Worksheet - Additional problems and solutions

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Problems

1. Let a and b be real positive numbers such that:

$$\frac{a}{b} + \frac{b}{a} = 8$$
$$\frac{a^2}{b} + \frac{b^2}{a} = 10.$$

   Find $\frac{1}{a} + \frac{1}{b}$.

2. Let a and b be real positive numbers such that:

$$\frac{a}{b} + \frac{b}{a} = c + 1$$
$$\frac{a^2}{b} + \frac{b^2}{a} = c.$$

   Find $\frac{1}{a} + \frac{1}{b}$ in terms of c.

3. Find the values of a an b for which the following doesn't hold:

$$\frac{1}{a} + \frac{1}{b} = \frac{\left(\frac{a}{b} + \frac{b}{a} + 2\right)\left(\frac{a}{b} + \frac{b}{a} - 1\right)}{\frac{a^2}{b} + \frac{b^2}{a}}.$$

4. Let a and b be real positive numbers such that:

$$\frac{a}{b} + \frac{b}{a} = 14$$
$$\frac{a^3}{b} + \frac{b^3}{a} = 194.$$

   Find $\frac{1}{a} + \frac{1}{b}$.

5.1. Find the closed form of $\dfrac{1}{a} + \dfrac{1}{b}$, given:

$$\frac{a}{b} + \frac{b}{a} = n$$
$$\frac{a^3}{b} + \frac{b^3}{a} = m.$$

5.2. Find the restrictions on a and b for the formula from 5.1. to hold.

Solutions

1.

$$\frac{1}{a}+\frac{1}{b}=\frac{(8+2)(8-1)}{10}=\frac{70}{10}=7$$

2.

$$\frac{1}{a}+\frac{1}{b}=\frac{(c+1+2)(c+1-1)}{c}=\frac{c(c+3)}{c}=c+3$$

3.
The given relationship doesn't hold when either a, b or $\frac{a^2}{b}+\frac{b^2}{a}$ is equal to 0. If $\frac{a^2}{b}+\frac{b^2}{a}$, then a = -b or a = b = 0. Therefore, the relationship doesn't hold when a = 0, b = 0 or a = -b.

4. This solution uses the result from the problem 5.1.

$$\frac{1}{a}+\frac{1}{b}=\sqrt{\frac{\left(\frac{a}{b}+\frac{b}{a}+2\right)\left(\left(\frac{a}{b}+\frac{b}{a}\right)^2-2\right)}{\frac{a^3}{b}+\frac{b^3}{a}}}=\sqrt{\frac{(14+2)(14^2-2)}{194}}=\sqrt{\frac{16*192}{192}}=\sqrt{16}=4$$

5.1. We have that:

$$a^2+b^2=nab$$
$$a^4+b^4=mab.$$

$$a^4+b^4=\left(a^2+b^2\right)^2-2a^2b^2$$

$$mab=(nab)^2-2a^2b^2=(n^2-2)a^2b^2$$

$$ab=\frac{m}{n^2-2}$$

$$(a+b)^2=a^2+2ab+b^2=nab+2ab=(n+2)ab$$

$$\frac{(a+b)^2}{(ab)^2}=\frac{(n+2)ab}{a^2b^2}=\frac{n+2}{ab}$$

$$\left(\frac{1}{a}+\frac{1}{b}\right)^2=\frac{(n+2)(n^2-2)}{m}$$

$$\frac{1}{a}+\frac{1}{b}=\sqrt{\frac{\left(\frac{a}{b}+\frac{b}{a}+2\right)\left(\left(\frac{a}{b}+\frac{b}{a}\right)^2-2\right)}{\frac{a^3}{b}+\frac{b^3}{a}}}$$

5.2.

Since $\frac{1}{0}$ is not defined, a, b and $\frac{a^3}{b}+\frac{b^3}{a}$ mustn't be equal to 0. If $\frac{a^3}{b}+\frac{b^3}{a}=0 \Longleftrightarrow a^4+b^4=0$ which is possible if and only if a = b = 0. Since we are looking only at the principle branch of the square root, $\frac{1}{a}+\frac{1}{b}\geq 0$. Also, the expression under the square root must be nonnegative.

$$\frac{\left(\frac{a}{b}+\frac{b}{a}+2\right)\left(\left(\frac{a}{b}+\frac{b}{a}\right)^2-2\right)}{\frac{a^3}{b}+\frac{b^3}{a}}\geq 0$$

$$\frac{\left(\frac{a}{b}+\frac{b}{a}+2\right)\left(\left(\frac{a}{b}+\frac{b}{a}\right)^2-2\right)}{\frac{a^4+b^4}{ab}}\geq 0$$

$$ab\left(\frac{a}{b}+\frac{b}{a}+2\right)\left(\left(\frac{a}{b}+\frac{b}{a}\right)^2-2\right)\geq 0$$

$$\left(a^2+b^2+2ab\right)\left(\left(\frac{a}{b}+\frac{b}{a}\right)^2-2\right)\geq 0$$

$$(a+b)^2\left(\left(\frac{a}{b}+\frac{b}{a}\right)^2-2\right)\geq 0$$

$$\left(\frac{a}{b}+\frac{b}{a}\right)^2-2\geq 0$$

$$\left(\frac{a^2+b^2}{ab}\right)^2-2\geq 0$$

$$\frac{a^4+2a^2b^2+b^4}{a^2b^2}-2\geq 0$$

$$\frac{a^4+2a^2b^2+b^4}{a^2b^2}-\frac{2a^2b^2}{a^2b^2}\geq 0$$

$$\frac{a^4+b^4}{a^2b^2}\geq 0$$

$$a^4+b^4\geq 0$$

This is true for real a and b, therefore the requirement is that $a\neq 0$, $b\neq 0$ and $\frac{1}{a}+\frac{1}{b}\geq 0$

This PDF worksheet is accompanying my article Generalized competition problems | Part 1 — An algebraic expression.