

Worksheet - Additional problems and solutions

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Problems

1. Let a and b be real positive numbers such that:

$$\begin{aligned}\frac{a}{b} + \frac{b}{a} &= 8 \\ \frac{a^2}{b} + \frac{b^2}{a} &= 10.\end{aligned}$$

Find $\frac{1}{a} + \frac{1}{b}$.

2. Let a and b be real positive numbers such that:

$$\begin{aligned}\frac{a}{b} + \frac{b}{a} &= c + 1 \\ \frac{a^2}{b} + \frac{b^2}{a} &= c.\end{aligned}$$

Find $\frac{1}{a} + \frac{1}{b}$ in terms of c .

3. Find the values of a and b for which does the following not hold:

$$\frac{1}{a} + \frac{1}{b} = \frac{\left(\frac{a}{b} + \frac{b}{a} + 2\right) \left(\frac{a}{b} + \frac{b}{a} - 1\right)}{\frac{a^2}{b} + \frac{b^2}{a}}.$$

4. Let a and b be real positive numbers such that:

$$\begin{aligned}\frac{a}{b} + \frac{b}{a} &= 14 \\ \frac{a^3}{b} + \frac{b^3}{a} &= 194.\end{aligned}$$

Find $\frac{1}{a} + \frac{1}{b}$.

- 5.1. Find the closed form of $\frac{1}{a} + \frac{1}{b}$, given:

$$\frac{a}{b} + \frac{b}{a} = n$$

$$\frac{a^3}{b} + \frac{b^3}{a} = m.$$

5.2. Find the restrictions on a and b for the formula from 5.1. to hold.

Solutions

1.

$$\frac{1}{a} + \frac{1}{b} = \frac{(8+2)(8-1)}{10} = \frac{70}{10} = 7$$

2.

$$\frac{1}{a} + \frac{1}{b} = \frac{(c+1+2)(c+1-1)}{c} = \frac{c(c+3)}{c} = c+3$$

3.

The given relationship doesn't hold when either a , b or $\frac{a^2}{b} + \frac{b^2}{a}$ is equal to 0. If $\frac{a^2}{b} + \frac{b^2}{a}$, then $a = -b$ or $a = b = 0$. Therefore, the relationship doesn't hold when $a = 0$, $b = 0$ or $a = -b$.

4. This solution uses the result from the [problem 5.1](#).

$$\frac{1}{a} + \frac{1}{b} = \sqrt{\frac{\left(\frac{a}{b} + \frac{b}{a} + 2\right) \left(\left(\frac{a}{b} + \frac{b}{a}\right)^2 - 2\right)}{\frac{a^3}{b} + \frac{b^3}{a}}} = \sqrt{\frac{(14+2)(14^2-2)}{194}} = \sqrt{\frac{16 * 192}{192}} = \sqrt{16} = 4$$

5.1. We have that:

$$a^2 + b^2 = nab$$

$$a^4 + b^4 = mab.$$

$$a^4 + b^4 = (a^2 + b^2)^2 - 2a^2b^2$$

$$mab = (nab)^2 - 2a^2b^2 = (n^2 - 2)a^2b^2$$

$$ab = \frac{m}{n^2 - 2}$$

$$(a+b)^2 = a^2 + 2ab + b^2 = nab + 2ab = (n+2)ab$$

$$\frac{(a+b)^2}{(ab)^2} = \frac{(n+2)ab}{a^2b^2} = \frac{n+2}{ab}$$

$$\left(\frac{1}{a} + \frac{1}{b}\right)^2 = \frac{(n+2)(n^2-2)}{m}$$

$$\frac{1}{a} + \frac{1}{b} = \sqrt{\frac{\left(\frac{a}{b} + \frac{b}{a} + 2\right) \left(\left(\frac{a}{b} + \frac{b}{a}\right)^2 - 2\right)}{\frac{a^3}{b} + \frac{b^3}{a}}}$$

5.2.

Since $\frac{1}{0}$ is not defined, a , b and $\frac{a^3}{b} + \frac{b^3}{a}$ mustn't be equal to 0. If $\frac{a^3}{b} + \frac{b^3}{a} = 0 \iff a^4 + b^4 = 0$ which is possible if and only if $a = b = 0$. Since we are looking only at the principle branch of the square root, $\frac{1}{a} + \frac{1}{b} \geq 0$. Also, the expression under the square root must be nonnegative.

$$\begin{aligned} & \frac{\left(\frac{a}{b} + \frac{b}{a} + 2\right) \left(\left(\frac{a}{b} + \frac{b}{a}\right)^2 - 2\right)}{\frac{a^3}{b} + \frac{b^3}{a}} \geq 0 \\ & \frac{\left(\frac{a}{b} + \frac{b}{a} + 2\right) \left(\left(\frac{a}{b} + \frac{b}{a}\right)^2 - 2\right)}{\frac{a^4 + b^4}{ab}} \geq 0 \\ & ab \left(\frac{a}{b} + \frac{b}{a} + 2\right) \left(\left(\frac{a}{b} + \frac{b}{a}\right)^2 - 2\right) \geq 0 \\ & (a^2 + b^2 + 2ab) \left(\left(\frac{a}{b} + \frac{b}{a}\right)^2 - 2\right) \geq 0 \\ & (a + b)^2 \left(\left(\frac{a}{b} + \frac{b}{a}\right)^2 - 2\right) \geq 0 \\ & \left(\frac{a}{b} + \frac{b}{a}\right)^2 - 2 \geq 0 \\ & \left(\frac{a^2 + b^2}{ab}\right)^2 - 2 \geq 0 \\ & \frac{a^4 + 2a^2b^2 + b^4}{a^2b^2} - 2 \geq 0 \\ & \frac{a^4 + 2a^2b^2 + b^4}{a^2b^2} - \frac{2a^2b^2}{a^2b^2} \geq 0 \\ & \frac{a^4 + b^4}{a^2b^2} \geq 0 \\ & a^4 + b^4 \geq 0 \end{aligned}$$

This is true for real a and b , therefore the requirement is that $a \neq 0$, $b \neq 0$ and $\frac{1}{a} + \frac{1}{b} \geq 0$