Topology Notes

March 2024

1 Definitions and Motivation

Definition 1. Let X be a set. $\tau \subseteq \mathcal{P}(X)$ is said to be a **topology on** X if

1. $\varnothing, X \in \tau$
2. τ is closed under unions.
3. τ is closed under finite intersections.

Equivalently, we can define a topology using its closed sets. See Exercise 17.1 in Monkres Topology.

Example 1.0.1. A topology where the only open sets are $\varnothing$ and X is called the **trivial (indiscrete) topology**.

Example 1.0.2. Let X be a set. Let τ be the set of all cofinite subsets of X. (X, τ) is called the **cofinite (finite complement)** topology of X.

Similar to the example above, we can define an open set as having a countable complement and get a topology on X.

Definition 2. Given an ordered set X, a subset Y of X is said to be **convex** if for all $a < b$ in Y, $(a, b) \subseteq Y$.

Intervals and rays are convex in X.

2 Basis of a Topology

Definition 3. Let X be a set. A basis $\mathcal{B}$ for (X, τ) is a collection of subsets of X such that

1. $\forall x \in X : \exists B \in \mathcal{B} : x \in B$
2. $\forall x \in X : \forall B_1, B_2 \in \mathcal{B} : x \in B_1 \cap B_2 \implies (\exists B_3 \in \mathcal{B} : x \in B_3 \wedge B_1 \cap B_2 \subseteq B_3)$

Given a basis $\mathcal{B}$, we can generate a topology of τ by defining the open sets as in the following lemma:

Lemma 2.1. $A \in X$ is open if and only if $\forall a \in A : \exists U \in \tau : a \in U \wedge U \subseteq A$.

In other words, A is open if and only if every point in A is contained in an open ball that's contained in A.

Proof. The forward implication is trivial, just pick $U = A$. Let's now prove the reverse implication. Assume a set A satisfies the condition on the right-hand side. Let U_a be the open set containing a. Notice that $A = \bigcup_{a \in A} U_a$, so A is a union of open sets and therefore open. $\square$

3 Subspaces

Lemma 3.1. Let X be an ordered set with the order topology and Y be a convex subset of X. Then, the order topology on Y is the same topology as the one Y inherits as a subspace of X.

4 Continuity

The continuity of a function $f : X \to Y$ doesn't depend only on the function but on the topologies put on X and Y.

Definition 4. A function $f : X \to Y$ is **continuous** if it preimages open sets of Y into open sets of X. More formally,

If $\mathcal{B}$ is a basis for Y, it suffices to show that f preimages every $B \in \mathcal{B}$ to an open set of X.

5 Homeomorphisms

Definition 5. Let X, Y be topological spaces and $f : X \to Y$ be a homeomorphism.

Definition 6. Let X, Y be topological spaces $f : X \to Y$ be an injective continuous function. If $f' : X \to f(X)$ is an homeomorphism, f is called a **topological imbedding**.

Lemma 5.1. Let X, Y be metric spaces and $f : X \to Y$ be an isometry. Then, f is an imbedding.

Proof. f is (uniformly) continuous and injective. Clearly, every restriction of f is also continuous. □

Lemma 5.2. Let $f : X \to Y$ be a homeomorphism and Z be a subspace of X. Then, f restricred to Z is still an homeomorphism.

Lemma 5.3. $[a, b] \simeq (c, d)$

Proof. Assume by contradiction that there exists a homeomorphism $f : [a, b] \to (c, d)$.

□

6 Closed Sets

Lemma 6.1. If A is closed in Y and Y is closed in X, A is closed in X.

7 Connectedness

Definition 7. Let (X, τ) be a topological space. X is said to be **disconnected** if there are disjoint open sets U, V such $U \cup V = X$.

Definition 8. X is said to be **connected** if it's not disconnected.

Lemma 7.1. *X is connected if and only if the only clopen sets in X are $\varnothing$ and X.*

Definition 9. X is said to be separated to disjoint sets A, B if A, B don't contain each other's limit points and $A \cap B = X$.

Lemma 7.2. *X is connected if and only if it doesn't separate into two sets A, B.*

Lemma 7.3. *Let C and D be a separation of X and Y be a connected subspace of X. Then, Y lies entirely in C or D.*

Proof. Notice that $C \cap Y$ and $D \cap Y$ are disjoint open sets in Y and notice that their union is Y. $\square$

Lemma 7.4. *The union of a collection of subspaces A_α that have a point in common is connected.*

7.1 Exercises

Let τ, τ' be two topologies on X. If τ is not connected, τ' is not connected. If τ' is connected, τ is also connected.

Lemma 7.5. *Let A_n be a sequence of connected subspaces of X. Assume that $A_n \cap A_{n+1} \neq \varnothing$. Prove that $A = \bigcup_{n=1}^{\infty} A_n$ is connected.*

The intuition is the following: the common point between consecutive subspaces acts as a bridge. I think this proof can be formalized by using contradiction and then getting two points in two sets, then arguing they have to be connected.

Proof. $\square$

Lemma 7.6. *Let X be a topological space and $A_{\alpha \in I}$ be a collection of connected subspaces. Let A be a connected subspace with a non-trivial intersection with all A_α. Then, $A \cup A_{\alpha \in I}$ is also connected.*

Here, it's as if we have a single connection point connecting all the other sets together.

Proof. $\square$

Let X be an infinite set in the cofinite topology. Then, X is connected since otherwise X would be finite.

Definition 10. A space is **totally disconnected** if the only connected subspaces are one-point sets.

Notice that if we equip a set with the discrete topology, it's totally disconnected.

8 Connected Subspaces of the Real Line

Definition 11. A simply ordered set L containing more than one element is called a **linear continuum** if the following hold:

1. L has the least upper bound property.

chimedean Property For any two elements $x < y$, $\exists z \in L : x < z < y$.

Theorem 8.1. If L is a linear continuum in the order topology, every convex subspace of L is connected.

Proof. □

Lemma 8.2. $(0,1), [0,1), [0,1]$ are not homemorphic.

Proof. Consider a homemorphism $f : [0,1) \to (0,1)$. Then, the restriction of this homemorphism on $(0,1)$ is still an homemorphism. However, the image of this map is no longer connected.

We can use a similar argument for the rest of the arguments. □

9 Compactness

Definition 12. Let X be a topological space. X is called **separable** if it has a countable dense subset.

10 Convergence in Topology

Definition 13. A sequence x_n is said to converge to x if for every neighborhood U of x, there's some $N \in \mathbb{N} : \forall n \geq N : x_n \in U$.

11 Separation Axioms

The separation axioms try to mimic the properties of metric spaces.

Definition 14. A topological space is said to be T_0 if any two distinct points in X are topologically distinguisable.

Definition 15. A topological space X satisfies T_1 if for any $x, y \in X$ we can find a neighborhood U of x such that $y \notin U$.

Lemma 11.1. A topological space X satisfies T_1 if and only if every singleton set is closed.

Corollary 11.1.1. If a topological space X satisfies T_1, every finite point set of X is closed.

Theorem 11.2. Let X be a space satisfying T_1 and $A \subseteq X$. Then, x is a limit point of A if and only if every neighborhood of x contains infinitely many points of A.

Proof. Assume x is a limit point of A and assume by contradiction that there's some neighborhood U of x such that U contains finitely many points of A. □

Definition 16. A topological space X satisfies T_2 or is **Hausdorff** if for any $x, y \in X$ we can find disjoint open sets U, V such that $x \in U$ and $y \in V$.

Lemma 11.3. Sequential limits in Hausdorff spaces are unique.

Proof. Let x_n be a sequence in X and assume x_n converges to x. Let $y \neq x$. Then, there's some U_x, U_y disjoint neighborhoods of x and y. Since x_n converges to x, there's some $N \in \mathbb{N} : \forall n \geq N : x_n \in U_x \implies x_n \notin U_y$. Thus, x_n doesn't converge to y. □

Lemma 11.4. Every simply ordered set is a Hausdorff space in the order topology.

Lemma 11.5. Hausdorff spaces are closed under products.

Lemma 11.6. Subspaces of Hausdorff spaces are Hausdorff.

Lemma 11.7. X is Hausdorff if and only if the diagonal $\Delta = \{(x, x) : x \in X\}$ is closed in $X \times X$.

Proof. □

Definition 17. A topological space X satisfies T_3 or is **regular** if for any point a and closed set $B \subseteq X$ there are disjoint open sets U, V such that $a \in U$ and $B \subseteq V$.

Example 11.7.1. The Zariski topology on $\mathbb{R}$ is not Hausdorff.

Definition 18. A topological space X satisfies T_4 or is **normal** if for any two closed sets $A, B \subseteq X$ there are disjoint open sets U, V such that $A \subseteq U$ and $B \subseteq V$.

Lemma 11.8.

Lemma 11.9. Every metric space is normal.

Proof. □

12 Countability Axioms

Definition 19. A topological space X is said to be **second-countable** if it has a countable basis.

Not every metric space has a second-countable basis.

Theorem 12.1. Assume X is second-countable. Then,

1. Every open covering of X has a countable subcover.
2. X is separable.

Proof. $\square$