Introduction to Audio and Music Engineering

Lecture 20

Topics:

- The Discrete Fourier Transform
- Interpretation of the DFT

The Discrete Fourier Transform

Time Domain → Frequency Domain

Signal sampled in time $x(n) \rightarrow X(k)$ Signal sampled in frequency

N samples → N samples

x(n) and X(k) both contain exactly the same information – simply different representations.

- DFT Uses:
  - Signal analysis and visualization tool
  - Many signal processing operations are simpler in the frequency domain

Definition of the DFT

x(n) are samples of the continuous signal x(t): (N samples)

k is the frequency index (N samples)

Time to frequency:

$$X(k) = \frac{1}{N} \sum_{n=0}^{N-1} x(n) e^{-2\pi jk \frac{n}{N}}$$

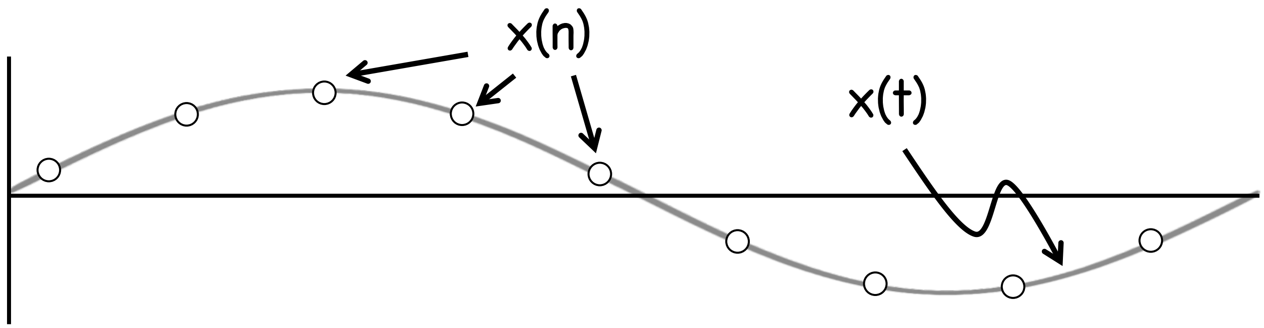
$$\begin{tikzcd} x(n) \arrow[r, bend left] & X(k) \arrow[l, bend left] \end{tikzcd}$$

Frequency domain back to the time domain:

$$x(n) = \sum_{k=0}^{N-1} X(k) e^{2\pi jn \frac{k}{N}}$$

How to compute the DFT

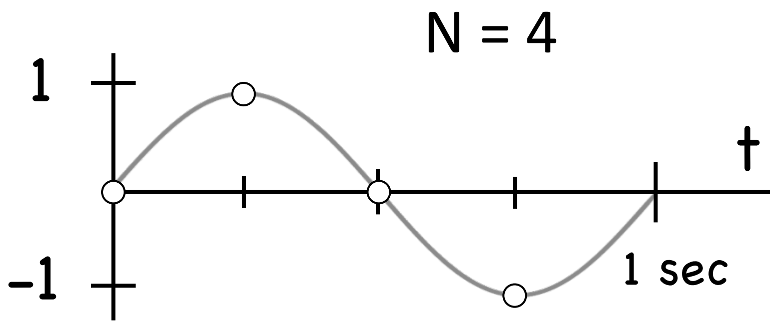
- Start with x(n), the samples of x(t)



- Start with k = 0
- Plug in x(n) values and complete sum in DFT
- Do for next value of k, continue up to k = N-1

A very simple example …

- Compute the DFT of a 1 Hz sine wave sampled 4x's per second for 1 second.



First sample is at t = 0.

$n = 0,1,2,3$

$x(n) = [0\ 1\ 0\ -1]$

Thus there are 4 values for k: $k = 0,1,2,3$

Re-write DFT using Euler's formula

$$X(k) = \frac{1}{N}\sum_{n=0}^{N-1} x(n)\left(\cos\left(2\pi k\frac{n}{N}\right) - j\sin\left(2\pi k\frac{n}{N}\right)\right)$$

DFT cont.

We need the samples of $\cos(2\pi kn/N)$ and $\sin(2\pi kn/N)$
for $k = 0,1,2,3$ and $n = 0,1,2,3$

k		n = 0,1,2,3
k = 0	$\cos(2\pi 0n/N)$	[1,1,1,1]
	$\sin(2\pi 0n/N)$	[0,0,0,0]
k = 1	$\cos(2\pi 1n/N)$	[1,0,-1,0]
	$\sin(2\pi 1n/N)$	[0,1,0,-1]
k = 2	$\cos(2\pi 2n/N)$	[1,-1,1,-1]
	$\sin(2\pi 2n/N)$	[0,0,0,0]
k = 3	$\cos(2\pi 3n/N)$	[1,0,-1,0]
	$\sin(2\pi 3n/N)$	[0,-1,0,1]

Now compute the sums …

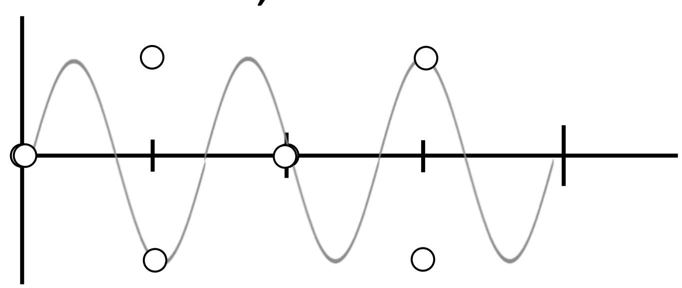
We need the samples of cos(2πkn/N) and sin(2πkn/N) for k = 0,1,2,3 and n = 0,1,2,3

k	$\frac{1}{N}\sum_{n=0}^{3} x(n)\cos\left(2\pi k \frac{n}{N}\right)$	$\frac{1}{N}\sum_{n=0}^{3} x(n)\sin\left(2\pi k \frac{n}{N}\right)$
k = 0	[0 1 0 -1] x [1 1 1 1] = 0	[0 1 0 -1] x [0 0 0 0] = 0
k = 1	[0 1 0 -1] x [1 0 -1 0] = 0	[0 1 0 -1] x [0 1 0 -1] = 1/2
k = 2	[0 1 0 -1] x [1 -1 1 -1] = 0	[0 1 0 -1] x [0 0 0 0] = 0
k = 3	[0 1 0 -1] x [1 0 -1 0] = 0	[0 1 0 -1] x [0 -1 0 1] = -1/2

sin, k = 3

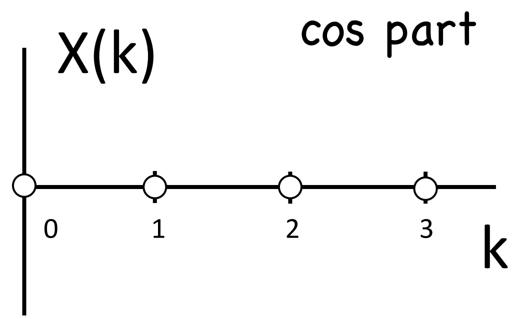
x(n)

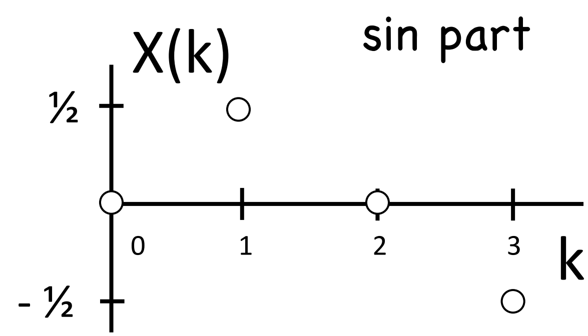
sin(2π3n/N)



= ¼ (-1 + -1) = - ½

How to interpret the result …

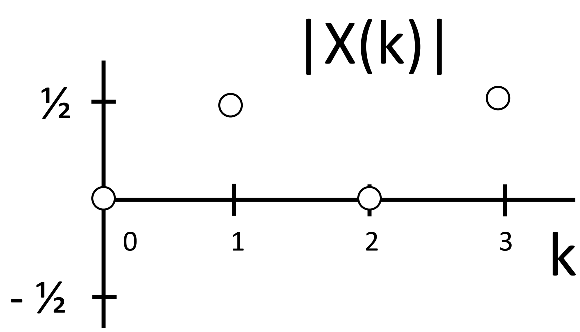




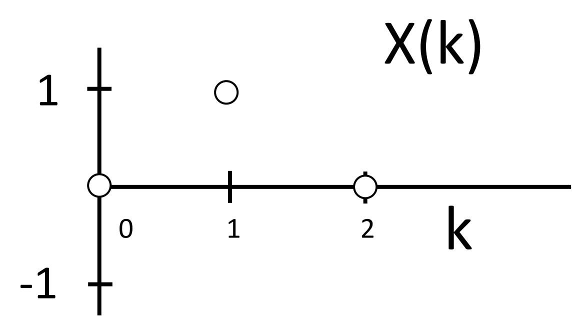
Note that sin part is multiplied by "j"

To compute the power spectrum we compute the "magnitude" of the DFT and "fold" it over the Nyquist frequency. Nyquist frequency = R/2, where R is the sample rate.

R = 4 samples/sec → R/2 = 2 Hz

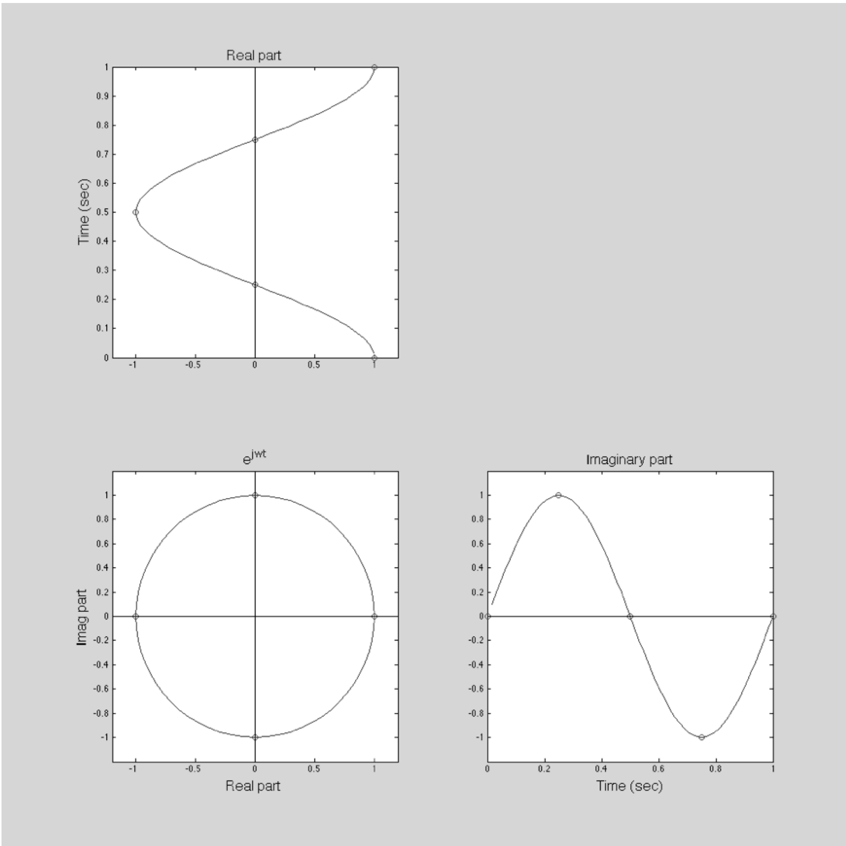


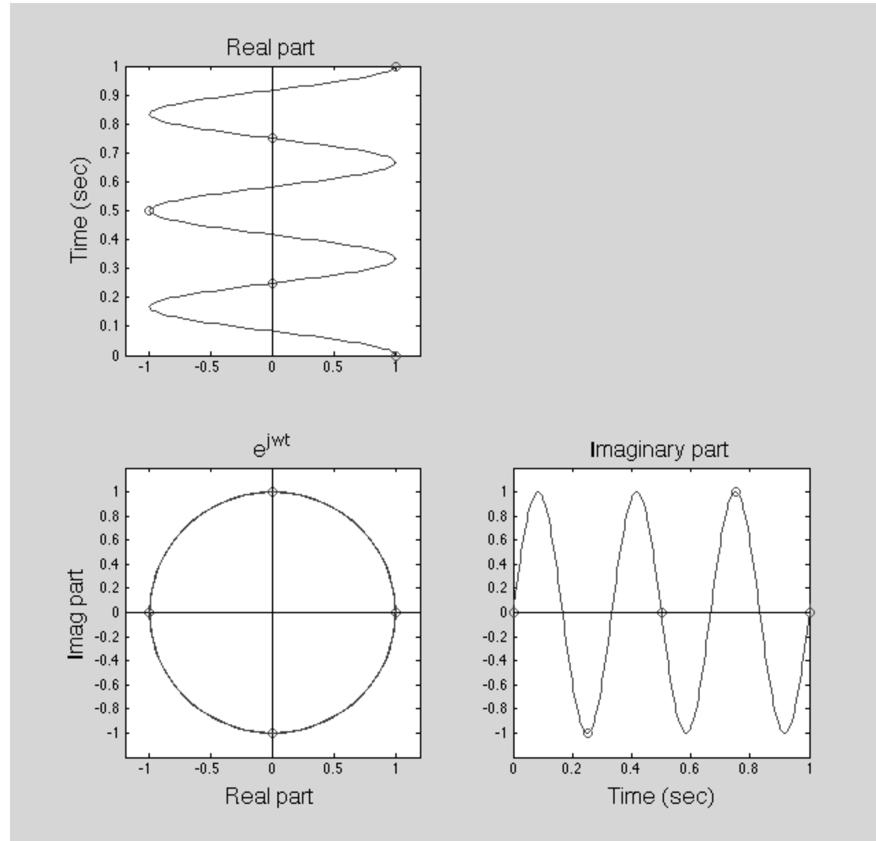
½ + ½ = 1



Folding means to add each frequency component above the Nyquist frequency to its complement below the Nyquist frequency

“Complementary” frequencies





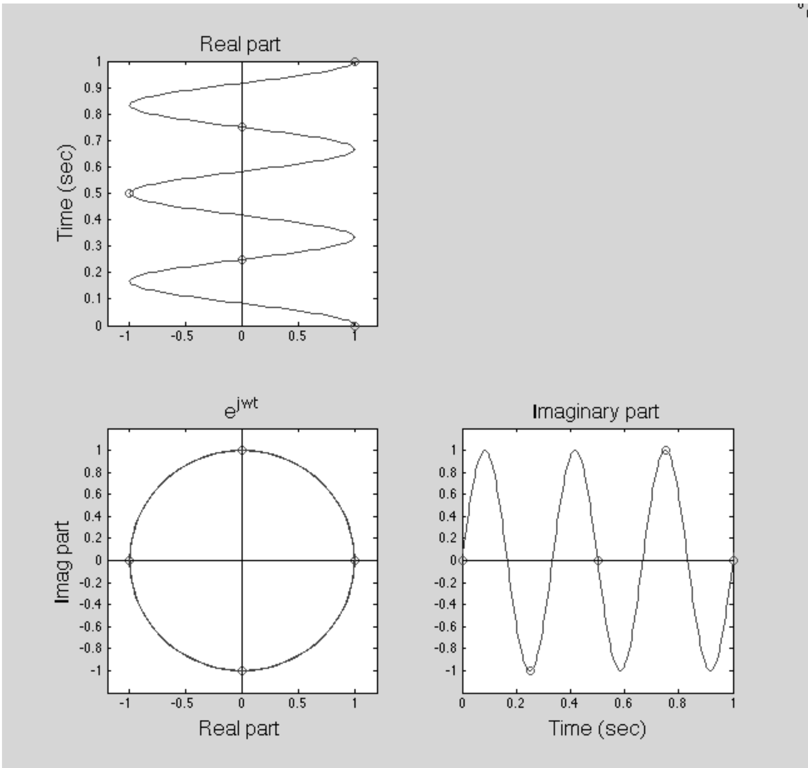
Samples of 1 Hz: [(1,0) (0,j) (–1,0) (0,–j)]

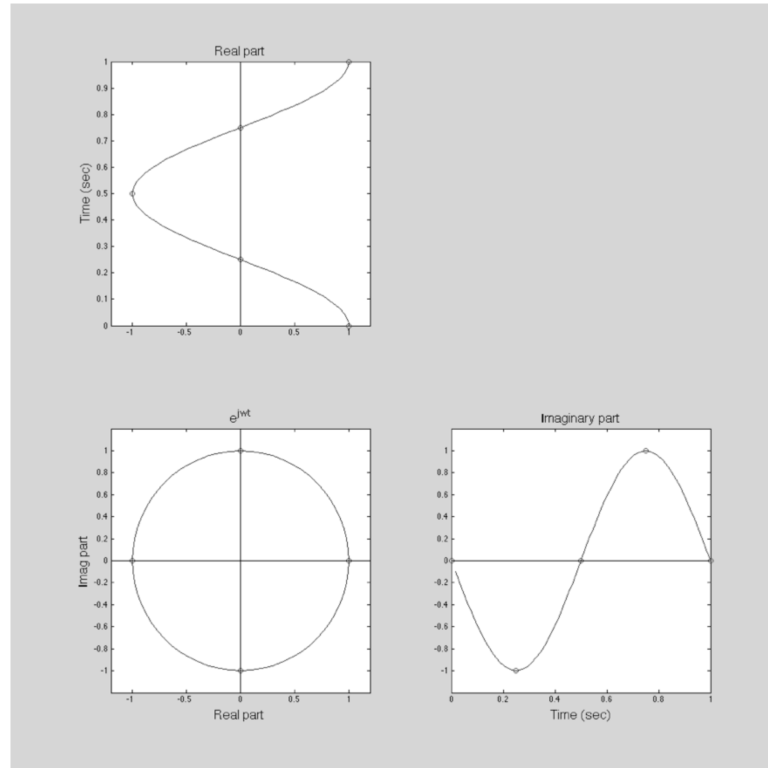
[1 j -1 –j]

Samples of 3 Hz: [1 –j –1 j]

These are just complex conjugates!

But look at this ...





Samples of 3 Hz: [1 −j -1 j]

Samples of -1 Hz: [1 −j -1 j]

The samples of the 3 Hz phasor are the same as the samples of the -1 Hz phasor!

Positive and negative frequencies

What is the relationship of the samples of 1 Hz to the samples of -1 Hz?

Phasor $$e^{j2\pi ft} = \cos(2\pi ft) + j\sin(2\pi ft)$$

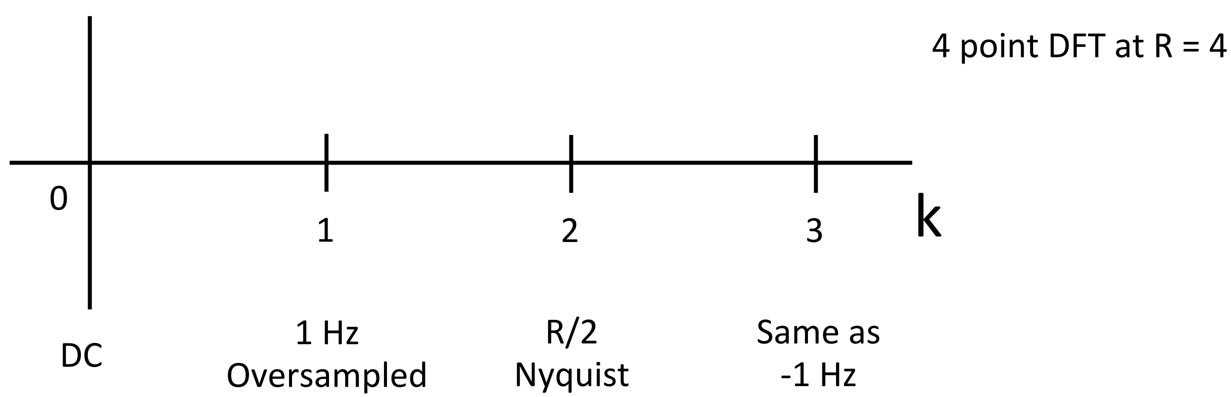
f = 1 $$e^{j2\pi t} = \cos(2\pi t) + j\sin(2\pi t)$$

f = -1 $$e^{-j2\pi t} = \cos(2\pi t) - j\sin(2\pi t)$$

The negative frequency phasor is just the C.C. of the positive frequency phasor.

And the samples of 3 Hz are the same as the samples of -1 Hz,

... so the samples of 3 Hz is just the C.C. of the samples of 1 Hz.



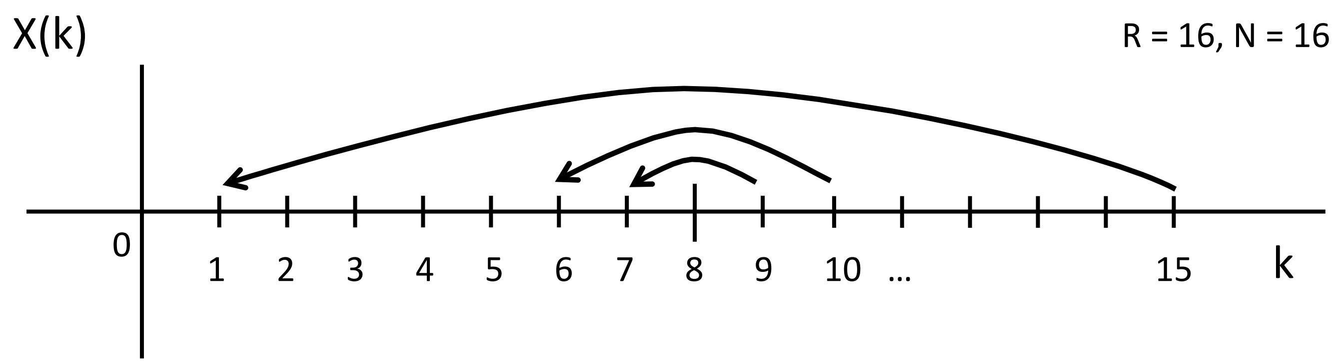
Positive and negative frequencies …

A phasor at -1 Hz that has the same real part (cos) as +1 Hz.

But the imaginary part of -1 Hz phasor is -1 x imaginary part of +1 Hz phasor.

So $X(k=3) = X^*(k=1)$

More generally – each X(k) above R/2 is the C.C. of an X(k) below R/2.



$X(9) = X^*(7)$
$X(10) = X^*(6)$
etc.