We need to move on to Chapter 8 - Sinusoidal Steady State Response - where we will learn how to represent how circuits ~~by their~~ respond to sinusoidal forcing functions that have been on forever and will remain on forever (hence: Steady State) This is extremely useful, because it is the foundation for concepts to come in future courses, for example:

Fourier Series: Any periodic function can be represented by an (infinite) sum of sines and cosines, each with different amplitudes, and all with frequencies that are multiples of the fundamental frequency.

$$f_{per}(t) = \sum_{i=1}^{\infty} A_i \cos(2\pi f_0 i t) + \sum_{i=1}^{\infty} B_i \sin(2\pi f_0 i t)$$

Or the Fourier Transform:

Any (bounded, etc.) function can be represented by summation (i.e. integral) of an infinite set of sinusoids with different frequencies and phases:

$$f(t) = \int_{-\infty}^{\infty} A(\omega)\cos(\omega t)\,d\omega + \int_{-\infty}^{\infty} B(\omega)\sin(\omega t)\,d\omega$$

If you can figure out what $A(\omega)$ and $B(\omega)$ are, then you know what $f(t)$ is going to be.

You will spend lots of time in later classes studying these things, we're going to start with the basics, for now.

We are going to <u>start</u> by analyzing ckts driven by a single frequency sinusoid, and work our way up.

First, we are going to represent sinusoids with complex numbers:

Euler's Relation: $e^{j\theta} = \cos\theta + j\sin\theta$

or $\cos\theta = \mathrm{Re}(e^{j\theta})$

$\sin\theta = \mathrm{Im}(e^{j\theta})$

A general sinusoid in the <u>Time Domain</u> is $v(t) = V_A \cos(\omega t + \phi)$

$$= V_A \,\mathrm{Re}\left(e^{j(\omega t + \phi)}\right)$$

$$= V_A \,\mathrm{Re}\left[e^{j\omega t} e^{j\phi}\right]$$

$$= \mathrm{Re}\left[V_A e^{j\phi} e^{j\omega t}\right]$$

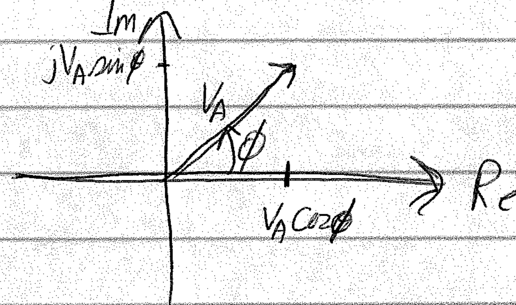
$$= \mathrm{Re}\left[\hat{V}_A e^{j\omega t}\right]$$

$\tilde{V}_A$ is the <u>phasor representation</u> (not Star Trek Phaser)

of a complex # $\tilde{V}_A = V_A e^{j\phi} = V_A[\cos\phi + j\sin\phi]$

This can be represented on a 2D coordinate

system:



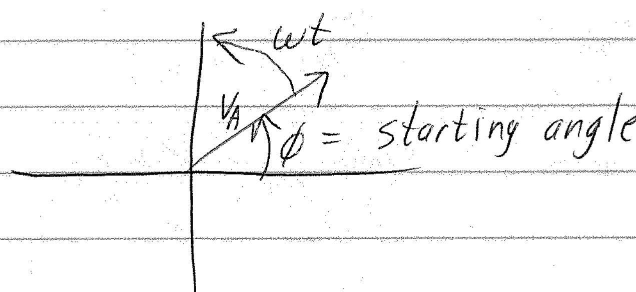
called the Phasor Diagram

Put the $e^{j\omega t}$ back in: $(V_A e^{j\phi})e^{j\omega t}$ leads to

$$V_A e^{j(\omega t+\phi)} = V_A e^{j\gamma(t)}$$

constant amplitude — Phase increases (wrt time)

or a <u>rotation</u> of the vector around the origin:



At any time, t, the Real part is just the

projection of the vector onto the horizontal axis.

Reverse the steps:

rotating phasor

$$v(t) = \mathrm{Re}\left[\tilde{V} e^{j\omega t}\right]$$

phasor at $t=0$, initial phase ϕ.

$$= \mathrm{Re}\left[V_A e^{j\phi} e^{j\omega t}\right]$$

$$= V_A \,\mathrm{Re}\left[\cos(\omega t+\phi) + j\sin(\omega t+\phi)\right]$$

$$= V_A \cos(\omega t+\phi) \text{ as we started with.}$$

We need to practice going from one end to the other without the intervening steps:

$$v(t) = 10\cos(377t - 17^\circ) \Rightarrow \tilde{V} = 10 e^{j(-17^\circ)}$$

usually want this in radians:

$$\frac{-17^\circ \pi}{180^\circ} = -0.297$$

$$\tilde{V} = 10 e^{-j0.297}$$

Another way of writing the phasor is Magnitude $\angle$ Phase

$$\tilde{V} = V_A e^{j\phi} = V_A \angle \phi \quad \text{or} \quad \tilde{V} = 10\angle -17^\circ$$

saves writing $e^{j(\phi)}$ each time

Do exercises 8-1 + 8-2.

Also, recall that $\underset{\text{radian frequency}}{w} = 2\pi \underset{\text{frequency (Hz)}}{f}$ (CY/sec)

f (Hz)	w
10	62.8
60	377
100	628
50	314

Recall that we multiply and ~~add~~ divide complex numbers by multiplying (dividing) amplitudes and adding (subtracting) phases:

$$(Ae^{j\phi})(Be^{j\gamma}) = AB\,e^{j(\phi+\gamma)}$$

Adding + Subtracting requires rectangular form:

$$\tilde{A} = \hat{A}e^{j\phi} = A\cos\phi + jA\sin\phi \quad \hat{B} = Be^{j\gamma} = B\cos\gamma + jB\sin\gamma$$

$$\tilde{A} + \tilde{B} = (A\cos\phi + B\cos\gamma) + j(A\sin\phi + B\sin\gamma)$$

$$= C + jD = Ee^{j\theta} \text{ where } E = \sqrt{C^2+D^2}$$

$$+\ \theta = \tan^{-1}\left(\frac{D}{C}\right)$$

We also use phasors to represent currents:

$$i(t) = \mathrm{Re}\left[\tilde{I} e^{j\omega t}\right]$$

Properties of Phasors:

Sum of phasors OF THE SAME FREQUENCY is the sum of the phasors:

$$v(t) = v_1(t) + v_2(t) + \cdots v_N(t) \quad \text{all same } f.$$

$$= \mathrm{Re}\left[\tilde{V}_1 e^{j\omega t}\right] + \mathrm{Re}\left[\tilde{V}_2 e^{j\omega t}\right] + \cdots \mathrm{Re}\left[\tilde{V}_N e^{j\omega t}\right]$$

$$= \mathrm{Re}\left[\left(\tilde{V}_1 + \tilde{V}_2 + \cdots \tilde{V}_N\right) e^{j\omega t}\right]$$

$$= \mathrm{Re}\left[\tilde{V} e^{j\omega t}\right]$$

Time Derivative:

$$\frac{d}{dt}\left[\mathrm{Re}\left(\tilde{V} e^{j\omega t}\right)\right] = \mathrm{Re}\left(\tilde{V} \frac{d}{dt}\left(e^{j\omega t}\right)\right)$$

$$= \mathrm{Re}\left(j\omega \tilde{V} e^{j\omega t}\right)$$

$$\cancel{= \mathrm{Re}(j\omega \tilde{V}}$$

$$= \mathrm{Re}\left(\left(j\omega \tilde{V}\right) e^{j\omega t}\right)$$

Recall

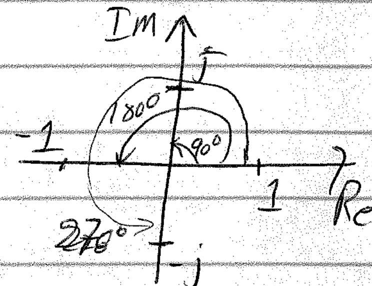
$j^0 = 1 = e^{j0}$

$j^1 = j = e^{j\pi/2} = e^{j90^\circ}$

$j^2 = -1 = e^{j\pi} = e^{j180^\circ}$

$j^3 = -j = e^{j3\pi/2} = e^{j270^\circ}$

$j^4 = 1 = e^{j2\pi} = e^{j360^\circ}$



$j^{-1} = \frac{1}{j} = -j = e^{-j\pi/2} = e^{-j90^\circ}$

$j^{-2} = \frac{1}{j^2} = -1$

from $\tilde{V} = V_A e^{j\phi}$

$$j\omega\tilde{V} = e^{j\frac{\pi}{2}}\omega V_A e^{j\phi}$$

$$= \omega V_A e^{j(\phi+\pi/2)}$$

$$= \omega V_A e^{j(\phi+90^\circ)}$$

derivative gives $\omega V_A \angle +90^\circ$

Integral (Inverse Derivative)

$$\int v(t)\,dt = \mathrm{Re}\left(\frac{1}{j\omega}\tilde{V}e^{j\omega t}\right)$$

$$= \mathrm{Re}\left(-j\frac{1}{\omega}\tilde{V}e^{j\omega t}\right)$$

$$= \mathrm{Re}\left(\underbrace{\frac{1}{\omega}V_A e^{j(\phi-90^\circ)}} e^{j\omega t}\right)$$

Integral gives $\frac{V_A}{\omega} \angle -90^\circ$

Example:

$$i_1(t) = 50\cos(100t)\ \text{mA}$$

$$i_2(t) = 20\cos(100t+60^\circ)\ \text{mA}$$

$\omega = 100\ s^{-1}$

$\tilde{I}_1 = 50\angle 0^\circ$ $\qquad$ $\tilde{I}_2 = 20\angle 60^\circ$

$= 50\cos(0) + j\,50\sin(0^\circ)$ $\qquad$ $= 10 + j\,17.32$

$= 50 + j\,0$

$$i_1(t) + i_2(t) = Re\left((\tilde{I}_1 + \tilde{I}_2) e^{j\omega t}\right)$$

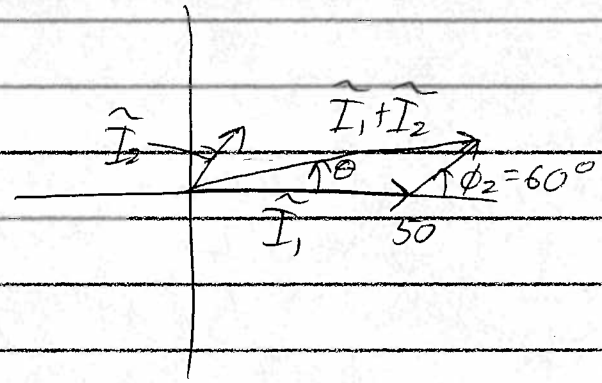
$$= Re\left((50 + j0 + 10 + j\,17.32) e^{j\omega t}\right)$$

$$= Re\left((60 + j\,17.32) e^{j\omega t}\right)$$

$$= Re\left(62.45 \angle 16.10^\circ \, e^{j\omega t}\right)$$

$$= 62.45 \cos(100t + 16.10^\circ)$$

Pictorially:



$$\frac{d}{dt}\left(i_1(t)\right) = Re\left(\left(100(50)\angle 90^\circ\right) e^{j\omega t}\right)$$

$$= Re\left(5000\, e^{j(\omega t + 90^\circ)}\right) \ mA/s$$

We can show that KCL + KVL still hold for phasors:

KCL: The algebraic sum of phasor currents leaving any node is zero.

KVL: The algebraic sum of phasor voltage drops around a loop is zero.

And Device Constraints:

$v_R = R i_R$ Resistors: $\boxed{\tilde{V}_R = R\hat{I}_R}$

$i_c = C\frac{d}{dt}v_c$ Capacitors: $i_c(t) = C\frac{dv_c}{dt}$

$$\hat{I}_c = C\,j\omega\,\tilde{V}_c$$
$$= j\omega C\,\tilde{V}_c$$

$v_L = L\frac{d}{dt}i_L$ or $\boxed{\tilde{V}_c = \frac{1}{j\omega C}\hat{I}_c}$

Inductors: $v_L = L\frac{di_L}{dt}$

$$\hat{V}_L = L(j\omega\hat{I}_L)$$

$$\boxed{\tilde{V}_L = j\omega L\,\hat{I}_L}$$

These all look like Ohm's Law:

$$\tilde{V} = \tilde{Z}\tilde{I}$$ where $\tilde{Z}$ is called the <u>impedance</u>

$\tilde{Z}_R = R$ (purely real) does not change with frequency

$\tilde{Z}_C = \frac{1}{j\omega C} = -j\frac{1}{\omega C}$ (purely imaginary)

$\tilde{Z}_L = j\omega L$ (purely imaginary)

change with frequency ω.

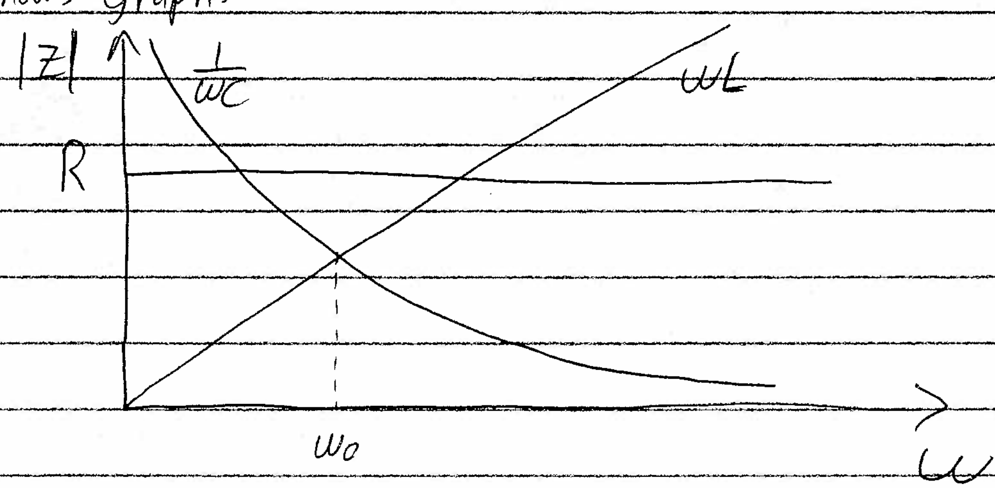
In general, $\tilde{Z}$ can be a complex number:

$$\tilde{Z} = R + j\chi$$

impedance = resistance, reactance

(reactance > 0 ⇒ inductive
reactance < 0 ⇒ capacitive)

Text shows graph:



ω_0 is when $|Z_C| = |Z_L|$ or

$$\frac{1}{\omega_0 C} = \omega_0 L$$

$$\frac{1}{LC} = \omega_0^2$$

$$\omega_0 = \frac{1}{\sqrt{LC}}$$ as we saw before.

Note behavior at low + high frequencies:

$\tilde{Z}_C = \frac{1}{j\omega C}$ $\quad |\tilde{Z}_C| = \frac{1}{\omega C}$

$\lim_{\omega \to 0} |\tilde{Z}_C| = \infty$ (an open ckt at DC)

$\lim_{\omega \to \infty} |\tilde{Z}_C| = 0$ (a short ckt at High F)

$\tilde{Z}_L = j\omega L$ $\quad |\tilde{Z}_L| = \omega L$

$\lim_{\omega \to 0} |\tilde{Z}_L| = 0$ (a short ckt at DC)

$\lim_{\omega \to \infty} |\tilde{Z}_L| = \infty$ (an open ckt at high F)