Introduction to Audio and Music Engineering

Lecture 14

Topics:

- Complex number review
- Complex exponential function
- Phasors
- Capacitors
- RC circuits
- High-pass and Low-pass filters

Complex Numbers

The need for imaginary numbers ...

$x^2 - 4 = 0 \quad \rightarrow \quad x^2 = 4$, so $x = +2$ and $x = -2$ are solutions

How about: $x^2 + 4 = 0$

$x^2 = -4 = (-1)(4)$ so $x = \sqrt{-1}\ \sqrt{4}$ $\quad x = \pm 2\sqrt{-1}$ $\quad x = \pm 2j$

$j = \sqrt{-1}$

$j^1 = j \quad j^2 = -1 \quad j^3 = -j \quad j^4 = 1 \quad j^5 = j \quad \ldots$

Complex Number: $z = \underbrace{x}_{\text{real part}} + j\,\underbrace{y}_{\text{imaginary part}}$

$Re(z) = x$

$Im(z) = y$

Complex Number Arithmetic

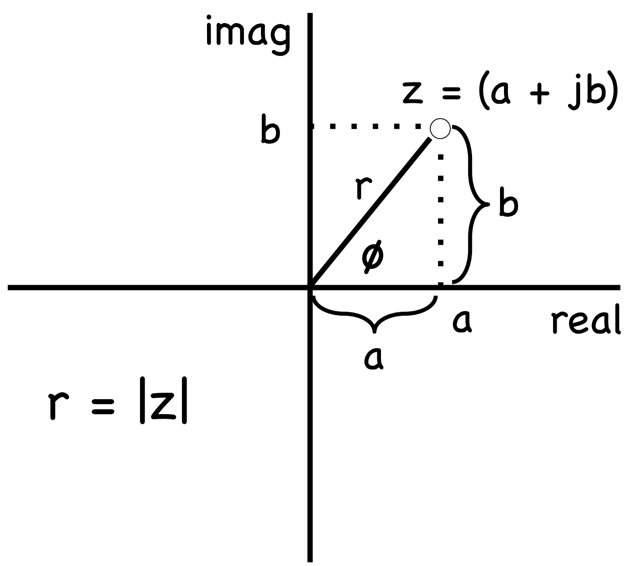
Addition: $(a + jb) + (c + jd) = (a+c) + j(b+d)$

Multiplication: $(a + jb) \times (c + jd) = (ac-bd) + j(bc+ad)$

Complex Conjugate: $(a + jb) \xrightarrow[cc]{} (a - jb)$ $\quad z = a + jb \quad z^* = a - jb$

$|z|^2 = z \times z^* = a^2 + b^2$

Graphing Complex Numbers:



$r = |z|$

$$\sin\phi = \frac{b}{r}, \quad \boxed{b = r\sin\phi}$$

$$\cos\phi = \frac{a}{r}, \quad \boxed{a = r\cos\phi}$$

$$r^2 = a^2 + b^2, \quad \boxed{r = \sqrt{a^2 + b^2}}$$

$$\tan\phi = \frac{b}{a}, \quad \boxed{\phi = \tan^{-1}\frac{b}{a}}$$

Complex Exponential Function

Taylor Series Expansions

https://en.wikipedia.org/wiki/Taylor_series

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$$

n! = n(n-1)(n-2)...1

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \ldots$$

Consider ... $e^{j\varphi}$

$$e^{j\phi} = 1 + \frac{j\phi}{1!} + \frac{(j\phi)^2}{2!} + \frac{(j\phi)^3}{3!} + \frac{(j\phi)^4}{4!} + \frac{(j\phi)^5}{5!} + \ldots$$

$$e^{j\phi} = \left(1 - \frac{\phi^2}{2!} + \frac{\phi^4}{4!} - \ldots\right) + j\left(\frac{\phi}{1!} - \frac{\phi^3}{3!} + \frac{\phi^5}{5!} - \ldots\right)$$

$$e^{j\phi} = \cos\phi + j\sin\phi$$ Euler's Formula

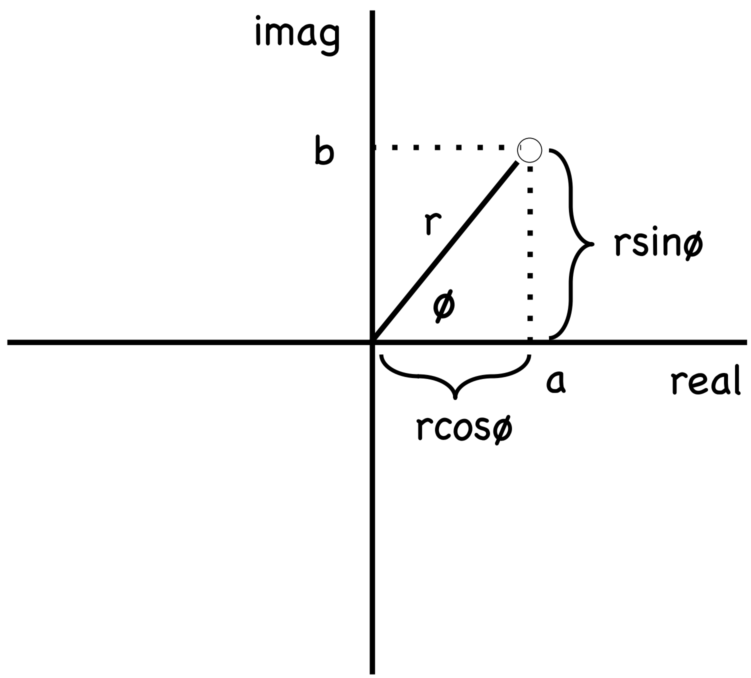
A remarkable equation!

$$e^{j\pi} = \boxed{-1} + \boxed{0}$$

or equivalently

$$e^{j\pi} + 1 = 0$$

$$re^{j\phi} = r\cos\phi + jr\sin\phi$$



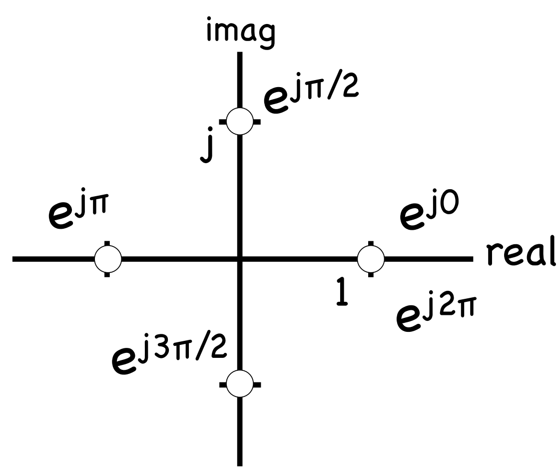
$$e^{j0} = \cos 0 + j\sin 0 = 1$$

$$e^{j\frac{\pi}{2}} = \cos\frac{\pi}{2} + j\sin\frac{\pi}{2} = j$$

$$e^{j\pi} = \cos\pi + j\sin\pi = -1$$

$$e^{j\frac{3\pi}{2}} = \cos\frac{3\pi}{2} + j\sin\frac{3\pi}{2} = -j$$

$$e^{j2\pi} = \cos 2\pi + j\sin 2\pi = 1$$



Complex Arithmetic revisited

Simplifications using complex exponential notation:

Multiplication: $r_1 e^{j\varphi_1} \times r_2 e^{j\varphi_2} = r_1 r_2 e^{j(\varphi_1+\varphi_2)}$

Division: $\dfrac{r_1 e^{j\varphi_1}}{r_2 e^{j\varphi_2}} = \dfrac{r_1}{r_2} e^{j(\varphi_1-\varphi_2)}$

Magnitude: $\left|re^{j\varphi}\right|^2 = r^2$

$\left|re^{j\varphi}\right| = r$

Useful Identities

$$\frac{e^{j\varphi} + e^{-j\varphi}}{2} = \cos\phi$$

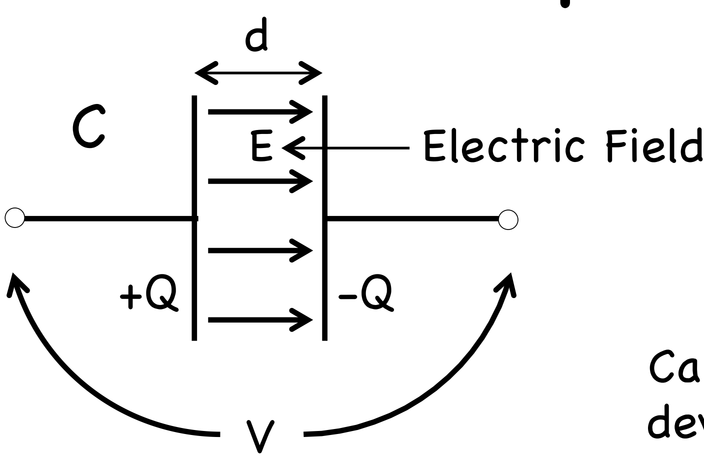
$$\frac{e^{j\varphi} - e^{-j\varphi}}{2} = j\sin\phi$$

Complex phasors and sinusoidal oscillations

$$e^{j\omega t} = \cos\omega t + j\sin\omega t$$

Circular motion in the complex plane

Capacitors



Energy is stored by the electric field in the capacitor.

$E = V/d$ or $V = E\,d$

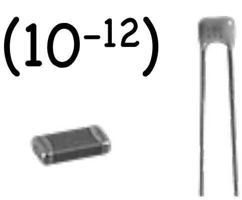
Capacitance is the capacity of the device to store charge ...

$C = Q/V$ = Coulomb/Volt (Farads)

$Q = C\,V$ 1 Farad capacitor → 1 Volt / 1 Coulomb

Typical capacitor values: picoFarads (pF) to milliFarads (mF)

(10^{-12}) (10^{-3})



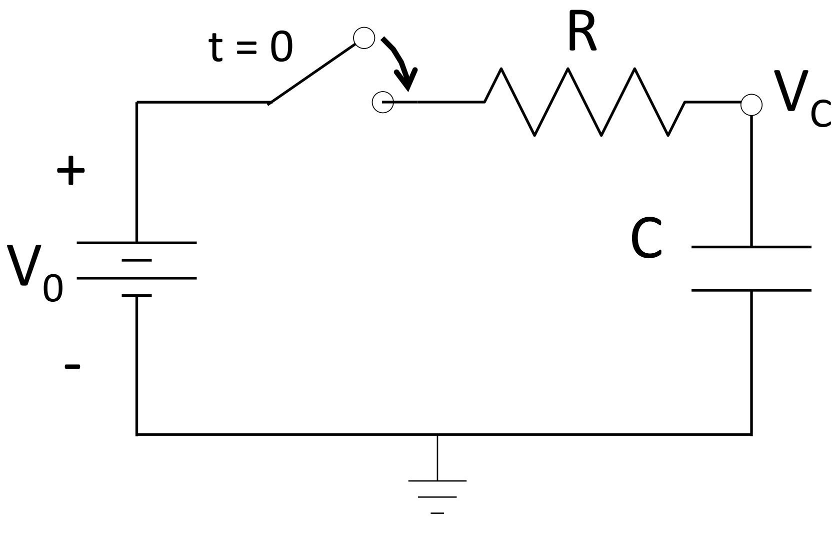
Current / Voltage relationship for a capacitor

$$Q = CV \qquad I = \frac{dQ}{dt} = C\frac{dV}{dt}$$

Assume that C has a fixed value, it does not change in time

Transient behavior of RC circuit:

Find the charge Q(t) and voltage $V_C(t)$ for the capacitor as a function of time



Kirchhoff's voltage law:

$$V_0 - V_R - V_C = 0 \rightarrow V_R + V_C = V_0$$

$$R\frac{dQ}{dt} + \frac{1}{C}Q = V_0$$

Initial Condition: Q(t=0) = 0

Final State: I = 0 so $Q(t\rightarrow\infty) = CV_0$

Solution …

$$R\frac{dQ}{dt}+\frac{1}{C}Q=V_0$$

Initial Condition: $Q(t=0) = 0$

First, write this as two equations …

1) $$\frac{dQ}{dt}+\frac{1}{RC}Q=\frac{V_0}{R}$$

2) $$\frac{dQ}{dt}+\frac{1}{RC}Q=0$$

Find any particular solution to (1), Q_p: $$\frac{1}{RC}Q_p=\frac{V_0}{R}\quad\rightarrow\quad Q_p=CV_0$$

Then add the general solution of (2), Q_g: $$Q_g(t)=Ae^{-\frac{t}{RC}}$$

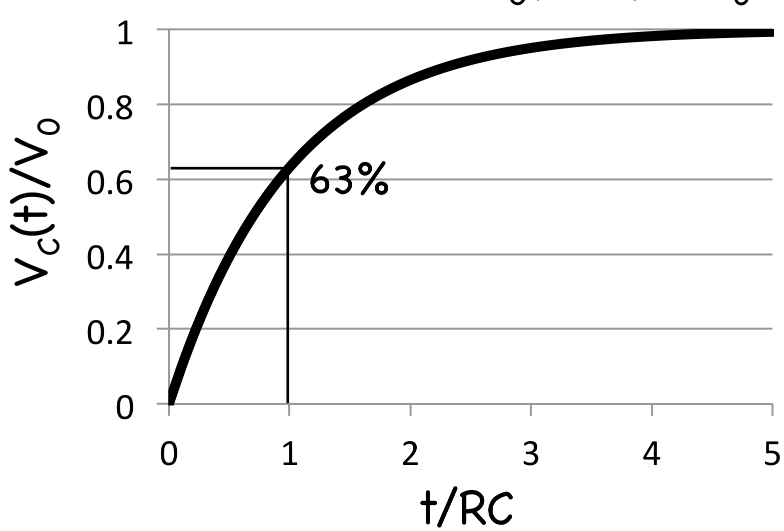
$$Q(t)=Q_p+Q_g=CV_0+Ae^{-\frac{t}{RC}}$$

but $Q(t)=0$ at $t=0$

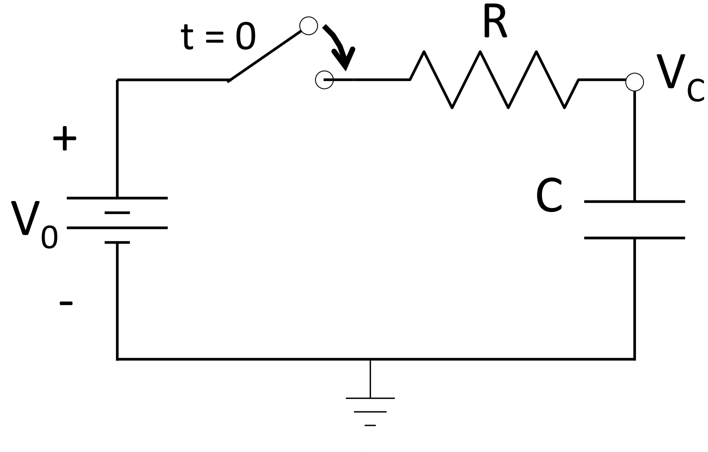
so $$Q(t)=CV_0\left(1-e^{-\frac{t}{RC}}\right)$$

and $$V_C(t)=V_0\left(1-e^{-\frac{t}{RC}}\right)$$

Final State: $V_C(t\rightarrow\infty) = V_0$

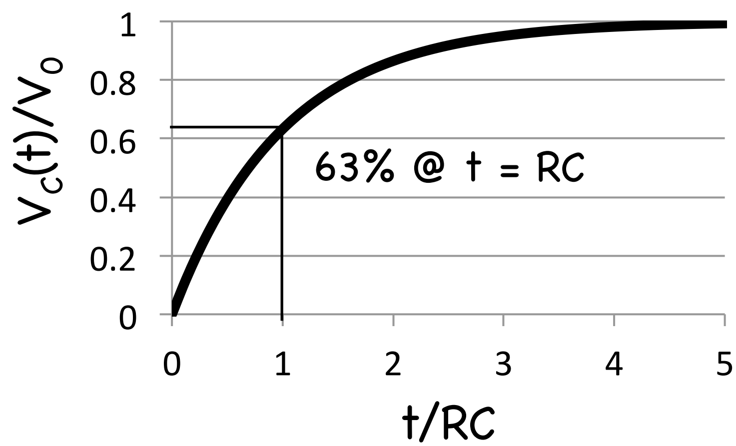


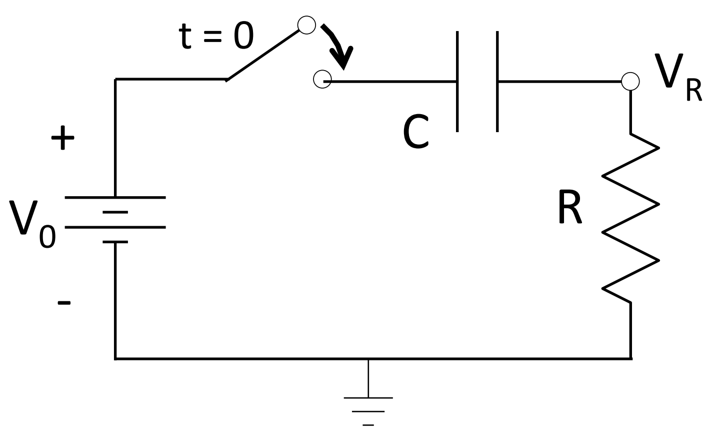
Two variations of the RC circuit …



Open and close switch repeatedly.
Rapidly: $V_C \approx 0$ Slowly: $V_C \approx V_0$

Low Pass





Open and close switch repeatedly.
Rapidly: $V_R \approx V_0$ Slowly: $V_R \approx 0$

High Pass