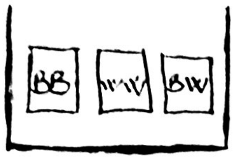
7.1.1. BERTRAND'S BOX PARADOX



$$P[x \text{ is drawn}] = \frac{1}{3} \quad \text{where } x \in \{BB, WW, BW\}$$

a)

TOTAL PROBABILITY

$$\begin{aligned}
P[\text{card shows } B] &= P[BB \text{ is drawn}] \cdot P[\text{card shows } B \mid BB \text{ is drawn}] \\
&+ P[WW \text{ is drawn}] \cdot P[\text{card shows } B \mid WW \text{ is drawn}] \\
&+ P[BW \text{ is drawn}] \cdot P[\text{card shows } B \mid BW \text{ is drawn}] \\
&= \frac{1}{3}\cdot 1 + \frac{1}{3}\cdot 0 + \frac{1}{3}\cdot\frac{1}{2} = \frac{\cancel{3}}{2}\cdot\frac{1}{\cancel{3}} = \frac{1}{2}
\end{aligned}$$

$$\begin{aligned}
P[\text{card shows } W] &= P[BB \text{ is drawn}] \cdot P[\text{card shows } W \mid BB \text{ is drawn}] \\
&+ P[WW \text{ is drawn}] \cdot P[\text{card shows } W \mid WW \text{ is drawn}] \\
&+ P[BW \text{ is drawn}] \cdot P[\text{card shows } W \mid BW \text{ is drawn}] \\
&= \frac{1}{3}\cdot 0 + \frac{1}{3}\cdot 1 + \frac{1}{3}\cdot\frac{1}{2} = \frac{\cancel{3}}{2}\cdot\frac{1}{\cancel{3}} = \frac{1}{2}
\end{aligned}$$

b)

DISTINGUISH BETWEEN BLACK SIDES

$$\begin{aligned}
&P[\text{other side is } B \mid \text{card shows } B] \\
&= P[BB \text{ is drawn}] \cdot \{P[\text{other side is } B_2 \mid \text{card shows } B_1, BB \text{ is drawn}] \\
&\qquad + P[\text{other side is } B_1 \mid \text{card shows } B_2, BB \text{ is drawn}]\} \\
&+ P[BW \text{ is drawn}] \cdot P[\text{other side is } B \mid \text{card shows } B, BW \text{ is drawn}] \\
&+ P[WW \text{ is drawn}] \cdot P[\text{other side is } B \mid \text{card shows } B, WW \text{ is drawn}] \\
&= \frac{1}{3}\cdot\{1+1\} + \frac{1}{3}\cdot 0 + \frac{1}{3}\cdot 0 = \frac{2}{3}
\end{aligned}$$

c) $P[\text{other side is } B \mid \text{card shows } W] =$

$$\begin{aligned}
&P[BB \text{ is drawn}] \cdot P[\text{other side is } B \mid \text{card shows } W, BB \text{ is drawn}] \\
&+ P[BW \text{ is drawn}] \cdot P[\text{other side is } B \mid \text{card shows } W, BW \text{ is drawn}] \\
&+ P[WW \text{ is drawn}] \cdot P[\text{other side is } B \mid \text{card shows } W, WW \text{ is drawn}] \\
&= \frac{1}{3}\cdot 0 + \frac{1}{3}\cdot 1 + \frac{1}{3}\cdot 0 = \frac{1}{3}
\end{aligned}$$

7.1.2 ENTROPY AND KULLBACK-LEIBLER DIVERGENCE

a) entropy of normal distribution

$$h[\mathcal{N}(\mu,\sigma^2)] = -\int_{-\infty}^{\infty} \mathcal{N}(x|\mu,\sigma^2)\ln \mathcal{N}(x|\mu,\sigma^2)\,dx =$$

$$= -\frac{1}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}}\cdot\left[\ln\frac{1}{\sqrt{2\pi\sigma^2}} + \left(-\frac{1}{2\sigma^2}(x-\mu)^2\right)\right]dx =$$

$$= -\frac{1}{\sqrt{2\pi\sigma^2}}\left[\ln\frac{1}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}dx - \frac{1}{2\sigma^2}\int_{-\infty}^{\infty}(x-\mu)^2 e^{-\frac{1}{2\sigma^2}(x-\mu)^2}dx\right] =$$

$$= -\frac{1}{\sqrt{2\pi\sigma^2}}\left[\ln\frac{1}{\sqrt{2\pi\sigma^2}}\cdot\sqrt{\frac{\pi}{\frac{1}{2\sigma^2}}} - \frac{1}{2\cancel{\sigma^2}}\cdot\sqrt{\frac{\pi}{\frac{1}{2\sigma^2}}}\cdot\frac{1}{\frac{1}{\cancel{\sigma^2}}}\right] =$$

$$= -\frac{1}{\cancel{\sqrt{2\pi\sigma^2}}}\left[-\cancel{\sqrt{2\pi\sigma^2}}\cdot\ln\sqrt{2\pi\sigma^2} - \frac{1}{2}\cancel{\sqrt{2\pi\sigma^2}}\right] =$$

$$= \ln\sqrt{2\pi\sigma^2} + \frac{1}{2} =$$

$$= \frac{1}{2}\ln(2\pi\sigma^2) + \frac{1}{2}\ln(e) =$$

$$= \frac{1}{2}\ln(2\pi\sigma^2 e)$$

b) KL divergence between two normal distributions

$$KL[\mathcal{N}(\mu_1,\sigma_1^2)\,\|\,\mathcal{N}(\mu_2,\sigma_2^2)] = \int_{-\infty}^{\infty}\mathcal{N}(x|\mu_1,\sigma_1^2)\ln\frac{\mathcal{N}(x|\mu_1,\sigma_1^2)}{\mathcal{N}(x|\mu_2,\sigma_2^2)}dx =$$

$$= \int_{-\infty}^{\infty}\mathcal{N}(x|\mu_1,\sigma_1^2)\ln\mathcal{N}(x|\mu_1,\sigma_1^2)dx - \int_{-\infty}^{\infty}\mathcal{N}(x|\mu_1,\sigma_1^2)\ln\mathcal{N}(x|\mu_2,\sigma_2^2)dx =$$

$$= -h[\mathcal{N}(\mu_1,\sigma_1^2)] - \int_{-\infty}^{\infty}\mathcal{N}(x|\mu_1,\sigma_1^2)\left[-\frac{1}{2\sigma_2^2}(x-\mu_2)^2 - \ln\sqrt{2\pi\sigma_2^2}\right]dx =$$

$$= -\frac{1}{2}\ln(2\pi\sigma_1^2 e) + \frac{1}{2\sigma_2^2}\left[\mathbb{E}_x^{(1)}(x^2) - 2\mu_2\mathbb{E}_x^{(1)}(x) + \mathbb{E}_x^{(1)}(\mu_2^2)\right] + \mathbb{E}_x^{(1)}\sqrt{2\pi\sigma_2^2}$$

$$\sigma_1^2 = \mathbb{E}_x^{(1)}(x^2) - \mathbb{E}_x^{(1)}(x) \rightarrow \mathbb{E}_x^{(1)}(x^2) = \sigma_1^2 + \mu_1^2$$

$$= -\frac{1}{2}\ln(2\pi\sigma_1^2 e) + \frac{\sigma_1^2 + \mu_1^2 - 2\mu_1\mu_2 + \mu_2^2}{2\sigma_2^2} + \ln\sqrt{2\pi\sigma_2^2} =$$

$$= -\frac{1}{2}\ln(2\pi\sigma_1^2 e) + \frac{\sigma_1^2 + (\mu_1-\mu_2)^2}{2\sigma_2^2} + \frac{1}{2}\ln(2\pi\sigma_2^2) =$$

$$= \frac{1}{2}\ln\frac{\cancel{2\pi}\sigma_2^2}{\cancel{2\pi}\sigma_1^2 e} + \frac{\sigma_1^2 + (\mu_1-\mu_2)^2}{2\sigma_2^2} =$$

$$= \ln\frac{\sigma_2}{\sigma_1} - \frac{1}{2}\ln(e) + \frac{\sigma_1^2 + (\mu_1-\mu_2)^2}{2\sigma_2^2} =$$

$$= \ln\frac{\sigma_2}{\sigma_1} + \frac{\sigma_1^2 + (\mu_1-\mu_2)^2}{2\sigma_2^2} - \frac{1}{2}$$