Rented Today, Bought Tomorrow: Buyout Pricing in the Circular Economy

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Online rental services that allow customers to purchase rented goods present a complex pricing challenge: setting buyout prices that balance immediate sales revenue with future rental income, while accounting for item-specific factors such as condition, popularity, and customer preferences. In this paper, we develop, estimate, and validate a data-driven framework to inform buyout prices in this setting. Leveraging a Markov Decision Process (MDP), our framework assesses individual item value based on rental demand, product damage, and customer purchase likelihood. We use real-world data from a leading fashion rental company to demonstrate that our methodology significantly improves profitability compared to existing practices and alternative benchmarks. We estimate that the proposed pricing policy increases earnings by 3.1% over the company's current practice. Our analysis also shows that operating a rental-only business model leaves revenue opportunities untapped, underscoring the strategic value of buyout options in managing inventory and generating additional income.

Key words: retail operations; pricing policy; circular economy

1. Introduction

Online rental services that offer customers the option to purchase rented goods present a complex challenge for managers: setting prices in the presence of competing revenue streams. Unlike traditional retailers, which generate income exclusively from sales, these firms must balance revenue from both renting and selling the same goods. Pricing in this context requires assessing each individual item's future revenue potential and translating that assessment into a tailored buyout price. Because rental value is shaped by item-specific characteristics such as condition, usage history, and customer appeal, pricing decisions cannot be made at the SKU level; managers must evaluate the expected performance of each item individually. This challenge is further complicated by factors such as wear-and-tear, rental duration, and shifting consumer preferences, all of which affect both an item's remaining rental value and its attractiveness for purchase.

In this paper, we propose a data-driven framework that provides a structured approach to analyzing and solving this problem. Our framework evaluates item-specific attributes, enabling managers to assess the rental value of each individual item and implement an effective pricing strategy in this context. To validate the effectiveness of this approach, we evaluate the model using real-world data from a leading subscription-based fashion rental service, demonstrating how our framework enhances decision-making and drives profitability. While our industry partner operates on a subscription model, the framework is broadly applicable to any rental firm, whether subscription-based or one-off, that offers customers the option to purchase rented items.

Rental subscription services allow customers to rent multiple products for a specified period of time in exchange for a recurring fee, offering a flexible and cost-effective alternative to ownership. By choosing to rent rather than purchase, customers enjoy access to a variety of products at a fraction of the cost while avoiding the "burdens" of ownership, which include higher upfront costs, limited product options, responsibility for maintenance and repairs, and the challenges of storage or disposal (Berry and Maricle 1973, Moeller and Wittkowski 2010).

The rental subscription model has expanded rapidly across industries, reflecting a broader consumer shift from ownership to flexibility and access. In the fashion industry, companies like ThredUp, Nuuly, and Rent the Runway offer customers the opportunity to rent clothing and accessories on a monthly basis, allowing for a dynamic wardrobe without committing to long-term ownership. Similarly, in electronics, companies such as Grover provide subscription-based rentals for devices like laptops, smartphones, and gaming consoles, enabling consumers to stay updated with the latest technology without incurring substantial upfront costs.

Beyond fashion and technology, subscription rental businesses are flourishing in a range of other industries. Furniture companies like Feather and Fernish allow customers to rent high-quality furniture pieces for flexible time durations, catering to those who relocate frequently or desire variety without long-term commitments. Similarly, the fitness industry has embraced this model with companies like Peloton and Tonal offering rentals for equipment alongside subscription-based access to digital workout programs. Even in outdoor recreation, these services provide camping and sporting equipment on a subscription basis, enabling adventure enthusiasts to access costly gear without purchasing it outright.

Also known as *rentailers* (Knox and Eliashberg 2009), rental firms with a purchase option face a pricing problem defined by the tension between rental and sales revenues. Selling an item generates immediate income, but retaining it preserves the potential to earn through future rentals. In this context, buyout prices serve as both a revenue lever and a mechanism for inventory management, making careful calibration essential to balance short-term gains with long-term profitability.

To address this tension, we develop a structured framework that prescribes buyout prices at the individual item level. Our approach integrates item-specific attributes with customer behavior insights and leverages a Markov Decision Process (MDP) to capture the dynamic interplay of rental demand, product damage, and purchase likelihood. This methodology enables firms to balance immediate sales revenue with the long-term value of recurring rentals in a systematic and data-driven way.

Although the framework is broadly applicable across product categories, we evaluate and validate it in the fashion industry, an ideal setting given the rapid depreciation of items (which we refer to as *garments*), their sensitivity to condition and customer perception, and the operational constraints that firms face when managing inventory. Partnering with a leading fashion rental company, we use extensive operational and transactional data to test the model and demonstrate its ability to guide buyout pricing decisions in practice.

Our study advances both the theory and practice of rental operations and dynamic pricing. First, we introduce a new class of pricing problems where a firm must simultaneously account for two competing revenue streams: rentals and sales. In contrast to traditional retail, pricing in this context requires dynamically balancing short-term sales revenue with long-term rental income – a fundamental tension not previously studied in the literature.

Second, we propose a tractable pricing framework tailored to the unique requirements of hybrid rental-sale firms. A key innovation of our approach is that it prescribes buyout prices at the individual item level rather than at the broader product or product-class level. Because garments within the same product (referred to as *SKU*) can vary substantially in condition and usage history, their economic value is not interchangeable. By pricing at the garment level, our framework captures these differences directly, ensuring that buyout decisions reflect the true rental and sales potential of each garment.

Third, the framework is designed to be scalable. Exact dynamic programming formulations quickly become infeasible as inventory increases, but our methodology remains tractable by using approximations that preserve the core economics of the problem. This scalability enables firms to apply item-level pricing across thousands of SKUs and millions of garments, making the approach practical for large-scale operations. In this way, the framework balances analytical rigor with operational implementability.

Finally, we validate the framework with a sample of real-world products from our industry partner. The results demonstrate our methodology's ability to increase earnings in hybrid rental-sale operations, outperforming the company's current practice by 3.1%. The benefits are most pronounced under inventory imbalances: when stock is low, the framework discourages purchases to preserve rental capacity, and when stock is high, it encourages sales to free space for more profitable styles. We further show that a rental-only model would significantly reduce revenues, highlighting buyouts' dual role as both an income source and a strategic tool for inventory management.

The rest of the paper is organized as follows. Sections 2 and 3 review the relevant literature and describe the business setting. Section 4 illustrates the fundamental trade-off in this problem and outlines the full, albeit intractable, formulation of a rentailer's buyout pricing problem. In Section 5, we develop a practical and tractable framework to optimize buyout prices in the single-garment setting, which we later extend to accommodate multiple garments in Section 6. We discuss the empirical estimation of the model parameters and assess the performance of our proposed framework in Sections 7 and 8, respectively. Section 9 concludes the paper.

2. Related Work

Our study explores a dual-revenue business model where the timing of sales critically impacts future rental revenues, creating a trade-off that traditional markdown pricing models do not address. The foundational work by Gallego and Van Ryzin (1994) introduced dynamic pricing for inventory management under stochastic demand, forming the basis of much subsequent research in revenue management and pricing. However, these models typically assume a single revenue stream, where income is earned solely through sales, and the timing of sales is not critical beyond inventory constraints.

Dynamic pricing has been applied extensively to rental services, which involve shared resources and introduce complexities such as multiple customer classes (Gans and Savin 2007), heterogeneous resources (Rusmevichientong et al. 2023), deterministic or stochastic rental durations (Lei and Jasin 2020), and advance reservations (Chen et al. 2017b). In a rental-only settings, Besbes et al. (2019) explore static pricing policies for reusable resources, while Balseiro et al. (2025) show that a simple two-price strategy – depending on whether the inventory level is high or low – can approximate optimal solutions despite the computational challenges. These studies primarily address the tension of allocating resources between current and future customers (with a potentially higher willingness to pay), a trade-off relevant in contexts such as cloud computing (Xu and Li 2013).

Our focus diverges from rental-only models by incorporating the option to sell resources, creating trade-off between rental and sales revenues that makes the pricing challenge fundamentally different. Rent-to-own (RTO) firms operate in a similar context, allowing customers to rent products with an option to purchase them over time. Armaghan et al. (2023) analyze optimal buyout pricing paths for RTO firms, addressing the tension between rental and sales revenues. However, there are key differences between RTO and subscription rental models. RTO involves repeated interactions with the same customer and typically applies prior rental payments toward purchase, effectively functioning as a financing plan where returned products are not re-rented to other customers in the future. In contrast, in subscription rental models, customers get one chance to purchase the product and must return it at the end of the rental period if they choose not to, ensuring the good remains available for other renters.

Knox and Eliashberg (2009) examine rent-versus-buy decisions in the video rental industry, emphasizing how pricing strategies shape customer choices. While their model considers trade-offs from both retailer and customer perspectives, it assumes infinite product supply and overlooks factors like inventory constraints, depreciation, and damage, which are critical aspects in a fashion rental service.

Beyond pricing, the sharing economy has garnered attention for its ability to expand access to goods, improve resource efficiency, and generate income for owners (Botsman and Rogers 2010, Edelman and Geradin 2015, Sundararajan 2016). The studies by Berry and Maricle (1973) and Moeller and Wittkowski (2010) emphasize that renting offers flexibility and affordability while avoiding ownership burdens, such as maintenance and storage. This flexibility is particularly important in trend-driven industries like fashion, where demand fluctuates rapidly based on style and seasonality (Lovelock and Gummesson 2004, Manning et al. 1995, Oliva and Kallenberg 2003). Varian (2000) and Filippas et al. (2020) analyze models that explain how service providers and end users can benefit from the sharing economy. Relatedly, Jain et al. (2025) characterize the market conditions under which a firm is better off running a subscription-based rental operation versus a pure-sales retail model.

Rental business models often involve complex supply chain dynamics. Cachon and Lariviere (2005), Gerchak et al. (2006), Dana and Spier (2001) highlight how revenue-sharing contracts can align supplier and retailer incentives, while Mortimer (2008) empirically demonstrates the profitability of such contracts in the home video rental industry. Inventory management in closed-loop systems presents further challenges, as items are continuously rented and returned. Tainiter (1964) developed early models to optimize inventory under uncertain demand, which were later expanded by Chen et al. (2017a) to address multi-item inventory control with advance demand information. Randhawa and Kumar (2008) show that a rental company's profits can be higher with subscription-based services compared to pay-per-use models and examine how usage restrictions in such services affect system efficiency, balancing customer satisfaction with operational performance.

While our framework is broadly applicable across product categories, our analysis focuses on the fashion apparel industry. Fashion goods depreciate quickly due to wear-and-tear, making their value highly sensitive to condition, customer perception, and the risk of irreparable damage. Prior work offers valuable insights into managing these dynamics. Slaugh et al. (2016) model usage-based loss from damage or purchase in rental settings, while Altug and Ceryan (2022) study how to allocate inventory between rental and sale in industries with short product lifecycles such as fashion.

Nageswaran et al. (2023) provide empirical evidence that incorporating consumer preferences into assortment and inventory planning improves outcomes for fashion rentailers, and Xu et al. (2025) develop a theoretical framework for personalized assortment recommendations that accounts for fit uncertainty. Collectively, these studies underscore the importance of preference-aware assortment strategies in enhancing customer satisfaction and retention – critical drivers of success in fashion rental businesses.

3. Business Setting and Fashion Rental Operations

The fashion rental industry has undergone remarkable growth since its emergence in the late 2000s, with pioneers such as Rent the Runway, ThredUp, Le Tote, GlamCorner, Armoire, HURR, Vivrelle, and Nuuly – to name a few – leading the way. Despite differences in geographic reach, product offerings, pricing structures, and rental policies, these companies share a common business model that prioritizes flexibility, access, and circularity over long-term ownership.

When customers subscribe to a fashion rental service, they gain access to a curated online catalog from which they can select a number of garments, determined by the firm's subscription plan. Because a portion of the firm's inventory is rented out at any given time, customers can only rent products that are in stock at the distribution center when they place the order.

Once an order is placed, the garments are professionally packaged and shipped in branded containers. Customers may use them for the duration of the rental term (one month for our partner), after which the garments are returned to the firm. At the distribution center, garments undergo professional cleaning and damage inspection; those that pass are reintroduced into inventory and made available for future rentals.

Garments can leave the system through two primary channels. First, those that fail the inspection due to wear-and-tear, stains, or missing components (e.g., buttons or belts) are offloaded to partner thrift stores, generating salvage revenue. Second, customers have the option to purchase rented garments. Upon receiving their order, customers are informed of a buyout price for each garment, which they can pay to keep it permanently. Customers must return to the firm the garments they do not desire to purchase, so that they can be rented to other customers in the future. This cyclical process (see Figure 1) reflects the core principles of the circular economy, with a focus on extending the lifecycle of garments through repeated use and providing customers with the flexibility to experiment with fashion trends at an accessible price, while minimizing waste.

Fashion rental companies operate on a hybrid business model with two distinct revenue streams: rental and sales. These streams are inherently in conflict, as selling a garment to a customer means the firm can no longer rent it in the future, while retaining the garment for rental foregoes immediate sales income. This creates a fundamental tension that lies at the heart of the buyout

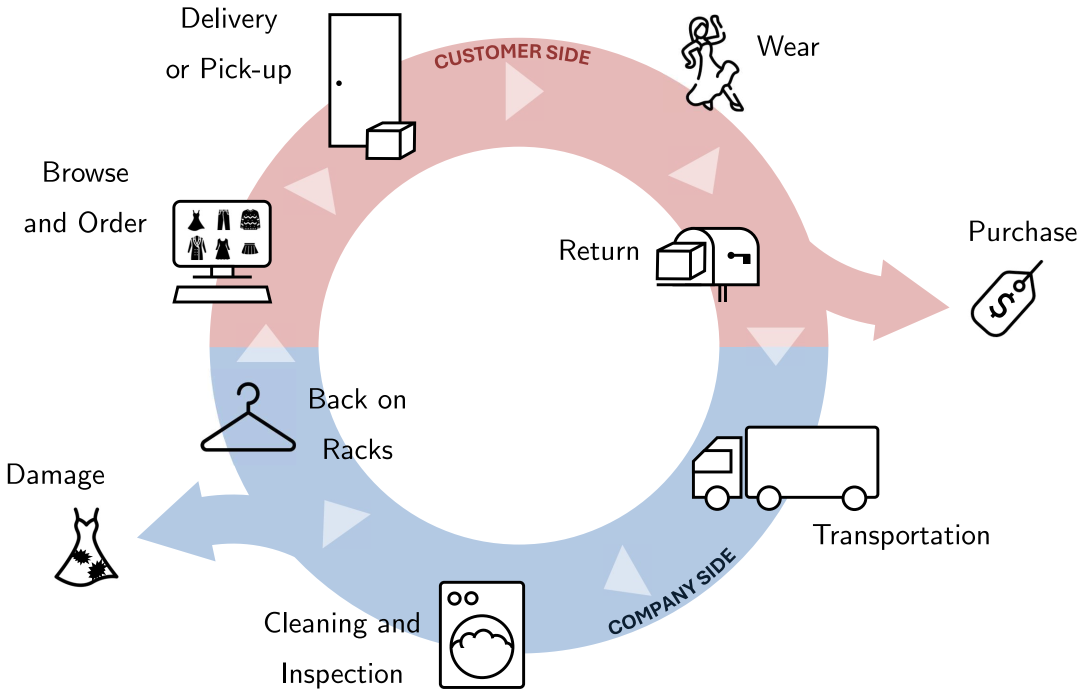

Figure 1 **Representation of the firm's business model.**

pricing problem. If the buyout price is set too low, customers are more likely to purchase the garment, leading to a loss of potential rental income. Conversely, if the buyout price is set too high, customers may be discouraged from purchasing the garment, leaving it in the system and potentially underutilized. Striking the right balance is crucial, as the optimal buyout price must account for both current and future revenue opportunities.

The pricing challenge is further complicated by the semi-closed-loop nature of the business model. In this system, garments remain in inventory while they undergo multiple rental cycles before eventually exiting the system, either due to damage or through customer purchase. This structure introduces operational complexities, as the firm must continuously maximize the value extracted from each garment during its lifecycle and effectively manage inventory replenishment when garments are bought out by customers or disposed of by the firm.

A further complication is that each garment accumulates a unique rental history that directly influences its condition – and therefore its economic value. Successive rentals can increase the risk of damage, diminish customer appeal, and shorten a garment's useful life. As a result, garments within the same SKU are not interchangeable: for instance, a blouse that has never been rented is more valuable than an otherwise-identical one that has endured a dozen rentals and shows visible wear. Effective pricing must therefore be determined at the individual garment level, rather than for the SKU as a whole.

Finally, fashion rental companies face practical constraints, such as budget limits and finite warehouse space, which make efficient inventory management essential. Buyout pricing can serve as a strategic lever in this process by shaping customer behavior. Lower prices on less popular or less durable garments encourage purchases that free space for more profitable styles, while higher prices on high-demand garments discourage immediate buyouts and preserve rental opportunities that generate greater long-term revenue.

4. The Buyout Pricing Problem

The coexistence of two competing revenue streams lies at the core of the buyout pricing problem. In this section, we begin with a simple model to illustrate this tension and then present the full characterization of the problem, which – though intractable – serves as the foundation for the framework developed in Section 5.

4.1. Motivating Example

Consider a fashion rental firm that rents out a single product and owns $q = 2$ identical garments of this product. For simplicity, we illustrate the model with two garments, though extending these results for $q > 2$ is straightforward.

Time is divided into discrete periods, each corresponding to the duration of the firm's rental term. At the beginning of each period, rental demand is realized according to a random variable with cumulative distribution function $F(\cdot)$ and probability mass function $f(\cdot)$. If rental demand exceeds the number of garments in inventory, excess rental demand is lost. Suppose that garments do not age as they fulfill rentals (i.e., this model is agnostic of the condition of the garments) and never get damaged, so they can only exit the system when they are purchased by a customer. We later relax these assumptions.

Every time a garment is rented, the firm collects a rental revenue r and assigns the garment a buyout price p that the customer can pay to keep the garment after the rental. Let $d(p)$ denote the probability that the customer purchases the garment. We assume that the firm must select one of two possible buyout prices, p_L and p_H, where $p_L < p_H$, such that the customer always purchases the garment when offered the low buyout price p_L, and never does so when offered the high buyout price p_H – that is, $d(p_L) = 1$ and $d(p_H) = 0$.

The firm's goal is to determine a buyout pricing policy that maximizes the discounted (with factor γ) earnings generated from renting and selling its inventory of the product. Table 1 presents the solution to this problem. The full mathematical details can be found in the e-companion to this paper.

Table 1 **Optimal policy in the two-product, condition-agnostic buyout pricing problem with determinist exits.**

Number of garments owned	1 owned	2 owned	
Number of garments rented	1 rented	1 rented	2 rented
Probability	$\bar{F}(0)$	$f(1)$	$\bar{F}(1)$
$p_L \leq \frac{r}{1-\gamma}\bar{F}(1)$	p_H	p_H	p_H, p_H
$p_L \in \left[\frac{r}{1-\gamma}\bar{F}(1),\ \frac{r}{1-\gamma}\bar{F}(0)\right]$	p_H	p_L	p_L, p_H
$p_L \geq \frac{r}{1-\gamma}\bar{F}(0)$	p_L	p_L	p_L, p_L

The optimal buyout pricing policy depends on the price p_L that customers are willing to pay to purchase a garment. Specifically, the firm must compare the one-time reward from selling the garment, p_L, to the rental value it could obtain from retaining it. The *marginal* rental value of the q^{th} garment, that is, incremental rental revenue the firm earns from retaining a garment in its inventory given that it already has $q-1$ other garments of the same product, is given by:

$$\frac{r}{1-\gamma} \times \bar{F}(q-1) \tag{1}$$

where $\bar{F}(q-1) = 1 - F(q-1)$ denotes the probability that the rental demand in a given period exceeds $q-1$ garments.

This trade-off gives rise to a threshold structure in the optimal policy: depending on the value of p_L and the distribution of rental demand, it becomes optimal for the firm to retain a certain number of garments for rentals and sell the rest. If $p_L \leq \frac{r}{1-\gamma}\bar{F}(1)$, the selling price is so low that the firm is better off keeping both garments for rentals. In contrast, if $p_L \geq \frac{r}{1-\gamma}\bar{F}(0)$, the selling price is sufficiently high that the firm would rather sell both garments. For intermediate prices, it is optimal for the firm to hold on to one garment and sell the other – which is achieved by initially pricing one garment at p_L and the other at p_H, and then always setting the high price p_H when only one garment is left in the system.

This example uncovers two key differences between pricing in this context as opposed to traditional retail. First, it can be optimal for two identical garments rented at the same time to receive different buyout prices. Second, the average buyout price can increase over time, since garments become more valuable for rentals as inventory levels decline.

More critically, the threshold policy in Table 1 highlights that the firm is willing to sell garments only when doing so does not compromise its ability to satisfy future rental demand. With abundant inventory, selling has little effect on rental capacity, making it financially justifiable. By contrast, when inventory is scarce, the firm faces a trade-off: selling yields immediate revenue but eliminates the option to earn future rental income, creating an opportunity cost that may warrant setting buyout prices high enough to discourage purchases altogether.

Unlike the classic dynamic pricing problem introduced by Gallego and Van Ryzin (1994), where selling a garment now only prevents the retailer from selling it later, our setting involves two competing revenue streams – rentals and sales – with interdependent value implications. In their framework, when a customer purchases a garment, it generates immediate sales revenue and the retailer is relieved of any holding costs for that garment in the future, making the (negative) contribution of the existing inventory to the revenues linear in the inventory level. In contrast, in our setting, the extent to which selling a garment reduces future rental earnings depends on the inventory level: when stock is plentiful, future rentals remain largely unaffected, but as inventory tightens, each sale directly reduces the firm's ability to meet rental demand. For this reason, instead of an extension to their classic problem, we argue that our study introduces a distinct class of dynamic pricing problems characterized by a fundamentally different revenue structure – one in which the optimal pricing decision must balance dynamically evolving trade-offs between recurring rental revenues and one-off sales income, thereby necessitating new modeling tools and solution approaches.

While the model presented in this section offers valuable intuition, it relies on simplifying assumptions that limit its realism. Specifically, it treats garment exits as deterministic rather than stochastic, restricts prices to discrete values, and ignores how garments depreciate with use – that is, how condition affects both the likelihood of damage and a customer's willingness to buy. In the next section, we introduce a more comprehensive model that relaxes these assumptions to better capture the buyout pricing challenges faced by fashion rental businesses, and the complexities that render the problem intractable.

4.2. The Full Buyout Pricing Problem: A General Formulation

Consider a fashion rental service that operates a single product with q identical garments. As in the motivating example in Section 4.1, time is divided into discrete periods equal to the firm's rental term and, in each period, rental demand is a random variable with cumulative distribution function $F(\cdot)$. If rental demand exceeds the number of garments in inventory, excess demand is lost.

We refer to the number of rentals previously served by a garment as its *condition*. A garment's condition may affect the probability it incurs damage during a rental, as well as its desirability in the eyes of the customer – and hence the probability that she purchases it.

A garment in condition n that rents during a certain period becomes damaged with probability δ_n, in which case it is discarded and generates salvage revenue for the firm. In addition, when a garment is rented, the firm assigns it a buyout price that the customer can pay to keep it permanently. Let $d_n(p)$ denote the probability that the renter purchases a garment in condition n

at price p. The customer only considers purchasing a garment if it has not been damaged during the rental.

The sequence of events is as follows: at the beginning of each period, rental demand is realized, and the firm determines how much of this demand can be met using its existing inventory of garments. Consistent with practices observed at our industry partner, we assume that garments are randomly selected from the available inventory to fulfill rental orders. The selected garments are then sent out to customers, who use them throughout the period. Before the period ends, if a garment was not damaged during the rental, the customer must decide whether to purchase it. If the customer chooses not to buy the garment, it is returned to the firm, which will dispose of the damaged garments and make the ones returned in good condition available for customers to rent in the following period. Hence, at the end of the period, all rented garments are (1) irreparably damaged during the rental and removed from inventory by the firm, (2) purchased by the customer, or (3) returned in good condition and made available for other customers to rent in the next period.

The firm's problem can be expressed as a discounted, discrete-time Markov Decision Process (MDP) where the firm aims to find the buyout pricing policy that maximizes the earnings generated by operating – i.e., renting and selling – its inventory. The state of the system is represented by $\mathbf{q}$, a q-dimensional vector that contains the condition of each garment owned by the firm. We denote with $\boldsymbol{\sigma} \in \{0,\ 1\}^q$ the selection of garments that are rented in a period where the inventory state is $\mathbf{q}$. Note that, if the firm owns $q = |\mathbf{q}|$ garments, there are 2^q possible selections of garments that go out on a rental. In principle, the firm could choose which garments to allocate for rentals as a function of realized demand and the current inventory state $\mathbf{q}$. However, consistent with the practice of our partner, we treat $\boldsymbol{\sigma}$ not as a decision variable but as a random selection from the available inventory, with the likelihood of each possible selection depending on the distribution of demand.

The firm's goal is to determine the pricing policy $\mathbf{p}(\mathbf{q}, \boldsymbol{\sigma})$ that maximizes the discounted (with parameter γ) expected future revenue from rentals and sales, given by:

$$V(\mathbf{q}) = R(\mathbf{q}) + \gamma \sum_{\boldsymbol{\sigma}} \sum_{\mathbf{q}'} P(\boldsymbol{\sigma}|\mathbf{q}) \times P(\mathbf{q}' \mid \mathbf{q}, \boldsymbol{\sigma}, \mathbf{p}) \times V(\mathbf{q}') \tag{2}$$

The term $R(\mathbf{q})$ represents the immediate revenue that the firm collects from being in state $\mathbf{q}$. This includes rental, sales, and salvage revenues. We denote the rental income per garment per period as r, and the firm collects revenue s every time it disposes of a garment.

The term $P(\boldsymbol{\sigma}|\mathbf{q})$ denotes the likelihood of the selection of garments $\boldsymbol{\sigma}$ being picked to fulfill rental orders when the inventory state is $\mathbf{q}$, and $P(\mathbf{q}' \mid \mathbf{q}, \boldsymbol{\sigma}, \mathbf{p})$ represents the probability that the system transitions from state $\mathbf{q}$ to state $\mathbf{q}'$, given that the selection of garments $\boldsymbol{\sigma}$ was rented and the firm assigned buyout prices $\mathbf{p}$ to those garments. Consequently, the product $P(\boldsymbol{\sigma}|\mathbf{q}) \times P(\mathbf{q}' \mid \mathbf{q}, \boldsymbol{\sigma}, \mathbf{p})$ captures the transition probabilities of the MDP. These transition probabilities are hard to express and compute, as they must account for (1) all the possible values of rental demand that can occur, (2) all the possible combinations of garments that can be selected to fulfill that demand, and (3) all the possible combinations of damage and purchases that can take place.

The complexity of the transition probabilities arises from the fact that, at any point in time, each of the q garments in inventory may be in a different condition. For a garment in condition n, there are three possible outcomes in a given period: if it is not rented, it remains in condition n; if it is rented and returned undamaged, it transitions to condition $n+1$; and if it is rented and either damaged or purchased, it exits the system. Since each garment evolves independently across these outcomes, the system can transition to as many as 3^q possible states in each period.

Because a fashion rental firm may operate over $q = 300$ garments for a single SKU, the dimensionality of the state space renders exact dynamic programming solutions infeasible. To overcome this, we focus first on the single-garment case and develop a framework that captures the core rental-sales trade-off while remaining computationally tractable and implementable in practice. For an SKU with a single garment, the methodology yields the exact optimal buyout price; for SKUs with multiple garments, it provides a tractable approximation, as we elaborate in Section 6.

5. Buyout Pricing Framework for a Single Garment

Suppose a fashion rental firm must assign a buyout price to garment i in condition n. In each period, corresponding to the rental term, the rental demand for the garment is random with cumulative distribution function $F(\cdot)$.

In a given period, garment i rents with probability $\rho_i = 1 - F(0)$, and with probability $1 - \rho_i$ it does not rent. If garment i is not rented in a given period, it remains in the same condition into the next period. Conversely, when the garment is rented, the firm collects a rental revenue r and ships the garment to the customer. During this rental, the garment may incur irreparable damage with probability $\delta_{i,n}$. If that happens, the customer will not consider purchasing garment i and will return it to the firm, who will dispose of it at a salvage price s_i and replace it with a different brand new garment, referred to as the *outside good*.

If garment i is not damaged during the rental, which happens with probability $1 - \delta_{i,n}$, the customer has the option to purchase it at the buyout price p set by the firm. Let $d_{i,n}(p)$ represent the probability that the customer purchases garment i in condition n at price p. If the customer buys the garment, the firm collects the buyout price p and replaces garment i with the outside good. If the customer does not purchase the garment, which happens with probability $1 - d_{i,n}(p)$, it is returned to the firm in condition $n + 1$, ready to be rented to other customers in the future.

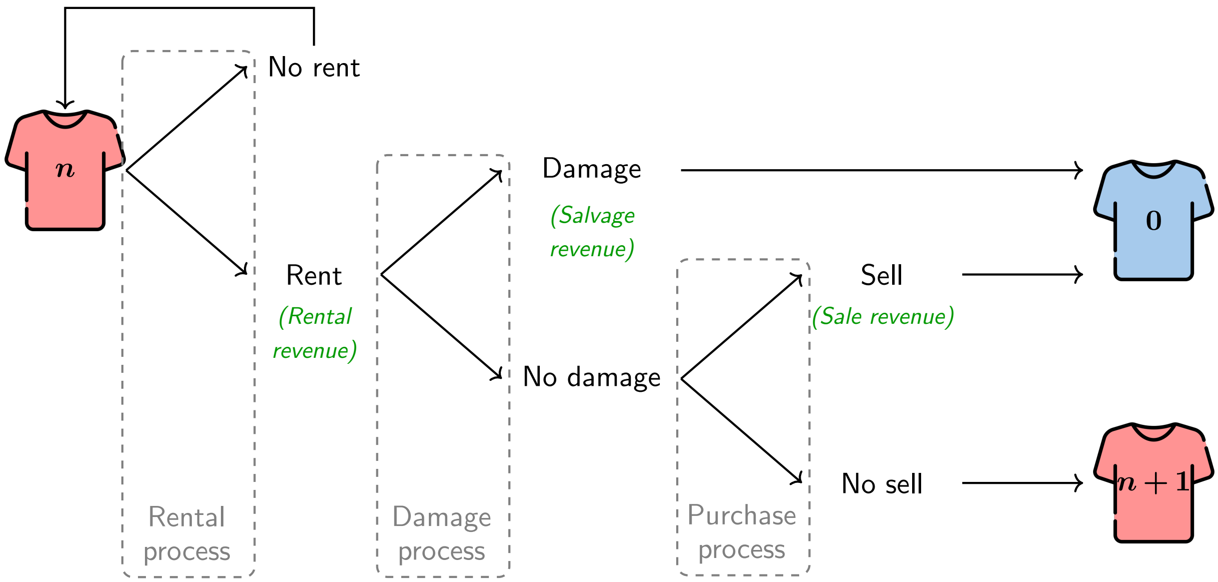

Figure 2 **MDP representation of one step of the firm's operation at the garment level. The red t-shirt represents the focal garment, while the blue t-shirt represents the outside good. The number on each t-shirt indicates the number of rentals each garment has served previously.**

Figure 2 illustrates this operational process, where the red and blue t-shirts represent garment i and the outside good, respectively.

The firm's operation involves three stochastic processes: rental, damage, and purchase. Given that garment i is in condition n, four outcomes are possible, and the probability and expected reward for each of these outcomes can be expressed in terms of the variables in Table 2. First, if the garment is not rented in a given period, it will be in the same condition in the next period, so the firm will earn $\gamma V_{i,n}$, where γ is the discount factor and $V_{i,n}$ represents the expected discounted future value of garment i in condition n. This happens with probability $1-\rho_i$. Second, if the garment rents and becomes damaged while it serves that rental, the customer will not buy it, and the firm will salvage it and replace it with the outside good. In this case, the firm collects $r+\gamma(s_i+V_{out})$ – note r is collected in the current period, and the salvage value s_i and the value of the outside good V_{out} are collected in the next time period. This happens with probability $\rho_i\delta_{i,n}$. Third, if the garment rents and does not get damaged during that rental, the customer will consider buying it. If she does, which happens with probability $\rho_i(1-\delta_{i,n})d_{i,n}(p)$, the firm collects $r+\gamma(p+V_{out})$. Finally, if the garment rents, it does not get damaged while it serves that rental, and the customer does not buy it, then the firm earns $r+\gamma V_{i,n+1}$. This happens with probability $\rho_i(1-\delta_{i,n})(1-d_{i,n}(p))$.

Table 2 Summary of Notation.

Variable	Description
$V_{i,n}$	Expected discounted future value of garment i in condition n
V_{out}	Expected discounted future value of the outside good
p	Buyout price
r	Net rental revenue per garment
s_i	Salvage value of garment i
γ	Discount factor
ρ_i	Probability that garment i rents in a given period
$\delta_{i,n}$	Probability that garment i in condition n gets damaged in the next rental
$d_{i,n}(p)$	Probability that garment i in condition n is purchased by the customer in the next rental, if the firm assigns buyout price p

By the law of total expectation, the expected discounted future value of garment i in condition n can be expressed as:

$$\begin{aligned} V_{i,n} =& (1-\rho_i)\times\gamma V_{i,n} && \text{(not rented)} \\ &+\rho_i\delta_{i,n}\times\Big(r+\gamma(s_i+V_{out})\Big) && \text{(rented and damaged)} \\ &+\rho_i(1-\delta_{i,n})d_{i,n}(p)\times\Big(r+\gamma(p+V_{out})\Big) && \text{(rented and purchased)} \\ &+\rho_i(1-\delta_{i,n})\Big(1-d_{i,n}(p)\Big)\times(r+\gamma V_{i,n+1}) && \text{(rented and returned)} \end{aligned} \tag{3}$$

This expression can be rewritten as:

$$\begin{aligned} V_{i,n}(p\mid\Theta) =& \frac{\rho_i}{1-\gamma+\gamma\rho_i}\times(r+\gamma V_{i,n+1}) \\ &+\frac{\rho_i}{1-\gamma+\gamma\rho_i}\delta_{i,n}\times\gamma(s_i+V_{out}-V_{i,n+1}) \\ &+\frac{\rho_i}{1-\gamma+\gamma\rho_i}(1-\delta_{i,n})d_{i,n}(p)\times\gamma(p+V_{out}-V_{i,n+1}) \end{aligned} \tag{4}$$

where Θ is a set of parameters, i.e., $\Theta=\{V_{i,n+1},V_{out};\ r,s_i;\ \rho_i,\delta_{i,n};\ \gamma\}$.

Supposing that the firm sets value-maximizing prices yields the following Bellman equations:

$$V^*_{i,n}=\max_{p\in\mathbb{R}_+}\ V_{i,n}(p) \tag{5}$$

The firm's problem consists of finding the optimal buyout price $p^*_{i,n}$ that maximizes the expected discounted future value of garment i in condition n, i.e.:

$$p^*_{i,n}=\arg\max_{p\in\mathbb{R}_+}\ V_{i,n}(p) \tag{6}$$

Note that only the last row in Equation (4) depends on the buyout price. Hence, the price that maximizes $V_{i,n}$ in Equation (4) must maximize $d_{i,n}(p)\times(p+V_{out}-V_{i,n+1})$, which implies that the optimal buyout price depends on the functional form of the purchase probability $d_{i,n}(p)$.

Proposition 1 presents a sufficient set of conditions on $d_{i,n}(p)$ under which the firm's pricing problem in Equation (6) has a unique, interior solution.

Proposition 1. *Let $d_{i,n} : \mathbb{R}_+ \to [0,1]$ be a differentiable, strictly decreasing, and log-concave function, that tends to 0 faster that $1/p$ as $p \to \infty$. Then, $\max_{p \in \mathbb{R}_+} V_{i,n}(p)$ has a unique, interior solution, i.e., $p^*_{i,n} := \arg\max_{p \in \mathbb{R}_+} \; V_{i,n}(p) < \infty$.*

Furthermore, if the above conditions are satisfied, it will always be the case that $p^*_{i,n} > V^*_{i,n+1} - V_{out}$.

Proof: See the e-companion to this paper.

We give $d_{i,n}(p)$ a logit structure, i.e.:

$$d_{i,n}(p) = \frac{\exp(\mu_{i,n} - \beta_i p)}{1 + \exp(\mu_{i,n} - \beta_i p)} \tag{7}$$

where $\mu_{i,n}$ is the utility that a customer derives from purchasing the garment i that has served n prior rentals, and β_i is her sensitivity to garment i's price, so that $-\beta_i p$ represents the disutility the customer incurs from purchasing garment i at price p. This specification allows for $\mu_{i,n}$ to depend on the condition of garment i through the number of rentals n it has undergone previously, and the price sensitivity β_i to be garment-class-specific (dress vs. jeans vs. sweater...).

Since Equation (7) satisfies the conditions in Proposition 1, there exists a unique optimal buyout price $p^*_{i,n}$ for a garment i in condition n, given by:

$$p^*_{i,n} = \Delta^*_{i,n+1} + \frac{1 + W\Big(\exp(\mu_{i,n} - \beta_i \Delta^*_{i,n+1} - 1)\Big)}{\beta_i} \tag{8}$$

where $W(\cdot)$ is Lambert's W function[1] (Corless et al. 1996), with

$$\Delta^*_{i,n+1} = V^*_{i,n+1} - V_{out} \tag{9}$$

Equations (8) and (9) indicate that the buyout prices prescribed by this framework balance the trade-off between retaining garment i for future rentals, or selling it and replacing it with the outside good. This is because $\Delta^*_{i,n+1}$ captures the net value of keeping the garment as opposed to selling it: if the firm keeps the garment, it obtains value $V^*_{i,n+1}$, whereas if the garment is sold and replaced by the outside good, the firm obtains V_{out}. Therefore, the buyout price prescription is influenced by (1) the value of the garment being priced, represented by $V^*_{i,n+1}$, and (2) the opportunity cost associated with the outside good that would replace it if the garment gets damaged or purchased, represented by V_{out}. This is precisely the merit of our framework: it quantifies the expected contribution of each individual garment to overall profitability, and prescribes a buyout price that reflects the firm's willingness to relinquish that value and transition to the outside good in exchange for immediate sales revenue.

[1] Lambert's W function, $W(x)$, denotes the solution w to the equation $w \exp(w) = x$. It is increasing, concave and positive for all $x > 0$.

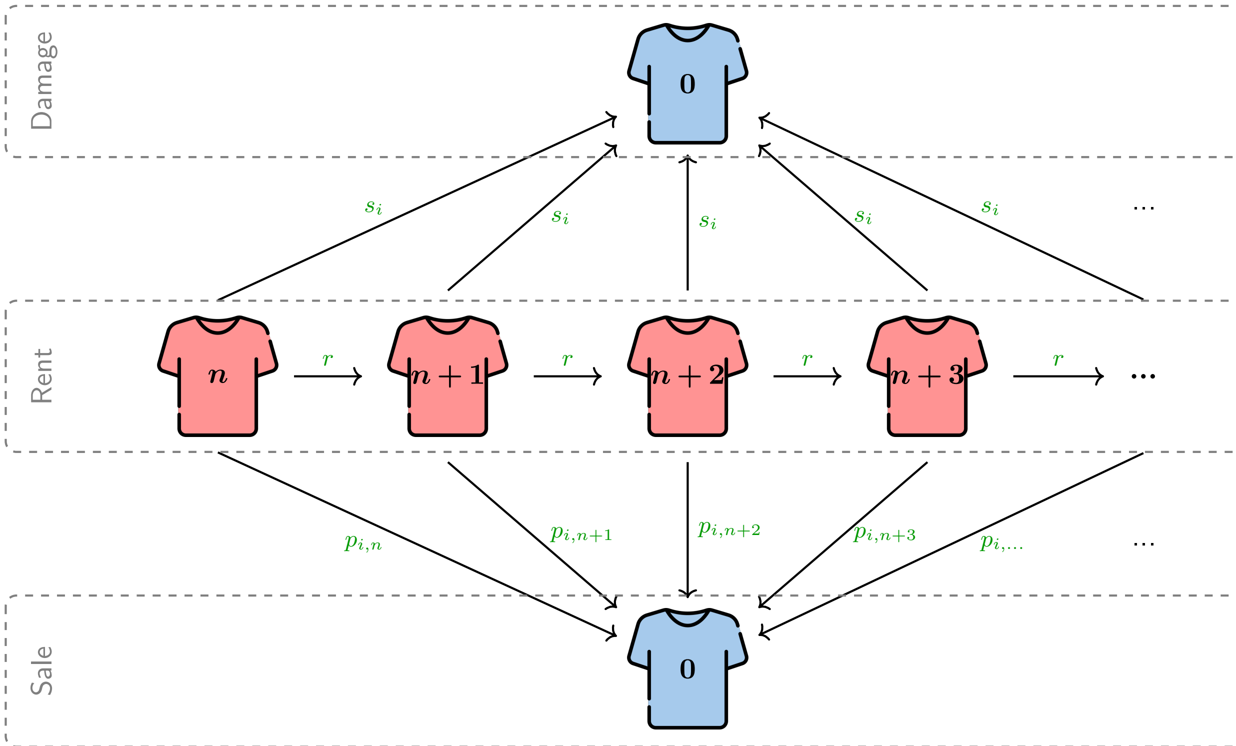

Figure 3 MDP representation of the firm's operation at the garment level.

The partial derivatives of the optimal buyout price $p^*_{i,n}$ with respect to $\Delta^*_{i,n+1}$, $\mu_{i,n}$ and β_i have the signs that economic principle would dictate. In particular, as we present in Lemma 1, the framework prescribes a higher price when the garment is very valuable to the firm or very desirable in the eyes of the customer, but firm should price it lower the more price-sensitive customers are.

LEMMA 1. *The optimal buyout price $p^*_{i,n}$ given by Equation (8) is increasing in both $\Delta^*_{i,n+1}$ and $\mu_{i,n}$, and decreasing in β_i.*

One key challenge in solving the proposed framework for the optimal buyout price prescription lies in managing its infinite state space, depicted in Figure 3. In our context, the likelihood of a garment surviving an indefinitely large number of rentals without being damaged or purchased is extremely low. This practical reality motivates our approach of truncating the Markov chain at a sufficiently high state, following Altman (2021).

Specifically, we set $V^*_{i,100} = s_i + V_{out}$, what implies that the garment can survive a maximum of 100 rentals before incurring damage. At that point, the firm replaces the garment with the outside good and collects the salvage value s_i. Based on the distribution of expected garment lifetimes observed in our data partner's inventory, shown in Figure 4, we find evidence that a truncation level of 100 rentals is sufficiently high to maintain accuracy while ensuring computational tractability.[2]

Using $V^*_{i,100} = s_i + V_{out}$ as a starting point, we use Equation (10) recursively to solve for $V^*_{i,n+1}$ by backward induction. Once $V^*_{i,n+1}$ is computed, Equations (8)-(9) yield the optimal buyout price for garment i in condition n, i.e., $p^*_{i,n}$.

$$\begin{aligned} V^*_{i,j} =& \frac{\rho_i}{1-\gamma+\gamma\rho_i}\Big(r + \gamma V^*_{i,j+1} + \delta_{i,j}\gamma(s_i + V_{out} - V^*_{i,j+1})\Big) \\ &+ \frac{\rho_i}{1-\gamma+\gamma\rho_i}\Big(1-\delta_{i,j}\Big)\frac{\gamma\, W\Big(\exp(\mu_{i,j} - \beta_i(V^*_{i,j+1} - V_{out}) - 1)\Big)}{\beta_i} \end{aligned} \tag{10}$$

6. A Tractable Approach for Multi-Garment Settings

The framework we have presented is designed to optimize buyout prices at the individual garment level. In practice, however, fashion rental services typically release multiple garments from each SKU, with interdependent rental dynamics: when a garment exits the system through damage or sale, some of the rental demand it was serving can be taken up by the remaining garments, effectively increasing their rental rates and thereby altering both their expected value to the firm and the buyout prices they should be assigned. The computational intractability of jointly modeling inventory levels – which determine rental rates – and garment conditions (see Section 4.2) calls for an alternative approach for setting buyout prices in multi-garment settings.

To address this, we extend our framework to the case where the firm owns $q > 1$ garments of an SKU by decoupling the multi-garment problem into q single-garment problems, with each garment assigned an appropriate rental probability. The value (and hence, the buyout price) our framework assigns to garment i depends on the inventory level of the SKU it belongs to through the garment's rental probability. Letting the discrete random variable $D \sim F(\cdot)$ represent the rental demand for the SKU garment i belongs to, and motivated by the result in Section 4.1 where the rental value of the q^{th} garment from a certain SKU is calculated as if the garment would only rent when all other $q-1$ garments are busy, we posit that the q^{th} garment is priced by replacing the rental probability ρ_i in our framework with:

$$\rho_{i,q} = \text{Prob}(D \geq q) = 1 - F(q-1) \tag{11}$$

This parameterization mirrors the logic of the M/M/Q queueing system, where the impact of removing a server is measured as the resulting decrease in service rate. Analogously, $\rho_{i,q}$ represents the expected number of rentals per period that the firm would forgo if garment i were removed, leaving only $q-1$ garments available. This approach accounts for inventory pooling and incorporates

[2] From our partner's data, the maximum number of rentals served by a single garment is 30. The SKU with the longest ex-ante expected lifetime (estimated as described in Section 7.2.1) can survive 40.8 rentals.

the principle of diminishing marginal returns, as each additional garment contributes progressively less value to the system.

In each time period, the firm observes the realization of rental demand and randomly selects garments from the available inventory of the corresponding SKU to fulfill customer orders. If a single garment is rented during that period, its price is computed based on $\rho_{i,q}$, which implies that letting the garment go would reduce the inventory level from q to $q-1$ and the firm would forgo $\rho_{i,q}$ rentals per period as a result. When multiple garments are rented in the same period, we adopt a sequential pricing approach: the garments are priced as if they correspond to inventory positions q, $q-1$, $q-2$, and so on. According to this sequential approach, the remaining garments become progressively more valuable as inventory level declines – a pattern consistent with the intuition from the motivating example also supported by our simulation results in Section 8.

Because it uses the same value of $\rho_{i,q}$ when computing the focal garment's value by backward iteration, this approach implicitly assumes that the garment contributes $\rho_{i,q}$ expected rentals per period to the overall system for as long as it remains in inventory, implying that the focal garment will be the next one from its SKU to exit the system. In practice, this is an approximation, as it is generally not possible to determine in advance which specific garment within an SKU will exit the firm's operations next. Our approach becomes exact and optimal if the departure order of garments is known in advance – e.g., if there is a single garment in inventory ($q=1$). In that case, the single garment is necessarily the next to leave the system and rents at a rate $\rho_{i,1}$ per period while active. For the more general case with $q>1$, the framework assumes that the focal garment contributes a constant $\rho_{i,q}$ rentals per period while in inventory, and that it will be the next garment from its SKU to leave the system.

Although approximate, this extension preserves the core economics of the buyout pricing problem while providing a tractable and implementable solution for the multi-garment case. To assess the performance of our framework in settings where it yields exact and approximate solutions, we examine these two cases separately in Section 8: one involving a single garment, and another involving multiple garments, based on a sample of our industry partner's assortment.

7. Empirical Estimation of Model Parameters

We have introduced a buyout pricing framework tailored to the specific characteristics of the hybrid rental-sale business model. In this section, we present the dataset provided by our partner and explain how we use it to characterize and estimate the key processes underlying the framework – damage, rental, and purchase. These estimates allow us to compute the parameters required for our pricing prescriptions: the rental probability $\rho_{i,q}$, the sequence of damage probabilities $\{\delta_{i,j}\}_{j\geq n+1}$, the sequence of purchase utilities $\{\mu_{i,j}\}_{j\geq n}$, and the price sensitivity β_i.

7.1. Data

We obtained rentals and inventory records spanning the 3-year period from January 2021 to December 2023. Throughout this period, the firm served 460,942 unique customers that placed a total of 3,023,570 orders, and owned 5,114,426 unique garments from 162,564 different SKUs. Of the rentals fulfilled by these garments, 4.5% resulted in the customer purchasing the garment, while 2.9% resulted in the garment being damaged. The remaining 92.6% of the times, the garment was returned to the firm in good condition.

Garments are organized in a hierarchical product tree that begins with broad categories and progressively narrows to specific, individual items. The broadest categories are referred to as *departments*, which group products based on their overall type or purpose. The firm has six departments: Tops (representing 42.24% of the firm's inventory in the period under study), One Pieces (37.28%), Bottoms (17.37%), Maternity (2.73%), Collaboration (0.36%), and Vintage (0.02%).

Within each department, products are further categorized into *classes* based on shared general features or functional similarities. For example, "T-shirts" and "Sweaters" are two classes within the "Tops" department, and "Jumpers + Rompers" and "Dresses" fall into the "One Pieces" department. Each class is further divided into *styles*, which consist of specific aesthetic variations of a product, representing consumer-facing variations in design and color.

The *SKU* is the most granular level and represents a specific combination of style and size. For example, if the firm carries a style in multiple sizes, each size constitutes a different SKU. The firm may own multiple physical copies of a given SKU. Each physical copy of an SKU is referred to as a *garment* and is uniquely identified with a garment ID, which allows the firm to track it over time as customers rent it out.

For each rental transaction during the observed period, the dataset contains: a unique order ID, the timestamp when the order was placed, the unique ID and demographics of the customer that placed the order, the unique IDs of the SKUs ordered by the customer, the unique IDs of the garments that were used to serve this rental, the buyout prices offered by the firm, and the outcomes of the rental – i.e., purchase, damage, or return, at the garment level. Table EC.1 shows some descriptive statistics for these variables.

The dataset also records end-of-day (EOD) inventory levels for every SKU owned by the firm, reflecting changes caused by rentals, purchases, and removals due to damage or obsolescence. Table EC.2 summarizes the number of distinct styles, SKUs and garments that were owned by the firm or available at the daily level.

7.2. Empirical Specifications

7.2.1. Damage Process: We postulate that the expected number of rentals that garment i can survive *without* being damaged follows a Poisson distribution with rate ξ_i gamma-distributed at the product style level. Equivalently, the number of rentals served by this garment before damage, denoted with Y_i, follows a negative binomial distribution with shape parameter $k_{t(i)}$ and rate parameter $\theta_{t(i)}$, where $t(i)$ denotes the style that garment i belongs to, i.e.,:

$$Y_i \sim \mathrm{NB}(k_{t(i)}, \theta_{t(i)}) \tag{12}$$

The interpretation of this probability distribution is as follows: garment i has a latent quality ξ_i, which represents the number of rentals that the garment is expected to survive without damage. We allow for heterogeneous qualities at the style level by letting the latent quality parameters follow a gamma distribution with shape parameter $k_{t(i)}$ and rate parameter $\theta_{t(i)}$. The choice to model damage at the style level is motivated by the intuition that the quality of the garment depends on factors like materials, fabric, style, brand, build, and color of the garments – which are shared across garments at the style level.

The data reveal that 9.5% of the garments operated by our partner have been removed from inventory due to damage, while the remaining garments were either purchased prior to being damaged or are still active in the firm's inventory. To account for this right-censoring, we employ a censored negative binomial regression (Hilbe 2011), an extension of the standard negative binomial regression that accommodates censored observations and functions similarly to a discrete-time survival model.

If garment i undergoes n rentals and is removed from the firm's inventory due to damage, then it served $Y_i = n-1$ rentals without damage, which happens with probability:

$$\Pr(Y_i = n-1 \mid k_{t(i)}, \theta_{t(i)}) = \frac{\Gamma(k_{t(i)} + n - 1)}{\Gamma(k_{t(i)})(n-1)!} \left(\frac{\theta_{t(i)}}{\theta_{t(i)}+1}\right)^{k_{t(i)}} \left(\frac{1}{\theta_{t(i)}+1}\right)^{n-1} \tag{13}$$

where $\Gamma(\cdot)$ denotes the gamma function given by $\Gamma(z) = \int_0^\infty x^{z-1}e^{-x}dx$.

Instead, if we observe this garment serve n rentals and be purchased by a customer or remain active, we know that it can serve $Y_i \geq n$ rentals without damage, which happens with probability:

$$\Pr(Y_i \geq n \mid k_{t(i)}, \theta_{t(i)}) = \frac{\Gamma(k_{t(i)} + n)}{\Gamma(k_{t(i)})n!} \left(\frac{\theta_{t(i)}}{\theta_{t(i)}+1}\right)^{k_{t(i)}} \left(\frac{1}{\theta_{t(i)}+1}\right)^{n} {}_2F_1(1, k_{t(i)}+n; n+1; \frac{1}{\theta_{t(i)}+1}) \tag{14}$$

where $\Gamma(\cdot)$ denotes the gamma function and ${}_2F_1(a,b;c;d)$ is the Gaussian hypergeometric function.

When setting up the (log-)likelihood function for the censored negative binomial regression, we use Equation (13) for garments with observed lifetimes and Equation (14) for those whose lifetimes are right-censored. The estimation process involves maximizing this likelihood function. To implement this, we use the *CENSORNB* package in *Stata* (Hilbe 2005), which is specifically designed for fitting (censored) negative binomial regressions.

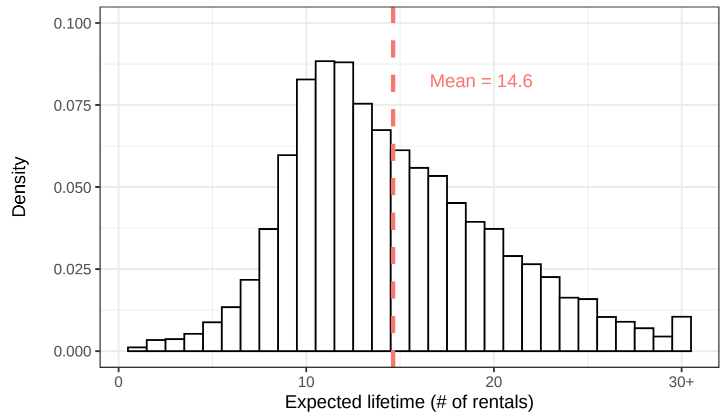

Figure 4 **Empirical distribution of ex-ante expected lifetimes of 16,292 styles.**

The negative binomial specification for the damage process enables us to calculate the probability that a garment, having survived n rentals without damage, becomes damaged during its next rental, i.e., the $n+1^{\text{th}}$ rental. This quantity, which we refer to as the *damage hazard* of garment i in condition n, is $\delta_{i,n}(k_{t(i)}, \theta_{t(i)}) = \Pr(Y_i = n \mid Y_i \geq n, k_{t(i)}, \theta_{t(i)})$ and can be expressed as:

$$\delta_{i,n}(k_{t(i)}, \theta_{t(i)}) = \frac{1}{{}_2F_1\left(1, k_{t(i)} + n; n+1; \frac{1}{\theta_{t(i)}+1}\right)} \tag{15}$$

In addition, since the unconditional mean of the negative binomial distribution is obtained by dividing the shape parameter by the rate parameter, a brand new garment from style $t(i)$ has an ex-ante expected lifetime of $1 + E(Y_i) = 1 + k_{t(i)}/\theta_{t(i)}$ rentals. Figure 4 displays the ex-ante expected lifetimes for 16,292 styles for which at least one garment experience damage, which follow a right-skewed distribution with a mean of 14.6 rentals and reveal significant variability, reflecting differences in durability across various garment styles.

7.2.2. Rental Process: We assume that rental demand for an SKU follows a Poisson process. This assumption is based on the observation that customers place orders by requesting an SKU, rather than a specific garment within that SKU – in fact, the website presents representative images of the SKU and customers do not know which specific garment they will receive, as the company randomly selects one available garment to fulfill orders when an order for that SKU is received.

Estimating the rental process poses a challenge similar to that encountered in modeling the damage process. Specifically, for the SKU $u(i)$ that garment i belongs to, the data records the daily number of realized rentals, denoted as $R_{u(i)}$, rather than the true rental demand, $D_{u(i)}$, resulting in right-censored observations. An observation that reinforces the presence of right-censoring in the data is that, as shown in Table EC.2, 39.2% of SKU-day pairs have no available inventory at the end of the day, suggesting that customers may encounter stockouts when attempting to rent certain SKUs. To account for this, we use end-of-day inventory data to identify potentially censored rental observations. We then fit a censored Poisson regression to estimate the daily rental demand rate for each SKU, $\lambda_{u(i)}$.

If there is available inventory of $u(i)$ at the end of a given day, we can be certain that the rental demand that day equals the number of realized rentals, i.e., $D_{u(i)} = R_{u(i)}$, which happens with probability:

$$\Pr(D_{u(i)} = R_{u(i)} \mid \lambda_{u(i)}) = \frac{(\lambda_{u(i)})^{R_{u(i)}} \exp(-\lambda_{u(i)})}{R_{u(i)}!} \tag{16}$$

Conversely, if no inventory of that SKU is available at the end of the day, it is possible the true rental demand $D_{u(i)}$ may have exceeded the realized number of rentals $R_{u(i)}$ and some customers that tried to rent this SKU found it out of stock. Hence:

$$\Pr\left(D_{u(i)} \geq R_{u(i)} \mid \lambda_{u(i)}\right) = \frac{\Gamma\left(R_{u(i)}\right) - \Gamma\left(R_{u(i)}, \lambda_{u(i)}\right)}{\Gamma\left(R_{u(i)}\right)} \tag{17}$$

where $\Gamma(\cdot)$ denotes the gamma function and $\Gamma(\cdot,\cdot)$ represents the incomplete gamma function.

Depending on whether there was available inventory of SKU $u(i)$ left at the end of each day, we construct the (log-)likelihood function using Equations (16) and (17), and maximize it to estimate a censored Poisson regression at the SKU level (Hilbe 2011), yielding the Maximum Likelihood Estimator (MLE) of the daily rental demand rate for each SKU, $\hat{\lambda}_{u(i)}$.

Since *daily* rental demand for SKU $u(i)$ follows a Poisson process with rate $\lambda_{u(i)}$, *monthly* rental demand for the same SKU is also Poisson with rate $\Lambda_{u(i)} = \frac{365.25}{12} \times \lambda_{u(i)}$. Consequently, based on Equation (11) and the discussion in Section 6, we parameterize the probability that garment i rents in a given period as:

$$\rho_{i,q_{u(i)}} = 1 - F\left(q_{u(i)} - 1\right) = \frac{\Gamma\left(q_{u(i)}\right) - \Gamma\left(q_{u(i)}, \hat{\Lambda}_{u(i)}\right)}{\Gamma\left(q_{u(i)}\right)} \tag{18}$$

where $\hat{\Lambda}_{u(i)}$ represents the MLE of the monthly rental demand rate for SKU $u(i)$, and $q_{u(i)}$ is the inventory level of that SKU.

7.2.3. Purchase Process: We give the purchase probability $d_{i,n}(p)$ a logit structure, i.e.:

$$d_{i,n}(p) = \frac{\exp(\mu_{i,n} - \beta_i p)}{1 + \exp(\mu_{i,n} - \beta_i p)} \tag{19}$$

where $\mu_{i,n}$ is the utility that a customer derives from purchasing the garment i that has served n rentals previously, and β_i is her sensitivity to garment i's price.

This specification allows for $\mu_{i,n}$ to depend on garment-specific characteristics and vary with garment i's condition. To incorporate these features, we include additional covariates in the logistic regression – namely, controls for garment class, size, color and tier,[3] as well as the garment condition n and a dummy that equals 1 if the garment is brand new and 0 otherwise.

$$\mu_{i,n} = \alpha_{class(i)} + \alpha_{size(i)} + \alpha_{color(i)} + \alpha_{tier(i)} + \eta n + \nu \mathbb{1}_{n=0} \tag{20}$$

[3] Our data partner classifies its garments in three tiers (good, better, and best), in an increasing order of sophistication.

Because our data partner prices 5% of its garments randomly, we estimate the purchase decision model free of bias using the subset of the data with exogenous variation in buyout prices. The estimates presented in Table EC.3 indicate that customers are sensitive to garment condition: *ceteris paribus*, newer garments are associated with a higher likelihood of purchase.

8. Model Assessment

In this section, we evaluate the performance of our buyout pricing framework through simulations and compare it with two benchmark models. This comparison enables us to quantify both the strategic value of buyouts in fashion rental operations and the incremental gains of our approach relative to the legacy pricing policy used by our partner and other industry players. Following the discussion in Section 6, we assess the performance of our model in (1) a single-garment setting, where our methodology is optimal, and (2) a multi-garment setting based on our industry partner's assortment, where the methodology provides a tractable approximation. The results highlight the strategic value of buyouts for managing inventory and boosting revenue, with our framework consistently outperforming current industry practice, especially under inventory imbalances.

8.1. Methodology

We simulate the performance of our framework and alternative benchmarks at the SKU level. For each SKU, we take the monthly rental demand arrival rate (Λ) and the initial inventory (Q_0) as given. In the single-garment simulations, we set the initial inventory to $Q_0 = 1$ garment and vary the rental demand rate Λ to recreate scenarios in which the firm is either overstocked or understocked. For the multi-garment simulations, we use a subsample of our partner's assortment: the initial inventory for each SKU $u(i)$ is set to the number of garments released by the firm at the time of its introduction, and the demand rate is the one estimated from the data, i.e., $\Lambda = \hat{\Lambda}_{u(i)}$. We generate multiple sample paths of customer arrivals, letting rental demand in each period follow a Poisson distribution with rate Λ.

At the beginning of each period, the firm observes the realization of rental demand and checks inventory availability. If demand exceeds the number of garments in stock, all available inventory is rented, and excess demand is lost. In contrast, if inventory exceeds demand, the firm randomly selects which garments to send out to fulfill rental requests. Every time a garment rents, the firm assigns a buyout price according to our proposed framework or an alternative policy.

During the rental period, three outcomes are possible: a garment may get damaged, the customer may purchase it at the assigned buyout price, or the garment may be returned to the firm in good condition for future rentals. When a garment exits the system due to damage or buyout, the firm replaces it in its warehouse with an *outside good*. The simulation continues until all inventory for the focal SKU is depleted. We assess the performance of each buyout pricing policy based on the (discounted) earnings generated from operating the inventory of the SKU.

The revenue structure incorporates various sources of income. For each realized rental, the firm earns rental revenue r. If a garment is purchased, the firm collects the assigned buyout price and the value of the outside good, V_{out}. In the case of damage, the firm collects V_{out} along with the garment's salvage value, s_i. If a garment is returned in good condition, no additional revenue is earned. To reflect the time value of money, all revenues are discounted using a factor $\gamma = 0.99$ per period.

In our simulations, we set the salvage price s_i to \$0 for all garments and calculate the rental revenue r as the fraction of the monthly subscription fee divided by the number of garments a customer is entitled to rent. The value of the outside good V_{out} is set at \$600, representing the discounted future value the firm would obtain if it operated a "representative" garment indefinitely.

To ensure that buyout prices remain appealing, the firm enforces a price cap policy, limiting buyout prices between 10% and 100% of a garment's MSRP.[4] We apply these bounds consistently across the different policies to maintain the comparability of the simulation results.

8.2. Benchmarks

We evaluate the performance of our framework against two benchmark policies: the legacy buyout pricing strategy currently used by our partner, and a counterfactual rental-only operation in which garments are not offered for sale. These comparisons allow us to separately quantify the revenue uplift generated by our methodology and the broader strategic contribution of buyouts to the rental business model.

8.2.1. Current Practice: Our partner has historically set buyout prices based on the assumption that garments depreciate linearly over a fixed number of 12 rentals. Consequently, denoting the procurement cost of garment i with $Cost_i$, each rental is associated with a depreciation of $\frac{1}{12} \times Cost_i$ in the garment's value. Under this policy, a garment that has completed 12 or more rentals is considered to have an implicit residual cost of zero. This approach is related to the notion of amortization in accounting, where the cost of an asset is evenly spread over its useful life to systematically allocate its value.

To recreate this pricing policy, we say that the net value of a garment returning after its $n+1^{\text{th}}$ rental is:

$$\Delta_{i,n+1}^{CP} = \frac{Cost_i}{12} \times \max(12 - n - 1,\ 0) \tag{21}$$

[4] Our partner operates products that are widely available for sale, brand new, from third-party apparel brands. We cap the buyout price at the garment's retail price to avoid charging customers more than the prevailing retail option.

and, in the same vein as Equation (8), the buyout price $p_{i,n}^{CP}$ prescribed by this policy is:

$$p_{i,n}^{CP} = \Delta_{i,n+1}^{CP} + \frac{1 + W\Big(\exp(\mu_{i,n} - \beta_i \Delta_{i,n+1}^{CP} - 1)\Big)}{\beta_i} \tag{22}$$

This method evaluates a garment's value based on its implicit residual cost rather than the potential future revenue it could contribute. As a result, a garment's value is effectively capped at its procurement cost, even though, in practice, we observe that high-durability, high-demand garments often generate substantially greater earnings over their lifetime. Furthermore, the legacy buyout pricing policy assumes a linear depreciation over a fixed horizon of 12 rentals, which could be problematic given that (1) the ex-ante expected lifetime of a garment is 14.6 rentals, and (2) there is substantial variation in the expected lifespans of different SKUs, as illustrated in Figure 4.

In contrast to our buyout pricing framework, current practice overlooks key factors related to the damage and rental processes. It does not consider the possibility of a garment incurring damage, nor does it account for factors such as the customer demand rate for renting an SKU or the available inventory to meet that demand. Moreover, it ignores the notion of V_{out}, which our framework uses as a reference it compares the value of the focal garment to when determining how much the firm should be willing to accept in order to replace the focal garment with the outside good.

8.2.2. Rental-Only Operation: To quantify the economic contribution of buyouts in the rentailer business model, we consider a scenario where garments are not made available for purchase. In this benchmark, the firm operates as a pure rental business, generating revenue solely through recurring rentals and eventual salvages. Because buyouts are not offered, garments exit the system only when they are damaged beyond repair – forcing the firm to retain all garments until they get damaged, removing the revenue benefits and inventory flexibility associated with buyouts.

Formally, we implement this policy by setting the purchase probability $d_{i,n}(p) = 0$ for all garments, regardless of condition or price.

8.3. Simulation Results with a Single Garment

Our buyout pricing framework captures the dynamics of an individual garment's condition but does not consider how the price assigned to one garment may influence the rental rates – and thus the value – of other garments within the same SKU. This limitation is irrelevant when an SKU contains only a single garment, allowing us to focus solely on garment-level dynamics, where our framework yields the optimal buyout price.

Figure 5 illustrates the performance of our framework compared to the benchmark policies for an SKU with a single garment. In this example, the garment has an expected lifetime of 12 rentals, an MSRP of \$160, and a unit cost of \$40. For each rental demand rate Λ, we simulate 10,000

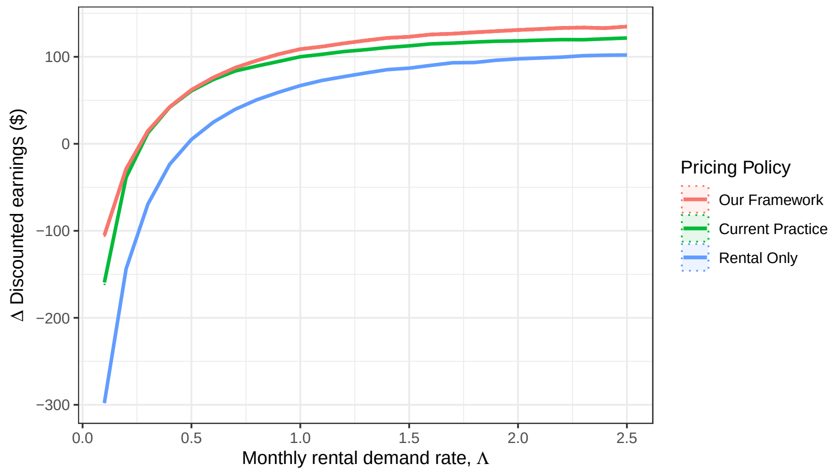

Figure 5 **Pricing policy performance, for an SKU with $q = 1$.**

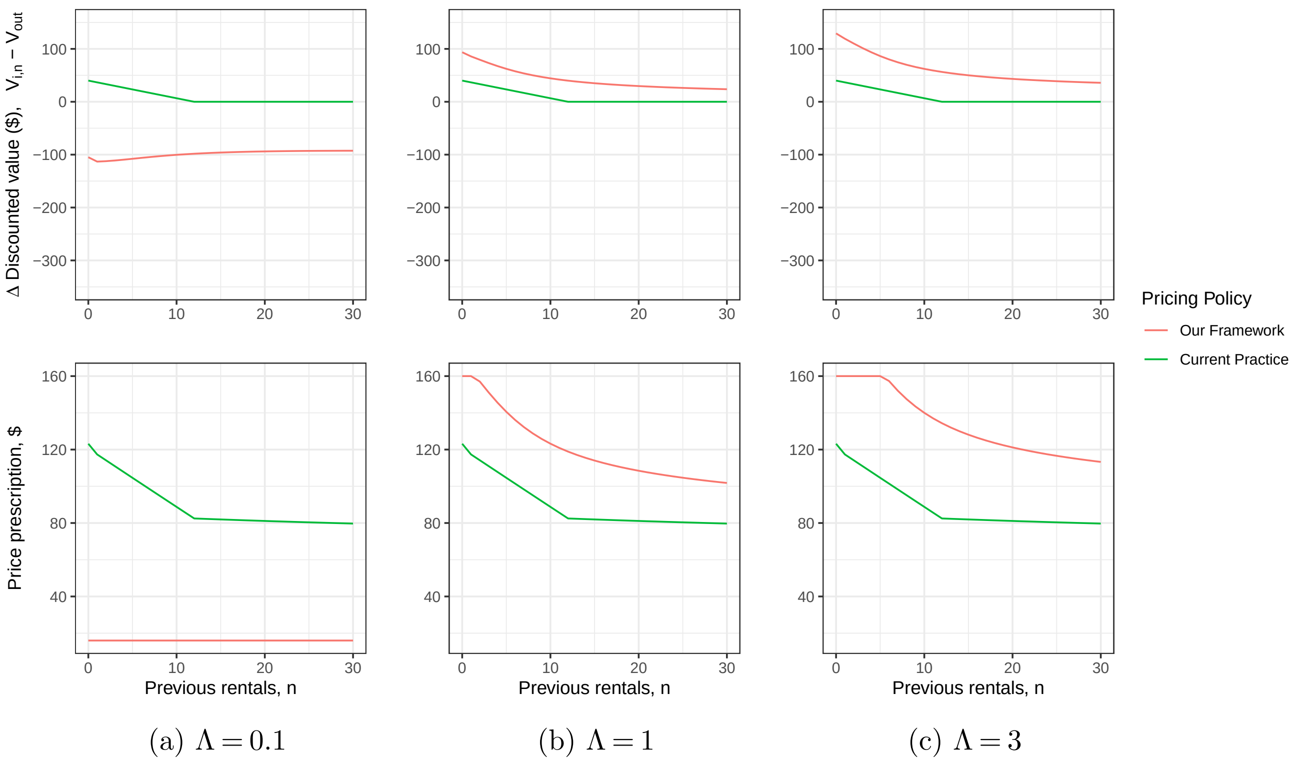

Figure 6 **Expected net discounted value (top) and buyout price prescription (bottom) at three values of the monthly rental demand rate, for an SKU with $q = 1$.**

sample paths and evaluate performance of each pricing policy based on the net discounted earnings accrued by the firm, relative to the value of the outside good. Notably, the y axis is standardized by V_{out}, so a positive value implies the firm generates more revenue than V_{out}, while a negative value means the firm earns less revenue than that baseline. The figure highlights several insights.

First, a rental-only policy – where garments are never offered for sale and remain in circulation until damaged – eliminates sales revenue and performs poorly across the full range of rental demand values shown in Figure 5. The shortcomings are most pronounced in low-demand scenarios, where the ability to sell excess inventory would help offset limited rental activity and free space for more profitable styles.

Second, the legacy pricing policy employed by our partner, which assumes linear depreciation over a fixed horizon of 12 rentals and excludes rental demand considerations from buyout pricing, prescribes identical prices for garments with the same number of prior rentals, regardless of the rental demand rate Λ. This rigidity, illustrated in Figure 6, limits the policy's ability to adapt to different demand scenarios.

Third, our pricing framework consistently outperforms both benchmarks in the single-garment setting, by tailoring prices to garment condition and the rental demand. Interestingly, there is a narrow band of demand rates (approximately $\Lambda = 0.3$ to $\Lambda = 0.7$) where the legacy policy performs nearly as well as our framework. This occurs because the legacy approach, which ignores rental demand, produces valuations similar to ours in this intermediate range. By contrast, our framework explicitly adjusts valuations – and therefore buyout prices – based on rental demand, leading current practice to overprice garments when demand is low (e.g., $\Lambda = 0.1$), and underprice them when demand is high (e.g., $\Lambda = 3$). This example underscores when our framework is most valuable: in high-demand environments, where elevated buyout prices preserve garments with high rental value, and in low-demand settings, where lower prices can help accelerate sales and free space for more profitable inventory.

8.4. Simulation Results for Our Partner's Assortment

We now use data from real products operated by our partner during the study period to simulate our pricing framework and the previously defined benchmarks in a multi-garment setting, where our methodology provides an approximate prescription. For this analysis, we draw a random sample of 1,000 SKUs from the total of 116,074 for which data is available and the framework parameters can be estimated. For each SKU, we take the starting inventory and the monthly rental demand rate as given – the former is observed directly in the data, while the latter can be estimated as described earlier. We then simulate 100 sample paths, with rental demand in each period drawn from a Poisson distribution with mean $\hat{\Lambda}_{u(i)}$. Buyout prices are assigned according to each policy, and performance is measured as discounted earnings relative to the current practice.

Table 3 presents the results. If our partner were to stop offering its garments for sale, effectively operating as a pure rental business, earnings would decline by 6.38% relative to the current practice. This result highlights the dual role of buyouts as both a source of income and a strategic lever for

Table 3 Pricing policy performance simulated on a random sample of SKUs operated by our partner. The results are presented as a % difference with respect to the earnings accrued by the Current Practice.

Policy	**% ΔEarnings**	**95% CI**
Current Practice (Ref.)	—	—
Rental-Only	− 6.38%	[−6.53%, −6.23%]
Our Framework	3.10%	[3.00%, 3.19%]

inventory management. Our proposed framework outperforms the legacy policy by 3.1%, which constitutes a meaningful gain in a low-margin industry such as fashion rental.

Interestingly, the average buyout price and the average number of months garments stay in circulation (i.e., from release to damage or purchase) under our proposed methodology remain stable relative to the current pricing policy. This stability suggests that the gains are not driven by a uniform shift in overall price levels, but rather by the framework's ability to make garment-level retention and release decisions and to set individualized buyout prices. In essence, the improvement stems from tailoring prices to garment-specific attributes and market conditions – that is, setting higher prices for high-demand or understocked products, and lower prices for overstocked, low-demand, or older garments. By adjusting prices in this manner, the firm can strategically influence customer purchase behavior in ways that enhance overall profitability.

Figure 7 displays our framework's buyout price prescriptions in cases where a six-garment SKU[5] is either overstocked (top) or understocked (bottom). When the firm has excess inventory, the framework initially prescribes low buyout prices to stimulate customer purchases and regulate stock levels, and then gradually raises prices once an adequate inventory level is reached – yielding a price path that increases over time, consistent with the motivating example in Section 4.1. Conversely, in scenarios where inventory is limited, our framework prescribes high prices initially to preserve inventory and only reduces prices gradually as the garments age.

9. Conclusion

The rental business model has grown rapidly in recent years, driven by consumer demand for access, affordability, and flexibility. Industries such as fashion have embraced this shift, with companies pioneering innovative rental strategies to meet evolving preferences while reducing waste and maximizing asset utilization. This trend reflects not only a reimagining of consumption patterns but also the ability of businesses to cater to diverse customer needs through sustainable and circular practices.

A central feature of this model is buyout pricing, which involves balancing immediate sales revenue and long-term rental income. This task requires firms to consider aspects such as rental

[5] As before, we analyze an SKU with an MSRP of $160, where each garment has an expected lifespan of 12 rentals.

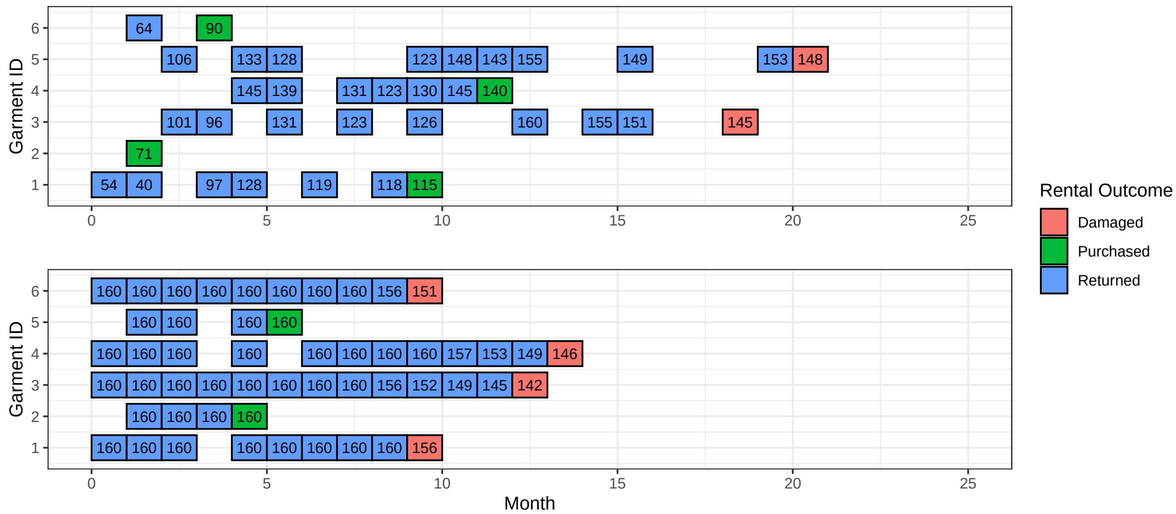

Figure 7 Sample path of buyout prices prescribed by our framework, for a SKU with six garments that is over-stocked (top) and under-stocked (bottom). Each box represents a rental, and the number inside denotes the buyout price prescription, in U.S. dollars.

demand, product lifecycles, and customer purchase behavior. In this paper, we have introduced a robust, data-driven framework to address this challenge, equipping firms with the tools needed to inform buyout pricing decisions.

At the heart of our approach is a Markov Decision Process (MDP) that captures the intricate interplay of three key probability processes – rental demand, garment damage, and customer purchase decisions –, offering actionable insights for dynamically setting buyout prices. Using real-world data from a leading fashion rental company, we have validated the effectiveness of the model and demonstrated its potential to increase revenue in a hybrid rental-sale operation. Importantly, the framework is adaptable to diverse business contexts, making it a valuable tool for practitioners operating subscription rental models across a wide range of industries.

Our findings highlight several key contributions. First, we show that the strategic value of buyouts is twofold: they allow the firm to manage inventory while simultaneously generating additional income. Second, our simulations demonstrate that the proposed methodology outperforms existing industry practices and alternative benchmarks, delivering substantial revenue improvements. Third, we find that our framework is most valuable when inventory is either scarce or abundant, and buyout prices can be used to discourage sales to preserve rental capacity or encourage purchases to offload excess stock.

Beyond these contributions, our work also advances the academic literature by introducing a new class of dynamic pricing problems that explicitly account for dual revenue streams from rentals and sales, extending prior research on both rental operations and markdown pricing. From a managerial perspective, our proposed framework provides actionable decision rules for setting garment-level buyout prices, enabling firms to capture additional revenue while aligning pricing strategies with real-time inventory conditions. Although our analysis is grounded in fashion, the approach is broadly applicable to other industries where products circulate through repeated use.

This study opens several avenues for future research. Rental demand often fluctuates due to seasonal trends. Incorporating seasonality into the model could enhance its accuracy and further refine pricing strategies. In addition, our framework assumes that pricing parameters are known a priori, but in practice, firms must learn these parameters over time. Future research could explore adaptive algorithms, such as bandit approaches, to dynamically update buyout pricing decisions while balancing exploration and exploitation. Lastly, maximizing revenue through higher buyout prices on high-demand products may inadvertently affect customer acquisition and retention. Future work could examine how buyout pricing strategies influence broader aspects of the business, such as customer satisfaction and loyalty.

By bridging academic insight and real-world application, our work equips firms to navigate the trade-offs between rental and sales revenues and support the development of more sustainable, customer-centric practices in the growing rental economy.

Acknowledgments

The authors gratefully acknowledge the financial support provided by the Baker Retailing Center at The Wharton School, University of Pennsylvania.

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E-companion to *Rented Today, Bought Tomorrow: Buyout Pricing in the Circular Economy*

EC.1. Additional Tables and Figures

Table EC.1 Buyout price and Rental outcome descriptive statistics.

Statistic	N	Mean	St. Dev.	Min	Max
MSRP ($)	16,453,310	151.1	69.8	14.0	1,000.0
Buyout price ($)	16,453,310	96.7	41.1	5.2	1,000.0
Buyout discount (%)	16,453,310	31.7	17.2	0	75.0
# of previous rentals	16,453,310	2.6	2.9	0	29
Purchased	16,453,310	0.045	0.206	0	1
Damaged	16,453,310	0.027	0.163	0	1

Table EC.2 Inventory descriptive statistics, at the daily level.

Statistic	N	Mean	St. Dev.	Min	Max
# of styles owned	1,095	12,601.4	4,687.3	5,150	19,600
# of styles available	1,095	10,542.0	3,928.6	3,335	16,550
# of SKUs owned	1,095	82,906.9	34,388.6	33,420	156,767
# of SKUs available	1,095	51,104.0	21,625.8	14,426	83,827
# of garments owned	1,095	1,686,912.0	1,156,360.0	360,127	4,158,860
# of garments available	1,095	771,780.0	539,463.8	119,000	1,894,428

Table EC.3 Logistic regression results for purchase decision, using the subset of data with random buyout price variation.

	Dependent variable:
	Purchase
	(1)
Brand new	0.339***
	(0.015)
Previous rentals	−0.027***
	(0.002)
Buyout price, Blouses + Shirts	−0.021***
	(0.001)
Buyout price, Collab Bottoms	−0.009
	(0.006)
Buyout price, Collab One Pieces	−0.011*
	(0.006)
Buyout price, Collab Tops	−0.013
	(0.014)
Buyout price, Dresses	−0.016***
	(0.0004)
Buyout price, Jackets + Coats + Blazers	−0.014***
	(0.0004)
Buyout price, Jeans + Denim	−0.015***
	(0.0005)
Buyout price, Jumpers + Rompers	−0.013***
	(0.001)
Buyout price, Maternity Bottom	−0.019***
	(0.002)
Buyout price, Maternity One Pieces	−0.023***
	(0.003)
Buyout price, Maternity Tops	−0.016***
	(0.003)
Buyout price, Pants + Leggings	−0.017***
	(0.001)
Buyout price, Shorts	−0.021***
	(0.001)
Buyout price, Skirts	−0.019***
	(0.001)
Buyout price, Sweaters + Sweatshirts	−0.019***
	(0.0005)
Buyout price, Tees + Tanks	−0.030***
	(0.002)
Buyout price, Vintage Bottoms	0.033
	(6.702)
Buyout price, Vintage Tops	−0.001
	(0.688)
Product class FEs	✓
Product size group FEs	✓
Product color FEs	✓
Product tier FEs	✓
Observations	619,946
Log Likelihood	−133,202.700
Akaike Inf. Crit.	266,527.400
Note:	*p<0.1; **p<0.05; ***p<0.01

EC.2. Formulation and Solution of the Two-Product, Condition-Agnostic Model with Deterministic Exits

In this section we formulate and solve the firm's buyout pricing problem with $q=2$ garments, stochastic rental demand, and deterministic garment exits (i.e., damage and sales). This model is based on Apaolaza et al. (2025).

Consider a fashion rental service that operates (i.e., rents and sells) a single product and owns $q=2$ identical garments of this product. For simplicity, we assume these garments do not age or get damaged as they fulfill rentals. Hence, garments can only exit the system via buyouts.

Time is divided into discrete periods of the duration of the firm's rental term. At the beginning of each period, rental demand is realized according to a random variable with cumulative distribution function $F(\cdot)$ and probability mass function $f(\cdot)$. Let $\bar{F}(x)=1-F(x)$. If rental demand exceeds the number of garments owned by the firm, excess demand is lost.

Every time a garment is rented, the firm collects a rental revenue r and assigns the garment a buyout price p. We assume that the firm must select one of two possible buyout prices, p_L and p_H, with $p_L < p_H$.

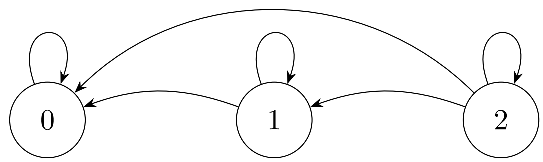
Let $d(p)$ denote the probability that the customer purchases a garment at buyout price p. We assume the customer always (resp., never) purchases the garment when offered buyout price p_L (resp., p_H), that is, $d(p_L)=1$ and $d(p_H)=0$.

The firm's goal is to determine a buyout pricing policy that maximizes the discounted (with parameter γ) earnings generated from renting and selling its inventory of the product.

We model this system as a Markov Decision Process (MDP), where the state $q \in \mathcal{Q}$ represents the number of garments the firm owns, with $\mathcal{Q}=\{0,1,2\}$. Motivated by the operational constraints of our industry partner – who does not restock inventory within the product's lifecycle – we assume that inventory cannot be replenished. As a result, the Markov chain can only transition from a state q to a state q' such that $q' \leq q$, reflecting the depletion of inventory through customer purchases. Figure EC.1 illustrates the possible state transitions in this MDP. Given this structure, we solve the MDP sequentially to determine the buyout prices to assign to garments when the inventory level is q=1 or 2.

When there is $q=1$ garment in inventory, the firm must determine a single buyout price, denoted with $p_A^{(1,1)}$, which will be assigned to the garment whenever it rents. To generalize the notation, we let $p_X^{(q,j)}$ represent the buyout price assigned when there are $q \in \{1,\ 2\}$ garments in inventory and $j \in \{1,\ \ldots,\ q\}$ of those garments are rented. The subscript $X \in \{A,\ B\}$ indexes individual garments when more than one is rented. Thus, $p_A^{(1,1)}$ represents the buyout price that the firm assigns to the first (and only) garment that rents – hence the subindex "A"–, when it owns 1 garment and 1 garment is rented – hence the superindex $(1,1)$.

Figure EC.1 Transitions in the two-garment buyout pricing problem.



Let $V_1\big(p_A^{(1,1)}\big)$ represent the discounted future earnings that the firm collects if it owns $q=1$ garment and it assigns the buyout price $p_A^{(1,1)}$ to the garment whenever a customer rents it.

$$\begin{aligned} V_1\big(p_A^{(1,1)}\big) =& f(0)\gamma V_1\big(p_A^{(1,1)}\big) + \bar{F}(0)\Big(r + d\big(p_A^{(1,1)}\big)\gamma p_A^{(1,1)} + \big(1 - d(p_A^{(1,1)})\big)\gamma V_1\big(p_A^{(1,1)}\big)\Big) \\ =& \frac{r + d\big(p_A^{(1,1)}\big)\gamma p_A^{(1,1)}}{1 - f(0)\gamma - \bar{F}(0)\big(1 - d(p_A^{(1,1)})\big)\gamma}\bar{F}(0) \end{aligned} \tag{EC.1}$$

There are two possible prices, p_L and p_H, such that $d(p_L)=1$ and $d(p_H)=0$. Plugging these prices into the value function $V_1\big(p_A^{(1,1)}\big)$, we obtain:

$$\begin{aligned} V_1(p_L) =& \frac{r + \gamma p_L}{1 - f(0)\gamma}\bar{F}(0) \\ V_1(p_H) =& \frac{r}{1 - f(0)\gamma - \bar{F}(0)\gamma}\bar{F}(0) = \frac{r}{1-\gamma}\bar{F}(0) \end{aligned} \tag{EC.2}$$

Note that $V_1(p_L) \geq V_1(p_H)$ if and only if:

$$\begin{aligned} \frac{r + \gamma p_L}{1 - f(0)\gamma} &\geq \frac{r}{1-\gamma} \\ r(1-\gamma) + \gamma(1-\gamma)p_L &\geq r(1 - f(0)\gamma) \\ \gamma(1-\gamma)p_L &\geq \gamma r\big(1 - f(0)\big) \\ p_L &\geq \frac{r}{1-\gamma}\big(1 - f(0)\big) = \frac{r}{1-\gamma}\bar{F}(0) \end{aligned} \tag{EC.3}$$

That is, when the selling price p_L exceeds a certain threshold $\frac{r}{1-\gamma}\bar{F}(0)$, the firm is better off selling the garment. By contrast, if p_L falls below this threshold, the firm earns more by continuing to rent the garment.

$$p_A^{(1,1)*} = \arg\max_{p\in\{p_L,p_H\}} V_1(p) = \begin{cases} p_L & \text{if } p_L \geq \frac{r}{1-\gamma}\bar{F}(0) \\ p_H & \text{otherwise} \end{cases} \tag{EC.4}$$

$$V_1^* = \max_{p\in\{p_L,p_H\}} V_1(p) = \begin{cases} \frac{r+\gamma p_L}{1-f(0)\gamma}\bar{F}(0) & \text{if } p_L \geq \frac{r}{1-\gamma}\bar{F}(0) \\ \frac{r}{1-\gamma}\bar{F}(0) & \text{otherwise} \end{cases} \tag{EC.5}$$

When the firm owns $q=2$ garments, it is concerned about three prices: when one garment rents, the firm assigns it buyout price $p_A^{(2,1)}$; when both rent, the firm sets prices $p_A^{(2,2)}$ and $p_B^{(2,2)}$. We define $\mathbf{p}^{(2)} = \big(p_A^{(2,1)}, p_A^{(2,2)}, p_B^{(2,2)}\big)$, and let $V_2(\mathbf{p}^{(2)})$ represent the discounted future earnings that the firm collects when it owns two garments and uses pricing policy $\mathbf{p}^{(2)}$:

$$\begin{aligned} V_2\big(\mathbf{p}^{(2)}\big) =& f(0)\gamma V_2\big(\mathbf{p}^{(2)}\big) \\ &+ f(1)\Big(r + d\big(p_A^{(2,1)}\big)\gamma\big(p_A^{(2,1)} + V_1^*\big) + \big(1-d(p_A^{(2,1)})\big)\gamma V_2\big(\mathbf{p}^{(2)}\big)\Big) \\ &+ \bar{F}(1)\Big(2r + d(p_A^{(2,2)}) \times d(p_B^{(2,2)}) \times \gamma(p_A^{(2,2)} + p_B^{(2,2)}) \\ &\qquad + d(p_A^{(2,2)}) \times \big(1-d(p_B^{(2,2)})\big) \times \gamma(p_A^{(2,2)} + V_1^*) \\ &\qquad + \big(1-d(p_A^{(2,2)})\big) \times d(p_B^{(2,2)}) \times \gamma(V_1^* + p_B^{(2,2)}) \\ &\qquad + \big(1-d(p_A^{(2,2)})\big) \times \big(1-d(p_B^{(2,2)})\big) \times \gamma V_2\big(\mathbf{p}^{(2)}\big)\Big) \end{aligned} \tag{EC.6}$$

which can be rewritten as:

$$V_2\big(\mathbf{p}^{(2)}\big) = \frac{\begin{aligned} & f(1)\Big(r + d\big(p_A^{(2,1)}\big)\gamma\big(p_A^{(2,1)} + V_1^*\big)\Big) \\ & + \bar{F}(1)\Big(2r + d(p_A^{(2,2)}) \times d(p_B^{(2,2)}) \times \gamma(p_A^{(2,2)} + p_B^{(2,2)}) \\ &\qquad + d(p_A^{(2,2)}) \times \big(1-d(p_B^{(2,2)})\big) \times \gamma(p_A^{(2,2)} + V_1^*) \\ &\qquad + \big(1-d(p_A^{(2,2)})\big) \times d(p_B^{(2,2)}) \times \gamma(V_1^* + p_B^{(2,2)})\Big) \end{aligned}}{1 - f(0)\gamma - f(1)\big(1-d(p_A^{(2,1)})\big)\gamma - \bar{F}(1)\big(1-d(p_A^{(2,2)})\big)\big(1-d(p_B^{(2,2)})\big)\gamma} \tag{EC.7}$$

When $q = 2$, the firm is concerned about three buyout prices and each price can take one of two values, leading to $2^3 = 8$ unique buyout pricing policies for the firm to consider. However, when garment exits are deterministic (as is the case), it can be proven that the optimal strategy is nested – i.e., if it is optimal to sell w garments when z garments are rented, then it is also optimal to sell $w' \geq w$ garments when $z' > z$ garments are rented. Additionally, the value function $V_2\big(\mathbf{p}^{(2)}\big)$ is symmetric in $p_A^{(2,2)}$ and $p_B^{(2,2)}$, indicating that both garments are perfect substitutes. Hence, the firm only needs to consider the non-symmetric, nested pricing strategies: $(p_H; p_H, p_H)$, $(p_L; p_L, p_H)$ and $(p_L; p_L, p_L)$.

We first analyze the case where $p_L \geq \frac{r}{1-\gamma}\bar{F}(0)$. In this case, we know that $p_A^{(1,1)*} = p_L$ and $V_1^* = \frac{r+\gamma p_L}{1-f(0)\gamma}\bar{F}(0)$. The three candidate buyout pricing strategies yield:

$$\begin{aligned} V_2\big(p_H; p_H, p_H\big) =& \frac{f(1)r + \bar{F}(1)2r}{1 - f(0)\gamma - f(1)\gamma - \bar{F}(1)\gamma} = \frac{f(1)r + \bar{F}(1)2r}{1-\gamma} \\ V_2\big(p_L; p_L, p_H\big) =& \frac{f(1)\Big(r + \gamma\big(p_L + \frac{r+\gamma p_L}{1-f(0)\gamma}\bar{F}(0)\big)\Big) + \bar{F}(1)\Big(2r + \gamma\big(p_L + \frac{r+\gamma p_L}{1-f(0)\gamma}\bar{F}(0)\big)\Big)}{1 - f(0)\gamma} \\ V_2\big(p_L; p_L, p_L\big) =& \frac{f(1)\Big(r + \gamma\big(p_L + \frac{r+\gamma p_L}{1-f(0)\gamma}\bar{F}(0)\big)\Big) + \bar{F}(1)\Big(2r + \gamma 2p_L\Big)}{1 - f(0)\gamma} \end{aligned} \tag{EC.8}$$

Note that $V_2\big(p_L;p_L,p_L\big) \geq V_2\big(p_L;p_L,p_H\big)$ if and only if:

$$\begin{aligned}
2r+\gamma 2p_L &\geq 2r+\gamma\Big(p_L+\frac{r+\gamma p_L}{1-f(0)\gamma}\bar{F}(0)\Big) \\
p_L &\geq \frac{r+\gamma p_L}{1-f(0)\gamma}\bar{F}(0) \\
(1-f(0)\gamma-\bar{F}(0)\gamma)p_L &\geq r\bar{F}(0) \\
(1-\gamma)p_L &\geq r\bar{F}(0) \\
p_L &\geq \frac{r}{1-\gamma}\bar{F}(0)
\end{aligned} \tag{EC.9}$$

which is true in the case we are analyzing.

To see whether $V_2\big(p_L;p_L,p_L\big) \geq V_2\big(p_H;p_H,p_H\big)$ we need to check if:

$$\frac{f(1)\Big(r+\gamma\big(p_L+\frac{r+\gamma p_L}{1-f(0)\gamma}\bar{F}(0)\big)\Big)+\bar{F}(1)\big(2r+\gamma 2p_L\big)}{1-f(0)\gamma} \geq \frac{f(1)r+\bar{F}(1)2r}{1-\gamma} \tag{EC.10}$$

or, equivalently, we can check that the factors that multiply $f(1)$ and $\bar{F}(1)$ on the expression on the RHS are greater or equal than those on the expression on the LHS. That is:

$$\begin{aligned}
\frac{r+\gamma\big(p_L+\frac{r+\gamma p_L}{1-f(0)\gamma}\bar{F}(0)\big)}{1-f(0)\gamma} &\geq \frac{r}{1-\gamma} \\
(1-\gamma)r+\gamma(1-\gamma)\Big(p_L+\frac{r+\gamma p_L}{1-f(0)\gamma}\Big) &\geq \big(1-f(0)\gamma\big)r \\
(1-\gamma)\Big(p_L+\frac{r+\gamma p_L}{1-f(0)\gamma}\Big) &\geq \big(1-f(0)\big)r \\
(1-\gamma)p_L\Big(1+\frac{\gamma}{1-f(0)\gamma}\Big) &\geq r\Big(1-f(0)-\frac{1-\gamma}{1-f(0)\gamma}\Big) \\
p_L &\geq \frac{r}{1-\gamma}\times\frac{1-f(0)-\frac{1-\gamma}{1-f(0)\gamma}}{1+\frac{\gamma}{1-f(0)\gamma}}
\end{aligned} \tag{EC.11}$$

is guaranteed in this case, since $\frac{1-f(0)-\frac{1-\gamma}{1-f(0)\gamma}}{1+\frac{\gamma}{1-f(0)\gamma}} < 1$, and

$$\begin{aligned}
\frac{2r+\gamma 2p_L}{1-f(0)\gamma} &\geq \frac{2r}{1-\gamma} \\
\big(1-\gamma\big)\big(r+\gamma p_L\big) &\geq r\big(1-f(0)\gamma\big) \\
\gamma(1-\gamma)p_L &\geq r\gamma\big(1-f(0)\big) \\
p_L &\geq \frac{r}{1-\gamma}\bar{F}(0)
\end{aligned} \tag{EC.12}$$

is always true in the case we are analyzing.

Hence, if the firm owns $q=2$ garments and the selling price is such that $p_L \geq \frac{r}{1-\gamma}\bar{F}(0)$, the firm is better off selling its entire inventory of the product, i.e., the optimal buyout policy is $\mathbf{p}^{(2)*} = (p_L;p_L,p_L)$.

Now, let's turn to the case where $p_L < \frac{r}{1-\gamma}\bar{F}(0)$. In this case, we know that $p_A^{(1,1)*} = p_H$ and $V_1^* = \frac{r}{1-\gamma}\bar{F}(0)$. The three candidate buyout pricing strategies yield:

$$\begin{aligned}
V_2\big(p_H;p_H,p_H\big) =& \frac{f(1)r+\bar{F}(1)2r}{1-f(0)\gamma-f(1)\gamma-\bar{F}(1)\gamma} = \frac{f(1)r+\bar{F}(1)2r}{1-\gamma} \\
V_2\big(p_L;p_L,p_H\big) =& \frac{f(1)\Big(r+\gamma\big(p_L+\frac{r}{1-\gamma}\bar{F}(0)\big)\Big)+\bar{F}(1)\Big(2r+\gamma(p_L+\frac{r}{1-\gamma}\bar{F}(0))\Big)}{1-f(0)\gamma} \\
V_2\big(p_L;p_L,p_L\big) =& \frac{f(1)\Big(r+\gamma\big(p_L+\frac{r}{1-\gamma}\bar{F}(0)\big)\Big)+\bar{F}(1)\Big(2r+\gamma 2p_L\Big)}{1-f(0)\gamma}
\end{aligned} \tag{EC.13}$$

Since $p_L < \frac{r}{1-\gamma}\bar{F}(0)$, it follows trivially that $V_2\big(p_L;p_L,p_H\big) > V_2\big(p_L;p_L,p_L\big)$. In addition, note that $V_2\big(p_H;p_H,p_H\big) > V_2\big(p_L;p_L,p_H\big)$ if and only if:

$$\begin{aligned}
\frac{f(1)r+\bar{F}(1)2r}{1-\gamma} &> \frac{f(1)\Big(r+\gamma\big(p_L+\frac{r}{1-\gamma}\bar{F}(0)\big)\Big)+\bar{F}(1)\Big(2r+\gamma\big(p_L+\frac{r}{1-\gamma}\bar{F}(0)\big)\Big)}{1-f(0)\gamma} \\
\big(1-f(0)\gamma\big)\big(f(1)r+\bar{F}(1)2r\big) &> \big(1-\gamma\big)\big(f(1)r+\bar{F}(1)2r\big)+\big(f(1)+\bar{F}(1)\big)\gamma(1-\gamma)\Big(p_L+\frac{r}{1-\gamma}\bar{F}(0)\Big) \\
\gamma\big(1-f(0)\big)\big(f(1)r+\bar{F}(1)2r\big) &> \bar{F}(0)\gamma(1-\gamma)\Big(p_L+\frac{r}{1-\gamma}\bar{F}(0)\Big) \\
\big(f(1)r+\bar{F}(1)2r\big) &> (1-\gamma)\Big(p_L+\frac{r}{1-\gamma}\bar{F}(0)\Big) \\
\big(f(1)+2\bar{F}(1)-\bar{F}(0)\big)\frac{r}{1-\gamma} &> p_L \\
\big(1-f(0)-f(1)\big)\frac{r}{1-\gamma} &> p_L \\
\bar{F}(1)\frac{r}{1-\gamma} &> p_L
\end{aligned} \tag{EC.14}$$

This result indicates that, if $p_L \in \big[\frac{r}{1-\gamma}\bar{F}(1),\ \frac{r}{1-\gamma}\bar{F}(0)\big)$, it is justified for the firm to retain just one garment for rentals. In contrast, if $p_L < \frac{r}{1-\gamma}\bar{F}(1)$, the firm will retain both garments for rentals. Combining with the results for the case where $p_L \geq \frac{r}{1-\gamma}\bar{F}(0)$, we obtain:

$$\mathbf{p}^{(2)^*} = \begin{cases} (p_L;\ p_L, p_L) & \text{if } p_L \geq \frac{r}{1-\gamma}\bar{F}(0) \\[2ex] (p_L;\ p_L, p_H) & \text{if } p_L \in \big[\frac{r}{1-\gamma}\bar{F}(1),\ \frac{r}{1-\gamma}\bar{F}(0)\big) \\[2ex] (p_H;\ p_H, p_H) & \text{if } p_L < \frac{r}{1-\gamma}\bar{F}(1) \end{cases} \tag{EC.15}$$

$$V_2^* = \begin{cases} \frac{f(1)\left(r+\gamma\left(p_L+\frac{r+\gamma p_L}{1-f(0)\gamma}\bar{F}(0)\right)\right)+\bar{F}(1)\left(2r+\gamma 2p_L\right)}{1-f(0)\gamma} & \text{if } p_L \geq \frac{r}{1-\gamma}\bar{F}(0) \\ \frac{f(1)\left(r+\gamma\left(p_L+\frac{r}{1-\gamma}\bar{F}(0)\right)\right)+\bar{F}(1)\left(2r+\gamma(p_L+\frac{r}{1-\gamma}\bar{F}(0))\right)}{1-f(0)\gamma} & \text{if } p_L \in \left[\frac{r}{1-\gamma}\bar{F}(1),\ \frac{r}{1-\gamma}\bar{F}(0)\right) \\ \frac{f(1)r+\bar{F}(1)2r}{1-\gamma} & \text{if } p_L < \frac{r}{1-\gamma}\bar{F}(1) \end{cases} \tag{EC.16}$$

EC.3. Mathematical Proofs

EC.3.1. Derivation of the MDP expressions. Proof of Equations (8) and (10).

The firm wants to determine

$$p^*_{i,n} = \arg\max_{p\in\mathbb{R}_+} V_{i,n} \tag{EC.17}$$

where

$$\begin{aligned} V_{i,n}(p \mid \Theta) = & \frac{\rho_i}{1-\gamma+\gamma\rho_i} \times (r + \gamma V_{i,n+1}) \\ & + \frac{\rho_i}{1-\gamma+\gamma\rho_i} \delta_{i,n} \times \gamma(s_i + V_{out} - V_{i,n+1}) \\ & + \frac{\rho_i}{1-\gamma+\gamma\rho_i} (1-\delta_{i,n}) d_{i,n}(p) \times \gamma(p + V_{out} - V_{i,n+1}) \end{aligned} \tag{EC.18}$$

Because only the last line depends on the buyout price p, we focus on maximizing $d_{i,n}(p) \times (p + V_{out} - V_{i,n+1})$. From the logistic assumption placed on $d_{i,n}(p)$, and assuming the firm will price optimally into the future, we rewrite the firm's problem as:

$$p^*_{i,n} = \arg\max_{p\in\mathbb{R}_+} \frac{\exp(\mu_{i,n} - \beta_i p)}{1 + \exp(\mu_{i,n} - \beta_i p)} \times (p + V_{out} - V^*_{i,n+1}) \tag{EC.19}$$

The price $p^*_{i,n}$ must satisfy the first-order condition, which is:

$$\beta_i (p^*_{i,n} + V_{out} - V^*_{i,n+1}) = 1 + \exp(\mu_{i,n} - \beta_i p^*_{i,n}) \tag{EC.20}$$

Rearranging the terms and adding $\mu_{i,n}$ on both sides:

$$\mu_{i,n} + \beta_i(V_{out} - V^*_{i,n+1}) - 1 = \mu_{i,n} - \beta_i p^*_{i,n} + \exp(\mu_{i,n} - \beta_i p^*_{i,n}) \tag{EC.21}$$

Exponentiating on both sides:

$$\exp\Big(\mu_{i,n} + \beta_i(V_{out} - V^*_{i,n+1}) - 1\Big) = \exp(\mu_{i,n} - \beta_i p^*_{i,n}) \times \exp\Big(\exp(\mu_{i,n} - \beta_i p^*_{i,n})\Big) \tag{EC.22}$$

Lambert's W function (Corless et al. 1996) can be used to express the solution to this equation. Lambert's W function $W(x)$ denotes the solution w to the equation $w\exp(w) = x$. It is increasing, concave and positive for all $x \in (0, \infty)$. In terms of Lambert's W function:

$$\exp(\mu_{i,n} - \beta_i p^*_{i,n}) = W\Big(\exp\big(\mu_{i,n} - \beta_i(V^*_{i,n+1} - V_{out}) - 1\big)\Big) \tag{EC.23}$$

Finally, substituting $\exp(\mu_{i,n} - \beta_i p^*_{i,n})$ back into Equation (EC.20), by basic algebraic manipulation we obtain:

$$p^*_{i,n} = V^*_{i,n+1} - V_{out} + \frac{1 + W\Big(\exp\big(\mu_{i,n} - \beta_i(V^*_{i,n+1} - V_{out}) - 1\big)\Big)}{\beta_i} \tag{EC.24}$$

This proves Equation (8).

Now, replacing $p^*_{i,n}$ into $d_{i,n}(p)\times(p+V_{out}-V^*_{i,n+1})$ yields:

$$\begin{aligned}
d_{i,n}(p^*_{i,n})\times(p^*_{i,n}+V^*_{out}-V_{i,n+1}) &= \frac{\exp(\mu_{i,n}-\beta_i p^*_{i,n})}{1+\exp(\mu_{i,n}-\beta_i p^*_{i,n})}\times(p^*_{i,n}+V_{out}-V^*_{i,n+1}) \\
&= \frac{W\Big(\exp\big(\mu_{i,n}-\beta_i(V^*_{i,n+1}-V_{out})-1\big)\Big)}{1+W\Big(\exp\big(\mu_{i,n}-\beta_i(V^*_{i,n+1}-V_{out})-1\big)\Big)}\times\frac{1+W\Big(\exp\big(\mu_{i,n}-\beta_i(V^*_{i,n+1}-V_{out})-1\big)\Big)}{\beta_i} \\
&= \frac{W\Big(\exp\big(\mu_{i,n}-\beta_i(V^*_{i,n+1}-V_{out})-1\big)\Big)}{\beta_i}
\end{aligned} \tag{EC.25}$$

So the recursive equation for backward induction is:

$$\begin{aligned}
V^*_{i,n}(\Theta) =& V_{i,n}(p^*_{i,n}\mid\Theta) \\
=& \frac{\rho_i}{1-\gamma+\gamma\rho_i}\times\big(r+\gamma V^*_{i,n+1}+\delta_{i,n}\gamma(s_i+V_{out}-V^*_{i,n+1})\big) \\
&+\frac{\rho_i}{1-\gamma+\gamma\rho_i}(1-\delta_{i,n})\gamma\frac{W\Big(\exp\big(\mu_{i,n}-\beta_i(V^*_{i,n+1}-V_{out})-1\big)\Big)}{\beta_i}
\end{aligned} \tag{EC.26}$$

This proves Equation (10).

EC.3.2. Proof of Proposition 1.

Let $d:\mathbb{R}_+\to[0,1]$ be a differentiable, strictly decreasing, log-concave function.

Let $J(p)$ denote the part of the objective function that depends on the buyout price p, after we drop the subindices for ease of notation:

$$J(p)=d(p)\times(p+V_{out}-V) \tag{EC.27}$$

First, it is easy to see that if $\lim_{p\to\infty} d(p)=D>0$, then $J(p)$ will tend to infinity.

$$\lim_{p\to\infty} J(p)=\lim_{p\to\infty} d(p)\times(p+V_{out}-V)=\lim_{p\to\infty} D\times p=\infty \tag{EC.28}$$

Hence, for $J(p)$ to have a maximizer, then it must be the case that $d(p)$ tends to 0 as $p\to\infty$. Because $d(p)$ is assumed strictly decreasing for all $p\in\mathbb{R}_+$, then it cannot touch the horizonal axis, so $d(p)$ is always positive.

Next, the first-order condition of the maximization problem is:

$$J'(p)=d(p)+d'(p)\times(p+V_{out}-V)=0 \tag{EC.29}$$

To show that $p\le V-V_{out}$ will never satisfy the first order condition, we show that $J'(p)\ne 0$ for all $p\le V-V_{out}$:

$$\begin{aligned}
p &\le V-V_{out} \\
p+V_{out}-V &\le 0 \\
d'(p)\times(p+V_{out}-V) &\ge 0 \quad \text{(because } d(p) \text{ is strictly decreasing)} \\
J'(p)=d(p)+d'(p)\times(p+V_{out}-V) &> 0 \quad \text{(because } d(p) \text{ is always positive)}
\end{aligned} \tag{EC.30}$$

So, if a maximizer p^* of the function $J(p)$ exists, then it must be $p^* > V - V_{out}$.

Now, we show that, if $d(p)$ is log concave, then $J(p)$ is log-concave as well, and hence it can have only one critical point. From the evidence shown before, we will just focus on the range $p > V - V_{out}$. If $d(p)$ is log-concave:

$$\frac{d^2}{dp^2} \log d(p) = \frac{d''(p) \times d(p) - d'(p)^2}{d(p)^2} < 0 \tag{EC.31}$$

The partial derivatives of $\log J(p) = \log d(p) + \log(p + V_{out} - V)$ are:

$$\frac{d}{dp} \log J(p) = \frac{d'(p)}{d(p)} + \frac{1}{p + V_{out} - V} \tag{EC.32}$$

and

$$\frac{d^2}{dp^2} \log J(p) = \frac{d''(p) \times d(p) - d'(p)^2}{d(p)^2} - \frac{1}{(p + V_{out} - V)^2} < 0 \tag{EC.33}$$

where the inequality follows because $d(p)$ is assumed log-concave. This proves $J(p)$ is log-concave, and consequently it can only have one critical point.

Finally, we show that if $d(p)$ tends to 0 faster than $1/p$, then $J(p)$ will have an interior maximizer $p^* < \infty$ – which will be unique as per the previous point. Note that $J'(V - V_{out}) = d(V - V_{out}) > 0$. If $J(p)$ has an interior critical, it $J(p)$ will be decreasing afterwards all as p tends to ∞ or, equivalently, $\lim_{p \to \infty} J'(p) < 0$

Without loss of generality, let $d(p)$ be asymptotically equivalent to p^{-a} as $p \to \infty$ with $a > 0$ a constant, i.e., $d(p) \sim p^{-a}$. Then, $d'(p) \sim -a \cdot p^{-(a+1)}$ as $p \to \infty$.

Becasue $J'(p) = d(p) + d'(p) \times (p + V_{out} - V)$, it follows that:

$$J'(p) \sim p^{-a} - a \cdot p^{-(a+1)} p = (1-a)p^{-a} \begin{cases} > 0 & \text{if } a < 1 \\ = 0 & \text{if } a = 1 \\ < 0 & \text{if } a > 1 \end{cases} \tag{EC.34}$$

This shows that if $d(p)$ tends to 0 faster than a polynomial of order 1, then $\lim_{p \to \infty} J'(p) < 0$, so there will exist an interior solution to the FOC which will be the maximizer of $J(p)$.

EC.3.3. Proof of Lemma 1.

For ease of notation, let us denote:

$$p\ (\Delta, \mu, \beta) = \Delta + \frac{1 + W\Big(\exp(\mu - \beta\Delta - 1)\Big)}{\beta} \tag{EC.35}$$

A useful result regarding Lambert's W function states that:

$$\frac{d}{dx}\ W(x) = \frac{W(x)}{x\Big(1 + W(x)\Big)} \tag{EC.36}$$

Then, the partial derivative of $p\ (\Delta,\mu,\beta)$ with respect to Δ is:

$$\begin{aligned}\frac{\partial}{\partial\Delta}\ p\ (\Delta,\mu,\beta) =& 1+\frac{1}{\beta}\ \frac{W(e^{\mu-\beta\Delta-1})}{e^{\mu-\beta\Delta-1}\left(1+W(e^{\mu-\beta\Delta-1})\right)}e^{\mu-\beta\Delta-1}(-\beta)\\ =&1-\frac{W(e^{\mu-\beta\Delta-1})}{1+W(e^{\mu-\beta\Delta-1})}\\ =&\frac{1}{1+W(e^{\mu-\beta\Delta-1})}\\ >&0\end{aligned} \tag{EC.37}$$

Similarly, the partial derivative of $p\ (\Delta,\mu,\beta)$ with respect to μ is:

$$\begin{aligned}\frac{\partial}{\partial\mu}\ p\ (\Delta,\mu,\beta) =& \frac{1}{\beta}\ \frac{W(e^{\mu-\beta\Delta-1})}{e^{\mu-\beta\Delta-1}\left(1+W(e^{\mu-\beta\Delta-1})\right)}e^{\mu-\beta\Delta-1}\\ =&\frac{1}{\beta}\ \frac{W(e^{\mu-\beta\Delta-1})}{1+W(e^{\mu-\beta\Delta-1})}\\ >&0\end{aligned} \tag{EC.38}$$

Finally, the partial derivative of $p\ (\Delta,\mu,\beta)$ with respect to β is:

$$\begin{aligned}\frac{\partial}{\partial\beta}\ p\ (\Delta,\mu,\beta) =& \frac{1}{\beta^2}\left(\frac{W(e^{\mu-\beta\Delta-1})}{e^{\mu-\beta\Delta-1}\left(1+W(e^{\mu-\beta\Delta-1})\right)}e^{\mu-\beta\Delta-1}(-\beta\Delta)-1-W(e^{\mu-\beta\Delta-1})\right)\\ =&\frac{1}{\beta^2}\left(\frac{W(e^{\mu-\beta\Delta-1})}{1+W(e^{\mu-\beta\Delta-1})}(-\beta\Delta)-1-W(e^{\mu-\beta\Delta-1})\right)\\ =&\frac{-1}{\beta^2}\left(\frac{\beta\Delta W(e^{\mu-\beta\Delta-1})}{1+W(e^{\mu-\beta\Delta-1})}+1+W(e^{\mu-\beta\Delta-1})\right)\\ <&0\end{aligned} \tag{EC.39}$$

The inequalities follow because $W(x)>0$ for all $x>0$.

EC.3.4. Derivation of the Negative Binomial expressions for garment damage. Proof of Equations (14) and (15).

Suppose Y is a random variable that follows a negative binomial distribution with shape parameter k and rate parameter θ, i.e., $Y\sim \mathrm{NB}(k,\theta)$.

The probability mass function for the NBD is:

$$\Pr(Y=n\mid k,\theta)=\frac{\Gamma(k+n)}{\Gamma(k)n!}\left(\frac{\theta}{\theta+1}\right)^k\left(\frac{1}{\theta+1}\right)^n \tag{EC.40}$$

where $\Gamma(\cdot)$ denotes the gamma function given by $\Gamma(z)=\int_0^\infty t^{z-1}e^{-t}dt$.

The survival probability is:

$$
\begin{aligned}
\Pr(Y \geq n \mid k, \theta) &= \sum_{i=n}^{\infty} \Pr(Y = i \mid k, \theta) \\
&= \sum_{i=n}^{\infty} \frac{\Gamma(k+i)}{\Gamma(k) i!} \left(\frac{\theta}{\theta+1}\right)^k \left(\frac{1}{\theta+1}\right)^i \\
&= \frac{1}{\Gamma(k)} \left(\frac{\theta}{\theta+1}\right)^k \sum_{i=n}^{\infty} \frac{\Gamma(k+i)}{i!} \left(\frac{1}{\theta+1}\right)^i \\
&\overset{(ii)}{=} \frac{1}{\Gamma(k)} \left(\frac{\theta}{\theta+1}\right)^k \left(\frac{1}{\theta+1}\right)^n \frac{\Gamma(k+n)}{n!} \; {}_2F_1(1, k+n; n+1; \frac{1}{\theta+1}) \\
&= \frac{\Gamma(k+n)}{\Gamma(k) n!} \left(\frac{\theta}{\theta+1}\right)^k \left(\frac{1}{\theta+1}\right)^n \; {}_2F_1(1, k+n; n+1; \frac{1}{\theta+1}) \\
&= \Pr(Y = n \mid k, \theta) \; {}_2F_1(1, k+n; n+1; \frac{1}{\theta+1})
\end{aligned}
\tag{EC.41}
$$

where $\Gamma(\cdot)$ denotes the gamma function and ${}_2F_1(a,b;c;d)$ is the Gaussian hypergeometric function.

The equality marked with $^{(ii)}$ follows because $\sum_{i=n}^{\infty} \frac{\Gamma(k+i)}{i!} \left(\frac{1}{\theta+1}\right)^i$ can be expressed as $\left(\frac{1}{\theta+1}\right)^n \frac{\Gamma(k+n)}{n!} \; {}_2F_1(1, k+n; n+1; \frac{1}{\theta+1})$. To see this:

$$
\begin{aligned}
\sum_{i=n}^{\infty} \frac{\Gamma(k+i)}{i!} \left(\frac{1}{\theta+1}\right)^i &= \sum_{i=n}^{\infty} \frac{(k+i-1)!}{i!} \left(\frac{1}{\theta+1}\right)^i \\
&= \sum_{j=0}^{\infty} \frac{(k+n+j-1)!}{(n+j)!} \left(\frac{1}{\theta+1}\right)^{n+j} \\
&= \left(\frac{1}{\theta+1}\right)^n \sum_{j=0}^{\infty} \frac{(k+n+j-1)!}{(n+j)!} \left(\frac{1}{\theta+1}\right)^j \\
&= \left(\frac{1}{\theta+1}\right)^n \sum_{j=0}^{\infty} \frac{(k+n-1)! \frac{(k+n+j-1)!}{(k+n-1)!}}{n! \frac{(n+j)!}{n!}} \left(\frac{1}{\theta+1}\right)^j \\
&\overset{(iii)}{=} \left(\frac{1}{\theta+1}\right)^n \frac{(k+n-1)!}{n!} \sum_{j=0}^{\infty} \frac{(k+n)_j}{(n+1)_j} \left(\frac{1}{\theta+1}\right)^j \\
&\overset{(iv)}{=} \left(\frac{1}{\theta+1}\right)^n \frac{\Gamma(k+n)}{n!} \; {}_2F_1(1, k+n; n+1; \frac{1}{\theta+1})
\end{aligned}
\tag{EC.42}
$$

The equality marked with $^{(iii)}$ follows because, from the rising Pochhammer factorial notation $(q)_i = \prod_{k=0}^{i-1}(q+k) = \frac{(q+i-1)!}{(q-1)!}$, we can rewrite:

$$
\frac{(k+n+j-1)!}{(k+n-1)!} = (k+n)_j \tag{EC.43}
$$

and

$$
\frac{(n+j)!}{n!} = (n+1)_j \tag{EC.44}
$$

The equality marked with $^{(iv)}$ follows because $\Gamma(x) = (x-1)!$ and because the Gaussian hypergeometric function is defined as:

$$ {}_2F_1(a,b;c;z) = \sum_{i=0}^{\infty} \frac{(a)_i (b)_i}{(c)_i} \frac{z^i}{i!} \tag{EC.45} $$

where $(q)_i$ denotes the rising Pochhammer factorial of q.

The hazard function $\delta_n(k,\theta) = \Pr(Y = n \mid Y \geq n, k, \theta)$ can be computed as:

$$ \begin{aligned} \delta_n(k,\theta) &= \Pr(Y = n \mid Y \geq n, k, \theta) \\ &= \frac{\Pr(Y = n \cap Y \geq n \mid k, \theta)}{\Pr(Y \geq n \mid k, \theta)} \\ &= \frac{\Pr(Y = n \mid k, \theta)}{\Pr(Y \geq n \mid k, \theta)} \\ &\overset{(v)}{=} \frac{1}{{}_2F_1(1, k+n; n+1; \frac{1}{\theta+1})} \end{aligned} \tag{EC.46} $$

The last equality, marked with $^{(v)}$ follows because $\Pr(Y \geq n \mid k,\theta) = \Pr(Y = n \mid k, \theta)\ {}_2F_1(1, k+n; n+1; \frac{1}{\theta+1})$, derived earlier.

EC.3.5. Derivation of the Poisson expressions for garment rentals. Proof of Equation (17).

Let $D \sim \text{Poisson}(\Lambda)$. Then:

$$ \Pr(D = R \mid \lambda) = \frac{\lambda^R \exp(-\lambda)}{R!} \text{ for } R = 0, 1, 2, ... \tag{EC.47} $$

and

$$ \Pr(D \leq R \mid \lambda) = \frac{\Gamma(R+1, \lambda)}{\Gamma(R+1)} \text{ for } R = 0, 1, 2, ... \tag{EC.48} $$

where $\Gamma(\cdot)$ denotes the gamma function and $\Gamma(\cdot,\cdot)$ represents the incomplete gamma function.

Then, the survival probability is:

$$ \Pr(D \geq R \mid \lambda) = 1 - \Pr(D \leq R-1 \mid \lambda) = 1 - \frac{\Gamma(R,\lambda)}{\Gamma(R)} = \frac{\Gamma(R) - \Gamma(R,\lambda)}{\Gamma(R)} \tag{EC.49} $$