Homework 4

Graded

Student

NATHAN LEUNG

Total Points

29 / 30 pts

Question 1

1 | **5** / 5 pts

- ✔ **– 0 pts** Correct
- **– 1 pt** Minor issue(s)
- **– 3 pts** Major issue(s)
- **– 5 pts** Completely incorrect

Question 2

2 | **4** / 5 pts

- **– 0 pts** Correct
- ✔ **– 1 pt** Minor issue(s)
- **– 3 pts** Major issue(s)
- **– 5 pts** Completely incorrect

1 Explain how this is justified using continuity of probability

Question 3

3 | **5** / 5 pts

- ✔ **– 0 pts** Correct
- **– 1 pt** Minor issue(s)
- **– 3 pts** Major issue(s)
- **– 5 pts** Completely incorrect

Question 4

4

5 / 5 pts

✔ **– 0 pts** Correct

– 1 pt Minor issue(s)

– 3 pts Major issue(s)

– 5 pts Completely incorrect

Question 5

5

5 / 5 pts

✔ **– 0 pts** Correct

– 1 pt Minor issue(s)

– 3 pts Major issue(s)

– 5 pts Completely incorrect

Question 6

6

5 / 5 pts

✔ **– 0 pts** Correct

– 1 pt Minor issue(s)

– 3 pts Major issue(s)

– 5 pts Completely incorrect

Question assigned to the following page: 1

MATH 170A Homework 4

Nathan Leung, UID 005835316

2 February 2024

Problem 1

For each $k \in \mathbb{N}$, let E_k denote some event.

Part (a)

Show that

$$\{ \text{ infinitely many } E_k \text{ occur } \} = \bigcap_{n=1}^{\infty} \left[\bigcup_{k=n}^{\infty} E_k \right]$$

This event is known as $\limsup_{k\to\infty} E_k$.

In order to show the equality of sets, we need to show the inclusion both ways. Denote

$$A = \{ \text{ infinitely many } E_k \text{ occur } \}$$
$$B = \bigcap_{n=1}^{\infty} \left[\bigcup_{k=n}^{\infty} E_k \right]$$

First, let $\omega \in A$. Then $\omega \in E_k$ for infinitely many $k \in \mathbb{N}$. To show $\omega \in B$, by the definition of set intersection, we need to show that $\omega \in \bigcup_{k=n}^{\infty} E_k$ for all $n \in \mathbb{N}$. Suppose for contradiction that there existed some $m \in \mathbb{N}$ such that $\omega \notin \bigcup_{k=m}^{\infty} E_k$. Then, by the definition of set union, $\omega \notin E_k$ for all $k \geq m$. Thus, $\omega \in E_j$ for $1 \leq j < m$ only, a finite number, contradicting the fact that $\omega \in A$ (and hence ω is in infinitely many E_k, by the definition of A).

Thus, our assumption that there existed some $m \in \mathbb{N}$ such that $\omega \notin \bigcup_{k=m}^{\infty} E_k$ must have been false, and in fact, it must be the case that $\omega \in \bigcup_{k=m}^{\infty} E_k$ for all $m \in \mathbb{N}$. Hence, the definition of set intersection tells us that $\omega \in \bigcap_{n=1}^{\infty} [\bigcup_{k=n}^{\infty} E_k] = B$. Since $\omega \in A$ was arbitrary, this allows us to conclude that $A \subset B$.

For the other inclusion, let $\omega \in B$. By the definition of B and set intersection, we know that $\omega \in \bigcup_{k=n}^{\infty} E_k$ for all $n \in \mathbb{N}$. Suppose for contradiction that $\omega \notin A$, i.e. $\omega \in E_k$ for only finitely many $k \in \mathbb{N}$. Then we can form a finite set containing all E_ks that occur, i.e.

$$\{E_{k_1}, E_{k_2}, ..., E_{k_m}\}$$

Since the set is finite, we can take the maximum of all subscripts k_j. Let $K = \max\{k_1, ..., k_m\}$. But then $\omega \notin \bigcup_{k=K+1}^{\infty} E_k$, contradicting the definition of B which says that $\omega \in \bigcup_{k=n}^{\infty} E_k$ for all

Question assigned to the following page: 1

$n \in \mathbb{N}$. Hence, our assumption that $\omega \notin A$ must have been false, and in fact $\omega \in A$. Since $\omega \in B$ was arbitrary, we can conclude that $B \subset A$.

Since the inclusion goes both ways, $A = B$ and we can conclude that

$$\{ \text{ infinitely many } E_k \text{ occur } \} = A = B = \bigcap_{n=1}^{\infty} \left[\bigcup_{k=n}^{\infty} E_k \right]$$

Part (b)

Show that the indicator random variable of $\limsup_{k\to\infty} E_k$ is given by

$$\limsup_{k\to\infty} 1_{E_k}(\omega)$$

By **Part (a)**, we essentially want to show that $\limsup_{k\to\infty} 1_{E_k}(\omega)$ is the indicator random variable of $A = \{ \text{ infinitely many } E_k \text{ occur } \}$, i.e. that

$$\limsup_{k\to\infty} 1_{E_k}(\omega) = \begin{cases} 1 & \omega \in \{ \text{ infinitely many } E_k \text{ occur } \} \\ 0 & \omega \notin \{ \text{ infinitely many } E_k \text{ occur } \} \end{cases}$$

or in other words,

$$\limsup_{k\to\infty} 1_{E_k}(\omega) = \begin{cases} 1 & \omega \in \{ \text{ infinitely many } E_k \text{ occur } \} \\ 0 & \omega \in \{ \text{ only finitely many } E_k \text{ occur } \} \end{cases}$$

There are two cases, and we will handle each in turn. First, suppose that $\omega \in \{ \text{ infinitely many } E_k \text{ occur } \}$. Suppose for contradiction that $\limsup_{k\to\infty} 1_{E_k}(\omega) = 0$. By the definition of lim sup, that means that $\lim_{n\to\infty} \left(\sup_{k\geq n} 1_{E_k}(\omega)\right) = 0$. By the definition of lim, that means for all $\epsilon > 0$, there exists $N_\epsilon \in \mathbb{N}$ such that $\left|\sup_{k\geq N_\epsilon}(1_{E_k}(\omega)) - 0\right| < \epsilon$, i.e. $\left|\sup_{k\geq N_\epsilon} 1_{E_k}(\omega)\right| < \epsilon$.

Pick $\epsilon = \frac{1}{2}$. Then, by the definition of a limit, there exists $N_{1/2} \in \mathbb{N}$ such that $\left|\sup_{k\geq N_{1/2}} 1_{E_k}(\omega)\right| < \frac{1}{2}$. Since 1_{E_k} by definition only outputs either a 0 or a 1, this must mean that $1_{E_k}(\omega) = 0$ for all $k \geq N_{1/2}$. That is, $1_{E_k}(\omega) = 1$ only for $1 \leq k < N_{1/2}$, a finite number. By the definition of an indicator function, this means that only a finite number of E_k occur, contradicting the fact that $\omega \in \{ \text{ infinitely many } E_k \text{ occur } \}$. Thus, our assumption that $\limsup_{k\to\infty} 1_{E_k}(\omega) = 0$ must have been false.

Note that (unlike a standard limit) $\limsup_{k\to\infty} 1_{E_k}(\omega)$ must exist because $\{1_{E_k}(\omega)\}_{k=1}^{\infty}\}$ is a bounded sequence in $\mathbb{R}$, and $\mathbb{R}$ has the least-upper bound property. Since 1_{E_k} is always either 0 or 1, $\limsup_{k\to\infty} 1_{E_k}(\omega) \in \{0, 1\}$. We showed it could not be 0 above, hence it must be the case that $\limsup_{k\to\infty} 1_{E_k}(\omega) = 1$, which is what we needed to show for this case.

In the other case, suppose that $\omega \notin \{ \text{ infinitely many } E_k \text{ occur } \}$, i.e. $\omega \in \{ \text{ only finitely many } E_k \text{ occur } \}$. We claim that then $\limsup_{k\to\infty} 1_{E_k}(\omega) = 0$. First, since only finite many E_k occur, we can form a finite set

$$\{E_{k_1}, E_{k_2}, ..., E_{k_m}\}$$

Question assigned to the following page: 1

containing all the E_j which occur. Then, since the set is finite, we can take the maximum of all subscripts k_j. Let $K = \max\{k_1, ..., k_m\}$. Then, for all $N > K$, $\sup_{k \geq N} 1_{E_k}(\omega) = 0$ since all events E_k with $k > K$ do not occur (hence the indicator function $E_k(\omega) = 0$); essentially, we are taking the sup of a sequence of 0s.

With that, fix an arbitrary $\epsilon > 0$. Let $N_\epsilon = K + 1$ from above. Then for all $k \geq N_\epsilon$, by the logic above, we get $\sup_{k \geq N_\epsilon} 1_{E_k}(\omega) = 0$ and consequently $\left|\sup_{k \geq N_\epsilon} 1_{E_k}(\omega) - 0\right| = 0 < \epsilon$. Since $\epsilon > 0$ was arbitrary, the definition of a limit is satisfied and we can write $\lim_{n \to \infty}(\sup_{k \geq n} 1_{E_k}(\omega)) = 0$. Then, by the definition of lim sup we can write $\limsup_{k \to \infty} 1_{E_k}(\omega) = 0$ which is what we needed to show.

Since $\limsup_{k \to \infty} 1_{E_k}(\omega) = 1$ when $\omega \in \{$ infinitely many E_k occur $\}$ and $\limsup_{k \to \infty} 1_{E_k}(\omega) = 0$ when $\omega \notin \{$ infinitely many E_k occur $\}$, it is indeed the indicator function of $\{$ infinitely many E_k occur $\}$.

Part (c)

We define $\liminf_{k \to \infty}(E_k)$ to be the event that all but finitely many E_k occur. Find and prove analogues of **Part (a)** and **Part (b)** for this event (use De Morgan's laws).

For the analogue of **Part (a)**, we claim that

$$\{ \text{ all but finitely many } E_k \text{ occur } \} = \bigcup_{n=1}^{\infty} \left[\bigcap_{k=n}^{\infty} E_k \right]$$

By De Morgan's laws, we can write

$$\begin{aligned}
\bigcup_{n=1}^{\infty} \left[\bigcap_{k=n}^{\infty} E_k \right] &= \bigcup_{n=1}^{\infty} \left[\bigcup_{k=n}^{\infty} E_k^C \right]^C \\
&= \left[\bigcap_{n=1}^{\infty} \left[\left[\bigcup_{k=n}^{\infty} E_k^C \right]^C \right]^C \right]^C \\
&= \left[\bigcap_{n=1}^{\infty} \left[\bigcup_{k=n}^{\infty} E_k^C \right] \right]^C && \text{Complement is an involution} \\
&= \left[\limsup_{k \to \infty} E_k^C \right]^C && \text{Definition of lim sup from } \textbf{Part (a)} \\
&= \left[\{ \text{ infinitely many } E_k^C \text{ occur } \} \right]^C && \text{Shown in } \textbf{Part (a)} \\
&= \{ \text{ only finitely many } E_k^C \text{ occur } \} \\
&= \{ \text{ only finitely many } E_k \text{ do not occur } \} && \text{Definition of complement} \\
&= \{ \text{ all but finitely many } E_k \text{ occur } \}
\end{aligned}$$

which is what we wanted to show.

For the analogue of **Part (b)**, we claim that the indicator random variable of $\liminf_{k \to \infty} E_k$ is given by

Question assigned to the following page: 1

Nathan Leung, UID 005835316
MATH 170A Homework 4
2 February 2024

$$\liminf_{k\to\infty} 1_{E_k}(\omega)$$

Essentially, we need to show that

$$\liminf_{k\to\infty} 1_{E_k}(\omega) = \begin{cases} 1 & \omega \in \{\text{ all but finitely many } E_k \text{ occur }\} \\ 0 & \omega \notin \{\text{ all but finitely many } E_k \text{ occur }\} \end{cases}$$

or in other words,

$$\liminf_{k\to\infty} 1_{E_k}(\omega) = \begin{cases} 1 & \omega \in \{\text{ all but finitely many } E_k \text{ occur }\} \\ 0 & \omega \in \{\text{ infinitely many } E_k \text{ do not occur }\} \end{cases}$$

Note that for any E_k, we have $1_{E_k^C}(\omega) = 1 - 1_{E_k}(\omega)$ (it is 1 when E_k does not occur and 0 when E_k does occur). Rearranging, we get $1_{E_k}(\omega) = 1 - 1_{E_k^C}(\omega)$. So alternatively, we can write

$$\begin{aligned} \liminf_{k\to\infty} 1_{E_k}(\omega) &= \liminf_{k\to\infty}(1 - 1_{E_k^C}(\omega)) \\ &= 1 - \liminf_{k\to\infty} 1_{E_k^C}(\omega) \end{aligned}$$

We want to show that this is 1 when all but finitely many E_k occur and 0 otherwise. First, suppose we have $\omega \in \{$ all but finitely many E_k occur $\}$. This means that finitely many E_k^C occur, i.e.

$$\{E_{k_1}^C, E_{k_2}^C, ..., E_{k_n}^C\}$$

are the E_k^C which occur. Take $K = \max\{k_1, ..., k_n\}$. Then for all $n > K + 1$, E_n^C does not occur and so $\liminf_{k\to\infty} 1_{E_k^C}(\omega) = 0$ (there will always be a zero in the tail of the sequence). Thus $1 = 1 - 0 = 1 - \liminf_{k\to\infty} 1_{E_k^C}(\omega) = \liminf_{k\to\infty}(1 - 1_{E_k^C}(\omega)) = \liminf_{k\to\infty} 1_{E_k}(\omega)$ which is what we wanted to show.

Next, suppose we have $\omega \in \{$ infinitely many E_k do not occur $\}$. Suppose for contradiction that $\liminf_{k\to\infty} 1_{E_k}(\omega) = 1$. By the definition of lim inf, that means $\lim_{n\to\infty}(\inf_{k\geq n} 1_{E_k}(\omega)) = 1$. Pick $\epsilon = \frac{1}{10}$. By the definition of a limit, this means that there exists $N_{1/10}$ such that for all $n \geq N_{1/10}$, we have $|\inf_{k\geq n} 1_{E_k}(\omega) - 1| < \frac{1}{10}$. By the definition of absolute value, we get

$$\begin{aligned} -\frac{1}{10} < \inf_{k\geq n} 1_{E_k}(\omega) - 1 &< \frac{1}{10} \\ \frac{9}{10} < \inf_{k\geq n} 1_{E_k}(\omega) &< \frac{11}{10} \end{aligned}$$

Since $1_{E_k}(\omega)$ is only ever 1 or 0 by definition, it must be the case that for all $k \geq n$, we have $1_{E_k}(\omega) = 1$ (if there were some E_k such that $1_{E_k}(\omega) = 0$, then the inf would be 0 as well, by the definition of inf). That is, all events E_k after a finite $N_{1/10} \in \mathbb{N}$ must occur. This means that

Questions assigned to the following page: 2 and 1

only a finite number of E_k do not occur (bounded above by $N_{1/10}$), contradicting the fact that $\omega \in \{$ infinitely many E_k do not occur $\}$. Thus, our assumption that $\liminf_{k\to\infty} 1_{E_k}(\omega) = 1$ must have been false, and in fact $\liminf_{k\to\infty} 1_{E_k}(\omega) = 0$, which is what we needed to show (again, as with lim sup, we note that lim inf must always exist because we are working with bounded sequences in $\mathbb{R}$).

Since $\liminf_{k\to\infty} 1_{E_k}(\omega) = 1$ when $\omega \in \{$ all but finitely many E_k occur $\}$ and $\liminf_{k\to\infty} 1_{E_k}(\omega) = 0$ when $\omega \in \{$ infinitely many E_k do not occur $\} = \{$ all but finitely many E_k occur $\}^C$, it is indeed the indicator function on $\{$ all but finitely many E_k occur $\}$.

Problem 2

I repeatedly attempt the same task. My probability of success on the kth attempt is $k(k+2)/(k+1)^2$, independent of the outcomes of all previous attempts. Here $k \in \{1, 2, 3, ...\}$. What is the probability that I never fail?

We have

$$\begin{aligned}\mathbb{P}(\{\text{ never fail }\}) &= \mathbb{P}(\{\text{ success on 1st trial }\} \cap \{\text{ success on 2nd trial }\} \cap \{\text{ success on 3rd trial }\} \cap ...)\\ &= \mathbb{P}(\{\text{ success on 1st trial }\}) \cdot \mathbb{P}(\{\text{ success on 2nd trial }\}) \cdot \mathbb{P}(\{\text{ success on 3rd trial }\}) \cdots\end{aligned}$$

where we got the last line from the independence of each attempt from the previous. Plugging in the given probability, we get

$$\mathbb{P}(\{\text{ never fail }\}) = \frac{1(3)}{2^2} \cdot \frac{2(4)}{3^2} \cdot \frac{3(5)}{4^2} \cdot \frac{4(6)}{5^2} \cdot \frac{5(7)}{6^2} \cdots$$

Note that every term in the denominator after and including 3^2 is cancelled by terms in the numerator to the left and right, e.g.

$$\mathbb{P}(\{\text{ never fail }\}) = \frac{1(3)}{2^2} \cdot \frac{2(4)}{3^2} \cdot \frac{3(5)}{4^2} \cdot \frac{4(6)}{5^2} \cdot \frac{5(7)}{6^2} \cdots$$

After cancelling all common factors, we are left with

$$\mathbb{P}(\{\text{ never fail }\}) = \frac{1}{2^2} \cdot 2 = \frac{1}{4} \cdot 2 = \boxed{\frac{1}{2}}$$

Question assigned to the following page: 3

Nathan Leung, UID 005835316
MATH 170A Homework 4
2 February 2024

Problem 3

Part (a)

Starting at the origin on the line, I take a step of one unit to the left or to the right with probability 1/2. I do this repeatedly with independent steps. If I take $2n$ steps, what is the probability that I find myself back at the origin?

In order to end up back the origin, one must have taken the same number of steps to the left and to the right. If we let our sample space $\Omega = \{L, R\}^{2n}$, with each outcome equiprobable (since L and R have equal probability each time), the problem reduces to finding how many tuples $(L, R, L, ...)$, etc. have the same number of Ls and Rs out of the total number of such tuples 2^{2n} (for each of the $2n$ steps, we have two choices: either L or R).

To find the total number of tuples of length $2n$ with n Ls and n Rs, we need to choose n steps to be L and make the rest R. Thus, there are $\binom{2n}{n}$ such tuples. Hence, our probability is

$$\mathbb{P}(\{\text{ end up at origin after } 2n \text{ steps }\}) = \boxed{\frac{\binom{2n}{n}}{2^{2n}}}$$

Part (b)

You and I independently complete this same random walk. What is the probability that we end up in the same location?

We define our sample space to be $\Omega = \{L, R\}^{2n} \times \{L, R\}^{2n}$ where the first $2n$-tuple is your random walk and the second $2n$-tuple is my random walk.

We can partition the sample space by the number of leftward and rightward moves in your random walk. The possible cases are $2n$ leftward steps and 0 rightward steps, $2n - 1$ leftward steps and 1 rightward step, etc. until we get to 0 leftward steps and $2n$ rightward steps. We denote

$$E_k = \{\text{ your random walk has } k \text{ leftward steps and } 2n - k \text{ rightward steps }\}$$
$$F_k = \{\text{ my random walk has } k \text{ leftward steps and } 2n - k \text{ rightward steps }\}$$

Then our probability becomes

$$\begin{aligned}
&\mathbb{P}(\{\text{end up in same location}\}) \\
&\quad = \mathbb{P}(\{\text{end up in same location}\} \mid E_0) \cdot \mathbb{P}(E_0) + \mathbb{P}(\{\text{end up in same location}\} \mid E_1) \cdot \mathbb{P}(E_1) + \\
&\quad\quad \mathbb{P}(\{\text{end up in same location}\} \mid E_2) \cdot \mathbb{P}(E_2) + \mathbb{P}(\{\text{end up in same location}\} \mid E_3) \cdot \mathbb{P}(E_3) + \\
&\quad\quad ... \\
&\quad\quad \mathbb{P}(\{\text{end up in same location}\} \mid E_{2n}) \cdot \mathbb{P}(E_{2n})
\end{aligned}$$

or more succinctly,

$$\begin{aligned}
&\mathbb{P}(\{\text{end up in same location}\}) \\
&\quad = \mathbb{P}(F_0 \mid E_0) \cdot \mathbb{P}(E_0) + \mathbb{P}(F_1 \mid E_1) \cdot \mathbb{P}(E_1) +
\end{aligned}$$

Questions assigned to the following page: 4 and 3

$$\mathbb{P}(F_2 \mid E_2) \cdot \mathbb{P}(E_2) + \mathbb{P}(F_3 \mid E_3) \cdot \mathbb{P}(E_3) +$$
$$...$$
$$\mathbb{P}(F_{2n} \mid E_{2n}) \cdot \mathbb{P}(E_{2n})$$

But the random walks are independent! Thus, the conditional probability is the same as the absolute probability. We get

$$\mathbb{P}(\{\text{end up in same location}\}) = \mathbb{P}(F_0) \cdot \mathbb{P}(E_0) + \mathbb{P}(F_1) \cdot \mathbb{P}(E_1) + \mathbb{P}(F_2) \cdot \mathbb{P}(E_2) + \mathbb{P}(F_3) \cdot \mathbb{P}(E_3) + ... + \mathbb{P}(F_{2n}) \cdot \mathbb{P}(E_{2n})$$

Note that $|E_k| = |F_k| = \binom{2n}{k}$ because the number of random walks with $2n$ total steps with k leftward steps and $2n - k$ rightward steps can be counted by counting the number of ways to choose k leftward steps out of $2n$ total steps. Thus we have

$$\mathbb{P}(E_k) = \mathbb{P}(F_k) = \frac{\binom{2n}{k}}{2^{2n}}$$

Plugging this formula into our expression above, we get

$$\begin{aligned}
&\mathbb{P}(\{\text{end up in same location}\}) \\
&= \left(\frac{\binom{2n}{0}}{2^{2n}}\right)^2 + \left(\frac{\binom{2n}{1}}{2^{2n}}\right)^2 + \left(\frac{\binom{2n}{2}}{2^{2n}}\right)^2 + \left(\frac{\binom{2n}{3}}{2^{2n}}\right)^2 + ... + \left(\frac{\binom{2n}{2n}}{2^{2n}}\right)^2 \\
&= \sum_{i=0}^{2n} \left(\frac{\binom{2n}{i}}{2^{2n}}\right)^2 \\
&= \sum_{i=0}^{2n} \frac{\binom{2n}{i}^2}{(2^{2n})^2} \\
&= \sum_{i=0}^{2n} \frac{\binom{2n}{i}^2}{2^{4n}} \\
&= \frac{1}{2^{4n}} \sum_{i=0}^{2n} \binom{2n}{i}^2 \\
&= \boxed{\frac{1}{2^{4n}} \binom{4n}{2n}} && \text{Vandermonde's identity}
\end{aligned}$$

Problem 4

Part (a)

Compute the probability that $X \sim \text{Geometric}(p)$ is even.

We can compute the probability that X is even by taking an infinite sum over all even numbers, i.e.

Question assigned to the following page: 4

Nathan Leung, UID 005835316
MATH 170A Homework 4
2 February 2024

$$
\begin{aligned}
\mathbb{P}(\{\ X \text{ is even }\}) &= \sum_{n=1}^{\infty} \mathbb{P}(X = 2n) \\
&= \sum_{n=1}^{\infty} p(1-p)^{2n-1} && \text{Definition of geometric distribution} \\
&= \sum_{n=1}^{\infty} p(1-p)^{2n} \cdot \frac{1}{1-p} \\
&= \frac{p}{1-p} \left(\sum_{n=1}^{\infty} (1-p)^{2n} \right) \\
&= \frac{p}{1-p} \cdot \frac{(1-p)^2}{1-(1-p)^2} && \text{Sum of geometric series} \\
&= \frac{p(1-p)}{1-(1-p)^2} && \text{Cancel common factors} \\
&= \frac{p(1-p)}{1-(1-2p+p^2)} \\
&= \frac{p(1-p)}{1-1+2p-p^2} \\
&= \frac{p(1-p)}{2p-p^2} \\
&= \frac{p(1-p)}{p(2-p)} \\
&= \frac{1-p}{2-p}
\end{aligned}
$$

Part (b)

Compute the probability that $X \sim \text{Poisson}(\lambda)$ is even.

Taking the infinite sum over all even numbers as we did in **Part (a)**, we get

$$
\begin{aligned}
\mathbb{P}(\{\ X \text{ is even }\}) &= \sum_{n=0}^{\infty} \mathbb{P}(X = 2n) \\
&= \sum_{n=0}^{\infty} \frac{\lambda^{2n}}{(2n)!} e^{-\lambda} && \text{Definition of Poisson distribution} \\
&= e^{-\lambda} \sum_{n=0}^{\infty} \frac{\lambda^{2n}}{(2n)!} \\
&= e^{-\lambda} \left(\frac{\lambda^0}{0!} + \frac{\lambda^2}{2!} + \frac{\lambda^4}{4!} + \frac{\lambda^6}{6!} + ... \right)
\end{aligned}
$$

Question assigned to the following page: 4

$$= e^{-\lambda}\left(1 + \frac{\lambda^2}{2!} + \frac{\lambda^4}{4!} + \frac{\lambda^6}{6!} + ...\right)$$

Thus, the problem reduces to determining a closed form for the power series $1 + \frac{\lambda^2}{2!} + \frac{\lambda^4}{4!} + \frac{\lambda^6}{6!} + ...$ (we know it converges because it is strictly less than the power series for e^λ).

We claim that this is the power series for $\frac{e^\lambda + e^{-\lambda}}{2}$. Indeed, we have

$$\begin{aligned}
e^\lambda &= 1 + \lambda + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \frac{\lambda^4}{4!} + \frac{\lambda^5}{5!} + ... \\
e^{-\lambda} &= 1 - \lambda + \frac{\lambda^2}{2!} - \frac{\lambda^3}{3!} + \frac{\lambda^4}{4!} - \frac{\lambda^5}{5!} + ... \\
e^\lambda + e^{-\lambda} &= 2 + 2\left(\frac{\lambda^2}{2!}\right) + 2\left(\frac{\lambda^4}{4!}\right) + ... && \text{Absolute convergence allows us to rearrange} \\
\frac{e^\lambda + e^{-\lambda}}{2} &= 1 + \frac{\lambda^2}{2!} + \frac{\lambda^4}{4!} + ...
\end{aligned}$$

Plugging this back into our expression for the probability, we get

$$\begin{aligned}
\mathbb{P}(\{\ X \text{ is even }\}) &= e^{-\lambda}\left(\frac{e^\lambda + e^{-\lambda}}{2}\right) \\
&= \boxed{\frac{1 + e^{-2\lambda}}{2}}
\end{aligned}$$

Part (c)

Compute the probability that $X \sim \text{Binomial}(n, p)$ is even.

Our calculation is

$$\begin{aligned}
\mathbb{P}(\{\ X \text{ is even }\}) &= \sum_{k=0}^{\lfloor n/2 \rfloor} \mathbb{P}(X = 2k) \\
&= \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} p^{2k}(1-p)^{n-2k} && \text{Definition of Binomial distribution}
\end{aligned}$$

To compute this sum, we need to find some way of only summing the binomial expansion terms with even exponent on p. Let $x = p$ and $y = 1 - p$. This reduces to finding some way to extract only the terms of even x-degree from $(x + y)^n$. By the Binomial Theorem, we have

$$(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

Questions assigned to the following page: 4 and 5

$$
\begin{aligned}
((-x)+y)^n &= \sum_{k=0}^{n} \binom{n}{k} (-x)^k y^{n-k} \\
&= \sum_{k=0}^{n} \binom{n}{k} (-1)^k x^k y^{n-k}
\end{aligned}
$$

Note that in the second binomial expansion, the terms are negative exactly when k is odd. Thus, we have

$$
\begin{aligned}
(x+y)^n + ((-x)+y)^n &= \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} + \sum_{k=0}^{n} \binom{n}{k} (-1)^k x^k y^{n-k} \\
&= \sum_{k=0,\ k \text{ even}}^{n} 2\binom{n}{k} x^k y^{n-k} \\
&= 2 \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} x^{2k} y^{n-2k}
\end{aligned}
$$

Thus we get

$$
\frac{(x+y)^n + (-x+y)^n}{2} = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} x^{2k} y^{n-2k}
$$

which contains the terms we are looking for. Plugging this formula back into our calculation for the probability with $x = p$ and $y = 1 - p$, we get

$$
\begin{aligned}
\mathbb{P}(\{\ X \text{ is even }\}) &= \sum_{k=0}^{\lfloor n/2 \rfloor} \mathbb{P}(X = 2k) \\
&= \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} p^{2k} (1-p)^{n-2k} \\
&= \frac{(p + (1-p))^n + (-p + (1-p))^n}{2} \\
&= \frac{1^n + (1-2p)^n}{2} \\
&= \boxed{\frac{1 + (1-2p)^n}{2}}
\end{aligned}
$$

Problem 5

I carry out a sequence of independent Bernoulli(p) trials. Independently of me, you carry out a sequence of independent Bernoulli(q) trials. Here $p, q \in (0, 1)$. What is the probability we both have our first success after the same number of trials?

Questions assigned to the following page: 5 and 6

Nathan Leung, UID 005835316
MATH 170A Homework 4
2 February 2024

Let X denote the random variable corresponding to the number of trials until my first success, and let Y denote the random variable corresponding to the number of trials until your first success. By the definition of a geometric distribution (the number of Bernoulli trials required for first success), we can write $X \sim \text{Geometric}(p)$ and $Y \sim \text{Geometric}(q)$. We want to compute

$$\begin{aligned}
\mathbb{P}(X = Y) &= \mathbb{P}(X = 1 \cap Y = 1) + \mathbb{P}(X = 2 \cap Y = 2) + ... \\
&= \sum_{i=1}^{\infty} \mathbb{P}(X = i \cap Y = i) \\
&= \sum_{i=1}^{\infty} \mathbb{P}(X = i)\mathbb{P}(Y = i) && \text{By independence} \\
&= \sum_{i=1}^{\infty} p(1-p)^{i-1} \cdot q(1-q)^{i-1} && \text{Definition of geometric distribution} \\
&= pq \sum_{i=1}^{\infty} (1-p)^{i-1}(1-q)^{i-1} \\
&= pq \sum_{i=1}^{\infty} \left((1-p)(1-q)\right)^{i-1} \\
&= pq \sum_{i=0}^{\infty} \left((1-p)(1-q)\right)^{i} && \text{Reindex sum} \\
&= pq \left(\frac{1}{1-(1-p)(1-q)}\right) && \text{Sum of geometric series} \\
&= \frac{pq}{1-(1-p-q+pq)} \\
&= \frac{pq}{1-1+p+q-pq} \\
&= \boxed{\frac{pq}{p+q-pq}}
\end{aligned}$$

Problem 6

Consider an infinite series of independent Bernoulli(p) trials, $0 < p < 1$. Given $r \in \mathbb{N}$, we define the random variable Y_r to be the number of trials completed when we achieve the rth success. Find the PMF for Y_r.

When $r = 1$, this reduces to the geometric distribution; i.e. $Y_r \sim \text{Geometric}(p)$ with $p_{Y_r}(k) = p(1-p)^{k-1}$.

Fixing some $k \in \mathbb{N}$ (the input to the PMF), when $r > 1$, the problem becomes one of finding the probability of getting $r-1$ successes in the first $k-1$ trials followed by the rth success in the kth trial. For $k \geq r$, the probability of getting $r-1$ successes in the first $k-1$ trials is given by

$$\mathbb{P}(\{r-1 \text{ successes in the first } k-1 \text{ trials}\})$$

Question assigned to the following page: 6

$$
\begin{aligned}
&= \underbrace{\binom{k-1}{r-1}}_{\text{ways to choose } r-1 \text{ successful trials out of } k-1 \text{ total}} \underbrace{p^{r-1}(1-p)^{(k-1)-(r-1)}}_{\text{probability of each run of } r-1 \text{ successes out of } k-1 \text{ trials}} \\
&= \binom{k-1}{r-1} p^{r-1}(1-p)^{k-1-r+1} \\
&= \binom{k-1}{r-1} p^{r-1}(1-p)^{k-r}
\end{aligned}
$$

Then, the probability of the kth trial being a success (the rth success, since $r-1$ successes already occurred), is simply p (since the trials are independent). Again invoking independence, we can get the overall pmf for Y_r by multiplying the two probabilities together:

$$
\begin{aligned}
p_{Y_r}(k) &= \binom{k-1}{r-1} p^{r-1}(1-p)^{k-r} \cdot p \\
&= \binom{k-1}{r-1} p^{r}(1-p)^{k-r}
\end{aligned}
$$

However, note that this only is defined for $k \geq r$. If $k < r$, then the probability is 0 (since it is impossible to get $r > k$ successes in only k trials). Thus, our final pmf is

$$
p_{Y_r}(k) = \begin{cases} \binom{k-1}{r-1} p^r (1-p)^{k-r} & k \geq r \\ 0 & \text{otherwise} \end{cases}
$$