Introduction to Audio and Music Engineering

Lecture 2

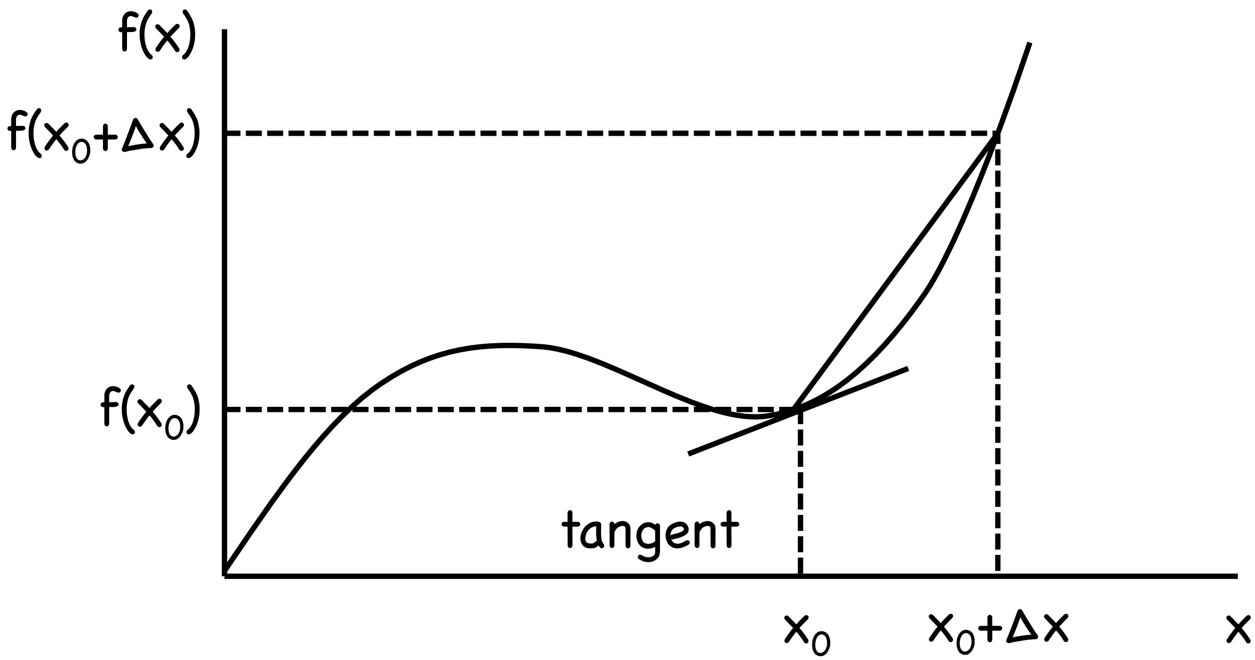
- A few mathematical prerequisites
- Limits and derivatives
- Simple harmonic oscillators
- Strings, Oscillations & Modes

Puzzler

The Early Commuter

A commuter is in the habit of arriving at his suburban station each evening exactly at 5 o'clock. His wife always meets the train and drives him home. One day he takes an earlier train, arriving at the station at four. The weather is pleasant so instead of calling home he starts walking along the route always taken by his wife. They meet somewhere on the way. He gets into the car and they drive home, arriving at their house <u>ten minutes earlier</u> than usual. Assuming that the wife always drives at a constant speed, and that on this occasion she left just in time to meet the five o'clock train, can you determine how long the husband walked before he was picked up?

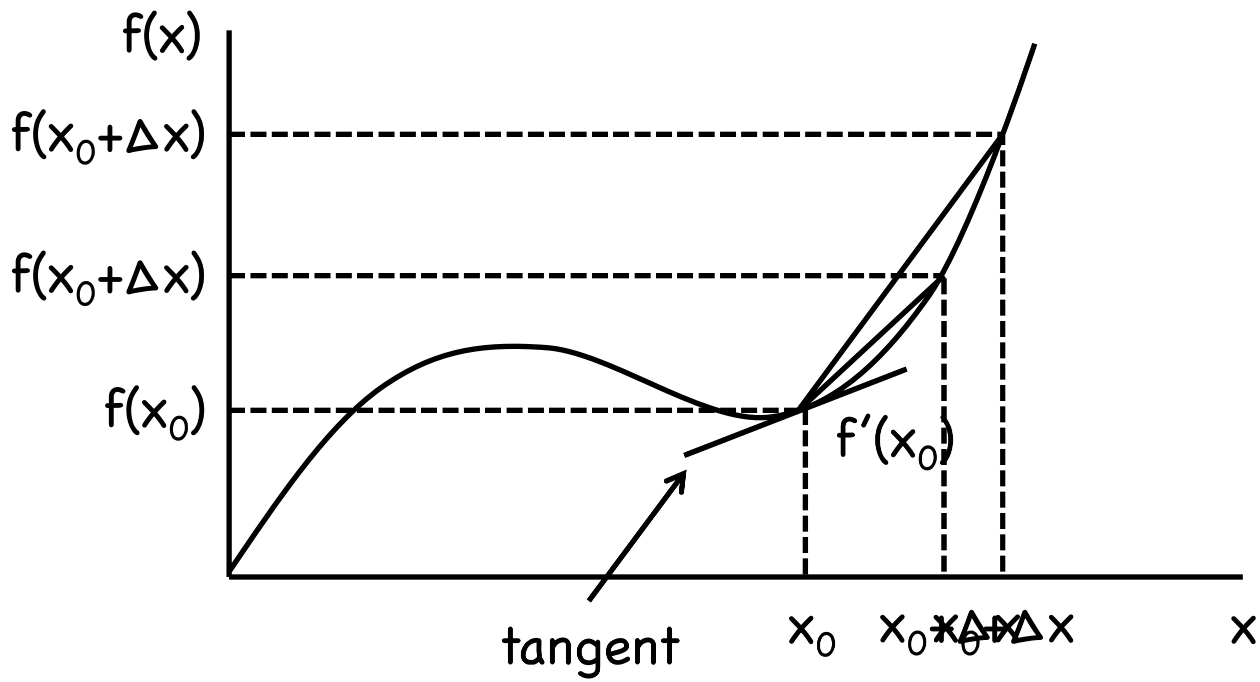
Limits and Derivatives



$$\text{Slope} = \frac{f(x_0+\Delta x) - f(x_0)}{x_0+\Delta x - x_0} = \frac{f(x_0+\Delta x) - f(x_0)}{\Delta x}$$

Make $\Delta x \rightarrow 0$

$$\lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} \equiv \frac{d}{dx} f(x) \Big|_{x_0} \equiv f'(x_0)$$



f(x)
f(x0+Δx)
f(x0+Δx)
f(x0)
f′(x0)
tangent
x0
x0+Δx
x0+Δx
x

A few simple derivatives we will need …

$$\frac{d}{dx}x^n = nx^{n-1}$$

$$\frac{d}{dx}\sin(x) = \cos(x)$$

$$\frac{d}{dx}Const = 0$$

$$\frac{d}{dx}\cos(x) = -\sin(x)$$

$$\frac{d}{dx}x = 1$$

$$\frac{d}{dx}e^x = e^x$$

Product rule: $$\frac{d}{dx}\left[f(x)\cdot g(x)\right] = f(x)\cdot g'(x) + f'(x)\cdot g(x)$$

$$\frac{d}{dx}\left[x\sin(x)\right] = x\cdot\cos(x) + 1\cdot\sin(x)$$

$$\frac{d}{dx}ax = a$$

Chain rule: $$\frac{d}{dx}\left[f(g(x))\right] = f'(g(x))\cdot g'(x)$$

$$\frac{d}{dx}\left[\sin(x^2)\right] = \cos(x^2)\cdot 2x$$

$$\frac{d}{dx}e^{ax} = ae^{ax}$$

Question

What is the derivative w.r.t. x of: $2x\sin(2x)$

a) $2x\cos(2x)$

b) $2\sin(2x)+2x\cos(2x)$

c) $2\sin(2x)+4x\cos(2x)$

d) $2\cos(2x)+4x\sin(2x)$

What's so special about e?

e = 2.71828 18284 59045 23536 02874 71352 66249 77572 47093 69995 …

$$\frac{d}{dx}e^x = e^x$$

The only exponential function (a^x) where the slope of the function equals the value of the function at every point.

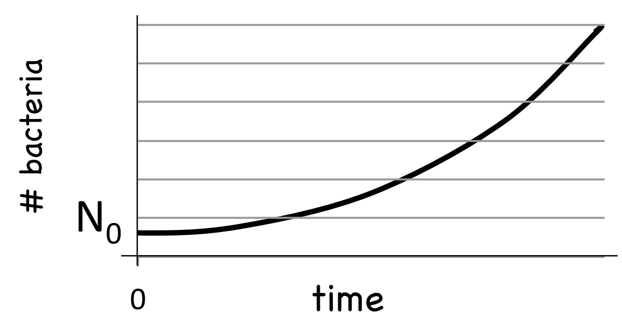
Bacteria colony growth: Let's say that there is a colony of bacteria growing in a petri dish and that the rate of increase of the number of bacteria is 2 times the number already present. How does the population grow over time?

Let y(t) = number of bacteria at time t, and y(t=0) = N_0 , (initial condition)

then $$\frac{d}{dt}y = 2 \cdot y$$ solution is … $$y(t) = Ae^{2t}$$

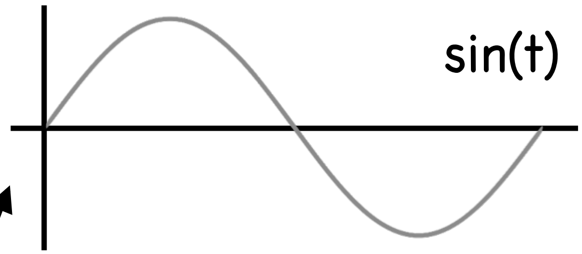
and since y(t=0) = N_0 → A = N_0

$$y(t) = N_0 e^{2t}$$



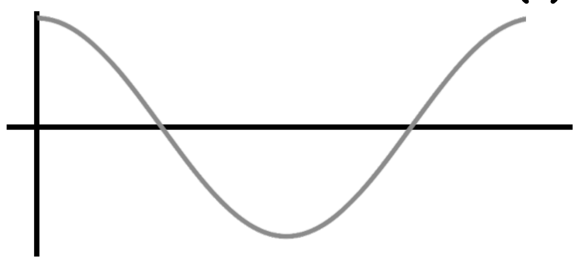
Derivatives of sin, cos

sin(t)



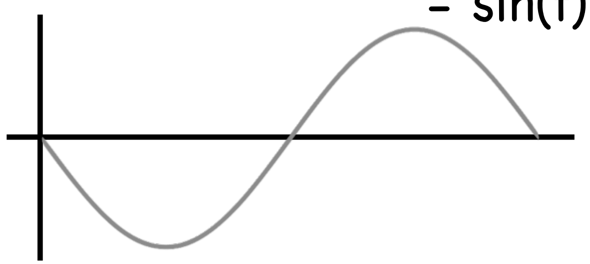
$$\frac{d}{dt}\sin(t) = \cos(t)$$

cos(t)



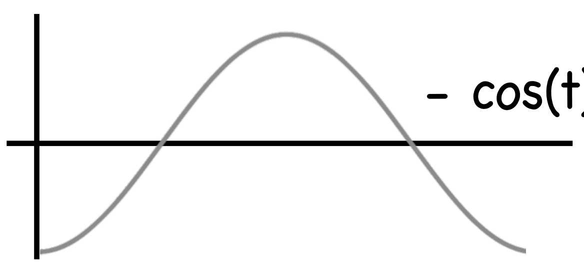
$$\frac{d}{dt}\cos(t) = -\sin(t)$$

- sin(t)



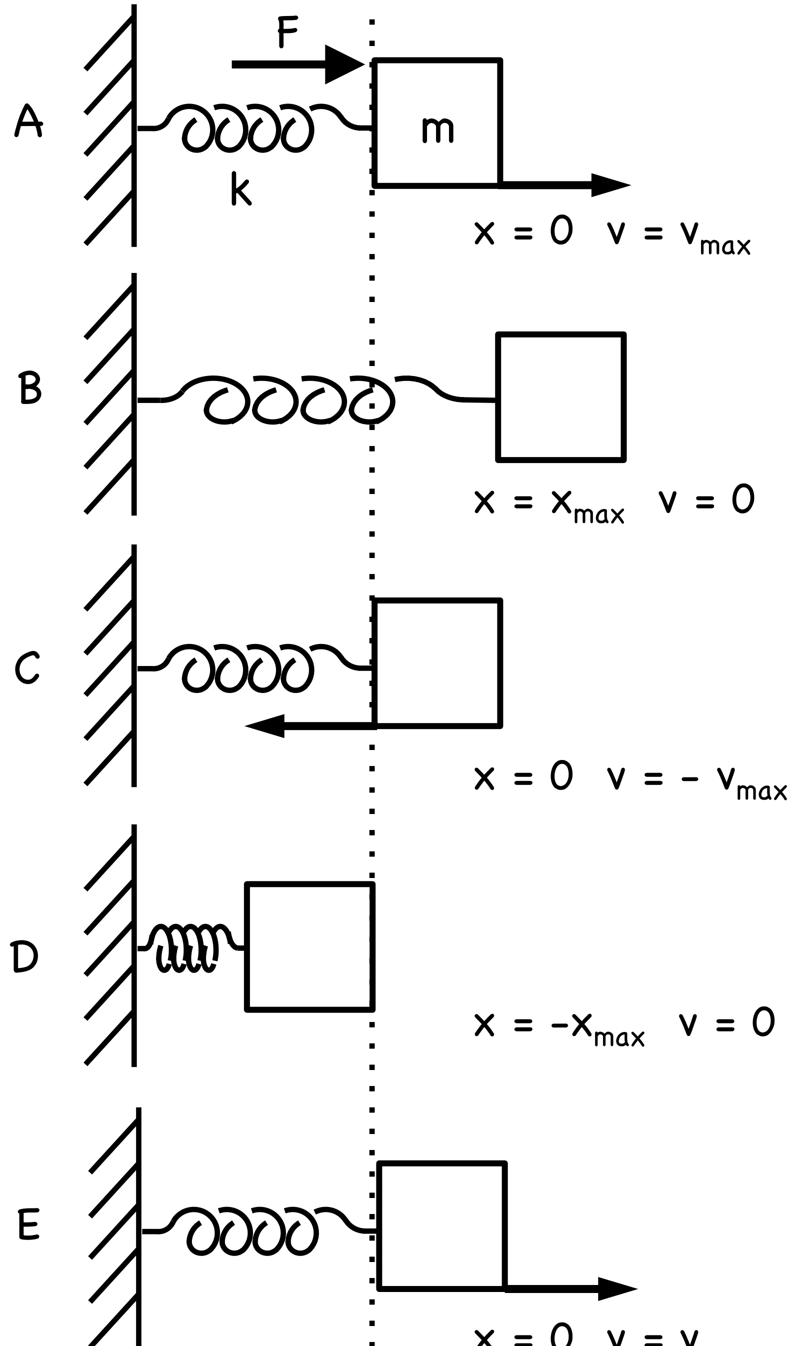
$$\frac{d}{dt}\left(-\sin(t)\right) = -\cos(t)$$

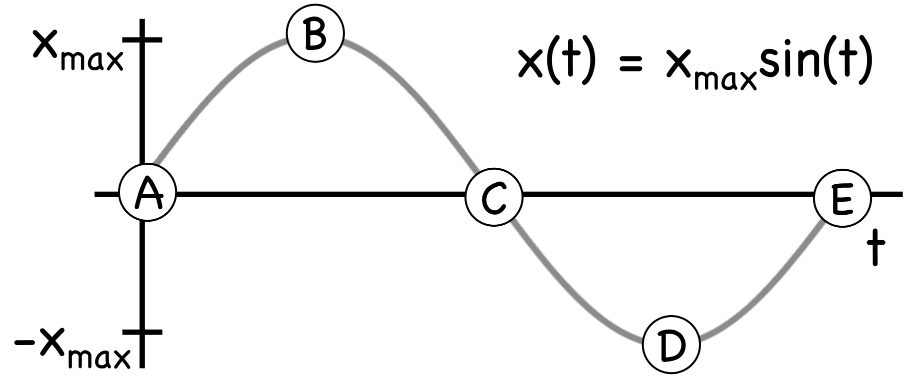
- cos(t)

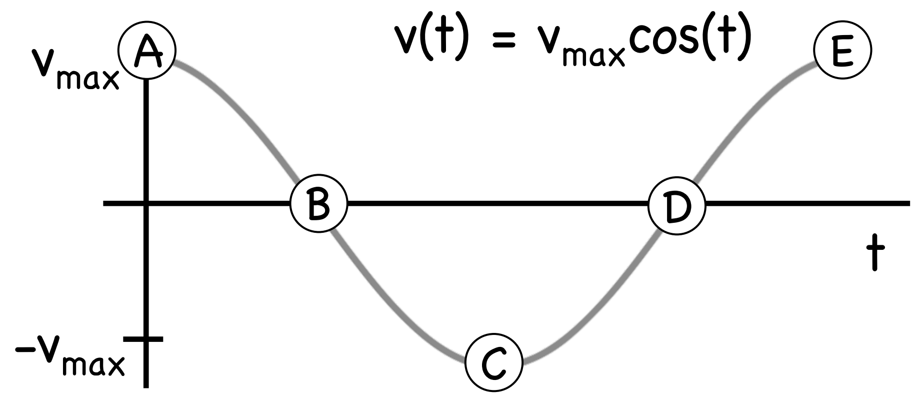


$$\frac{d}{dt}\left(-\cos(t)\right) = \sin(t)$$

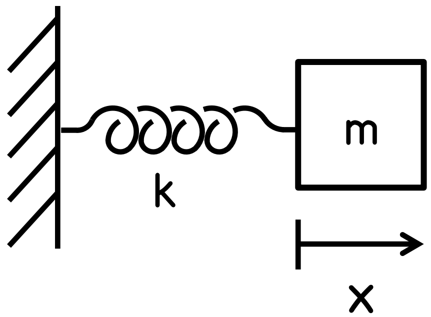
Simple Harmonic Oscillator







Simple Harmonic Oscillator



<u>Newton's 2nd Law</u>

F = ma

<u>Hooke's Law</u>

F = -kx

$$\frac{d}{dt}x = v \text{ , velocity}$$

$$\frac{d}{dt}v = a \text{ , acceleration}$$

$$\frac{d}{dt}\frac{dx}{dt} \equiv \frac{d^2x}{dt^2} = a$$

$$m\frac{d^2x}{dt^2} = -kx$$

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x \qquad \text{let } \omega^2 \equiv \frac{k}{m} \quad \text{, so } \frac{d^2x}{dt^2} = -\omega^2 x$$

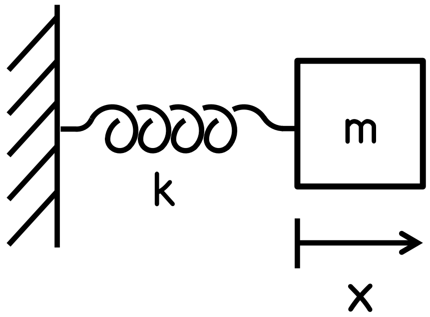
Can we find a function that satisfies this differential equation?

It works!

$$x = \boxed{x_0 \sin(\omega t)} \qquad \frac{d}{dt}x = x_0\omega\cos(\omega t) \qquad \frac{d^2}{dt^2}x = \boxed{-x_0\omega^2\sin(\omega t)}$$

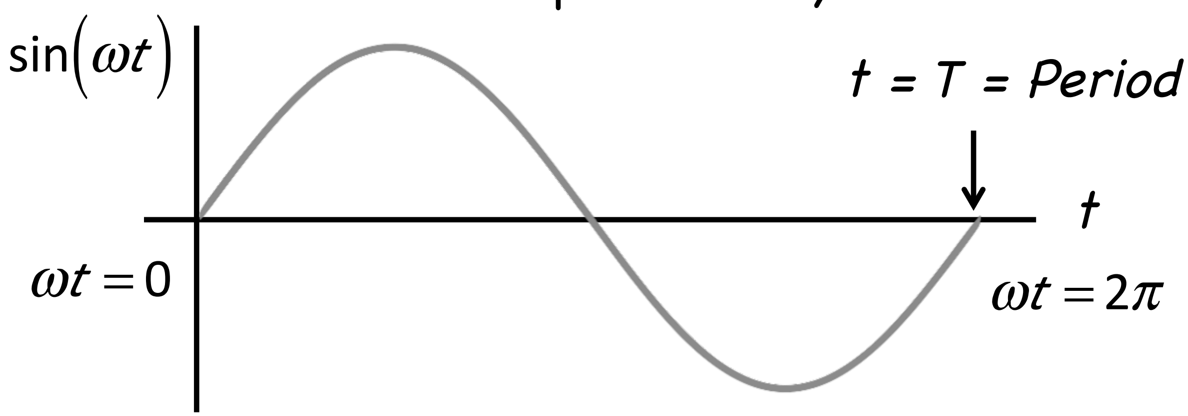
$$\text{so } x(t) = x_0\sin(\omega t) \qquad \text{where } \omega \equiv \sqrt{\frac{k}{m}}$$

Frequency and Period



$$x(t) = x_0 \sin(\omega t) \qquad \text{where} \quad \omega \equiv \sqrt{\frac{k}{m}}$$

Sine repeats every 2π



$$\omega T = 2\pi$$

$$T = \frac{2\pi}{\omega}$$ *Period (seconds per cycle)*

Angular frequency (radians per second)

Frequency (cycles per second)

Frequency (cycles per second) $$f = \frac{1}{T} = \frac{\omega}{2\pi}$$

$$\omega = 2\pi f$$

Question

What is approximate frequency (in Hertz) of a simple harmonic oscillator of mass 1 kg with a spring constant of 9 Nts/m?

a) 2 Hz

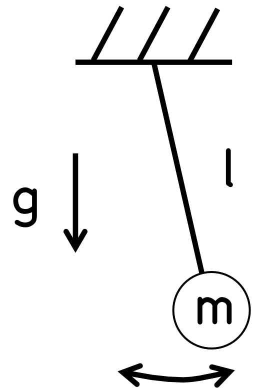
b) $\boxed{0.5 \text{ Hz}}$

c) 9 Hz

d) 1/9 Hz

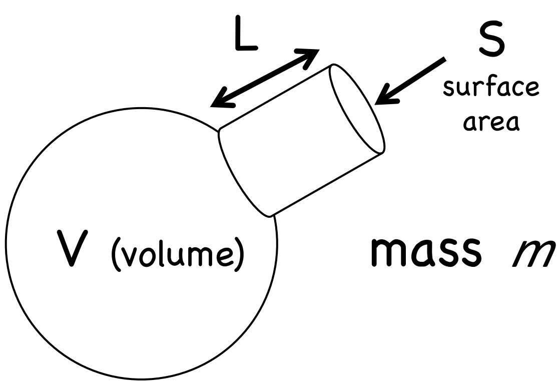
Other systems that display simple harmonic oscillation

Simple pendulum



$$\omega \equiv \sqrt{\frac{g}{l}} \qquad f \equiv \frac{1}{2\pi}\sqrt{\frac{g}{l}}$$

Helmholtz resonator



$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = \frac{c}{2\pi}\sqrt{\frac{S}{VL}}$$

mass $m = \rho S L$

↑
air density

for …

c = 340 m/sec
S = $\pi \times 10^{-4}$ m^2
V = 100 cc = 10^{-4} m^3
L = 3×10^{-2} m

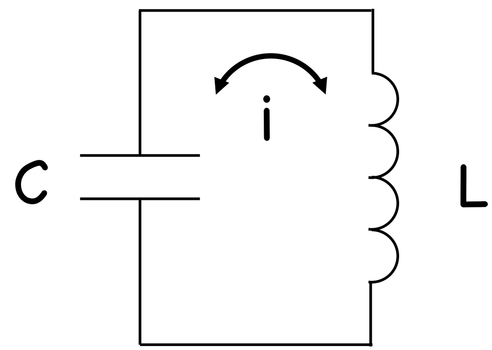
Spring

$$k = \rho \frac{S^2 c^2}{V}$$

c = sound velocity

f = 500 Hz

Electrical Oscillator:



Capacitor

"spring"

Inductor

"mass"

$$\omega = \frac{1}{\sqrt{LC}}$$

$$f = \frac{1}{2\pi}\frac{1}{\sqrt{LC}}$$

Resonant frequency