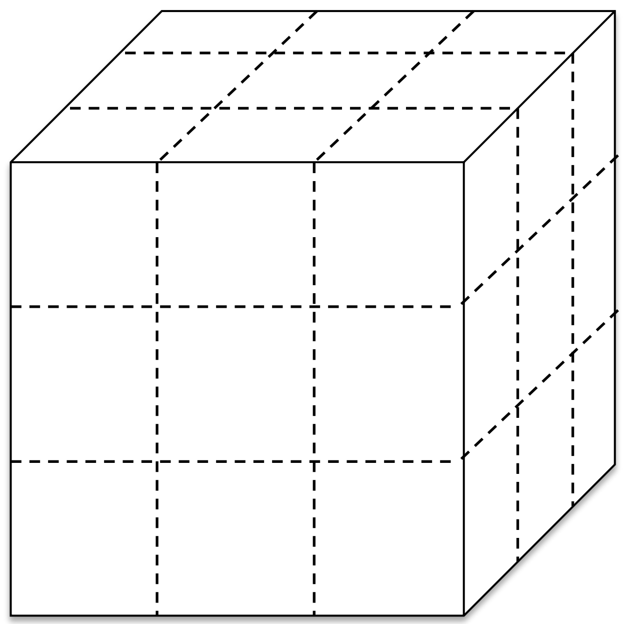
Puzzler

A carpenter, working with a buzz saw, wishes to cut a wooden cube, three inches on a side, into 27 one-inch cubes. He can do this easily by making six cuts through the cube, keeping the pieces together in the cube shape. Can he reduce the number of necessary cuts by rearranging the pieces after each cut?

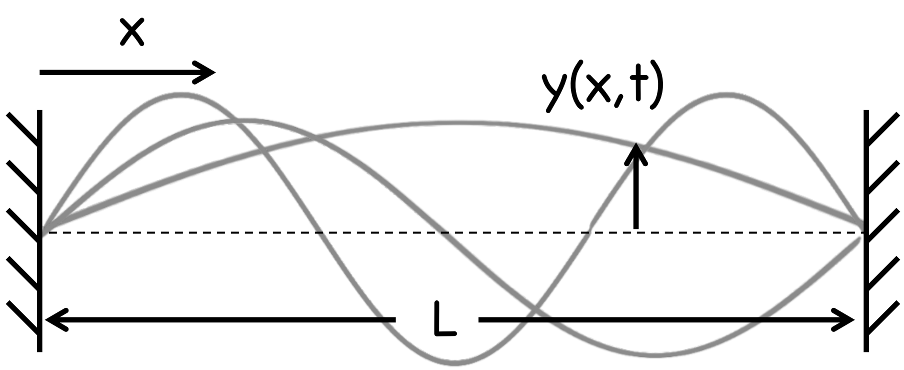


Introduction to Audio and Music Engineering

Lecture 5

- Superposition of modes
- Fourier series
- Plucked strings
- Fourier coefficients
- Plucking position and timbre
- Pickup position and timbre

String vibration review



$$\mu \frac{d^2 y(x,t)}{dt^2} = T \frac{d^2 y(x,t)}{dx^2}$$

T = tension

μ = mass/length

Speed of wave on string $\quad c \equiv \sqrt{\frac{T}{\mu}}$

Solution is sinusoidal in time and space: $\quad y(x,t) = \sin(n\omega_0 t) \cdot \sin(n\pi \frac{x}{L})$

where $\quad \omega_0 = \pi \frac{c}{L}$

$$f_0 = \frac{\omega_0}{2\pi} = \frac{c}{2L}$$

Spatial wavelength: $\quad \lambda_n = \frac{2L}{n}$

Temporal frequency: $\quad f_n = n \frac{\omega_0}{2\pi} = n \frac{c}{2L} = nf_0$

Superposition of modes

It is possible to build up any periodic shape from sine and cosine waves!

Fourier's Theorem

Examples: Run the m-file: Simple_Fourier_series.m

Square wave:
Only odd harmonics

$$x(t)=\sum_{n=1}^{\infty}\frac{1}{(2n-1)}\sin\left(2\pi(2n-1)ft\right)$$

Triangle wave:
Only odd harmonics

$$x(t)=\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos\left(2\pi(2n-1)ft\right)$$

Sawtooth wave:

$$x(t)=\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(2\pi nft\right)$$

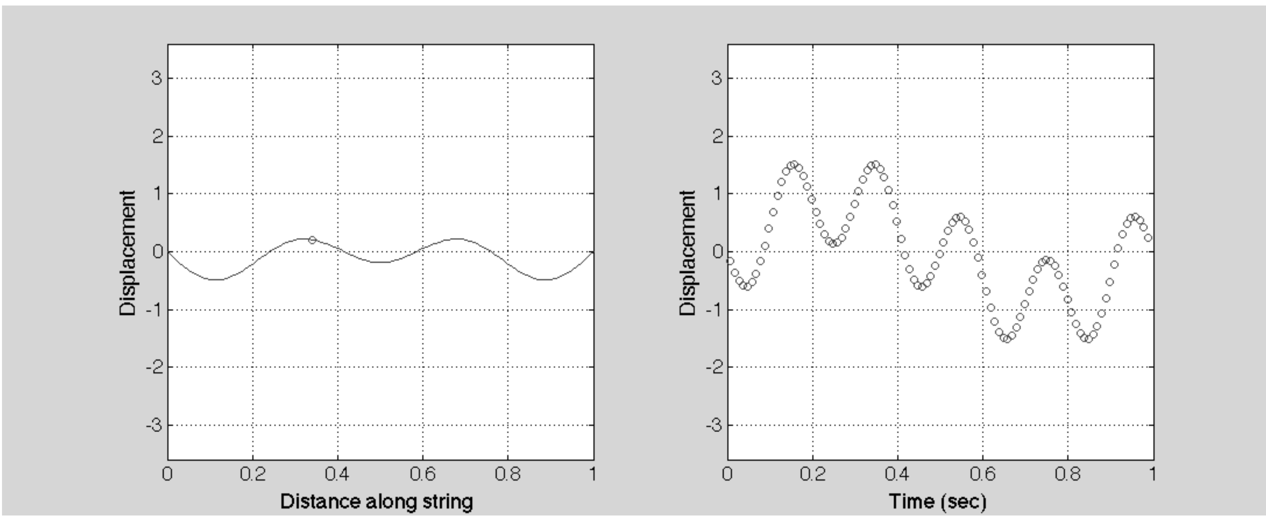
Composite waveforms in space and time

Each mode of the string oscillates at its own frequency.

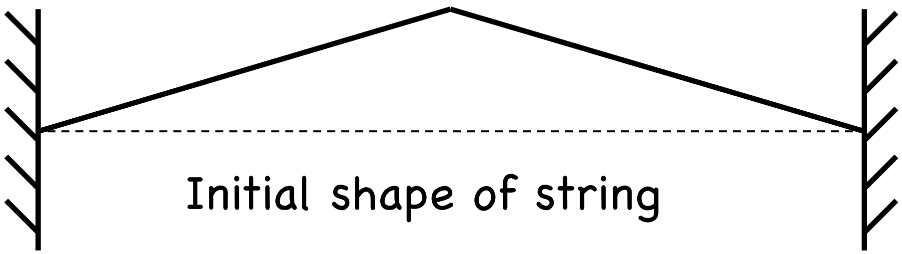
The time evolution of the combined waveform can become very complex!

Examples: Run the m-file: stringmodes2.m

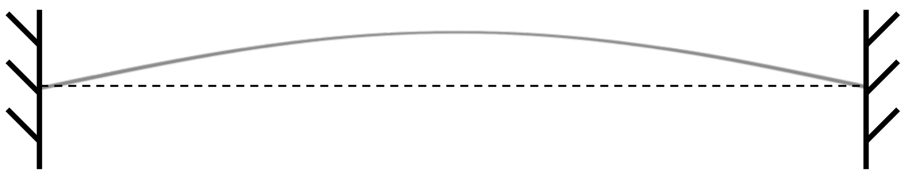
Both the overall shape of the string and the motion of any given point on the string in time are complex combinations of sine waves.



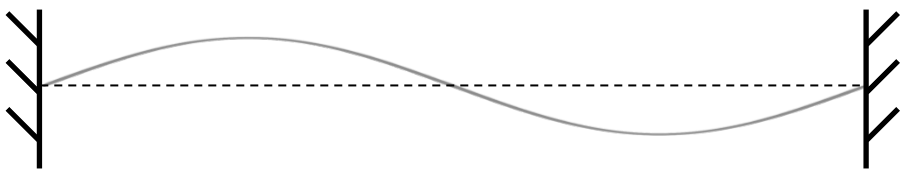
The Plucked String



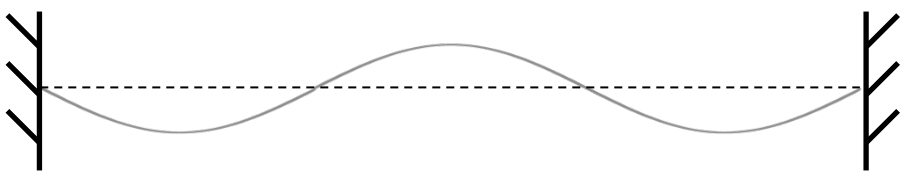
What combination of modes are needed to add up to this shape?



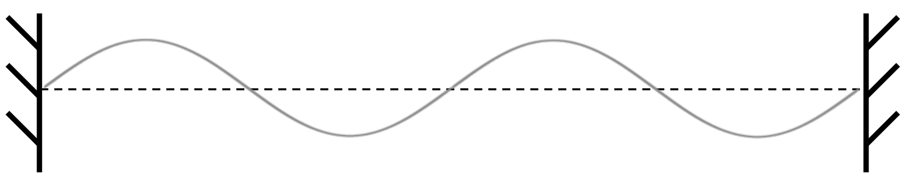
Yes



No – wrong symmetry



Yes



No – wrong symmetry

But how much of each one is needed?

Fourier Series

Fourier series for y(x,t) :

$$y(x,t) = \sum_n B_n \cos(\omega_n t)\cdot\sin(k_n x)$$

time* space

Amplitude of n'th mode
... Fourier Coefficient

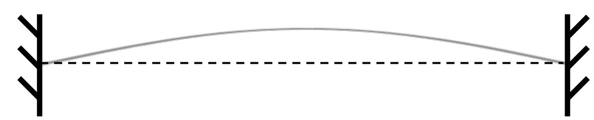
$$\omega_n = n\pi\frac{c}{L} = n2\pi\frac{c}{2L} = n2\pi f_0$$

$$k_n = \frac{\omega_n}{c} = n\frac{\pi}{L}$$

Re-write this as: $y(x,t) = \sum_n B_n \cos(n2\pi f_0 t)\cdot\sin(n\pi\frac{x}{L})$

$$y(x,0) = \sum_n B_n \sin(n\pi\frac{x}{L})$$ at t = 0

when n = 1 ... this is just the first mode of the string



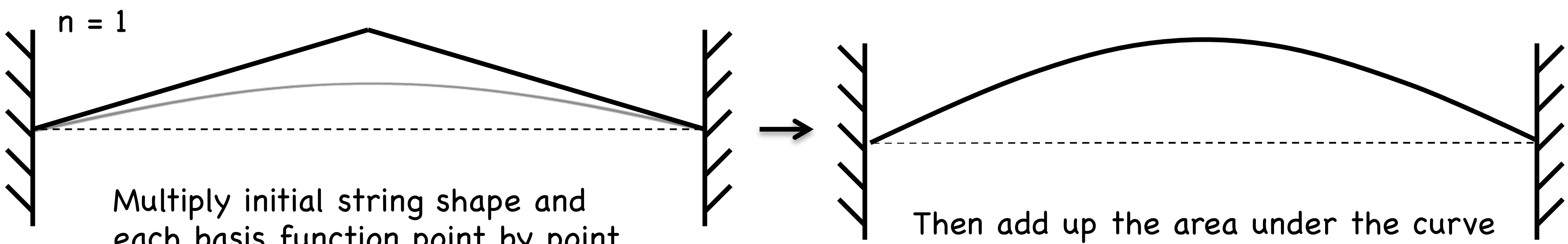
If you know y(x,0) there is a formula to find B_n !

$$B_n = \frac{2}{L}\int_0^L y(x,0)\sin(\frac{n\pi x}{L})dx$$

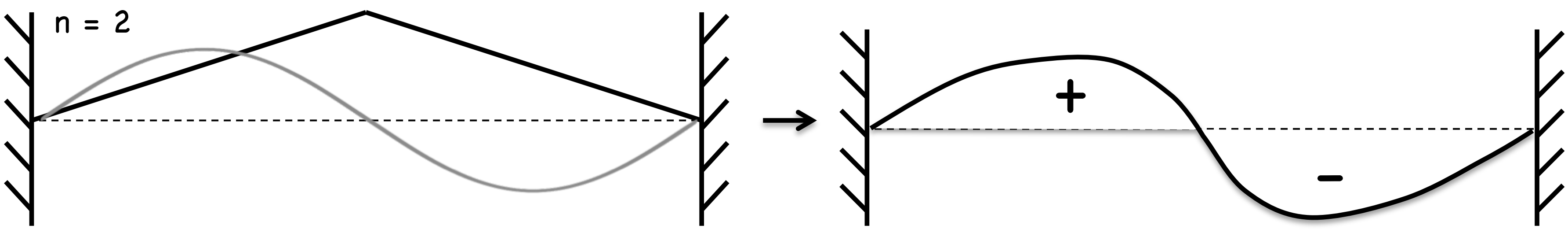
* The time function is a cosine because string starts at maximum displacement

Physical Interpretation of B_n formula

For each value of n: n = 1,2,3…



$B_1 > 0$

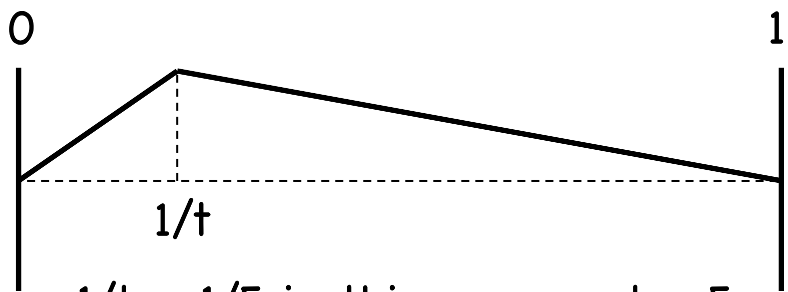


+ and - areas cancel each other.

$B_2 = 0$

See FourCoeff.m demonstration

Analytical Solution for Fourier Coefficients



1/t ≈ 1/5 in this case, so t = 5

$$B_n = \frac{2}{n^2\pi^2}\frac{t^2}{t-1}\sin\left(\frac{n\pi}{t}\right)$$

See pluckmodes.m demo

… computes the time evolution of the string shape and the amplitudes of each Fourier coefficient, B_n