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Pitch-Class Set Multiplication in Theory and Practice

Stephen Heinemann

Since its 1955 premiere, *Le Marteau sans maître* has been regarded as one of the most important compositions of the post-war era. Certainly it is the best-known work of Pierre Boulez. Although many attempts have been made to analyze *Le Marteau*'s serial organization, it is Lev Koblyakov who has most effectively clarified the results, even if not the process, of Boulez's technique.[1] Koblyakov demonstrates that an operation invented by Boulez called "multiplication" generates the pitch-class sets that form the basis of the cycle of "L'Artisanat furieux" (movements 1, 3 and 7) in *Le Marteau.* The composer himself, in his theoretical writings, has also delineated certain features of multiplication. Yet neither Boulez nor Koblyakov has described and examined the pitch-class sets that result from multiplicative operations. Boulez's reluctance to elaborate on his technique and the apparent inability of Koblyakov to decipher it can easily lead to the following misperceptions: the operation is arbitrary; the results are unpredictable; compositional and analytical choices are capricious; and important properties arise from arithmetical, as opposed to pitch-class, operations. And underlying such assumptions is a more fundamental question: What compelled Boulez to use the operation at all?

In this essay I will examine and explain Boulez's multiplicative operations and thereby correct each of these misperceptions. I will present three different but related operations called *simple, compound,* and *complex multiplication.* Simple and compound multiplication are treated cursorily (but neither differentiated as such nor usefully formalized) in Boulez's writings.[2] I will demonstrate, however, that complex multiplication, the mechanics of which were not divulged by Boulez, was employed to generate the pitch-class sets that constitute the first cycle of *Le Marteau.*[3] I will also attempt to clarify some aspects of Koblyakov's published analyses, as

This article expands on a paper presented at the annual meeting of the Society for Music Theory, Kansas City, 1992, and is primarily drawn from Stephen Heinemann, "Pitch-Class Set Multiplication in Boulez's *Le Marteau sans maître*" (D.M.A. diss., University of Washington, 1993).

[1]Lev Koblyakov, "P. Boulez 'Le marteau sans maître,' analysis of pitch structure," *Zeitschrift für Musiktheorie* 8/1 (1977): 24–39. These early findings were refined in Lev Koblyakov, "The World of Harmony of Pierre Boulez: Analysis of *Le marteau sans maître*" (Ph.D. diss., Hebrew University of Jerusalem, 1981), and *Pierre Boulez: a World of Harmony* (New York: Harwood Academic Publishers, 1990).

[2]The most important of these—at least as far as *Le Marteau* is concerned—is the chapter entitled "Musical Technique" that forms the bulk of Pierre Boulez, *Boulez on Music Today,* trans. Susan Bradshaw and Richard Rodney Bennett (Cambridge: Harvard University Press, 1971), 35–143. The title and intent of the chapter recall the treatise of Boulez's principal teacher Olivier Messiaen, *Technique de mon langage musical* (Paris: Alphonse Leduc, 1944).

[3]This operation's application will also be demonstrated with respect to Boulez's earlier, subsequently withdrawn, *Oubli signal lapidé.* Koblyakov lists nine works following *Le Marteau* in which multiplication is employed: the Third Sonata, *Structures II, Don, Tombeau, Eclat, Eclat/Multiples, Figures-Doubles-Prismes, Domaines,* and *cummings ist der Dichter* (Koblyakov, *Pierre Boulez,* 32).

well as posit a process-based listening strategy for sections of the first movement of *Le Marteau,* using techniques derived from analyses of music by Slonimsky, Lutoslawski, and Stravinsky.

Boulez's multiplication has been cited by Richard Cohn as an early and important formulation of transpositional combination.[4] This is accurate to the extent that both operations can be understood informally as the construction of one set upon each element of another. But while Cohn's theories concern multiple combinations of abstractly formulated T_n- and T_n/T_nI-type sets, the multiplicative operations presented here involve paired pitch-class sets drawn directly from specific musical representations. Regardless of such differences, the theory of transpositional combination provides an invaluable introduction to many of the workings and possibilities of multiplication as practiced by Boulez and, in different schemata, by other composers and theorists.[5]

[4]Richard L. Cohn, "Transpositional Combination in Twentieth-Century Music" (Ph.D. diss., Eastman School of Music, 1987), 48–50; "Inversional Symmetry and Transpositional Combination in Bartók," *Music Theory Spectrum* 10 (1988): 23.

[5]Cohn has observed that differences in terminology can camouflage closely related techniques, citing the resemblance of his "transpositional combination" and Boulez's "multiplication" to, among others, Howard Hanson's "projection" and Jonathan Bernard's "parallel symmetry." See Cohn, "Inversional Symmetry," 23. To these can be added Nicolas Slonimsky's "inter-, infra-, and ultrapolation" and Anatol Vieru's "composition." See Nicolas Slonimsky, *Thesaurus of Scales and Melodic Patterns* (New York: Schirmer Books/Macmillan, 1987; originally published by Charles Scribner's Sons, 1947), ii; and Anatol Vieru, "Modalism—A 'Third World,' " *Perspectives of New Music* 24/1 (1985): 65. George Perle appropriates Cohn's term but asserts having "formulated the concept" in 1962 (correspondence, *Music Theory Spectrum* 17/1 [1995]: 138), a claim contradicted by the earlier work of Slonimsky, Boulez, and Hanson; conversely, Slonimsky identifies Domenico Alaleona, Alois Haba, and Joseph Schillinger as his theoretical precursors. Transpositional combination serves ideally as the umbrella under which this variety of technical approaches can be understood. My own experience with Boulez's technique suggests that the term "multiple transposition" would be more accurate (if less concise), but to the originator goes the nomenclatorial prerogative.

In Cohn's formalization, the transpositional combination of two T_n-type sets can be calculated on an additive matrix of the type shown in Example 1: each element of one operand set is added to every element of the other, resulting in a set of integers analogous to pitch classes from which T_n- and T_n/T_nI-type sets are derived. The operation is signified by $*$ between operands. Undermining a tenet of much post-tonal theory, the greater abstraction of the T_n/T_nI-type set is of limited value here. Example 1 shows that the transpositional combination of [0126] and [015], both of which are inversionally asymmetrical, can be obtained in four different ways, and can even yield sets of different cardinalities.

SIMPLE MULTIPLICATION OF PITCH-CLASS SETS

Even the T_n-type is too abstract for purposes of pitch-class set multiplication; a theoretical tool is required at the next greater level of specificity. Since the T_n-type is derived from the normal form of an unordered pc set, this next level will be derived from a type of partially-ordered set called an *initially ordered pitch-class set,* or *io set.* This is a set in which one pitch class (selected according to contextual criteria) is ordered as the first; the remaining pcs are unordered with respect to each other, but succeed the first. The first pc of an io set is referred to as the *initial pitch class* and is designated by the letter r. The normal form of an io set is derived by listing pc integers in ascending order and rotating this order to begin with the initial pitch class; the resultant io set is notated with a combination of ordered and unordered set notations.[6] For example, given pc set {1479} and r = 4,

[6]Conventions of notation are taken principally from John Rahn, *Basic Atonal Theory* (New York: Longman, 1980). The conventions most important to this study include: C = 0; <xyz> = ordered pc set; {xyz} = unordered

Example 1. Four realizations of [0126] * [015]

6	6	7	e
2	2	3	7
1	1	2	6
0	0	1	5
*	0	1	5

(0126) * (015) →
{e0123567} →
(01234678) →
8-5 [01234678]

6	6	t	e
2	2	6	7
1	1	5	6
0	0	4	5
*	0	4	5

(0126) * (045) →
{te0124567} →
(012346789) →
9-5 [012346789]

6	6	7	e
5	5	6	t
4	4	5	9
0	0	1	5
*	0	1	5

(0456) * (015) →
{45679te01} →
(012356789) →
9-5 [012346789]

6	6	t	e
5	5	9	t
4	4	8	9
0	0	4	5
*	0	4	5

(0456) * (045) →
{45689te0} →
(01245678) →
8-5 [01234678]

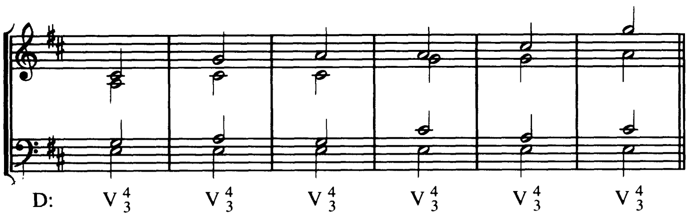
the io set is <4{791}>. The generalized form of such a set draws from the T_n-type concept and is called the *ordered pitch-class intervallic structure,* or *ois.* The ois lists the ordered pitch-class intervals from the initial pitch class to every element of an io set, including itself; for normal form, integers are underlined and listed in ascending order. For example, the ois of <4{791}> is 0359. The value of the ois formulation is apparent when calculations of multiplicative operations are made: unlike the transpositional combination of abstract T_n-type sets, they will be shown to permit the generation of pitch-class sets. At the same time, the initially ordered set eliminates the redundant calculations that result from dealing with ordered sets, since any set larger than a dyad will have fewer initial orderings than permutations.[7] Finally, these are hardly new concepts. Just as the "generic" dominant-seventh chord is an example of a T_n-type set in tonal practice, so also is the specification of root position or inversion within such a chord an example of an ois. When, for instance, a second-inversion dominant-seventh chord is constructed on the bass (that is, initial) pc E, an initially ordered pc set results—the V^4_3 chord in D major, shown in Example 2 in each of its six permutations beginning with E, the first of which corresponds to io set normal form.[8]

pc set; (xyz) = T_n-type set; [xyz] = T_n/T_nI-type set, or set class; i<x,y> = ordered pc interval from x to y (= y−x [mod 12]); i(x,y) = unordered pc interval, or interval class, between x and y (= lesser of y−x and x−y); |A| = cardinality of set A. Pitch classes 10 and 11 are represented by t and e respectively. It should be noted that Boulez's "multiplication" differs greatly from the familiar M_1, M_5, M_7, and M_{11} transforms, the "multiplicative operations" discussed in *Basic Atonal Theory.*

[7]The number of io sets that can be derived from pc set A equals |A|, while the number of ordered sets that can be derived from pc set A equals |A|−factorial. The number of unique oiss that can be derived from any T_n/T_nI-type set is double the set's cardinality, divided by its degree of symmetry.

[8]The ordered pitch-class intervallic structure has been formalized differently elsewhere. Alan Chapman calls this structure an "AB [for *a*bove *b*ass] set," abstracting it from a pitch-class set but using it as a means to establish relationships between different set classes. See Alan Chapman, "Some Intervallic Aspects of Pitch-Class Set Relations," *Journal of Music Theory* 25/2 (1981): 275–290. Robert Morris develops a general system of classification referred to as "FB [for *f*igured *b*ass] class" in his "Equivalence and Similarity in Pitch and their Interaction with Pcset Theory" (paper presented at the

Example 2. io set and ois in tonal theory



Unlike a T_n-type set, any ois can be constructed on a pitch class to form a single pc set. For example, while the note C could be a starting point for one of three members of (047)—forming major triads in which C is the root (C major), third (A♭ major), or fifth (F major)—the construction of 047 on C results only in C major. The construction of an ois on a given pitch class is signified by a circled multiplication symbol, ⊗ (a sign traditionally used for operations resembling arithmetical multiplication[9]), and is calculated in integer notation by adding the ois integers to the pitch-class integer, as shown in each matrix in Example 3. The construction of an ois on a pitch class forms the basis of pitch-class set multiplication.

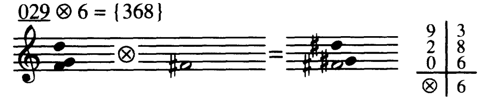
The simple multiplication of two pitch-class sets consists of the construction of the ois of one operand set, the *multiplicand,* on each pc of another operand set, the *multiplier.* The set comprising the union of all pcs resulting from these constructions is called the *product.* The operation is again signified by ⊗.

Example 4a shows one of Boulez's illustrations of simple multiplication. He gives two operand sets, labeled A and E, and the five products that can result from their simple multiplication (AE,1; AE,2; AE,3; EA,1; and EA,2).[10] The calculation of these products is shown in staff and integer matrix notations in Examples 4b through 4f. In Examples 4b, 4c, and 4d, the oiss of each of the three io sets derived from multiplicand A are constructed on each pc of multiplier E. In Examples 4e and 4f, the oiss of each of the two io sets derived from multiplicand E are constructed on each pc of multiplier A. The different orderings of pcs on the staff are addressed below; for now, products will be defined as unordered sets.

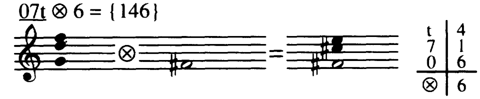
While all of this seems straightforward enough, there is discomfiting inelegance lurking here. Perhaps the most obviously inelegant feature is noncommutativity: A ⊗ E ≠ E ⊗ A. Further, Examples 4b, 4c, and 4d, collectively considered, show multiplicands and multipliers with identical pitch-class contents resulting in different products, as do 4e and 4f. An io set's pc content is therefore less significant to

Example 3. ois/pc constructions

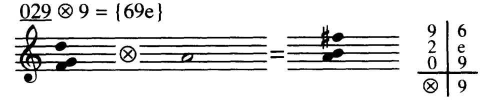
a. ois < 5{72} > = 029
029 ⊗ 6 = {368}



b. ois < 7{25} > = 07t
07t ⊗ 6 = {146}



c. ois < 5{72} > = 029
029 ⊗ 9 = {69e}



Annual Meeting of the Society for Music Theory, Tallahassee, 1994). As with figured bass, neither the AB set nor FB class includes the interval from r to itself (part of the T_n-type modeling) necessary to the present formalization.

[9]Cohn differently defines A ∗ B when A and B are pc sets. The use of Cohn's ∗ will be retained for operations involving T_n-type sets.

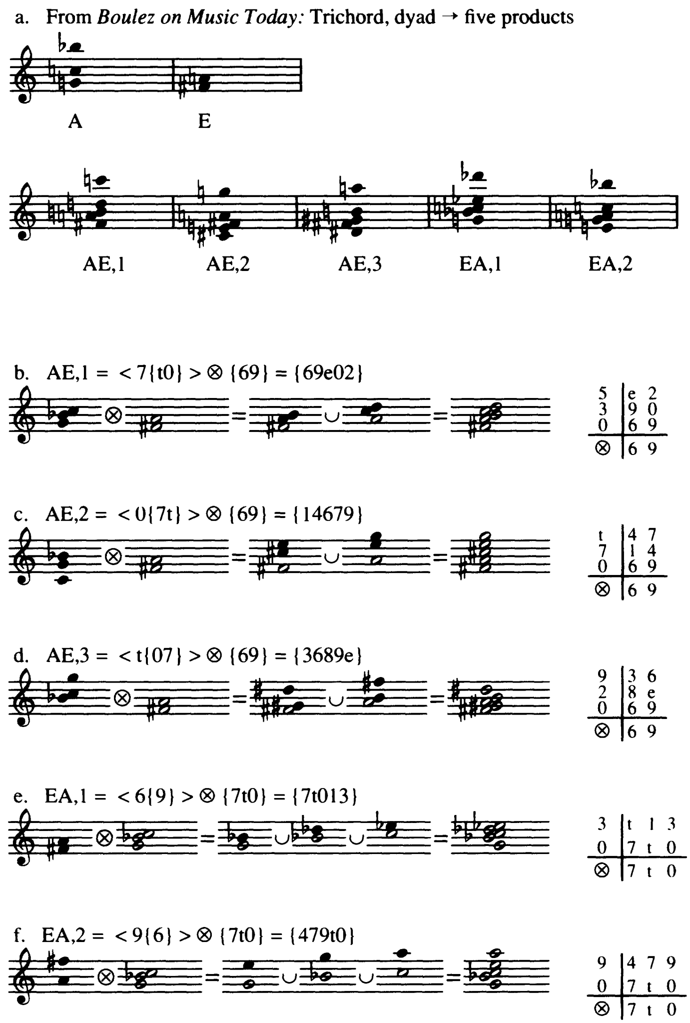
[10]Boulez, *Boulez on Music Today,* 79.

multiplication than the ois it spawns; in fact, any of the twelve transpositions of the io sets in any of these examples will leave each product unchanged, since every such transposition will yield the same ois—a property called *multiplicand redundancy.*

Based on the foregoing, the formula for the simple multiplication of pitch-class sets is: where A and B are pc sets and B = {b,c,...m}, $A \otimes B = (\mathrm{ois}(A) \otimes b) \cup (\mathrm{ois}(A) \otimes c) \cup \ldots (\mathrm{ois}(A) \otimes m)$. Of particular interest is the formula for the simple multiplication of pitch classes: where $a \in A$, $b \in B$, and r is the initial pc of A: $a \otimes b = i\langle r,a\rangle + b = (a - r) + b$. Through this formula, other thorny areas can be explored. One is *variety of products,* already well-demonstrated in Example 4. The initial pitch class r is variable, its only restriction being that it is an element of the multiplicand. Each of the five pcs in the operands under consideration can serve as r, thus resulting in five different products. Another is *multiplier replication,* the property that the multiplier set must be a subset of the product: since some a will equal r, $(a - r) + b = (r - r) + b = b$. (This is why the bottom row of each additive matrix duplicates the multiplier.) A composer striving for an equally weighted arrangement of pitch classes—Boulez, for an apparent example—will encounter a stumbling block when multiplier pcs recur continually. Another problematic area concerns initial ordering of the multiplicand: no criteria have been established which will prefer one initial ordering to another.

However, there is one quite elegant feature of simple multiplication that is predictable both through Cohn's theory and through the formula for pitch-class multiplication, and that was well-understood by Boulez. Although the products shown in Example 4a are different pitch-class sets, every one is transpositionally equivalent, or, in Boulez's words, "totally isomorphic,"[11] to each of the others. Boulez has arranged the

Example 4. Simple multiplication of pitch-class sets



[11]Ibid.

pitch-spatial representation of the products in Example 4a so that they reflect the property; perhaps this transpositional equivalence led him to experiment further with multiplication and ultimately to realize its potential.

SIMPLE MULTIPLICATION OF LINES

There is an invaluable aspect to the superficially negative property of multiplicand redundancy. The abstraction of an ois from a specific multiplicand pc set becomes possible, forming one of the most important features of simple multiplication theory; many passages from a variety of composers can be described with methods derived from the foregoing. Unlike Boulez, for whom complex multiplication is essentially a precompositional technique generating unordered pitch-class sets that will be given compositional order in a manner that refers obliquely (at most) to their source, the composers considered here have employed simple multiplication at the musical surface in varying degrees of immediacy and technical sophistication. Each case involves the construction of a multiplicand *linear ordered pitch-class intervallic structure* (*line ois*) on pcs within a multiplier ordered set. The construction of a line ois on a pitch class results in a line segment that can be elaborated in several different ways. Line ois construction on a multiplier consisting of a repeated pitch class, resulting in repetition of the segment, produces an *isomelos;* if the line ois is constructed on a scalewise or arpeggiated multiplier, a *pattern* is produced.[12] If consistent with respect to rhythmic and pitch-spatial realization, an isomelos becomes more specifically an *ostinato* and a pattern becomes a *sequence.*

[12]This nonstandard usage of "isomelos" follows Vincent Persichetti, *Twentieth-Century Harmony* (New York: Norton, 1961), 217. The definition of "pattern" parallels Slonimsky's but subsumes his definition of "scale" as an interpolated progression (see note 16). Both terms are here extended to include pitch class.

The sequences forming the bulk of Slonimsky's *Thesaurus of Scales and Melodic Patterns* constitute perhaps the earliest substantial and systematic application of simple multiplicative principles. Most of this speculative work (containing "a great number of melodically plausible patterns that are new"[13]) involves the construction of line oiss on pcs within multiplier ordered sets that divide the octave into equal parts (and are thus both transpositionally and inversionally symmetrical). Example 5 shows two patterns from the *Thesaurus* that will be discussed in terms of simple multiplication.

The theory of simple multiplication of pc sets is easily extended to construct patterns such as Slonimsky's; all that is required is a modification of the ois to account for linearity. A combination of Rahn's "line equivalence class" notation[14] and ois notation will represent ordered pc intervals—again, from an initial pitch class, but now defined as the first-occurring pc rather than the "bass"[15]—separated by dashes and underlined. These line oiss are multiplied by ordered pc sets by constructing the line ois on each pc of the multiplier pc set, resulting in lines of pcs. In Example 5 the production of the ascending line of Slonimsky's Pattern 196 is shown as $\underline{0–5–9} \otimes \langle 048 \rangle$ = 0–5–9–4–9–1–8–1–5, and the ascending line of Pattern 395 as $\underline{0–5} \otimes \langle 0369 \rangle$ = 0–5–3–8–6–e–9–2. (An appropriately ordered reading of the product in an additive matrix will display these lines. Further refinements of pitch and ordering would account for the octave repetition and line retrograde that characterize Slonimsky's patterns,

[13]Slonimsky, *Thesaurus of Scales and Melodic Patterns,* i.

[14]Rahn, *Basic Atonal Theory,* 139.

[15]Morris (in "Equivalence and Similarity in Pitch") again provides a similar formalization, but one based on the lowest pitch within a segment rather than the first-occurring. A pitch-class-based theory such as that presented here will trip over the "lowest pitch" requirement, especially when describing passages such as the first operation in the Stravinsky example cited below, in which an initial pc is realized in pitch space three times as the lowest pitch and twice as the highest.

Example 5. Slonimsky's Patterns Nos. 196 and 395 (*Thesaurus of Scales* © 1947, 1974)

but the essential qualities of his system are sufficiently described through the present schema.) Ostinatos and sequences very frequently conclude with an incomplete statement of the multiplicand line segment; this is indicated (in later examples) in integer notation by a parenthetical final element in the multiplier, in staff notation by a dashed beam.

As these patterns are lines, their characteristic use is melodic; in this context, their appearance becomes somewhat problematic. Multiplier replication is an obvious feature, particularly since the multipliers most extensively employed by Slonimsky, the interval-cyclic referential collections <06>, <048>, <0369>, <02468t>, and certain permutations thereof, are so familiar.[16] Since each multiplier is based on pitch class C, every pattern contains (and, moreover, begins and ends with) that pitch class, thus forming a C paratonality that is reinforced by the traditionally tonal associations of the sequential gesture.

Unlike the union of io set ois/pc constructions, line ois/pc constructions do not omit repetitions; Pattern 196, for instance, repeats pitch classes 9, 1, and 5. When repetitions result from otherwise nonrepetitive operands, their appearance can be predicted in the form of *pairings*. A single occurrence of a pitch class cannot be paired with itself. A twice-occurring pc, x_1 and x_2, forms one pairing, x_1/x_2. A thrice-occurring pc, x_1, x_2, and x_3, forms three pairings, x_1/x_2, x_1/x_3, and x_2/x_3. The number of pairings of *n* occurrences $= (n^2-2) \div 2$, so that four occurrences result in six pairings, five occurrences in ten pairings, and so forth. With one exception, an interval class that appears in both operands will result in one pairing in the product: 0–1 ⊗ <01> = 0–1–1–2, 0–2 ⊗ <02> = 0–2–2–4, and so on until the exception, ic 6, which results in two pairings: 0–6 ⊗ <06> = 0–6–6–0. By extension, pairings in line multiplication can be predicted via the interval vectors of the operands: *the sum of the arithmetical multiplicative products of the quantities of like in-*

[16]Equal division of the octave is inescapable for Slonimsky, who conceptualizes patterns as forming when pitches are temporally *inserted between* pitches of a "progression" (his term for the multiplier), not *constructed on* a pitch or pc. In pitch space, these insertions are either between progression pitches (called "interpolation"), below the lower progression pitch ("infrapolation"), or above the higher ("ultrapolation"); the intervallic consistency of such insertions depends upon the cyclic progressions. Pattern 196 exemplifies "ultrapolation of two notes" and Pattern 395 "ultrapolation of one note." (Many of his patterns involve combinations of these techniques and are described by lengthy hyphenates [such as "infra-inter-ultrapolation"], appealing, no doubt, to Slonimsky's propensity toward what he called "sesquipedalian macropolysyllabification.") Examples 6c–e below demonstrate the use of asymmetrical multipliers, a technique employed frequently by Boulez and well-explored by Cohn.

terval classes (with the product of ic 6 doubled) equals the number of pairings. Examples of the workings of the pairings theorem follow.

In Slonimsky's Pattern 196, the interval vector of the multiplicand 0–5–9 is <0,0,1,1,1,0> and the interval vector of the multiplier <048> is <0,0,0,3,0,0>:

$$\begin{array}{r} \langle 0,\ 0,\ 1,\ 1,\ 1,\ 0\rangle \\ \times\ \underline{\langle 0,\ 0,\ 0,\ 3,\ 0,\ 0\rangle} \\ 0+0+0+3+0+0\ =\ 3 \end{array}$$

Interval class 4 appears once in the multiplicand—i(5–9)—and thrice in the multiplier—i(0,4), i(4,8), and i(8,0)—resulting in three pairings, 1/1, 5/5, and 9/9.

Example 6 shows five operations that further demonstrate the pairings theorem as well as staff-notational conventions. Multiplicand pcs are joined by beams closer to the noteheads, while the more distant beams connect multiplier pcs. In Example 6a, 0–4–8 ⊗ <159> = 1–5–9–5–9–1–9–1–5 (each of the three pcs occurs three times—1/1/1, 5/5/5, 9/9/9—for a total of nine pairings):

$$\begin{array}{r} \langle 0,\ 0,\ 0,\ 3,\ 0,\ 0\rangle \\ \times\ \underline{\langle 0,\ 0,\ 0,\ 3,\ 0,\ 0\rangle} \\ 0+0+0+9+0+0\ =\ 9 \end{array}$$

Example 6b shows operands with no interval classes held in common. 0–1–3–6 ⊗ <048> = 0–1–3–6–4–5–7–t–8–9–e–2 (no pairings):

$$\begin{array}{r} \langle 1,\ 1,\ 2,\ 0,\ 1,\ 1\rangle \\ \times\ \underline{\langle 0,\ 0,\ 0,\ 3,\ 0,\ 0\rangle} \\ 0+0+0+0+0+0\ =\ 0 \end{array}$$

Example 6c shows operands with several interval classes held in common. 0–1–2–5 ⊗ <014> = 0–1–2–5–1–2–3–6–4–5–6–9 (four pairings: 1/1, 2/2, 5/5, 6/6):

$$\begin{array}{r} \langle 2,\ 1,\ 1,\ 1,\ 1,\ 0\rangle \\ \times\ \underline{\langle 1,\ 0,\ 1,\ 1,\ 0,\ 0\rangle} \\ 2+0+1+1+0+0\ =\ 4 \end{array}$$

As described above, the arithmetical multiplicative product of ic 6 must be doubled. Example 6d shows a tetrachord containing a tritone multiplied by itself (or "squared"). 0–1–5–7 ⊗ <0157> = 0–1–5–7–1–2–6–8–5–6–t–0–7–8–0–2 (nine pairings: 0/0/0, 1/1, 2/2, 5/5, 6/6, 7/7, 8/8):

$$\begin{array}{r} \langle 1,\ 1,\ 0,\ 1,\ 2,\ 1\rangle \\ \times\ \underline{\langle 1,\ 1,\ 0,\ 1,\ 2,\ 1\rangle} \\ 1+1+0+1+4+1\times 2\ =\ 1+1+0+1+4+2\ =\ 9 \end{array}$$

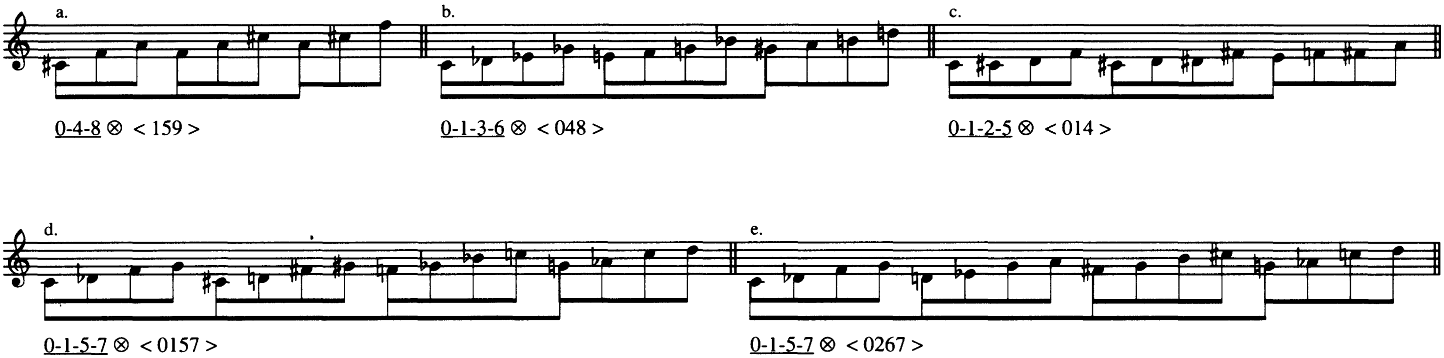
Just as other common-tone theorems[17] have their limitations, this one will predict neither the types of pairings (e.g., whether three pairings are realized as three twice-occurring pcs or as one thrice-occurring pc) nor, except as noted below, the cardinality of the unordered pc set comprising the union of resulting line pcs. Example 6e inverts the multiplier of the previous example, which does not change that operand's interval vector: 0–1–5–7 ⊗ <0267> = 0–1–5–7–2–3–7–9–6–7–e–1–7–8–0–2. Nine pairings still result (0/0, 1/1, 2/2, 7/7/7/7), but the cardinality of the resultant unordered set is greater by one.

The theorem does predict, however, that two operand sets A and B with no interval classes in common will produce a set with a cardinality equal to $|A| \times |B|$, since no pcs are duplicated. (This was demonstrated in Example 6b, a product cardinality of $4 \times 3 = 12$.) Further, when one pairing is predicted, it must be realized as one twice-occurring pc, so that $|A \otimes B| = (|A| \times |B|) - 1$; when two pairings are predicted, they must be realized as two twice-occurring pcs, so that $|A \otimes B| = (|A| \times |B|) - 2$. Conversely, when $|A| \times |B| > 12$ (as is the case when one tetrachord is multiplied by another), the operands must have at least one interval class in common.[18]

[17]For examples, see Rahn, *Basic Atonal Theory*, 97–123.

[18]This provides an explanation of the impossibility of a tetrachord lacking both ic 3 and ic 6, since [0369], the sole tetrachord limited to two interval classes, contains only those; the multiplication of [0369] by a hypothetical

Example 6. Other line multiplications



Another problematic area in line multiplication is that of the sequence itself: at what point does a pattern become overly predictable? Tonal practice would suggest that Pattern 196 be broken off after the seventh note, the start of the third statement of the sequence—the pattern starts to wear thin at the one-third point of its ascent. Atonal (or paratonal) practice is less clear on this issue; Slonimsky demonstrates that rhythmic variation within any pattern will, to a degree, compensate for sequential predictability.[19] Another solution is shown in Example 7: the upper line is a brief oboe passage from Witold Lutoslawski's *Concerto for Orchestra* (1954), and the lower line illustrates the two multiplicative operations 0–2–5–3 ⊗ <te1> = t–0–3–1–e–1–4–2–1–3–6–4 and 0–e–t ⊗ <174> = 1–0–e–7–6–5–4–3–2 that motivate the passage. In the analysis on the lower staff, these operations are beamed in the manner of Example 6. Lutoslawski breaks sequential predictability in four ways: through the juxtaposition of two different multiplicative operations in parallel symmetry; through the use of the asymmetrical multiplier <te1> in the first operation; through the multiplier's change of direction in the second operation; and through rhythmic variation of line segments in the second operation.

Despite the pitch-class-based nature of the line ois as formalized here, the Slonimsky and Lutoslawski examples are clearly realized in pitch space, as befits their sequential heritage. An increased complexity is discernible within Igor Stravinsky's *Symphony of Psalms* (1930), manifested in a wealth of plausible interpretations. The opening fugue subject of the second movement, shown in Example 8, can be shown to consist of two multiplicative operations. The first

tetrachord devoid of ic 3 and 6 would result in a product with sixteen different pitch classes. There are instances of every other possible pair of interval classes missing from at least one tetrachordal set class.

[19]Slonimsky, *Thesaurus,* iv. Any future studies of the possibilities and problems of Slonimsky's work might profitably inspect free-jazz improvisations recorded since the *Thesaurus* achieved unexpected popularity with performers influenced by saxophone virtuoso John Coltrane, who used it as a practice book. (See J. C. Thomas, *Chasin' The Trane* [New York: Da Capo, 1976], 102.) Similarly influential has been Oliver Nelson's *Patterns: An Aid to Improvisation* (Hollywood, Calif.: Noslen, 1966), which presents chromatically sequential material (albeit less extensively or systematically than does the *Thesaurus*) that is equally amenable to the theory presented here.

Example 7. Lutoslawski, *Concerto for Orchestra*, mm. 471–478, oboes; analysis of line multiplication

of these employs the line ois 0–3–e–2 (established in the first movement in various realizations[20]) multiplied by a repeated pc C which, after an opening C5, alternates between C5 and C6—a simple yet effective disguise of an isomelos. The other pitches in this first operation do not change octaves, but the successive statements of the multiplicand are varied rhythmically. The second operation, more obviously based in pitch space but disguised by ingenious rhythmic displacement, rotates the multiplicand of the first to the equivalent 0–8–e–9,[21] and multiplies it by a descending-wholetone tetrachord, <31e9>.

Another analysis of Stravinsky's subject is shown in Example 9. In contrast to the previous analysis's note-by-note

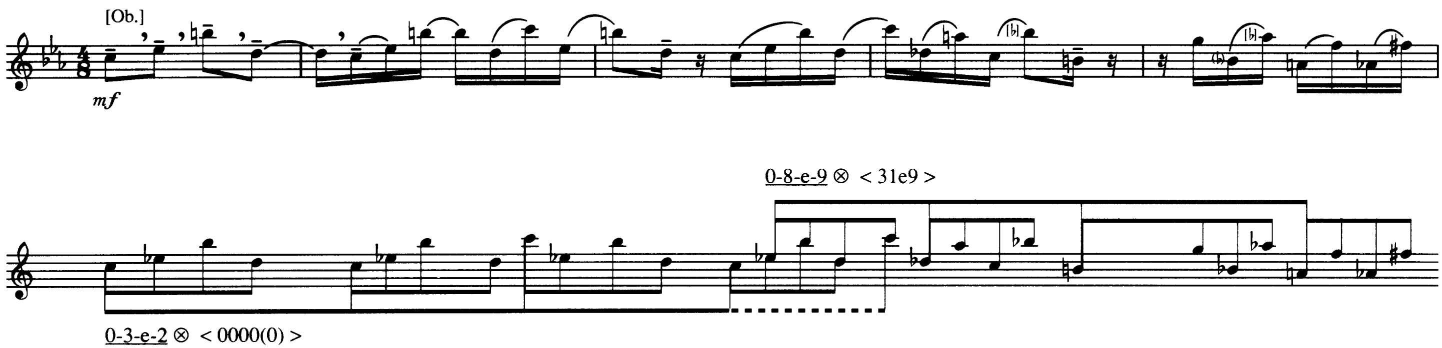
[20]According to Stravinsky, "The subject of the fugue was developed from the sequence of thirds already introduced as an ostinato in the first movement," a figure previously described as "the root idea of the whole symphony." Igor Stravinsky and Robert Craft, "A Quintet of Dialogues," *Perspectives of New Music* 1/1 (1962): 16. Statements of this ostinato in the first movement occur most prominently at rehearsal numbers 4 (oboes and English horn), 7 (oboes and English horn, with cellos and basses in augmented rhythm a tritone away, completing an octatonic collection), and 12+3 (like 7, but harp replaces double reeds which move to 0–3–2–5, the other octatonic tetrachordal segment in a rotated similar shape). Given its "root idea" status, the motivic line ois 0–3–e–2 is curiously absent from the third movement of the *Symphony of Psalms* (the first-composed of the three), but the composer regarded the ois as having been "derived from the trumpet-harp motive at the beginning of the *allegro*" of the third movement (ibid.), an ois of 0–3–1–5, equivalent to 0–4–e–2; see note 21. The significance of both motives' set classes [0134] and [0135] to other Stravinsky works is explore[illegible] Joseph N. Straus, "A Principle of Voice Leading in the Music of Stravinsky," *Music Theory Spectrum* 4 (1982): 106–24 passim.

[21]Equivalent oiss are those which, when constructed on a pc, produce sets of the same T_n-type. They are calculated by subtracting each ois integer from all ois integers (see Heinemann, "Pitch-Class Set Multiplication in Boulez," 29–30); for example, all the oiss equivalent to 023e, in their normal forms, are:

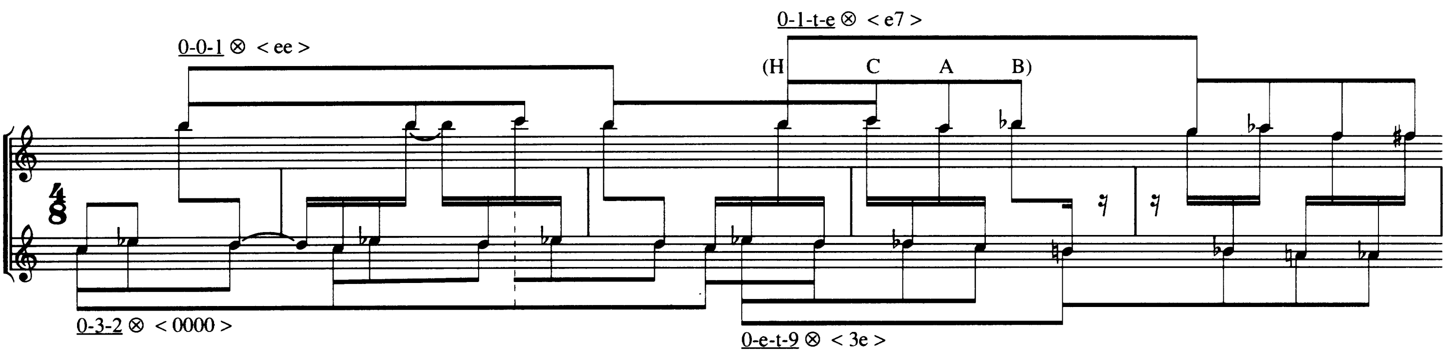
023e − 0 = 023e;
023e − 2 = 019t;
023e − 3 = 089e;
023e − e = 0134.

In any such listing, one of the equivalent oiss (or more than one, in the case of transpositionally symmetrical sets) will match the integer content of the common T_n-type—here (0134).

Example 8. Stravinsky, *Symphony of Psalms,* II, mm. 1–5; analysis of line multiplication



Example 9. Stravinsky, *Symphony of Psalms,* II, mm. 1–5; analysis of compound-melodic line multiplication



approach, this one is influenced by pitch register and regards the subject as a compound melody in which four separate line multiplications are found.[22] These operations are shown with the analytical beams and stems above the upper staff and is preserved between the staves. In both the upper and lower below the lower, again with the line oiss beamed closer to and the multiplier further from the noteheads; the original rhythm voices, the first and second operations overlap. Especially evocative here is the upper voice's second multiplication, where the operation <u>0–1–t–e</u> ⊗ <e7> yields the B-A-C-H motive in retrograde—first at "signature" pitch, then transposed.

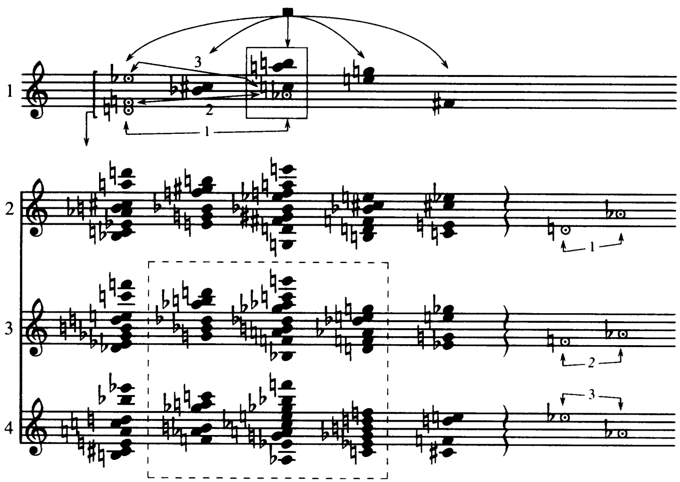
COMPOUND MULTIPLICATION

The second type of operation, compound multiplication, will be discussed only briefly because it appears to represent an intermediate stage in Boulez's thinking and is not particularly important to *Le Marteau.* This operation differs from simple multiplication in that the simple multiplicative product is transposed according to some schema. The example that Boulez gives of this is shown here as Example 10, which he describes as follows:

> If the ensemble of all the complexes [pc sets in staff 1] is multiplied by a given complex [the boxed pc set in staff 1], this will result in a series of complexes of mobile density [i.e., cardinality], of which, in addition, certain constituents will be irregularly reducible; although *multiple* and *variable,* these complexes are deduced from one another in the most functional way possible, in that they obey a logical, coherent structure.[23]

[22] This compound-melodic subject has also been presented as an example of octave displacement, a "modernization" of a more traditionally narrow-ranged melody. See, for example, Harold Owen, *Modal and Tonal Counterpoint* (New York: Schirmer, 1992), 346. Such an interpretation could be supported by the closely spaced treatment of the subject in the second flute, mm. 16–17, but this is easily accounted for: the lower melody (as analyzed here) dips below the range of the flute and is accordingly raised an octave. Owen observes that "much of the character would be lost" in narrowing the contour.

Example 10. Compound multiplication in *Boulez on Music Today*



There are several different ways in which this figure can be interpreted, but, in any case, initial ordering, conceptually problematic with just one operand, is now required of both. The initial pitch classes of two of the sets are represented here as invented note-heads—circles with dots inside. Transposition is determined by intervals between these initial pitch classes, as shown at the right of the sketch. The advantage of this operation is that multiplier replication is no longer an

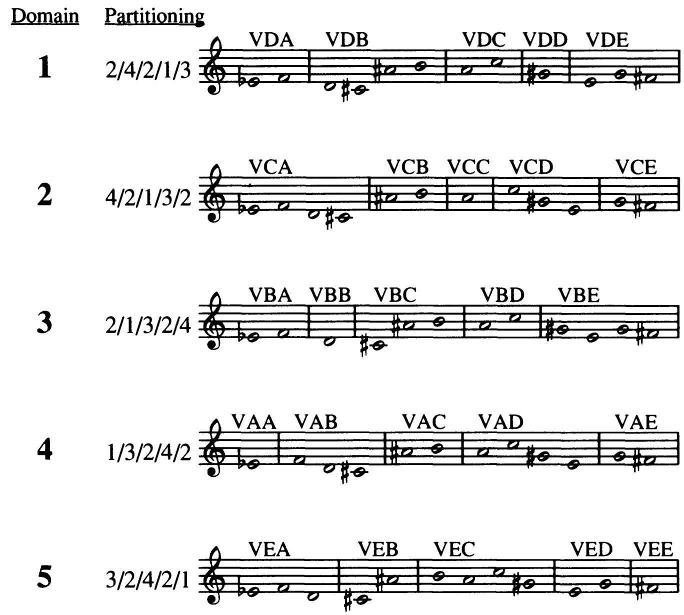
[23] Boulez, *Boulez on Music Today,* 39–40. Sets inside the dashed box in Example 10 are implied but not literally given by Boulez in the original, and the staff enumerations have been added here for reference purposes. Example 10 is discussed in greater detail in Heinemann, "Pitch-Class Set Multiplication in Boulez," 50–56.

issue; the multiplier, shown in this example as a boxed set in staff 1, is not necessarily a subset of the product. However, other pitfalls of simple multiplication remain: the operation is noncommutative; variety of products is evident from the different staves, and there is no rationale for preferring a product in one staff to a product in another; and initial ordering, problematic with only one operand, is now required of both. Regardless of these problems, the sketch in Example 10 was probably vital to Koblyakov's detective work: staff 1 consists of sets partitioned from the row that is the basis for the first cycle of *Le Marteau,* while staff 2 contains multiplicative products that appear in the composition itself, as will be exemplified presently.

COMPLEX MULTIPLICATION

As a prelude to discussing complex multiplication, it is necessary to delineate some organizational principles in the first cycle of *Le Marteau sans maître* as distilled and modified from Koblyakov's writings. The cycle is based on the twelve-tone row 3521te908476 that is partitioned into five subsets with cardinalities of, in order, 2/4/2/1/3. This cardinality series is rotated while the row itself remains constant, and each such rotation splits the row into different subsets. The process of partitioning by means of the rotating cardinality series is illustrated in Example 11. The partitioned subsets are called *V-sets* and are represented generally by the three-letter designation VXY: the V simply indicates that the set is a V-set;[24] X represents the first through fifth position of the monad in partitioning, and thus varies depending on the rotation of the

Example 11. The *Marteau* row; derivation of V-sets



cardinality series; Y represents the first through fifth position of the V-set within the cardinality series. V-sets are employed to generate *pitch-class domains,* precompositionally determined groups of pc sets from which is drawn the pitch-class material of the composition. The five rotations of the cardinality series each produce V-sets that in turn generate five domains: V-sets from the original series generate Domain 1, V-sets from the next rotation generate Domain 2, and so forth. Within the VXY designation, X and/or Y can be made specific by replacing either or both with letters A through E; for instance, as seen in Example 11, VXA is the first V-set in any domain (since Y = A, the first position in the cardinality series), VCY is any V-set in Domain 2 (since X =

[24]The "V" designation follows Koblyakov, "Analysis of pitch structure." He changes the designation of V-sets to a bracketing system in *Pierre Boulez: a World of Harmony,* but this change appears to be based on the incorrect notion that transpositions of V-sets supersede V-set function in *Le Marteau.* See ensuing discussion and Heinemann, "Pitch-Class Set Multiplication in Boulez," 63.

Example 12. Domain matrix (adapted from *Boulez on Music Today*)

VXA	VXB	VXC	VXD	VXE
AA	*AB*	*AC*	*AD*	*AE*
BA	BB	BC	BD	BE
CA	CB	CC	CD	CE
DA	DB	DC	DD	DE
EA	EB	EC	ED	EE

C, the position of the monad in the 4/2/1/3/2 rotation), and VCA refers specifically to the first V-set in Domain 2. The generalized structure of a domain is shown in Example 12, wherein V-sets are shown at the top of the matrix. The sets below these, each represented by a pair of letters, are called *domain sets;* these sets are in fact products formed by the multiplication of V-sets. Their two-letter designations are taken from the Y of each of the two VXY sets that generate them: for example, domain set DC is produced by the multiplicative operation VXD ⊗ VXC. The five actual pitch-class domains themselves are shown in Example 13. The realization of these sets in the musical surface can be traced through the domain matrix in a fairly systematic way that has been thoroughly explained by Koblyakov. The italicized sets—sets AA through AE in the top row of the domain matrix—are not used in the composition, for reasons explored below.

I contend that all of the domain sets in these pitch-class domains are generated by the V-sets by means of *complex multiplication.* This position differs significantly from that of Koblyakov, who believes that domain sets are generated from *transpositions* of the V-sets according to some operation he does not describe. His transposed V-sets do occur; they can be seen in each domain in the row that has the same A through E identification as the X in each VXY—for instance, in Domain 1, field D is a transposition of the Domain 1 VDY sets. However, these "transposed V-sets" are actually themselves products of complex multiplication.

The delineation of Example 10 and its compound multiplicative method introduced a new consideration: the possibility that a pc within a multiplicand can have importance beyond its ois-producing function. In Example 10, pc 8, the initial pc of the multiplicand, was paired with each element of {235} to determine transposition levels for each set in lines 2 through 4. An extension of this consideration might include the use of a transposition-determining pc which remains constant and a part of the operation *even if it is not an element of either operand.* Not coincidentally, this is exactly the "missing link" in the theory of multiplication, the crucial component that disposes of every problematic area of simple and compound multiplication. This component is called the *transposition-determining constant*—a pitch class, chosen according to some schema, that participates in a multiplicative operation. The transposition-determining constant, designated k, may or may not be an element of the multiplicand, multiplier, or product.

Complex multiplication combines the techniques of simple and compound multiplication into an operation that is elegant in its economy and constancy. This operation takes a product of simple multiplication, here called "simple AB," and transposes it via a method that seems likely to have evolved from compound multiplication. The formula for complex multiplication follows.

Where A and B are pitch-class sets, r is the initial pitch class of A, and k is the transposition-determining constant:

$A \otimes B = \text{ois}(A) \otimes B = \text{simple } AB$; $T_n(\text{simple } AB) = AB$, where $n = i\langle k,r \rangle$.

Example 13. Pitch-class domains for the first cycle of *Le Marteau sans maître*

(C = 0)

1

	A	B	C	D	E
VD	{35}	{te12}	{90}	{8}	{467}
A	*{135}*	*{89te012}*	*{79t0}*	*{68}*	*{24567}*
B	{89te012}	{3456789te}	{235689}	{1245}	{9te01234}
C	{79t0}	{235689}	{147}	{03}	{8te12}
D	{68}	{1245}	{03}	{e}	{79t}
E	{24567}	{9te01234}	{8te12}	{79t}	{356789}

2

	A	B	C	D	E
VC	{1235}	{te}	{9}	{048}	{67}
A	*{01234568}*	*{9te012}*	*{89t0}*	*{345789e01}*	*{56789t}*
B	{9te012}	{678}	{56}	{014589}	{234}
C	{89t0}	{56}	{4}	{37e}	{12}
D	{345789e01}	{014589}	{37e}	{26t}	{014589}
E	{56789t}	{234}	{12}	{014589}	{te0}

3

	A	B	C	D	E
VB	{35}	{2}	{te1}	{90}	{4678}
A	*{135}*	*{02}*	*{89te1}*	*{79t0}*	*{245678}*
B	{02}	{e}	{78t}	{69}	{1345}
C	{89t1}	{78t}	{345679}	{23568}	{9te01234}
D	{79t0}	{69}	{23568}	{147}	{8te0123}
E	{245678}	{1345}	{9te01234}	{8te0123}	{356789te}

4

	A	B	C	D	E
VA	{3}	{125}	{te}	{8904}	{67}
A	*{3}*	*{125}*	*{te}*	*{8904}*	*{67}*
B	{125}	{e01347}	{89t01}	{678te23}	{4,5,6,8,9}
C	{te}	{89t01}	{567}	{34578e0}	{1,2,3}
D	{8904}	{678te23}	{34578e0}	{123569t}	{e013478}
E	{67}	{45689}	{123}	{e013478}	{9te}

5

	A	B	C	D	E
VE	{235}	{t1}	{89e0}	{47}	{6}
A	*{234568}*	*{te124}*	*{89te0123}*	*{4578t}*	*{679}*
B	{te124}	{690}	{4578te}	{036}	{25}
C	{89te0123}	{4578te}	{23456789t}	{te1245}	{0134}
D	{4578t}	{036}	{te1245}	{690}	{8e}
E	{679}	{25}	{0134}	{8e}	{t}

The operation is thus a two-step process: first, the simple multiplication of the multiplicand ois by the multiplier, as described previously; and second, the transposition of the simple multiplicative product by the ordered pitch-class interval from the transposition-determining constant to the initial pitch class of the multiplicand.

Example 14 illustrates complex multiplication and, for sake of comparison, employs the same operands as in Example 4. Any pitch class could have been chosen as the transposition-determining constant; F is used here. The transposition-determining constant is shown first, followed by the ordered pitch-class interval from k to r; next is r, followed by the simple multiplication. Finally, the simple multiplicative product is transposed by the calculated interval to become the complex multiplicative product. An examination of Example 14 will demonstrate that, given two operand pc sets and a transposition-determining constant, only one product is possible. This operation is commutative. Unlike in simple multiplication, the pitch-class content of the multiplicand is now as vital as its ois, since the initial pc is a part of the transposition determination. Initial ordering of the multiplicand remains vital to the operation, but exactly which initial ordering is chosen is irrelevant, thus rendering moot the necessity for selection criteria. As in compound multiplication, except as noted below, multiplier replication is not an issue; in fact, in Example 14, the operands and the product do not intersect at all. In each operation shown, the transposition changes for the simple multiplicative product to result in the sole complex multiplicative product, with the transposition-determining constant acting as catalyst—not an element of the operands, but allowing the operation to take place.

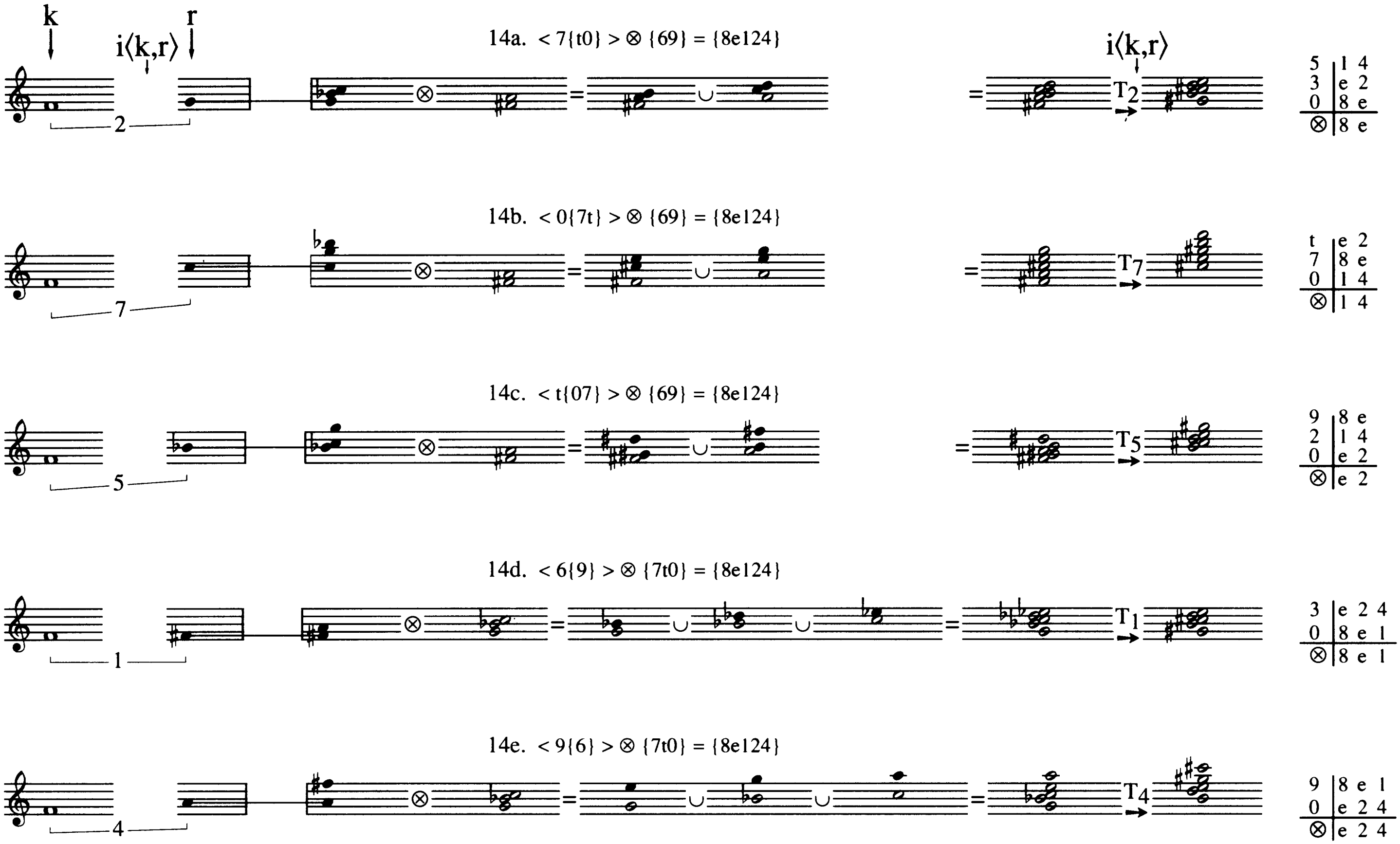
In Example 14, the integer matrices to the right of the staff notations use transpositions of the multiplier. These matrices demonstrate the complex multiplication corollary: $A \otimes B = \text{ois}(A) \otimes T_n(B) = AB$ where $n = i<k,r>$; the multiplier can be transposed by the calculated interval prior to simple multiplication in order to arrive at the same product.

The transposition-determining constant can be chosen by any means the composer deems appropriate. In *Le Marteau,* a single constant governs each domain, chosen according to this criterion: the transposition-determining constant is the rightmost of the first three pcs of the row appearing in set VXA, the first V-set partitioned from the row in each domain. Thus (as can be inferred from Example 11) in Domains 1 and 3, k equals F; in Domains 2 and 5, k equals D; in Domain 4, k equals E♭.

There is a significant parallel between simple multiplication and complex multiplication. In the former operation, there is multiplier replication—the property that the multiplier is always a subset of the product. In complex multiplication, there is the property of *k-set co-operand replication:* if one operand is a "k-set"—that is, has k as an element—the other operand will be a subset of the product. (Because complex multiplication is commutative, either the multiplicand or the multiplier may be a k-set—hence the use of the term "co-operand" here.) This property provides a reasonable explanation for Boulez's jettisoning of field A in every domain. Using the V-sets along with the domain sets establishes a literal connection with the original row and is thus conceptually sound. Since k is an element of each set VXA, these V-sets are necessarily subsets of the corresponding field A domain sets; eliminating the domain sets avoids unequal weighting of the V-set pitch classes.

Simple line multiplication produces inherently ordered pc sets. In contrast, the five complex multiplicative operations shown in Example 14 show four different orderings of the same product pcs. If the operand spacings are changed, still other orderings are possible. Each product pc can thus be shown in any order relative to every other product pc. Each operation validates the others, and none deserves any

Example 14. Complex multiplication: where k = 5, {7t0} ⊗ {69} = {8e124}



preference. Complex multiplicative products are inherently *unordered* sets by nature of their generation, and this may have been one of the operation's most appealing characteristics to Boulez. It is even possible that his compositional move toward what he termed "*local indiscipline* . . . a freedom to choose, to decide and to reject"[25] was a direct result of this operation.

The assumption has been implicit, but is, in fact, Boulez's own process accurately portrayed here? He alone could say with complete certainty, but it should be noted that, given the method for determining V-sets, the complex multiplication operation described herein, and the criterion for deriving the transposition-determining constant, *the domain sets shown in Example 13 are the only possible products;* the correspondence is complete. It has also been demonstrated that the process can be carried out using tools available to Boulez, which would have included staff notation but presumably not integer notation. Similarly, while the process is *complex,* it is not especially *complicated*—that is part of its elegance. Unfortunately, Boulez's sketches for *Le Marteau* have been lost—we must look elsewhere for further confirmation.

Example 15 shows a sketch from the Paul Sacher Foundation's Pierre Boulez Collection; a reproduction of this page appears in the tenth-anniversary *Festschrift* of the Arnold Schoenberg Institute, *From Pierrot to Marteau.*[26] Serving a precompositional function in the manner of a traditional row-table, the sketch has the same general appearance as Koblyakov's representations of the domains from *Le Marteau,* and was identified in the Schoenberg Institute publication as being from that work. However, according to Koblyakov, it is actually from a choral piece composed two years earlier than *Le Marteau* (and subsequently withdrawn) entitled *Oubli signal lapidé.*[27] These pitch-class domains are shown in integer notation in Example 16. An examination of these domains reveals many parallels with the theory already presented, including but not limited to the following. First, the generating row is the inversion of the *Marteau* row. Second, the cardinality sequence partitioning this row is the retrograde of that of *Le Marteau.* Third, the A field is again replaced by V-sets. Fourth, the transposition-determining constant is now the *leftmost* pc of set VXA—in other words, k equals E♭ throughout. Fifth, while the procedure for selecting k has been altered slightly, the complex multiplication operation itself is exactly the same as that described for *Le Marteau,* under which only these domain sets will result; the correspondence is again complete.

By focusing on individual pitch classes within each operand, we can prove the assertions that complex multiplication is commutative and generates a single product. Recall that, in simple multiplication, $a \otimes b = (a - r) + b$. In complex multiplication, this product is transposed by the ordered pitch-class interval from k to r. Therefore, in complex multiplication,

$$\begin{aligned} a \otimes b &= T_{i<k,r>}(\text{simple ab}) \\ &= T_{(r-k)}((a-r)+b) \\ &= ((a-r)+b)+(r-k) \\ &= ((a+b)-k)+(r-r) \\ &= a+b-k. \end{aligned}$$

The algebraic simplification of the complex multiplication of pitch classes to the equation $a \otimes b = a + b - k$ might dilute the operation's drama but does demonstrate its elegance. (It also provides a convenient shortcut for calculation—we can

[25] Boulez, *Conversations with Cèlestin Deliège* (London: Eulenberg Books, 1976), 66.

[26] Arnold Schoenberg Institute, *From Pierrot to Marteau* (Los Angeles: Arnold Schoenberg Institute, 1990), 21, reprinted here by permission of the Paul Sacher Foundation. Treble clef is implied in the sketch.

[27] Koblyakov to author, Oct. 18, 1990. His talks with Boulez and access to the Paul Sacher Foundation's holdings have convinced him that all sketch materials for *Le Marteau* have been lost.

Example 15. Boulez's notation of pitch-class domains for *Oubli signal lapidé*

Example 16. Pitch-class domains for *Oubli signal lapidé*

(C = 0)

1

	A	B	C	D	E
VB	{134}	{5}	{78}	{69t2}	{e0}
A	*{e12345}*	*{356}*	*{56789}*	*{6789te0234}*	*{9te01}*
B	{356}	{7}	{9t}	{8e04}	{12}
C	{56789}	{9t}	{e01}	{te12367}	{345}
D	{6789te0234}	{8e04}	{te12367}	{0134589}	{23567te}
E	{9te01}	{12}	{345}	{23567te}	{789}

2

	A	B	C	D	E
VA	{3}	{14}	{5789}	{6t}	{e02}
A	*{3}*	*{14}*	*{5789}*	*{6t}*	*{e02}*
B	{14}	{e25}	{356789t}*	{478e}	{9t013}
C	{5789}	{356789t}	{79te0123}	{8te0234}	{12345678}
D	{6t}	{478e}	{8te0234}	{159}	{235679}
E	{e02}	{9t013}	{12345678}	{235679}	{789te1}

* shown as {456789t} in sketch

3

	A	B	C	D	E
VE	{13}	{4578}	{69}	{te2}	{0}
A	*{e13}*	*{2345678}*	*{4679}*	*{89te02}*	*{t0}*
B	{2345678}	{56789te01}	{78te12}	{e0123467}	{1245}
C	{4679}	{78te12}	{903}	{12458}	{36}
D	{89te02}	{e0123467}	{12458}	{5679t1}	{78e}
E	{t0}	{1245}	{36}	{78e}	{9}

4

	A	B	C	D	E
VD	{1345}	{78}	{69t}	{2}	{e0}
A	*{e1234567}*	*{56789t}*	*{46789te0}*	*{0234}*	*{9te012}*
B	{56789t}	{e01}	{te123}	{67}	{345}
C	{46789te0}	{te123}	{901345}	{589}	{23567}
D	{0234}	{67}	{589}	{1}	{te}
E	{9te012}	{345}	{23567}	{te}	{789}

5

	A	B	C	D	E
VC	{13}	{458}	{7}	{69}	{te02}
A	*{e13}*	*{234568}*	*{57}*	*{4679}*	*{89te02}*
B	{234568}	{5679t1}†	{890}	{78te2}	{e0123457}
C	{57}	{890}	{e}	{t1}	{2346}
D	{4679}	{78te2}	{t1}	{903}	{1234568}
E	{89te02}	{e0123457}	{2346}	{1234568}	{56789te1}

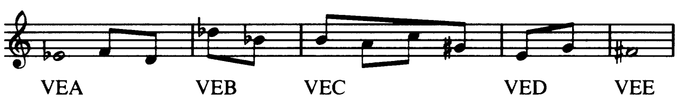
† shown as {35679te0} in sketch

simply add the operand integers and subtract k.) The only variable in the operation, the initial pc of the multiplicand, has canceled itself out, leaving just the constants—the operand pitch classes and the transposition-determining constant.

COMPLEX MULTIPLICATION AND THE MUSICAL SURFACE

An examination of the domain sets and their realization in the first movement of *Le Marteau* will reveal the outlines of a process-based listening strategy. Two sections of the movement use Domain 5, generated by the V-sets shown in Example 17. Occurrences of interval class 3 are prominent here, as their beaming demonstrates. Since multiplication preserves the unordered intervals of each operand, we can expect interval class 3 to be as prominent in the domain sets, and so it is; see the complete representation of Domain 5 in Example 18, where interval-class 3 relations are bracketed. The pitch classes that are isolated in this regard are shown as open noteheads. (The V-sets and field C in Domain 5 also appear in Example 10 above as, respectively, staves 1 and 2.) Furthermore, all except two of these domain sets are also subsets of the octatonic collection—or, to be more specific, of one or two of the three octatonic pitch-class collections. The roman numerals in Example 18 refer to these collections as labeled by Pieter van den Toorn, shown in Example 19.[28] Moreover, these subsets are either of the T_n/T_nI types [03]

Example 17. *Le Marteau:* Interval class 3 in Domain 5 V-sets



or [036], or are *segments* of an octatonic *scale;* as such, they are easily perceptible.

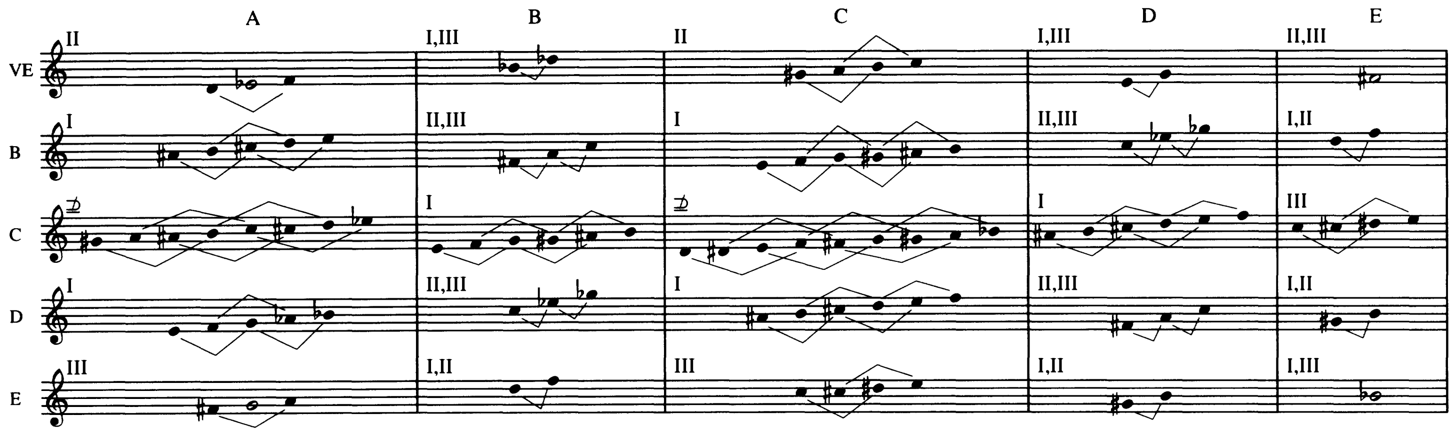
Well-documented octatonicism in Stravinsky provides an introduction to an analytical method that can be applied to certain passages in Boulez despite the obvious dissimilarities in musical surfaces. Example 20 shows another approach to the *Symphony of Psalms* fugue subject, one that emphasizes the interval-class 3 content and octatonic subset attribute of the motivic set class [0134]. Here, the segmentation of sets is indicated by barlines, and rhythms are preserved graphically by relating pitches to the evenly-spaced eighth-notes below each system, where measure numbers will also be found. The two mutually interactive listening strategies are illustrated here: interval-class 3 is shown with beams as in Example 17, and the octatonic analysis is shown above each system by a thick line indicating the significant appearance of a collection. The compound operation $T_{(t+t+t)}\{e023\} = T_6\{e023\} = \{5689\}$ results in the completion of Collection II.

Example 21 is a two-staff reduction of the first movement of *Le Marteau sans maître,* mm. 11–20. The notational conventions of Example 20 obtain here. Only the interval-class 3 relationships within domain sets are given, but others are easily found—for instance, F♯5 in the fourth set is repeated in the fifth, where it is associated with A3. The octatonic analysis is more fluid than in the Stravinsky, because a change of sets does not necessarily indicate a change of collection. Collection III can be safely ignored as an important organizing factor. The realization of set CC in m. 15 is particularly evocative: this chromatic nonachord is the only set employed

[28]Pieter van den Toorn, *The Music of Igor Stravinsky* (New Haven: Yale University Press, 1983), and *Stravinsky and* The Rite of Spring: *The Beginnings of a Musical Language* (Berkeley: University of California Press, 1987). Since van den Toorn does not describe how he chose this taxonomy, it is interesting to note that Collections I, II, and III can be constructed respectively on pcs 1, 2, and 3 with an ois that duplicates the T_n-type:

Collection I: $\underline{0134679t} \otimes 1 = \{124578te\}$
Collection II: $\underline{0134679t} \otimes 2 = \{235689e0\}$
Collection III: $\underline{0134679t} \otimes 3 = \{34679t01\}$

Example 18. *Le Marteau:* Interval class 3 in Domain 5, staff notation, with analytical markings

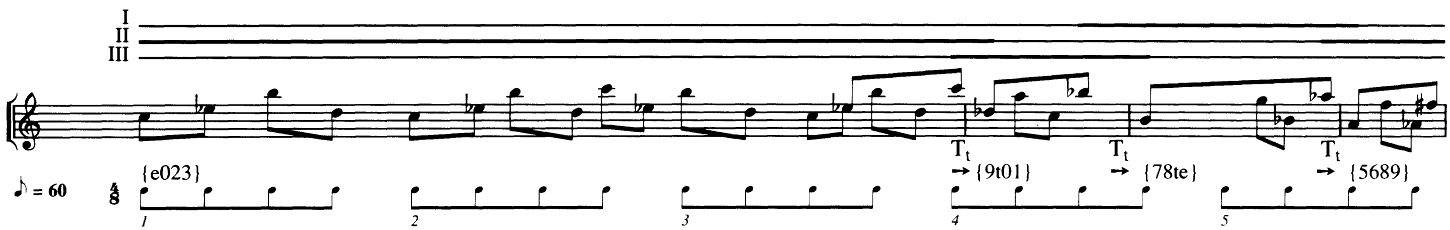


Example 19. Octatonic pitch-class collections (van den Toorn's taxonomy)

here which is not an octatonic subset, yet it is composed in such a way that it continues the previously-established Collection I for its first four pitches; when the foreign E♭ is introduced, the remaining five pitches are all contained in Collection II. The sets appearing here which are shown as Collection III subsets are composed in a similar manner. Koblyakov's matrix tracing of this passage—an analytical technique reminiscent of twelve-counting—is shown in Example 22. Examples 23 and 24 provide the same kinds of analysis of mm. 53–60; these measures are even more straightforward analytically than those of Example 21. It should be noted, however, that the other domains do not parse as easily as Domain 5 and that such an analytical approach is not without its obstacles. The aural "processing" in terms of interval-class 3 and octatonic structuring is complicated by tonal and neotonal associations in the Stravinsky and by the sheer rapidity of change in the Boulez. Nevertheless, these analyses are firmly rooted in the pitch-class content of each passage and constitute a natural outgrowth of the processes of their composition.

Boulez chose to use multiplication in other works of this era; perhaps now this choice is understandable. Complex multiplication is an elegant, logical extension of the serialist tradition that provides the composer with a great deal of flexibility at the local level. The resulting pitches are interrelated, but row-counting as a listening strategy is completely

Example 20. Stravinsky, *Symphony of Psalms,* II, mm. 1–5; octatonic/ic-3 analysis



inappropriate. The technique is individualistic, and Boulez was obviously no proselytizer for it. His work thus invites us to take it on its own terms—yet we can meaningfully bring to it our own experience as listeners. As such, it represents the best aims of the most significant music of our recent past.

Composer-percussionist William Kraft, who performed in the American premiere of *Le Marteau sans maître* (11 March 1957 in Los Angeles, Boulez conducting), wrote of the experience:

> The newness and originality of Boulez's style, and the consequent challenge it placed on the average listener, prompted me to ask him if he was at all concerned about public acceptance. His answer was, "Yes." I went on, "How long do you think it will take?" to which he replied, "Eighty years."[29]

Given the current state of "public acceptance," even this estimate might seem overly optimistic. But we have, as of this writing, just passed its midpoint. Composers who challenge our cognitive capabilities and composers who stay within the putative limits of these capabilities have led an often uneasy coexistence throughout the history of Western art music; perhaps the irreconcilability itself has incubated both the progression and refinement of the vocabularies of musical expression.[30] The reach has often exceeded the grasp but is all the more valuable for that; through this process of continual redefinition, the grasp has grown, not able to collect everything but better able to deal with that which has been collected.

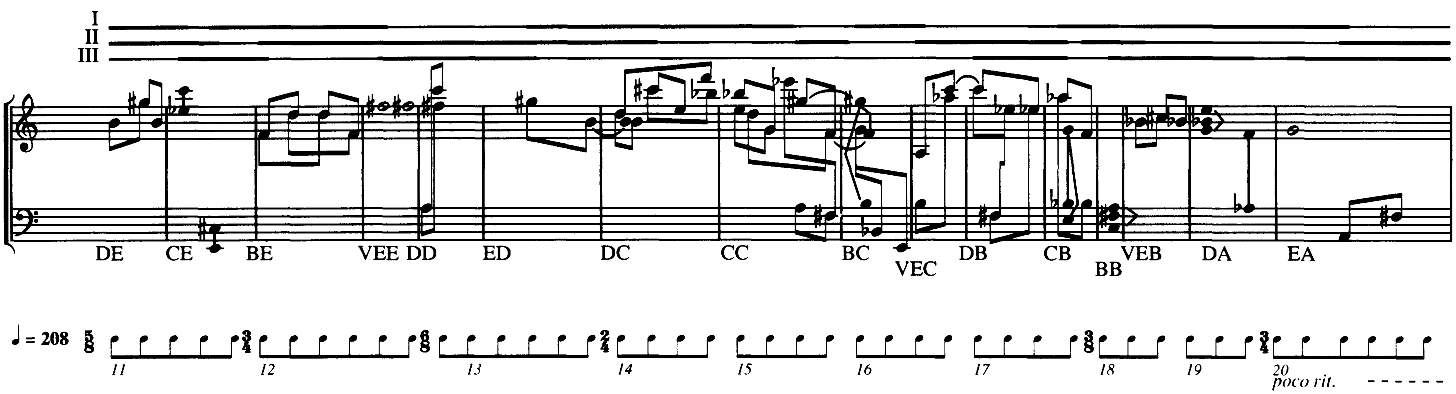
ABSTRACT

This article examines pitch-class set multiplication in its various manifestations, formalized as simple, compound, and complex multiplication. Simple multiplicative techniques are related to their origins in sequence and ostinato, and different technical approaches by Slonimsky, Lutoslawski, and Stravinsky are exemplified. Complex multiplication is posited as an elegant, uniquely Boulezian technique for generating unordered pitch-class sets. An analysis of sets within *Le Marteau sans maître* demonstrates process-based listening strategies.

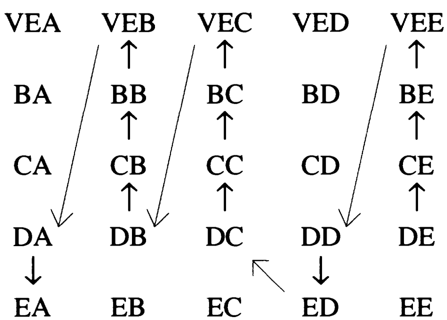
[29]*From Pierrot to Marteau,* 55–57.

[30]These issues are confronted in detail with respect to *Le Marteau* by Fred Lerdahl, "Cognitive Constraints on Compositional Systems," in *Generative Processes in Music: The Psychology of Performance, Improvisation, and Composition,* ed. John Sloboda (Oxford: Oxford University Press, 1988), 231–259, and by Ciro G. Scotto, "Can Non-Tonal Systems Support Music as Richly as the Tonal System?" (D.M.A. diss., University of Washington, 1995), especially in Chapter 1.

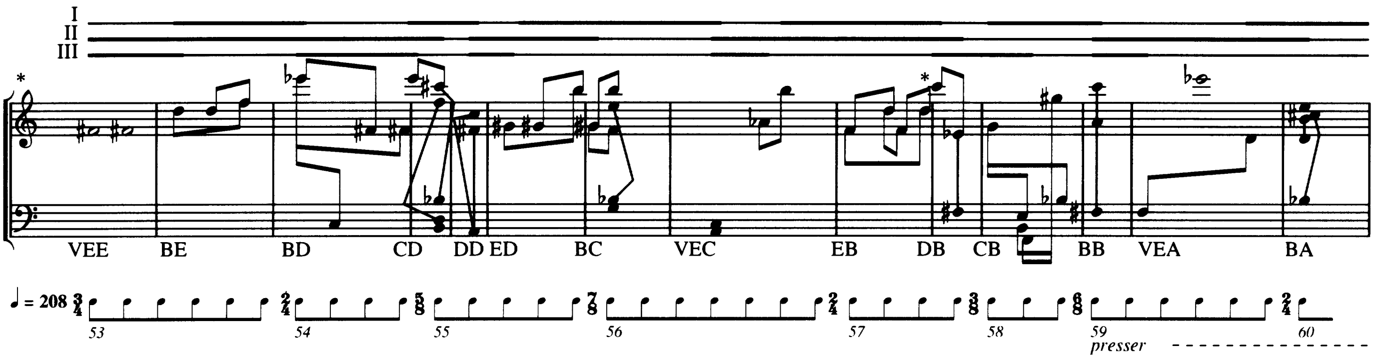
Example 21. *Le Marteau sans maître:* first movement, mm. 11–20
Accidentals apply only to notes they immediately precede.



Example 22. Koblyakov's matrix tracing of mm. 11–20 (*Pierre Boulez: a World of Harmony*); original uses lower-case letters



Example 23. *Le Marteau sans maître:* first movement, mm. 53–60
Accidentals apply only to notes they immediately precede.



*D5 shown as E5 in score

Example 24. Koblyakov's matrix tracing of mm. 53–60 (*Pierre Boulez: a World of Harmony*); original uses lower-case letters