Barry Truax, Tutorial for Frequency Modulation Synthesis[1]

*Note: this tutorial with the appropriate **sound examples** is available on the* Handbook for Acoustic Ecology *which is also included in the 2nd edition of* Acoustic Communication

Frequency Modulation (or FM) synthesis is a simple and powerful method for creating and controlling complex spectra, introduced by John Chowning of Stanford University around 1973. In its simplest form it involves a sine wave carrier whose instantaneous frequency is varied, i.e. modulated, according to the waveform (assumed here to be another sine wave) of the so-called modulator. This model then is often called simple FM or sine-wave FM. Other forms of FM are extensions of the basic model. The systematic properties of FM were used to compose Barry Truax's tape solo works *Arras*, *Androgyny*, *Wave Edge*, *Solar Ellipse*, *Sonic Landscape No. 3*, and Tape VII from *Gilgamesh,* as well as those involving live performers or graphics (*Aerial*, *Love Songs*, *Divan*, *Sonic Landscape No. 4*).

When the frequency of the modulator (which we'll call M) is in the sub-audio range (1-20 Hz), we can hear siren-like changes in pitch of the carrier. However, when we raise M to the audio range (above 30 Hz) then we hear a new timbre composed of frequencies called **SIDEBANDS**. To determine which sidebands are present, we have to control the ratio between the carrier frequency (C) and the modulating frequency (M). Instead of dealing with these frequencies in Hz, we'll refer to this relationship as the **C : M RATIO**, keeping C and M as integers.

Properties of C:M Ratios

First, about the properties of ratios, and some conventions we're using:

We will only deal with ratios that are called non-reducible, that is, those involving integers evenly divisible only by 1, and not by any other integer. For example, the ratio 2:2 is the same as 1:1 and can be reduced to it for all practical purposes. Likewise, 10:4 is the same as 5:2, and 9:6 is the same as 3:2, and so on.

Secondly, we're going to divide all possible ratios into some subgroups for ease of handling. One group will be those described as the 1:N ratios. This means ratios like 1:1, 1:2, 1:3, 1:4 etc. They will be found to have particular properties.

Another group will be those described as N:M where N,M are less than 10. In thise case, the restriction to single digit numbers is purely for ease of arithmetic calculation. The last group is called 'large number ratios', and this involves numbers 10 and up. Again, the division is arbitrary. We won't deal with ratios like 100:1 or 100:99. Those are legitimate ones and you can discover their properties through POD synthesis. C:M ratios are sometimes expressed with real numbers, e.g. the ratio 1:1.4, but these can be approximated by integers, in this case 5:7. In FM, a set of sidebands is produced around the carrier C, equally spaced at a distance equal to the modulating frequency M. Therefore, we often refer to the sidebands in pairs: 1st, 2nd, 3rd, and so on.

Calculating Sidebands

The so-called upper sidebands are those lying above the carrier. Their frequencies are:

C+M C+2M C+3M C+4M C+5M ....

1 http://www.sfu.ca/~truax/fmtut.html [26.11.2012]

For example, if C:M is 1:2, that is, the modulator is twice the frequency of the carrier, then the first upper sideband is: C+M = 1+2 = 3. The second upper sideband is: C+2M = 1+(2x2) = 1+4 = 5. Another way to get the second sideband is to add M=2 to the value of the first sideband which is 3; i.e. (C+M) + M = 3+2 = 5. It quickly becomes clear that the upper sidebands in this example are all the odd numbers, and since the carrier is 1, the upper sidebands are all the odd harmonics, with the carrier as the fundamental (i.e. the lowest frequency in the spectrum).

However, if our C:M were 2:5, the first upper sideband would be 2+5 = 7. Since 7 is not a multiple of 2, it would be termed inharmonic. But the second upper sideband would be 7+5 = 12, and that is the 6th harmonic. Therefore, we can see that sidebands can be harmonic or inharmonic.

The lower sidebands are: C-M C-2M C-3M C-4M C-5M ...

When the sideband is a positive number it will lie below the carrier, but at some point, its value will become negative. It is then said to be reflected because we simply drop the minus sign and treat it as a positive number, e.g. the sideband -3 appears in the spectrum as 3. Acoustically, however, this reflected process involves a phase inversion, i.e. the spectral component is 180 degrees out of phase. Mathematically, we express this reflection by using absolute value signs around the expression: /C-M/ to indicate that we drop the minus and treat the number as positive.

For example, for 1:2, the 1st lower sideband is: /C-M/ = /1-2/ = /-1/ =1.

The second lower sideband is: /C-2M/ = /1-(2x2)/ = /1-4/ = /-3/ = 3. However, to make things easier, we could have added 2 to the first lower sideband (1), which is already reflected, and have obtained 3.

For the ratio 1:1, the 1st lower sideband is 0 (inaudible) and the 2nd, 3rd and 4th lower sidebands are 1, 2, 3, respectively.

For 7:5, the lower sidebands are: 2 3 8 13 where 3 is the 1st reflected one. One concern we have in using a given C:M ratio is whether the carrier frequency is the lowest frequency in the spectrum, i.e. is it the fundamental? If it is, we can treat the carrier frequency as the principal pitch that will be heard in the resulting timbre.

First we have the case of the 1:1 ratio whose upper sidebands are 2,3,4,... and whose lower sidebands are 0,1,2,3,... Clearly the carrier is the lowest non-zero component, and all the sidebands are harmonics, i.e. multiples. Because 1:1 is the only ratio with a zero lower sideband, it is a special case. It is also the only ratio producing the entire harmonic series. For other ratios, we could work out their sidebands and decide if any of them were lower than the carrier. That is fine, but tedious, and we'd like to know in advance what to expect. First, we might note that in the 1:2 case the 1st lower sideband is /-1/ = 1; therefore it fell against the carrier. Further, if M is larger than twice C, e.g. 2:5, then the 1st reflected sideband will always be greater than the carrier.

Work out a few examples and you'll find that the rule is:

FOR THE CARRIER TO BE THE FUNDAMENTAL: M MUST BE GREATER THAN OR EQUAL TO TWICE C, OR ELSE BE THE 1:1 RATIO.

Another useful property to be familiar with is the occurrence, as you may have noticed already, of lower sidebands coinciding with the upper set. They are said to 'fall against' them.

What this means acoustically is that the amplitudes of the two sidebands will add together, influenced as well by the phase inversion of the reflected sideband. Since the amplitude of each sideband varies according to the strength of the modulation, as expressed by the Modulation Index, the sum of the contributions of each sideband becomes quite complex!

You've probably noticed that the sidebands of the 1:1 ratio have this property of falling against each other. You'll find a very similar pattern with all of the N:1 ratios, i.e. 2:1, 3:1, 4:1, 5:1 ...

The second type of ratio showing the same property is that like 1:2. The odd harmonics are found in both the upper and lower sidebands. Therefore we can extrapolate the second case, namely odd N:2, that is, the ratios 1:2, 3:2, 5:2, 7:2 ...

No other ratios except N:1 and odd N:2 have this property. All ratios other than N:1 and odd N:2 have an asymmetrical spacing of their sidebands. That is, the distance in frequency between adjacent sidebands is unequal. For instance, the ratio 2:5 with its sidebands 2 3 7 8 12 .... However, there is still a pattern to the spacing.

We want the distance between the first upper sideband (C + M) and the first reflected lower sideband (C - M) which is negative by definition and therefore can be made positive to give the expression (M - C). Subtract these two frequencies to get: (C + M) - (M - C) = 2C We find that the answer is 2C. Since the spacing of all upper or lower sidebands is M, then the remaining spacing is M - 2C.

Therefore the rule is:

WHEN C IS THE LOWEST FREQUENCY IN THE SPECTRUM, THE SPACING OF SIDEBANDS ABOVE IT (EXCEPT FOR 1:1) WILL BE: M - 2.C & 2.C

For the above example (2:5), the spacing is 2x2=4 and 5-4=1. Note that we have to start with C as the fundamental. This will be referred to in the next section as the Normal Form of the Ratio.

Normal Form of the C:M Ratio

Definition: *a C:M ratio is in Normal Form (N.F.) when the carrier is the fundamental in the spectrum it produces.*

Rule: *for a ratio to be in Normal Form, M must be greater than or equal to twice C, or else be the ratio 1:1*

What we are doing with the concept of Normal Form is providing the basis of a classification scheme for all ratios, and at the same time codifying our rule of thumb about the conditions for the carrier to be the fundamental. If we consider only ratios involving integers up to 9, we can list all those in Normal Form:

1:1 1:2 4:9 3:7 2:5 3:8 1:3 2:7 1:4 2:9 1:5 1:6 1:7 1:8 1:9

These have been listed in an order related to what's called the **Farey Series** where the value of M/C is increasing, or C/M is decreasing. Each N.F. ratio will have associated with it a family of ratios, as shown below.

Reducing a Non Normal Form Ratio to its Normal Form

When the M value in a ratio is less than twice the C value, it is not in Normal Form, but can be reduced to it by applying the operation:

C = /C - M/

What this means is that you subtract M from C (ignoring any minus sign) and treat the result as the new C value. You keep doing this until the ratio satisfies the Normal Form criterion.

Consider, for example, the ratio 3:1. We first take 3-1=2 and get the ratio 2:1, which is still not in Normal Form. Another application of the operation gives us 2-1=1 and the ratio 1:1 which is in Normal Form by definition.

A trickier example is 8:5. First we reduce 8-5=3 and get 3:5 (not N.F.) Reducing again, we get 3-5=-2, which becomes 2:5 (ignoring the minus!) and because 5 is greater than twice 2, we have a Normal Form ratio. Notice that the reduction operation is the reverse of generating sidebands.

To generate sidebands we add M to C; to reduce the ratio, we subtract.

Families of C:M Ratios

For each Normal Form ratio there exists a set of ratios which produce the same set of sidebands. We'll call this set of ratios a 'family', and identify the family by its Normal Form ratio.

Consider the ratios 2:5, 3:5 and 7:5. Here are their sidebands:

2:5 2 3 7 8 12 13 17 18 22 23 27 28

3:5 3 2 8 7 13 12 18 17 23 22 28 27

7:5 7 2 12 3 17 8 22 13 27 18 32 23

We can see that all 3 ratios produce the same sidebands but in a different order. 2:5 is the Normal Form ratio and so this is the 2:5 family. How can we generate the entire set of family members? The rule is:

For a C:M ratio in Normal Form, the set of family members is:

C + N.M : M & /C - N.M/ : M for N=1,2,3,4,...

We simply take each sideband in turn and use it as the C value of the ratio, keeping M constant. The 2:5 family is 3:5,7:5,8:5,12:5,13:5,17:5,18:5,...

Harmonic & Inharmonic C:M Ratios

When we showed how to generate sidebands, we noted that some were harmonic and some were inharmonic (i.e. not multiples of the fundamental). Acoustically, this is an important distinction because each ratio produces an inharmonic or inharmonic timbre - a property usable compositionally.

The rule for determining harmonic/inharmonic is easy for Normal Form Ratios:

Harmonic N.F. ratios are always of the form 1:N, and inharmonic ones aren't.

Harmonic Ratios: 1:1 1:2 1:3 1:4 1:5 1:6 1:7 1:8 1:9

Inharmonic Ratios: 2:9 2:7 3:8 2:5 2:7 4:9

When a ratio is not in Normal Form, and you want to know if it is harmonic or not, you can reduce it to Normal Form & find out. However, ratios with M = 1,2,3,4,6 are always harmonic; for M=5,7,8,9 check if C is a multiple of M plus or minus 1; if so, it's harmonic. For instance, 9:5, 11:5, 14:5 and 16:5 are members of the 1:5 family whose C values are 10-1,10+1,15-1,15+1.

Calculating The Fundamental Frequency

On the other hand, 2:5 family members are 7:5, 8:5, 12:5, 13:5 & are inharmonic. Once we are thinking along the lines of families of ratios producing the same set of sidebands, we may want to know how to calculate the right carrier frequency for a family member ratio that produces the same spectrum as the Normal Form ratio on a given fundamental-carrier, e.g. 100 Hz for ease of calculation. In the PODX system this can be done for you automatically when you ask for the frequencies to be treated as modulators (note: M is constant for a family. To do the calculation yourself, take a Normal Form ratio NFC:M with a given carrier frequency. Then for a given family member ratio C:M, what should its carrier frequency (FC) be to keep the fundamental the same?

The equation is:

FC = (C ratio) x (N.F. Carrier freq.) / (N.F. C ratio)

For example, if the N.F. carrier frequency is 100 Hz. for 1:1, then the corresponding carrier for 3:1 is (3x100/1) = 300 Hz. For any 1:N ratio we can get the answer by taking C x 100. For a NFC:M ratio, we take C x 100 / NFC, e.g. for 7:5 and 2:5, the carrier for 7:5 is 7x100/2 = 350 Hz. Acoustically we hear a non N.F. family member on a high carrier as having its energy distributed around high partials, rather than centred on the fundamental, at least for a low modulation index.

When we use a non Normal Form ratio with a specific carrier frequency, we often want to know what is its fundamental frequency, or we wish a specific fundamental and want to know how to calculate the appropriate carrier. This is done for you in PODX when you request that the frequency be treated as the fundamental. The program calculates the right carrier to produce the fundamental that is required.

To calculate the fundamental (FF) for a non Normal Form ratio C:M whose carrier is known, we need to know the Normal Form of the ratio NFC:M, and then we can use the equation:

FF = (Carrier frequency) x (N.F. C ratio) / (C ratio)

For example, with 5:2 and a carrier of 500 Hz, the Normal Form ratio is 1:2 and therefore the fundamental is (500 x 1 / 5) = 100 Hz. Similarly, for 7:5 and a carrier of 700 Hz., the Normal Form ratio is 2:5, and therefore the fundamental is (700 x 2 / 7) = 200 Hz.

If we know the fundamental and need the carrier, we can use the equation:

Carrier = (Fundamental) / (C ratio) / (N.F. C ratio)

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