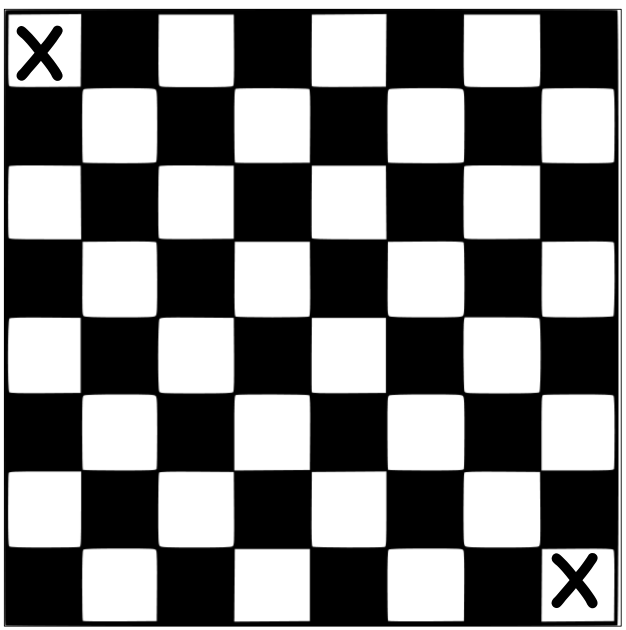
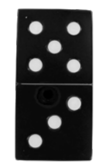
Puzzler

You are given a chess board and 32 dominoes. Each domino covers exactly two squares of the chess board. If two of the corners are removed from the board and we discard one of the dominoes, is it still possible to completely cover the remaining 62 squares of the chess board with the remaining 31 dominoes?





Introduction to Audio and Music Engineering

Lecture 4

- Musical Intervals
- Just Pentatonic Scale
- Pythagorean Scale
- Equally tempered scale
- Placing the frets on a guitar

Musical Intervals and Scales

The simplest musical interval is the octave: $\frac{f_2}{f_1} = \frac{2}{1}$

The next is the "perfect" fifth: $\frac{f_2}{f_1} = \frac{3}{2}$

then the perfect fourth: $\frac{f_2}{f_1} = \frac{4}{3}$

Note that fifth + fourth = octave $\frac{3}{2} \times \frac{4}{3} = 2$

Intervals are added by multiplying the frequency ratios.

Just Intervals

Harmonic:	f	2f	3f	4f	5f	6f
Frequency:	1	2	3	4	5	6
Ratio:		2/1	3/2	4/3	5/4	6/5
Interval:		Octave	M5	M4	M3	m3

Just Pentatonic Scale

1. Start with any note: C
2. Introduce the 4th: $\times 4/3 \rightarrow$ F
3. Introduce the 5th: $\times 3/2 \rightarrow$ G
4. Go down a 4th from G to reach D
   $3/2 \div 4/3 = 3/2 \times 3/4 = 9/8$; D $= 9/8 \times$ C
5. Go up a 5th from D to reach A
   $9/8 \times 3/2 = 27/16$; A $= 27/16 \times$ C

"I've Got Rhythm"

http://www.youtube.com/watch?v=vIpNepgmCQA

Pythagorean Diatonic Scale

Extend the Just Pentatonic Scale

6. Add E by going down a fourth from A

$$E = 27/16 \times 3/4 = 81/64 \text{ C}$$

7. Add B by going up a fifth from E

$$B = 3/2 \times 81/64 = 243/128$$

C	D	E	F	G	A	B
1	9/8	81/64	4/3	3/2	27/16	243/128

Pythagorean Dodecaphonic Scale (12 tones)

1. Start on the tonic (C)
2. Go up by fifths, but ÷ 2 to stay in octave

Note	C	G	D	A	E	B	F#	C#	G#	D#	A#	E#	B#
up by 5'ths	1	3	9	27	81	243	729	2187	6561	19683	59049	177147	531441
	1	2	4	8	16	32	64	128	256	512	1024	2048	4096
within	1	3	9	27	81	243	729	2187	6561	19683	59049	177147	531441
octave	1	2	8	16	64	128	512	2048	4096	16384	32768	131072	524288
ratio	1	1.5	1.125	1.6875	1.2656	1.8984	1.4238	1.0679	1.6018	1.2014	1.802	1.3515	1.0136433

Note that $(3/2)^{12} = 531{,}441/4096$

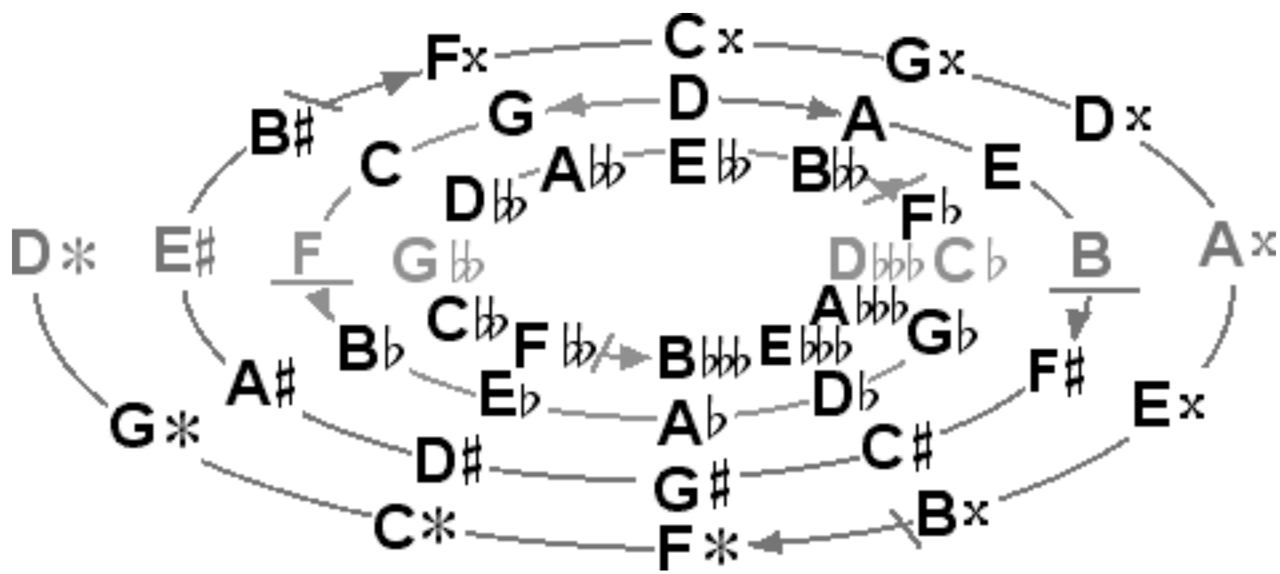
Bring back to octave:

$\div 128 \rightarrow 531{,}441/524{,}288 = 1.0136433$ (Pythagorean Comma)

Cycle of fifths does not come back to the same pitch.

Infinite spiral of fifths

Fifth = 3/2



$$\left(\frac{3}{2}\right)^{12} = \frac{129.7463}{2^7} = 1.0136$$

Pythagorean comma

Go up by a fifth 12x and you do NOT come back back to the same pitch!

Cycle of fourths

1. Start on the tonic (C)
2. Go up by fourths, (x 4/3), but ÷ 2 to stay in octave

Note	C	F	Bb	Eb	Ab	Db	Gb	B	E	A	D	G	C
up by 4'ths	1	4	16	64	256	1024	4096	16384	65536	262144	1048576	4194304	16777216
	1	3	9	27	81	243	729	2187	6561	19683	59049	177147	531441
within	1	4	16	32	128	256	1028	4096	8192	32768	65536	262144	1048576
octave	1	3	9	27	81	243	729	2187	6561	19683	59049	177147	531441
ratio	1	1.3333	1.7778	1.1852	1.5802	1.0535	1.4102	1.8729	1.2486	1.6648	1.109858	1.479811	1.9730807

This time $(4/3)^{12}$ = 16,777,216/531,441

Bring back into the octave:

÷16 → 1,048,576/531,441 = 1.974184 (Short of the octave)

Note that 2/1.974184 = 1.0130768

Compare to the Pythagorean comma 1.0136433

Pythagorean Scale

	Pythagorean (choose smaller ratios)			
	C	1	1	1
	Db	256	243	1.0535
	D	9	8	1.125
	Eb	32	27	1.1852
	E	81	64	1.2656
	F	4	3	1.3333
(Wolf Tone)	F#/Gb	729	512	1.4238
(Wolf Tone)	F#/Gb	1028	729	1.4102
	G	3	2	1.5
	Ab	128	81	1.5802
	A	27	16	1.6875
	Bb	16	9	1.7778
	B	243	128	1.8984
	C	2	1	2

Note	C	G	D	A	E	B	F#	C#	G#	D#	A#	E#	B#
up by 5'ths	1	3	9	27	81	243	729	2187	6561	19683	59049	177147	531441
	1	2	4	8	16	32	64	128	256	512	1024	2048	4096
within	1	3	9	27	81	243	729	2187	6561	19683	59049	177147	531441
octave	1	2	8	16	64	128	512	2048	4096	16384	32768	131072	524288
ratio	1	1.5	1.125	1.6875	1.2656	1.8984	1.4238	1.0679	1.6018	1.2014	1.802032	1.351524	1.0136433

Note	C	F	Bb	Eb	Ab	Db	Gb	B	E	A	D	G	C
up by 4'ths	1	4	16	64	256	1024	4096	16384	65536	262144	1048576	4194304	16777216
	1	3	9	27	81	243	729	2187	6561	19683	59049	177147	531441
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Just Tuning

The thirds in the Pythagorean Scale are somewhat dissonant:

Pyth: C→E : 81/64 = 1.265625

Just: C→E : 5/4 = 1.25

$81/64 \div 5/4 = 81/80$

Syntonic comma …

about 1/5 of a semitone

Adjust the thirds, and make intervals ratios of smaller numbers.

<u>Just Intonation</u>

C	1	1	1
C#	16	15	1.0667
Dd	10	9	1.1111
D	9	8	1.125
Eb	6	5	1.2
E	5	4	1.25
F	4	3	1.3333
F#	45	32	1.4063
Gb	64	45	1.4222
G	3	2	1.5
Ab	8	5	1.6
A	5	3	1.6667
A#	7	4	1.75
A#/Bb	16	9	1.7778
Bb	9	5	1.8
B	15	8	1.875
C	2	1	2

Choose tone according to key

Stop the madness!

Equal temperament – divide the octave into 12 equal intervals

<u>Semitone</u> $2^{1/12} = 1.059463...$

$$\left(2^{1/12}\right)^{12} = 2$$

Fixes the Pythagorean comma problem!

All of the intervals are a little out of tune, but transposition is not a problem any longer.

Cents ... 1200 cents per octave

One half step = 100 cents

Comparison of Pythagorean, Just and Equal Temperament

Pythagorean (choose smaller ratios)

C	1	1	1
Db	256	243	1.0535
D	9	8	1.125
Eb	32	27	1.1852
E	81	64	1.2656
F	4	3	1.3333
F#/Gb	729	512	1.4238
F#/Gb	1028	729	1.4102
G	3	2	1.5
Ab	128	81	1.5802
A	27	16	1.6875
Bb	16	9	1.7778
B	243	128	1.8984
C	2	1	2

Just Intonation

C	1	1	1
C#	16	15	1.0667
Dd	10	9	1.1111
D	9	8	1.125
Eb	6	5	1.2
E	5	4	1.25
F	4	3	1.3333
F#	45	32	1.4063
Gb	64	45	1.4222
G	3	2	1.5
Ab	8	5	1.6
A	5	3	1.6667
A#	7	4	1.75
A#/Bb	16	9	1.7778
Bb	9	5	1.8
B	15	8	1.875
C	2	1	2

Choose tone according to key

Equal Tempered

C	1
Db	1.0594631
D	1.122462
Eb	1.1892071
E	1.259921
F	1.3348399
F#	1.4142136
G	1.4983071
Ab	1.5874011
A	1.6817928
Bb	1.7817974
B	1.8877486
C	2

Guitar Fret Placement

Back to the string …

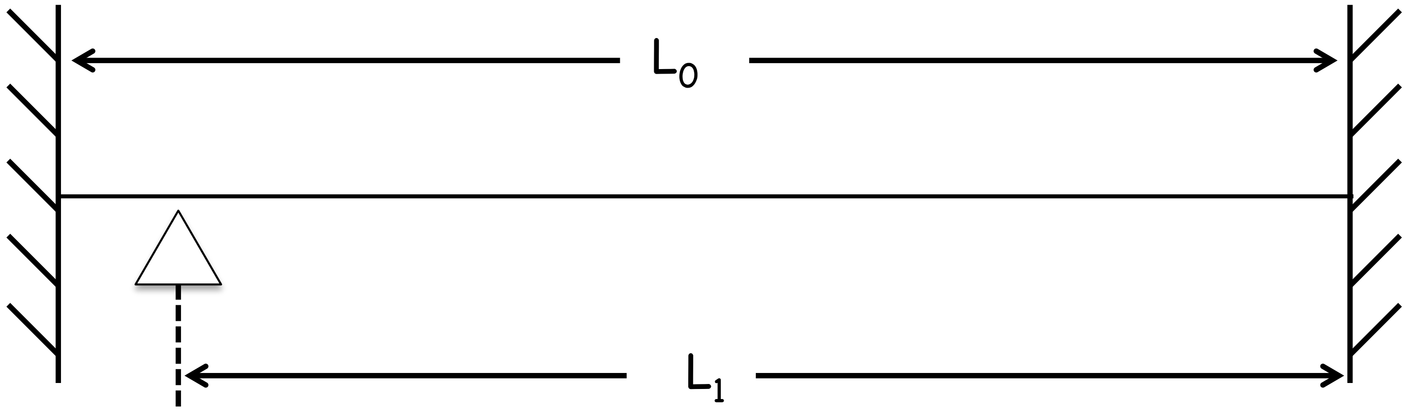
The speed of a bending wave propagation is determined by mass and tension of string.

$$c \equiv \sqrt{\frac{T}{\mu}}$$

The frequency is determined by c and the length of the string.

$$f_0 = \frac{c}{2L}$$

To raise the pitch by 1 half-tone shorten the string by a factor of $2^{1/12}$.



Raise the frequency ½ tone

$$\frac{f_1}{f_0} = \frac{2^{1/12} f_0}{f_0} = 2^{1/12}$$

The frequency is proportional to 1/length of the string

$$f \propto \frac{1}{L}$$

So the length must change in inverse proportion to the frequency

$$\frac{L_0}{L_1} = 2^{1/12}$$

$$L_1 = \frac{L_0}{2^{1/12}} = 0.94387\ L_0$$

Vincenzo Galileo's rule of 18

Reduce the length of a string by 1/18th for each half tone.

$$L_1 = \left(1 - \frac{1}{18}\right) L_0 = 0.9444\ L_0$$

Compare to: $L_1 = 0.94387\ L_0$ for equal temperament

Close, but the error accumulates: $(0.9444)^{12} = 0.5036$

Not 0.5 as it should be for the octave.