$. By the definition of a Gamma/chi-squared distribution,

$$f_W(w) = \begin{cases} \frac{1}{\Gamma(\frac{v+k}{2})\,2^{(v+k)/2}} w^{\frac{v+k}{2}-1}e^{-\frac{1}{2}w} & \text{if } w > 0 \\ 0 & \text{otherwise} \end{cases}$$

We can write f_Z in terms of f_W like so:

Question assigned to the following page: 3

Homework 9

3 cont

$z > 0$ from before

$$f_Z(z) = AC f_W(z) \cdot \Gamma\left(\frac{v+k}{2}\right) 2^{(v+k)/2}$$

Since $f_Z(z)$ is a PDF,

$$\int_{-\infty}^{\infty} f_Z(z) = 1$$

That is,

$$\int_{-\infty}^{\infty} AC f_W(z) \Gamma\left(\frac{v+k}{2}\right) \cdot 2^{(v+k)/2} dz = 1$$

$$AC \cdot \Gamma\left(\frac{v+k}{2}\right) \cdot 2^{(v+k)/2} \underbrace{\int_{-\infty}^{\infty} f_W(z) dz}_{\text{also } 1} = 1$$

$$AC \cdot \Gamma\left(\frac{v+k}{2}\right) \cdot 2^{\frac{(v+k)}{2}} = 1$$

$$\frac{\Gamma\left(\frac{v+k}{2}\right) \cdot \cancel{2^{(v+k)/2}}}{\cancel{2^{(k+v)/2}} \Gamma\left(\frac{v}{2}\right) \Gamma\left(\frac{k}{2}\right)} C = 1$$

$$C = \frac{\Gamma\left(\frac{v}{2}\right) \Gamma\left(\frac{k}{2}\right)}{\Gamma\left(\frac{v+k}{2}\right)}$$

$$\int_0^1 u^{\frac{v}{2}-1}(1-u)^{\frac{k}{2}-1}\,du = \frac{\Gamma\left(\frac{v}{2}\right)\Gamma\left(\frac{k}{2}\right)}{\Gamma\left(\frac{v+k}{2}\right)}$$

Question assigned to the following page: 3

HOMEWORK 9

3 cont. And

$$f_Z(z) = A\,C\,f_W(z)\,\Gamma\left(\frac{v+k}{2}\right)2^{(v+k)/2}$$

$$= \frac{\cancel{\Gamma\left(\frac{v+k}{2}\right)}\cancel{2^{(v+k)/2}}}{\cancel{2^{(k+v)/2}}\,\cancel{\Gamma\left(\frac{v}{2}\right)}\cancel{\Gamma\left(\frac{k}{2}\right)}} \cdot \frac{\cancel{\Gamma\left(\frac{v}{2}\right)}\cancel{\Gamma\left(\frac{k}{2}\right)}}{\cancel{\Gamma\left(\frac{v+k}{2}\right)}}\, f_W(z)$$

$$= f_W(z)$$

i.e. Z and W have the same PDF and hence are distributed identically.

Thus, $Z \sim \chi^2_{v+k}$.

Question assigned to the following page: 4

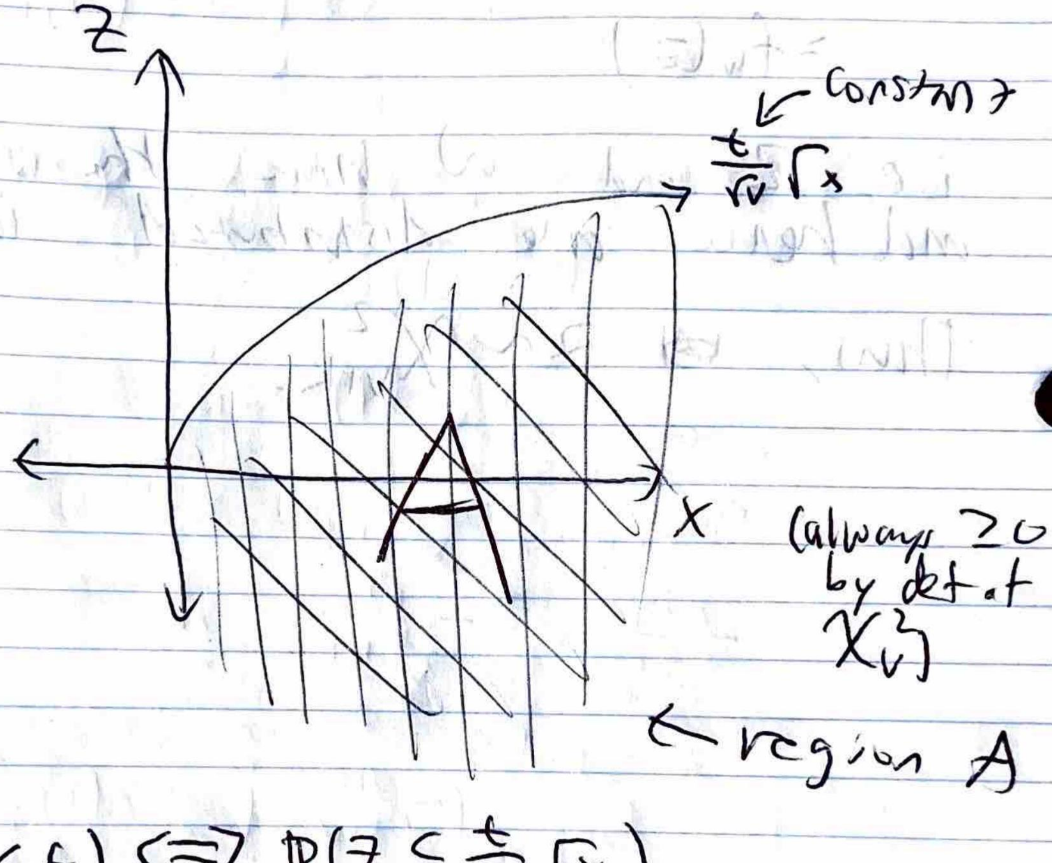
HOMEWORK 9

4 $X \sim \chi_v^2 \quad Z \sim N(0,1)$ independent

$T = Z\sqrt{\frac{v}{X}}$

$\mathbb{P}(T \le t) = \mathbb{P}\left(Z\sqrt{\frac{v}{X}} \le t\right)$

$= \mathbb{P}\left(Z \le t\sqrt{\frac{X}{v}}\right) = \mathbb{P}\left(Z \le \frac{t}{\sqrt{v}}\sqrt{X}\right)$



$\mathbb{P}(T \le t) \iff \mathbb{P}\left(Z \le \frac{t}{\sqrt{v}}\sqrt{X}\right)$

$\iff \mathbb{P}((Z,X) \in A)$

$\mathbb{P}((Z,X) \in A) = \iint_A f_Z(z) f_X(x)\, dz\, dx$ since independent

$\infty \quad \frac{t}{\sqrt{v}}\sqrt{x}$

$$= \int_{0} \int_{-\infty} f_Z(z) f_X(x)\, dz\, dx$$

Question assigned to the following page: 4

Homework 9

4 (cont) We can differentiate to find the PDF

$$f_T(t) = \frac{d}{dt} \mathbb{P}(T \leq t)$$

$$= \frac{d}{dt} \int_0^\infty \int_{-\infty}^{\frac{t}{\sqrt{v}}\sqrt{x}} f_Z(z) f_X(x) \, dz \, dx$$

$$= \int_0^\infty \frac{d}{dt} \int_{-\infty}^{\frac{t}{\sqrt{v}}\sqrt{x}} f_Z(z) f_X(x) \, dz \, dx$$ Leibniz integral rule

$$= \int_0^\infty f_X(x) \left(\frac{d}{dt} \int_{-\infty}^{\frac{t}{\sqrt{v}}\sqrt{x}} f_Z(z) \, dz \right) dx$$

$$= \int_0^\infty f_X(x) \cdot f_Z\left(\frac{t}{\sqrt{v}}\sqrt{x}\right) \cdot \frac{\sqrt{x}}{\sqrt{v}} \, dx$$ Leibniz integral rule / Chain rule

$$= \int_0^\infty \left(\frac{1}{2^{v/2}\Gamma(v/2)} x^{\frac{v}{2}-1} e^{-x/2} \right) \left(\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{t}{\sqrt{v}} \cdot \sqrt{x}\right)^2} \right) \left(\frac{\sqrt{x}}{\sqrt{v}} \right) dx$$

$$= \frac{1}{2^{(v+1)/2}\sqrt{\pi v}\,\Gamma(v/2)} \int_0^\infty x^{\frac{v+1}{2}-1} e^{-\frac{x}{2} - \frac{1}{2}\left(\frac{t^2 x}{v}\right)} dx$$

Question assigned to the following page: 4

HOMEWORK 9

4 cont.

$$f_T(t) = \frac{1}{2^{(v+1)/2}\Gamma(v/2)\sqrt{\pi v}} \int_0^\infty x^{\frac{v+1}{2}-1} e^{-\frac{x}{2}\left(1+\frac{t^2}{v}\right)} dx$$

Let $C = \frac{1}{2^{(v+1)/2}\Gamma(\frac{v}{2})\sqrt{\pi v}}$

Then $f_T(t) = C\int_0^\infty x^{\frac{v+1}{2}-1} e^{-\frac{x}{2}(1+\frac{t^2}{v})} dx$

We change variables:

$$\frac{-u}{2} = \frac{-x}{2}\left(1+\frac{t^2}{v}\right)$$

$$u = x\left(1+\frac{t^2}{v}\right)$$

$$x = \frac{u}{1+\frac{t^2}{v}} = \frac{uv}{v+t^2}$$

$$dx = \frac{du}{1+\frac{t^2}{v}} = \frac{v\,du}{v+t^2}$$

$$f_T(t) = C\int_0^\infty \left(\frac{uv}{v+t^2}\right)^{\frac{v+1}{2}-1} e^{-\frac{u}{2}} \left(\frac{v\,du}{v+t^2}\right)$$

$$= C\int_0^\infty \left(u^{\frac{v+1}{2}-1}\right)\left(\frac{v}{v+t^2}\right)^{\frac{v+1}{2}-1} e^{-\frac{u}{2}}\left(\frac{v}{v+t^2}\right)du$$

$$= C\int_0^\infty u^{\frac{v+1}{2}-1}\left(\frac{v}{v+t^2}\right)^{\frac{v+1}{2}} e^{-\frac{u}{2}} du$$

$$= c\left(\frac{v}{v+t^2}\right)^{\frac{v+1}{2}} \int_0^\infty u^{\frac{v+1}{2}} e^{-\frac{u}{2}} \, du$$

Question assigned to the following page: 4

Homework 9

4 cont. Let $W \sim \chi^2_{v+1}$. Then

$$f_W(w) = \begin{cases} \frac{1}{2^{(v+1)/2}\,\Gamma\left(\frac{v+1}{2}\right)} w^{\frac{v+1}{2}-1} e^{-w/2} & w \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

And so on $[0, \infty)$, $f_W(w) \cdot 2^{(v+1)/2}\Gamma\left(\frac{v+1}{2}\right) = w^{(v+1)/2-1} e^{-w/2}$. Substituting this into the integral, we get

$$f_T(t) = C\left(\frac{v}{v+t^2}\right)^{\frac{v+1}{2}} \int_0^\infty f_W(u) \cdot 2^{(v+1)/2}\Gamma\left(\frac{v+1}{2}\right) du$$

$$= C\left(\frac{v}{v+t^2}\right)^{\frac{v+1}{2}} 2^{(v+1)/2}\Gamma\left(\frac{v+1}{2}\right) \underbrace{\int_0^\infty f_W(u)\, du}_{=1 \text{ since pdf}}$$

$$= \frac{2^{(v+1)/2}\,\Gamma\left(\frac{v+1}{2}\right)}{2^{(v+1)/2}\,\Gamma\left(\frac{v}{2}\right)\sqrt{\pi v}} \left(\frac{v}{v+t^2}\right)^{\frac{v+1}{2}}$$

$$= \frac{\Gamma\left(\frac{v+1}{2}\right)}{\Gamma\left(\frac{v}{2}\right)\sqrt{\pi v}} \left(\frac{v}{v+t^2}\right)^{\frac{v+1}{2}}$$

$$= \frac{\Gamma\left(\frac{v+1}{2}\right)}{\Gamma\left(\frac{v}{2}\right)\sqrt{\pi v}} \left(\frac{v+t^2}{v}\right)^{-\left(\frac{v+1}{2}\right)}$$

$$= \frac{\Gamma\left(\frac{v+1}{2}\right)}{\Gamma\left(\frac{v}{2}\right)\sqrt{\pi v}} \left(1 + \frac{t^2}{v}\right)^{-\left(\frac{v+1}{2}\right)}$$

Question assigned to the following page: 4

HOMEWORK 9

4 cont. Since $f_T(t)$ is a PDF, we have

$$1 = \int_{-\infty}^{\infty} f_T(t) = \int_{-\infty}^{\infty} \frac{\Gamma\left(\frac{v+1}{2}\right)}{\Gamma\left(\frac{v}{2}\right)\sqrt{v\pi}} \left(1 + \frac{t^2}{v}\right)^{-\left(\frac{v+1}{2}\right)} dt$$

$$= \frac{\Gamma\left(\frac{v+1}{2}\right)}{\Gamma\left(\frac{v}{2}\right)\sqrt{\pi v}} \int_{-\infty}^{\infty} \left(1 + \frac{t^2}{v}\right)^{-\left(\frac{v+1}{2}\right)} dt$$

And so

$$\int_{-\infty}^{\infty} \left(1 + \frac{t^2}{v}\right)^{-\left(\frac{v+1}{2}\right)} dt = \frac{\Gamma\left(\frac{v}{2}\right)\sqrt{\pi v}}{\Gamma\left(\frac{v+1}{2}\right)}$$