

The solution to the bonus problem

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November 2, 2021

Solution

Let $N_2(n)$ be the number of *semi-pretty* square boards of size $n \times n$. The border of the board can be split into two identical sections containing $2(n-1)$ cells. Therefore, the number of possible borders is $2^{2(n-1)} = (2^{n-1})^2$. Therefore, for odd n :

$$N_2(n) = (2^{n-1})^2 \cdot N_2(n-2) = (2^{n-1})^2 \cdot (2^{n-3})^2 \dots (2^2)^2 N_2(1) = (2^{n-3} \cdot 2^{n-1} \dots 2^2)^2 N_2(1)$$

We know from the original problem that for odd n :

$$N(n) = (2^{n-1} \cdot 2^{n-3} \dots 2^2) N(1) \iff 2^{n-1} \cdot 2^{n-3} \dots 2^2 = \frac{N(n)}{N(1)}$$

Therefore,

$$N_2(n) = \left(\frac{N(n)}{N(1)} \right)^2 N_2(1) = \frac{N^2(n) N_2(1)}{N^2(1)}$$

Since $N_2(1) = N(1) = 2$,

$$N_2(n) = \frac{\left(2^{\frac{n^2+3}{4}} \right)^2}{2} = 2^{\frac{n^2+3}{2}} \cdot 2^{-1} = 2^{\frac{n^2+1}{2}}, n \text{ is odd.}$$

On the other hand, for even n :

$$N_2(n) = (2^{n-1})^2 \cdot N_2(n-2) = (2^{n-1})^2 \cdot (2^{n-3})^2 \dots (2^1)^2 N_2(0) = (2^{n-3} \cdot 2^{n-1} \dots 2^1)^2 N_2(0)$$

We know from the original problem that for even n :

$$N(n) = (2^{n-1} \cdot 2^{n-3} \dots 2^1) N(0) \iff 2^{n-1} \cdot 2^{n-3} \dots 2^1 = \frac{N(n)}{N(0)}$$

Therefore,

$$N_2(n) = \left(\frac{N(n)}{N(0)} \right)^2 N_2(0) = \frac{N^2(n) N_2(0)}{N^2(0)}$$

Since $N_2(0) = N(0) = 1$,

$$N_2(n) = \frac{\left(2^{\frac{n^2}{4}} \right)^2}{1} = 2^{\frac{n^2}{2}}, n \text{ is even.}$$

Finally,

$$N_2(n) = \begin{cases} 2^{\frac{n^2+1}{2}}, & n \text{ is odd} \\ 2^{\frac{n^2}{2}}, & n \text{ is even} \end{cases}$$