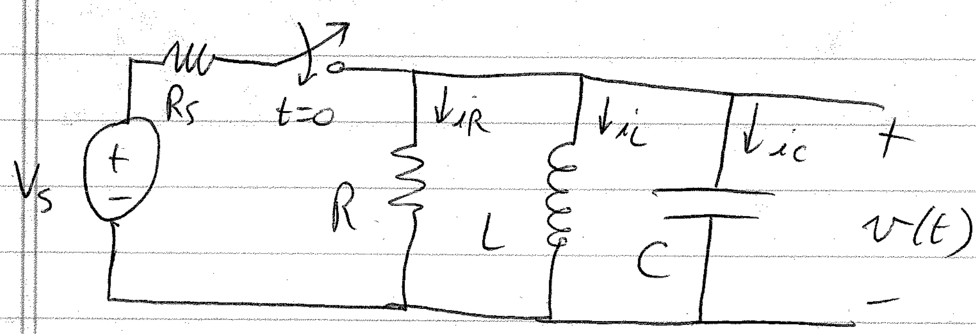
Let's combine Parallel RLC and RLC Step Response into some examples.

First, we have to allow for there to be a Forced Response (same type as the driving source) in addition to the Natural Response:

$$y(t) = \underbrace{\text{Natural Response}}_{\text{decays to zero as } t\to\infty} + \underbrace{\text{Forced Response}}_{\text{remains as } t\to\infty}$$

We have to evaluate constants (A+B, C+D, E+F) with ~~For example~~ the Forced Response included.

For example:

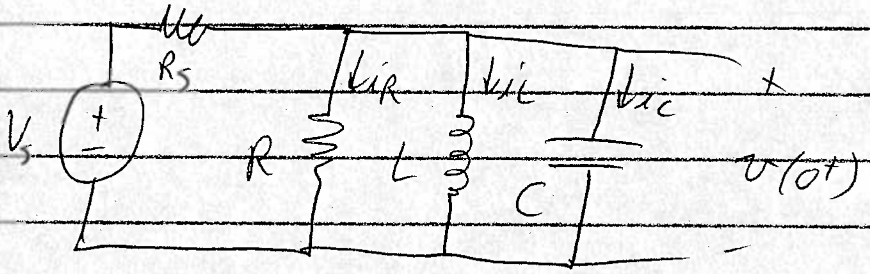


Find $v(t)$

$t=0^-$: Switch open, L a short, C an open ckt.

$v(0^-)=0$
$i_L(0^-)=0$ } ~~source-free~~

$t=0^+$: Switch closed, but no charges have moved yet.



$v(0^+)=0$ (b.c. v_C must be continuous)

Look at IV: $i_R = \frac{v(0^+)}{R}$ $\quad i_L(0^+) = \frac{1}{L}\int v(x)\,dx + i_L(0)$ (continuous) $\quad \underline{i_C = C\frac{dv}{dt}(0^+)}$

KCL: $\frac{v(t)-V_S}{R_S} + \frac{v(t)}{R} + i_L + C\frac{dv}{dt} = 0$ (General)

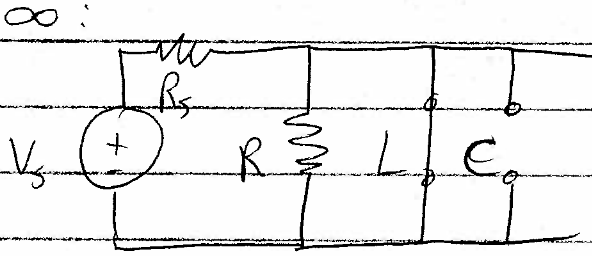
$t=0^+$: $\frac{\cancelto{0}{v(0^+)}-V_S}{R_S} + \frac{\cancelto{0}{v(0^+)}}{R} + \cancelto{0}{i_L(0^+)} + C\frac{dv}{dt}(0^+) = 0$

$$C\frac{dv}{dt}(0^+) = \frac{V_S}{R_S}$$

$$\underline{\frac{dv}{dt}(0^+) = \frac{1}{R_S C}V_S}$$

R_{eq}: Turn off V_S (short), ask what R_{eq} is seen by LC? $\underline{R_{eq} = R_S \| R}$

$t=\infty$:



+

$v(\infty)=0$ (L a short)

$i_L(\infty) = \frac{V_S}{R_S}$

Now we need to know values:

Let's use $V_s = 5V$, $R = 2k\Omega$, $R_s = 2k\Omega$,

$L = 1\,H$, $C = \frac{1}{4}\,\mu F$ (as before.)

So, as before, $\omega_0 = \frac{1}{\sqrt{LC}} = 2{,}000\ s^{-1}$

a) Pick $R = 2k\Omega$ + $R_s = 2k\Omega$, so $R_{eq} = 2k // 2k = 1k\Omega$

Now $\alpha_{par} = \frac{1}{2R_{eq}C} = \frac{1}{2(1k\Omega)(\frac{1}{4}\mu F)} = 2{,}000\ s^{-1}$

So ckt. is Critically Damped.

$$v(t) = (C + Dt)e^{-\alpha t} + v(\infty)$$

but $v(\infty) = 0$, so

$$v(t) = (C + Dt)e^{-\alpha t}$$

$v(0^+) = 0 = C$, so

$$v(t) = Dte^{-\alpha t}$$

$$\frac{dv}{dt} = D\left[e^{-\alpha t} - \alpha t e^{-\alpha t}\right]$$
$$= De^{-\alpha t}\left[1 - \alpha t\right]$$

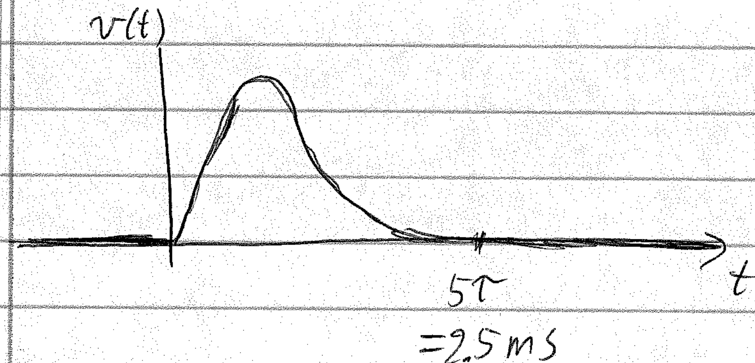
$$\frac{dv}{dt}(0) = D(1)[1] = D$$

$$D = \frac{1}{R_s C} V_s = \frac{1}{(2k)(\frac{1}{4}\mu)}(5V)$$

$$D = \frac{1}{\frac{1}{2}m} 5V = (2000)(5V) = 10{,}000\ V/s$$

So

$$v(t) = \left(10{,}000 \tfrac{V}{s}\right) t\, e^{-2{,}000 t} \quad \left(\tau = \tfrac{1}{2000}\ s = 500\mu s.\right)$$



Let's now pick $R = 10k\Omega$ + $R_s = 10k\Omega$, so

$R_{eq} = 5k\Omega$, AND let's ask for $i_L(t)$.

$t=0^-$: $v(0^-) = 0 = v(0^+)$

$i_L(0^-) = 0 = i_L(0^+)$

$t=0^+$: $i_L(0^+) = 0$ still

IV: $v_L = L\frac{di_L}{dt} \implies L\frac{di_L}{dt}(0^+) = v(0^+) = 0$

$$\underline{\frac{di_L}{dt}(0^+) = 0}$$

$R_{eq} = 5k\Omega$, so $\alpha = \frac{1}{2R_{eq}C} = \frac{1}{2(5k)(\frac{1}{4}\mu)} = \frac{2}{5}k$

$= 400\ s^{-1}$

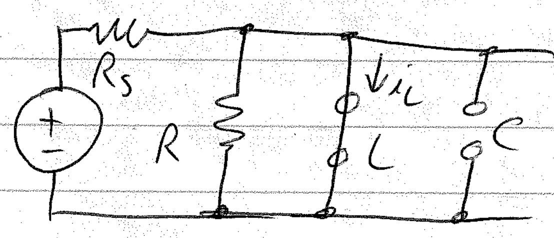
So underdamped: $\omega_d = \sqrt{(2000)^2 - (400)^2} = 1,960$

$$i_L(t) = e^{-\alpha t}\left[E\cos\omega_d t + F\sin\omega_d t\right] + i_L(\infty)$$

$$\cancel{i_L(0^+) = (1)\left[E(1) + F(0)\right] + i_L(\infty)}$$

$\cancel{0 = }$

$t = \infty$:



All current through L: $i_L(\infty) = \frac{V_S}{R_S}$

Substitute

$$i_L(0^+) = (1)\left[E(1) + F(0)\right] + \frac{V_S}{R_S} = 0$$

$$E = -\frac{V_S}{R_S} = -\frac{5V}{5k} = -1\times10^{-3}$$

$$\frac{di}{dt} = -\alpha e^{-\alpha t}\left[E\cos\omega_d t + F\sin\omega_d t\right] + \omega_d e^{-\alpha t}\left[-E\sin\omega_d t + F\cos\omega_d t\right] + 0$$

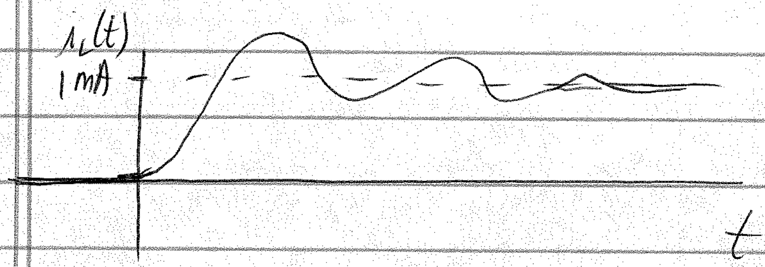
$$\frac{di}{dt}(0^+) = -\alpha\left[E\right] + \omega_d\left[F\right]$$

For ours: $\frac{di}{dt}(0^+) = 0 = -400\left(-\frac{V_S}{R_S}\right) + (1960)(F)$

$$1960\,F = -400\left(\frac{5V}{5k}\right)$$
$$= -0.4$$
$$F = -\frac{0.4}{1960} = -204\times10^{-6}$$

$$i_L(t) = e^{-400t}\left[-1\times10^{3}\cos(1960t) - 204\times10^{-6}\sin(1960t)\right] + 1\times10^{-3}$$

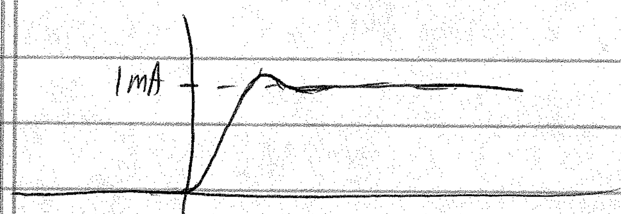
$$= e^{-400t}\left[-1\,mA\cos(1960t) - 0.204\,mA\sin(1960t)\right] + 1\,mA$$



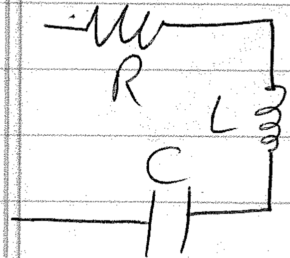
$$\tau = \frac{1}{400} = .25\times10^{-2} = 2.5\,ms$$

$$T = \frac{2\pi}{\omega_d} = \frac{2\pi}{1960} = 3.2\,ms = \underline{1.28\,\tau}$$

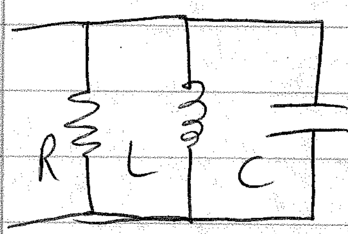
So should be:



Note a couple of things



Series $\alpha = \frac{R}{2L}$ $\uparrow R \ \uparrow \alpha$



parallel $\alpha = \frac{1}{2RC}$ $\uparrow R \ \downarrow \alpha$

In under damped ckt, energy moves back and forth between L + C, with some being dissipated in R on each move.

In series, i must move thru R, so increasing R "burns up" more energy, so $\alpha \uparrow$. R active when L has energy.

In parallel, i can go straight from L to C, some goes thru R because of voltage, most when v is largest, when C stores the energy.

We wrote $\omega_0 = \frac{1}{\sqrt{LC}}$

and $\alpha = \frac{R_{eq}}{2L}$ or $\frac{1}{2R_{eq}C}$

Text uses $\zeta\omega_0 = \alpha$ where ζ is the damping ratio

$$\zeta = \frac{\alpha}{\omega_0} = \frac{\sqrt{LC}\,R_{eq}}{2L} = \frac{1}{2}\sqrt{\frac{C}{L}}\,R_{eq} \text{ (Series)}$$

$$\text{or } \frac{\sqrt{LC}}{2R_{eq}C} = \frac{1}{2}\sqrt{\frac{L}{C}}\,\frac{1}{R_{eq}} \text{ (Parallel)}$$

Using this notation:

$\zeta > 1 \Rightarrow$ overdamped

$\zeta = 1 \Rightarrow$ critically damped

$\zeta < 1 \Rightarrow$ under damped

and $s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$

$$= -\zeta\omega_0 \pm \sqrt{(\zeta\omega_0)^2 - \omega_0^2}$$

$$= \omega_0\left[-\zeta \pm \sqrt{\zeta^2 - 1}\right]$$

I will stick to α + ω_0 and not use ζ.