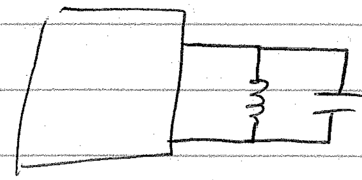
Do problems, Exercises, and Examples in 7-4! 224

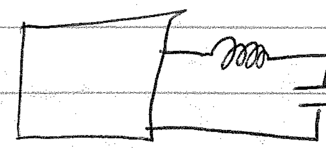
Now let's go the next step and put L+C together.

We can do this in 2 ways:



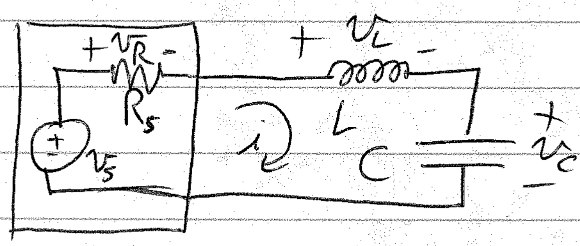
Parallel

or



Series

Text (and we) will start with Series:



Given $i(0^+) = I_o$
$v_c(0^+) = V_o$

KVL: $v_L + v_c + v_R - v_s = 0$

or $v_L + v_c + v_R = v_s$

Characteristic Eqns:

$$L\frac{di}{dt} + \frac{1}{C}\int_0^t i(x)dx + v_c(0) + R_s i = v_s$$

$\frac{d}{dt}$: $$L\frac{d^2 i}{dt^2} + \frac{1}{C}i(t) + R_s\frac{di}{dt} = \cancel{v_s}\frac{dv_s}{dt}$$

$\div L$: $$\frac{d^2 i}{dt^2} + \frac{R_s}{L}\frac{di}{dt} + \frac{1}{LC}i(t) = \frac{1}{L}\frac{dv_s}{dt}$$

Text writes equation in $v_c(t)$:

$$\frac{d^2 v_c}{dt^2} + \frac{R_s}{L}\frac{dv_c}{dt} + \frac{1}{LC}v_c = \frac{1}{LC}v_s$$

2nd order, inhomogeneous DE's w/constant coefficients

1.) Recall with 1st order we had to know the IC on the value itself? ($v_c(0^+)$ or $i(0^+)$)
Well now we will need IC's for the value and its 1st derivative ($\frac{dy}{dt}(0^+)$)
$y(0^+)$

Again, the response to an input is the sum of a Forced Response + Natural Response
(same as input) (decays to zero)

The natural response is the result of the Homogeneous Equation:

$$\frac{d^2 i}{dt^2} + \frac{R_s}{L}\frac{di}{dt} + \frac{1}{LC}i(t) = 0$$

Propose an exponential form:

$$i_N(t) = Ke^{st}$$

$$\frac{di_N}{dt} = sKe^{st}$$

$$\frac{d^2 i_N}{dt} = s^2 Ke^{st}$$

Or: $\cancel{s^2 + \frac{R}{L}}$

$$s^2Ke^{st} + \frac{R_s}{L}sKe^{st} + \frac{1}{LC}Ke^{st} = 0$$

Factor $\left[s^2 + \frac{R_s}{L}s + \frac{1}{LC}\right]Ke^{st} = 0$

Can only be zero if one or more of the 3 factors is zero:

$K = 0 \Rightarrow$ Trivial Solution

$e^{st} = 0$ only for $s \to -\infty$

So it must be that

$$s^2 + \frac{R_s}{L}s + \frac{1}{LC} = 0 \quad \text{(Characteristic Equation)}$$

Quadratic Formula: $ax^2 + bx + c = 0$

(2 values of s for which this is true)

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$s_{1,2} = \frac{-\frac{R_s}{L} \pm \sqrt{\left(\frac{R_s}{L}\right)^2 - 4(1)\left(\frac{1}{LC}\right)}}{2(1)}$$

$$= -\frac{R_s}{2L} \pm \frac{1}{2}\sqrt{\frac{R_s^2}{L^2} - \frac{4}{LC}}$$

$$= -\frac{R_s}{2L} \pm \sqrt{\left(\frac{R_s}{2L}\right)^2 - \frac{1}{LC}}$$

$$= -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} \quad \text{where } \alpha = \frac{R_s}{2L}$$

$$+ \ \omega_0 = \frac{1}{\sqrt{LC}}$$

α is called the "Damping Coefficient" and ω_0 is the "Natural Frequency" of the system.

We can now have 3 possible cases:

A) $\alpha > \omega_0$: $s_{1,2} = -\alpha \pm \underbrace{\sqrt{\alpha^2 - \omega_0^2}}_{\text{real number}}$ $\quad (\alpha^2 - \omega_0^2 > 0)$

call it γ, can prove $|\gamma| < |\alpha|$

$s_{1,2} = -\alpha \pm \gamma$ = a negative number

$i_N(t) = Ae^{s_1 t} + Be^{s_2 t}$ (called "over damping")

B) $\alpha = \omega_0$: $s_{1,2} = -\alpha \pm 0 = -\alpha$

"Critical Damping," more on this later

C) $\alpha < \omega_0$: $s_{1,2} = -\alpha \pm \sqrt{\underbrace{\alpha^2 - \omega_0^2}_{<0}}$

Factor out (-1)

$s_{1,2} = -\alpha \pm \sqrt{(-1)(\omega_0^2 - \alpha^2)}$

$= -\alpha \pm j\sqrt{\omega_0^2 - \alpha^2}$

$= -\alpha \pm j\,\omega_d$

damped frequency

more on this later, too.

Back to case (A.)

If the Forcing Function is $\underbrace{\text{zero}}_{\text{so } v_F(t)=0}$, and we are only looking for the Natural Response, then we can use IC's to find A + B:

$$i(0^+) = Ae^{s_1(0)} + Be^{s_2(0)} = A + B = I_0$$

But this is only one equation + 2 unknowns.

We need to relate the ICs in one more equation: We know $i(0^+) = I_0$ + $v_c(0^+) = V_0$ $\Big\}$ both are <u>state variables</u>!

Relate $i(t)$ to a <u>voltage</u> thru a <u>derivative</u>:

$v_L = L\frac{di}{dt}$. KVL: $v_R + v_L + v_c - v_s = 0$

$$v_L = v_s - v_c - v_R \quad \text{(Always)}$$

$$\frac{di}{dt} = \frac{1}{L}(v_s - v_c - v_R)$$

At 0^+: $\frac{di}{dt}(0^+) = \frac{1}{L}\left(v_s(0^+) - v_c(0^+) - Ri(0^+)\right)$

For our case $\frac{di}{dt}(0^+) = \frac{1}{L}(0 - V_0 - RI_0)$

$$i(t) = Ae^{s_1 t} + Be^{s_2 t}$$

$$i(0^+) = I_0 = A + B$$

$$\frac{di}{dt} = s_1 A_1 e^{s_1 t} + s_2 B e^{s_2 t}$$

$$\frac{di}{dt}(0^+) = s_1 A + s_2 B$$

$$-\frac{1}{L}(V_0 + RI_0) = s_1 A + s_2(I_0 - A)$$

$$= (s_1 - s_2)A + s_2 I_0$$

$$-\frac{1}{L}(V_0 + RI_0) - s_2 I_0 = (s_1 - s_2)A$$

recall: $s_{1,2} = -\alpha \pm \gamma$, so $s_1 - s_2 = (-\alpha + \gamma) - (-\gamma - \gamma)$

$$= 2\gamma$$

$$A = \frac{-\frac{1}{L}(V_0 + RI_0) - s_2 I_0}{2\gamma} = -\frac{\frac{1}{L}(V_0 + RI_0) + s_2 I_0}{2\gamma}$$

$$B = I_0 - A = I_0 - \frac{-\frac{1}{L}(V_0 + RI_0) - s_2 I_0}{2\gamma}$$

$$= \frac{2\gamma I_0 + \frac{1}{L}(V_0 + RI_0) + s_2 I_0}{2\gamma}$$

$$= \frac{\frac{1}{L}(V_0 + RI_0) + (s_2 + 2\gamma) I_0}{2\gamma}$$

$$= \frac{\frac{1}{L}(V_0 + RI_0) + (s_1) I_0}{2\gamma}$$

<u>To Case B</u> (read in text, pp 351)

It turns out the exponential form is not quite right, the solution <u>Should</u> be of the form

$$y(t) = Ce^{-\alpha t} + Dte^{-\alpha t}$$ ~~where $s = -\alpha$~~

$$y(0^+) = C$$

and we need to solve for C+D.

This is called "Critical Damping"

We will not spend much time on this,

because it represents only 1 very particular

set of parameters that achieves $\alpha = \omega_0$.

Depending on a circuit (or any system) to

be critically damped would be folly.

As soon as the temperature, or Relative

humidity, or any other factor changes, it will

no longer be critically damped.

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Critical Damping

$$y(t) = Ce^{-\alpha t} + Dte^{-\alpha t}$$

$$y(0^+) = C$$

$$\frac{dy}{dt} = -\alpha Ce^{-\alpha t} + D\left[-\alpha te^{-\alpha t} + e^{-\alpha t}\right]$$

$$\frac{dy}{dt}(0^+) = -\alpha C + D$$

$$D = \frac{dy}{dt}(0^+) + \alpha(y(0^+))$$

$$= \frac{dy}{dt}(0^+) + \alpha y(0^+)$$

$$y(t) = y(0^+)e^{-\alpha t} + \left[\frac{dy}{dt}(0^+) + \alpha y(0^+)\right]te^{-\alpha t}$$

Case C $\quad \alpha < \omega_o$,

$$s_1 = -\alpha - j\omega_d \qquad s_2 = -\alpha + j\omega_d$$

(text calls $\omega_d = \beta$)

To deal with this we will use ~~the~~ Euler's relations:

$$e^{\pm j\theta} = \cos\theta \pm j\sin\theta$$

Add + subtract: $\quad e^{j\theta} = \cos\theta + j\sin\theta$
$\qquad\qquad\qquad\quad e^{-j\theta} = \cos\theta - j\sin\theta$

$e^{j\theta} + e^{-j\theta} = 2\cos\theta \qquad\qquad e^{j\theta} - e^{-j\theta} = 2j\sin\theta$

or $\cos\theta = \frac{1}{2}(e^{j\theta} + e^{-j\theta}) \qquad\qquad \sin\theta = \frac{1}{2j}(e^{j\theta} - e^{-j\theta})$

Now, put s_1 + s_2 into the expression:

$$
\begin{aligned}
y(t) &= Ae^{s_1 t} + Be^{s_2 t} \\
&= Ae^{(-\alpha - j\omega_d)t} + Be^{(-\alpha + j\omega_d)t} \\
&= Ae^{-\alpha t}e^{-j\omega_d t} + Be^{-\alpha t}e^{j\omega_d t}
\end{aligned}
$$

(now we allow A+B to be complex numbers,

$$= e^{-\alpha t}\left[Ae^{-j\omega_d t} + Be^{j\omega_d t}\right]$$

$$\cancel{= e^{-\alpha t}\left[Ae^{-j\omega_d t} + Ae^{j\omega_d t} - Ae^{j\omega_d t} + Be^{-j\omega_d t} - Be^{-j\omega_d t} + Be^{j\omega_d t}\right]}$$

$$\cancel{= e^{-\alpha t}\left[A(e^{-j\omega_d t} + e^{j\omega_d t}) - Ae^{j\omega_d t} + Be^{-j\omega_d t} + B(e^{j\omega_d t} - e^{-j\omega_d t})\right]}$$

$$y(t) = e^{-\alpha t}\left[A\cos(\omega_d t) - jA\sin(\omega_d t) + B\cos(\omega_d t) + jB\sin(\omega_d t)\right]$$

$$= e^{-\alpha t}\left[(A+B)\cos(\omega_d t) + j(-A+B)\sin(\omega_d t)\right]$$

$$\boxed{y(t) = e^{-\alpha t}\left[E\cos(\omega_d t) + F\sin(\omega_d t)\right]}$$

new constants

$$y(0^+) = E$$

$$\frac{dy}{dt} = -\alpha e^{-\alpha t}\left[E\cos(\omega_d t) + F\sin(\omega_d t)\right] + e^{-\alpha t}\left[-\omega_d E\sin(\omega_d t) + \omega_d F\cos(\omega_d t)\right]$$

$$\frac{dy}{dt}(0^+) = -\alpha[E] + [\omega_d F] = -\alpha E + \omega_d F$$

$$\omega_d F = \frac{dy}{dt}(0^+) + \alpha y(0^+)$$

$$F = \frac{\frac{dy}{dt}(0^+) + \alpha y(0^+)}{\omega_d}$$

This $y(t)$ is a damped sinusoid. ~~The easiest case to consider is if~~ "Exponential decay" of the ~~Let's consider~~ natural response: