The solution to the bonus problem

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Solution

Let $N_2(n)$ be the number of *semi-pretty* square boards of size n x n. The border of the board can be split into two identical sections containing $2(n$ - 1) cells. Therefore, the number of possible borders is $2^{2(n-1)} = \left(2^{n-1}\right)^2$. Therefore, for odd n:

$$N_2(n) = \left(2^{n-1}\right)^2 \cdot N_2(n-2) = \left(2^{n-1}\right)^2 \cdot \left(2^{n-3}\right)^2 \dots \left(2^2\right)^2 N_2(1) = \left(2^{n-3} \cdot 2^{n-1} \dots 2^2\right)^2 N_2(1)$$

We know from the original problem that for odd n:

$$N(n) = \left(2^{n-1} \cdot 2^{n-3} \dots 2^2\right) N(1) \iff 2^{n-1} \cdot 2^{n-3} \dots 2^2 = \frac{N(n)}{N(1)}$$

Therefore,

$$N_2(n) = \left(\frac{N(n)}{N(1)}\right)^2 N_2(1) = \frac{N^2(n) N_2(1)}{N^2(1)}$$

Since $N_2(1) = N(1) = 2$,

$$N_2(n) = \frac{\left(2^{\frac{n^2+3}{4}}\right)^2}{2} = 2^{\frac{n^2+3}{2}} \cdot 2^{-1} = 2^{\frac{n^2+1}{2}}, \ n \text{ is odd.}$$

On the other hand, for even n:

$$N_2(n) = \left(2^{n-1}\right)^2 \cdot N_2(n-2) = \left(2^{n-1}\right)^2 \cdot \left(2^{n-3}\right)^2 \dots \left(2^1\right)^2 N_2(0) = \left(2^{n-3} \cdot 2^{n-1} \dots 2^1\right)^2 N_2(0)$$

We know from the original problem that for even n:

$$N(n) = \left(2^{n-1} \cdot 2^{n-3} \dots 2^1\right) N(0) \iff 2^{n-1} \cdot 2^{n-3} \dots 2^1 = \frac{N(n)}{N(0)}$$

Therefore,

$$N_2(n) = \left(\frac{N(n)}{N(0)}\right)^2 N_2(0) = \frac{N^2(n) N_2(0)}{N^2(0)}$$

Since $N_2(0) = N(0) = 1$,

$$N_2(n) = \frac{\left(2^{\frac{n^2}{4}}\right)^2}{1} = 2^{\frac{n^2}{2}}, \ n \text{ is even.}$$

Finally,

$$N_2(n) = \begin{cases} 2^{\frac{n^2+1}{2}}, n \text{ is odd} \\ 2^{\frac{n^2}{2}}, n \text{ is even} \end{cases}$$

This PDF worksheet is accompanying my article Generalized competition problems | Part 2 — A combinatorial problem.