Proceeding of the IEEE
International Conference on Robotics and Biomimetics
Dali, China, December 2019

Analysis of Random Walk Models in Swarm Robots for Area Exploration *

[1]Bao Pang and [2]Jiahui Qi
1. school of Control Science and Engineering
Shandong University
Jinan, Shandong, China
pangbao11@foxmail.com, qijiahui@mail.sdu.edu.cn

[2]Chengjin Zhang*, [2]Yong Song and [2]Runtao Yang
2. school of Mechanical, Electrical and Information Engineering
Shandong University at Weihai
Weihai, Shandong, China
cjzhang@sdu.edu.cn, songyong@sdu.edu.cn, runtao-sd@163.com

***Abstract*— The purpose of area exploration is to cover an area effectively and swarm robots are used widely for this type of area exploration because of their robustness, flexibility, and scalability. Meanwhile, due to the limited individual abilities of swarm robots, the random walk methods have been the generally used area-exploration strategy. Although random walk methods possess better performance in area exploration, there still exist some problems. For one thing, little work has been devoted to the theoretical analysis of the random walk models. For another, no effective measures are used to evaluate the searching efficiency of random walk methods and the searching efficiency is verified mainly by simulation experiments. Therefore, in order to make up for the deficiency, this paper presents the mathematical description of the random walk models by drawing on the experience of related researches in biology. The proposed mathematical theory can not only be used to aid our understanding of random walks but also be easy to analyze and control random motion of the robots. Also, the mean squared displacement (MSD) is introduced as the performance measure to evaluate the effectiveness of the random walk methods. In order to proof the effectiveness of the performance measure of MSD, the area-exploration missions of swarm robots are carried out in the simulation experiments and the the coverage rate is used to evaluate the searching efficiency. The experimental results prove that the MSD is an effective performance measure to evaluate the searching efficiency.**

***Index Terms*— Random Walk; Brownian Motion; Probability Density Function; Mean Squared Displacement; Von Mises Distribution.**

I. INTRODUCTION

Area exploration is the process of searching the environment which can be used for subsequent mapping or navigation [1][2]. In recent years, area exploration has attracted widespread attention and it can be used in various tasks such as planetary exploration [3], search-rescue mission [4], foraging for food [5]. In area exploration, the core research issue is how to traverse an unknown area effectively. In a very large environment, it is relatively inefficient to have just one robot traverse the entire area. Instead, the exploration can be collectively accomplished by a robot swarm, and swarm robots are used widely for this type of area exploration because of their robustness, flexibility, and scalability due to redundancy and locality of sensing and communication [6]. Most existing search methods depend on delicate systems of sensors (e.g., odometers and ultrasound radar) and sophisticated mapping algorithms [7][8]. However, swarm robots, with their limited individual abilities (i.e., local sensing and low processing power), do not support complex localization and mapping. Indeed, random walk is a simple behavior that can be easily implemented in a robot swarm and random walk is still the most commonly adopted behavior for exploration with robot swarms [9][10][11].

The random walk theory results from the irregular motion of particles suspended in a fluid, famously studied by the botanist Robert Brown, now known as Brownian motion [12]. This pattern of motion can alternate the particle's position randomly from one domain to another. In general, the random walk models consist of two parts: the random movement-direction and the changing step-size at each step. Based on the correlations between the directions of successive step, the random walk can be divided into two categories, namely, (i)uncorrelated random walk, where the direction of movement is completely independent of the previous directions moved, and (ii)correlated random walk, where there is a correlation between successive step orientations [13]. In correlated random walk, each step orientation is influenced by either the previous direction or the direction toward a given target. According to the difference in step length, the random walk can have either a fixed or variable step length.

In swarm robots, the most commonly used random walk methods are Brownion motion and Levy flight, which have been widely used for random searching due to the characteristic of being easy to realize [15][16][17]. Wagner et al. used Brownion motion as the exploration strategy to explore the unknown areacover [18]. Furthermore, a novel random walk method based on Brownian motion has been

*This work is supported by the National Natural Science Foundation of China under Grants (61573213, 61673245, 61603214, 61803227), National Key Research and Development Plan of China under Grant 2017YFB1300205, Shandong Province Key Research and Development Plan under Grants (2016ZDJS02A07, 2018GGX101039), China Postdoctoral Science Foundation under Grant 2018M630778 and Independent Innovation Foundation of Shandong University under Grant 2018ZQXM005.

978-1-7281-6321-5/19//$31.00 ©2019 IEEE

proposed to improve the area coverage ratio by allowing the movement of each robot to be influenced by landmarks installed in the environment [19]. In area exploration, many researchers used the Levy flight as the search strategy to improve the searching efficiency [20]. Fricke et al. evaluated the effectiveness of a Levy flight search strategy and used a genetic algorithm to map the relationship between the search parameters and target configurations [21]. Schroeder et al. proposed a control law that incorporates a combination of virtual-pheromone-based method and Levy flight for efficient area exploration [22]. Inspired by the unique dynamical characteristics of ant foraging, Deshpande et. al. proposed a control law for efficient area exploration in a robot swarm by using pheromone and by adaptively switching between Brownian motion and Levy flight [23].

Although random walk methods have been widely used in swarm robots, there still exist some problems. For one, little work has been devoted to the theoretical analysis of the random walk models. Furthermore, no effective performance measures are used to evaluate the searching efficiency of random walk methods. Therefore, combined with the related researches in biology, this paper introduces the mathematical theory behind random walks in a straightforward manner. In this paper, the random motion is regarded as diffusion process through analyzing the characteristics of random walks and the drift-diffusion equation of the random walk is deduced by solving the partial differential equation. Then, the equation of the probability density function (PDF) for the position of the robot is deduced based on the diffusive process. Moreover, the von Mises distribution is used to describe the random angles of step orientation for a two-dimensional random walk and the mean squared displacement (MSD) is introduced to evaluate the searching efficiency.

The following sections are organized as follows. Sect. 2 outlines the mathematical description of the random walk models and mathematical model of MSD is introduced. In Sect. 3, simulation experiments are carried out to proof the effectiveness of the proposed mathematical models.Finally, the conclusion is drawn in Sect. 4.

II. Analysis of Random Walk Models

First of all, this section presents the mathematical description of random walk in one dimension and gives the derivation process of the MSD. Then, the mathematical description of random walk in high dimensions and the equation of MSD are introduced. Finally, the mathematical description and the MSD of realistic random walk in two dimensions are given.

A. Random Walk in One Dimension

Suppose the robot moves on the one-dimensional lattice with probability of turning right or left being r and l. Let the robot moves a short distance δ in a short time τ from the original point $x = 0$. After each time step, the robot may move a distance δ to the right with probability r or to the left with probability l. If the robot is at location x at time $t + \tau$, then there are two possibilities for its location at time t: (i) it was at $x - \delta$ and then moved to the right, (ii) it was at $x + \delta$ and then moved to the left. Thus we have

$$p(x, t + \tau) = p(x - \delta, t)r + p(x + \delta, t)l \tag{1}$$

As the distance δ and time τ are small, (1) can be expressed in terms of Taylor series about (x, t). Based on the Taylor series, the partial differential equation can be deduced [14]

$$\frac{\partial p}{\partial t} = -\frac{\delta\varepsilon}{\tau}\frac{\partial p}{\partial x} + \frac{\delta^2}{2\tau}\frac{\partial^2 p}{\partial x^2} + O(\tau^2) + O(\delta^3) \tag{2}$$

where $\varepsilon = r - l$; $O(\tau^2)$ and $O(\delta^3)$ represent higher order terms.

Let $\delta,\tau \to 0$, so $O(\tau^2)$, $O(\delta^3)$ tend to zero and (2) can be transformed into

$$\frac{\partial p}{\partial t} = -u\frac{\partial p}{\partial x} + D\frac{\partial^2 p}{\partial x^2} \tag{3}$$

where $u = \lim\limits_{\delta,\tau\to 0} \frac{\delta\varepsilon}{\tau}$, $D = \lim\limits_{\delta,\tau\to 0} \frac{\delta^2}{2\tau}$. This equation is called the drift-diffusion equation which is a combination of the drift equation and diffusion equation. The first term $(-u\frac{\partial p}{\partial x})$ describes the drift of the robot in the preferred direction. The second term $(D\frac{\partial^2 p}{\partial x^2})$ is the diffusion of the robot to the surrounding area. Suppose the robot is at $x = 0$ at time $t = 0$, the solution of (3) can be expressed as

$$p(x, t) = \frac{1}{\sqrt{4\pi Dt}} \exp(\frac{-(x - ut)^2}{4Dt}) \tag{4}$$

From (4), the location of the robot satisfies normal distribution with $N(ut, 2Dt)$.

Generally speaking, the bigger the distance of the robot from the current position to original point is, the bigger the area which has been searched by the robots is. In order to evaluate the searching efficiency of random walk methods, the concept of MSD is introduced, which is a measure of the deviation of the position of a robot with respect to a reference position over time. MSD can be used to measure the area searched by the robots in area exploration. Moreover, the mean location $E(X_t)$ is also used to analyze the searching efficiency. The mean location $E(X_t)$ and MSD $E(X_t^2)$ are defined as

$$E(X_t) = \int_{-\infty}^{\infty} xp(x, t)dx \tag{5}$$

$$E(X_t^2) = \int_{-\infty}^{\infty} x^2 p(x, t)dx \tag{6}$$

Based on (4), it is easy to show that

$$E(X_t) = ut, E(X_t^2) = u^2t^2 + 2Dt \tag{7}$$

For correlated random walk, when there is a preferred direction (the probability $r \neq l$), the MSD can be seen as

$E(X_t^2) \sim t^2$ (t is large). In this case, the MSD of a diffusion process is proportional to t^2.

For uncorrelated random walk, when there is no preferred direction (the probability $r = l$), the mean location $E(X_t)$ and MSD $E(X_t^2)$ are as follows

$$E(X_t) = 0, E(X_t^2) = 2Dt \tag{8}$$

For such a process, the MSD of a diffusion process is linear in time.

B. Random Walk in Higher Dimensions

The derivation of random walk in N-dimensional lattice is similar to the one in one-dimensional lattice and the standard drift-diffusion equation is

$$\frac{\partial p}{\partial t} = -u \cdot \nabla p + D\nabla^2 p \tag{9}$$

where u is the average drift velocity (an N-dimensional vector); ∇ is the gradient operator; and ∇^2 is the Laplacian. Suppose the robot is at $x = 0$ at time $t = 0$, the solution of (9) can be expressed as [24]

$$p(x,t) = \frac{1}{(4\pi Dt)^{N/2}} \exp\left(\frac{-|x-ut|^2}{4Dt}\right) \tag{10}$$

As defined in (5), the mean location $E(X_t) = ut$. With $R_t = |x_t|$, the MSD is defined given by

$$\begin{aligned} E(R_t^2) &= \textstyle\int_{\mathcal{R}^N} |x|^2 p(x,t)dx \\ &= \textstyle\int_{\mathcal{R}^N} (x_1^2 + ... + x_N^2)^2 p(x,t)dx_1...dx_N \end{aligned} \tag{11}$$

Based on (10) and (11), it is easy to show that

$$E(R_t^2) = |u|^2 t^2 + 2NDt \tag{12}$$

For correlated random walk, the MSD of a diffusion process is proportional to t^2. For uncorrelated random walk, the MSD of a diffusion process is linear in time.

C. The Realistic Random Walk in Two Dimensions

The random walk model discussed in the above subsection is limited in the N-dimensional lattices, which means that the robot possesses finite step orientation. However, in practical application, the direction selected by the robot at each step is completely random and the robot can move to any direction θ on the unit circle. Therefore, in order to describe the random walk theoretically, the circular distribution is used to show the movement-direction. The von Mises distribution is a commonly used circular distribution and it is defined as [25]

$$f(\theta) = \frac{1}{2\pi I_0(\kappa)} e^{\kappa \cos(\theta - \theta_0)} \tag{13}$$

where I_0 denotes the modified Bessel function of the first kind and order 0, defined by $I_m(\kappa) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos(m\theta) e^{\kappa \cos\theta} d\theta$; θ_0 is the mean of the generated angle in the range $[-\pi, \pi]$ and κ is a reciprocal measure of dispersion of the generated angles, which can be used to measure the concentration of the generated angles to the mean. When the von Mises distribution is used in random walk models, it is the uncorrelated random walk if the $\kappa = 0$. If the $\kappa \neq 0$, it is the correlated random walk and the mean θ_0 is the preferred direction; moreover, if κ is large, the distribution becomes very concentrated about the θ_0.

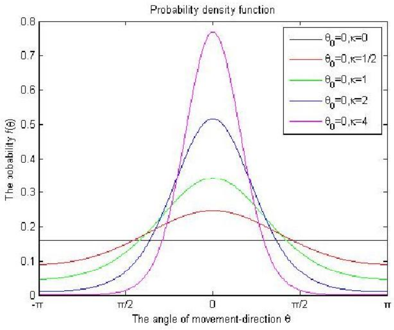

Fig. 1. The probability density function of the von Mises distribution.

Fig. 1 shows the probability density function of the von Mises distribution. As shown in Fig. 1, the von Mises distribution is transformed to a uniform distribution when $\kappa = 0$.

In correlated random walks, the robot should generate the step length from the step length PDF and generate the step direction θ from the von Mises distribution. In the circular distribution, the useful measures are the mean cosine c and mean sine s of the angle, which can be expressed as

$$\begin{aligned} c = E(\cos\theta) = \textstyle\int_{-\pi}^{\pi} \cos\theta f(\theta) d\theta \\ s = E(\sin\theta) = \textstyle\int_{-\pi}^{\pi} \sin\theta f(\theta) d\theta \end{aligned} \tag{14}$$

Using this model and standard drift-diffusion equation, the MSD of random walk after n steps is derived as [26]

$$\begin{aligned} E(R_n^2) = nE(L^2) + 2(E(L))^2 \\ \times\left(\tfrac{n(c-c^2-s^2)-c}{(1-c)^2+s^2} + \tfrac{2s^2+(c^2+s^2)^{(n+1)/2}\gamma}{((1-c)^2+s^2)^2}\right) \end{aligned} \tag{15}$$

where $\gamma = ((1-\mathrm{c})^2 - \mathrm{s}^2)\cos((\mathrm{n}+1)\theta_0) - 2\mathrm{s}(1-\mathrm{c})\sin((\mathrm{n}+1)\theta_0)$.

When robots search the environment randomly, clockwise and anticlockwise turns are often balanced which means that θ is symmetric about the preferred direction $\theta_0 = 0$ and the mean s is zero. Therefore, (15) can be reduced to a much simpler form with $s = 0$

$$E(R_n^2) = nE(L^2)\left(n\left(\frac{1+c}{1-c} + b^2\right) - \frac{2c(1-c^n)}{(1-c)^2}\right) \tag{16}$$

where $b^2 = \frac{E(L^2)}{(E(L))^2} - 1$; if the step length L is constant, $E(L^2)$ is equal to $(E(L))^2$ and $b = 0$; if the step L is variable, $E(L^2)$ is larger than $(E(L))^2$ and $b > 0$. Therefore, (16) shows that the MSD with the variable step length is always larger than the one with fixed step length. The variability in the step length has great influence on the searching efficiency of the random walks.

For uncorrelated random walk, $f(\theta)$ is a uniform distribution and both s and c are zero, and (17) reduced to $E(R_n^2) = nE(L^2)$. From the definition of MSD, we can conclude that $E(L^2)$ is equals to $D(L) + (E(L))^2$, where $D(L)$ is the variance of the step length. Therefore, the MSD of uncorrelated random walk is

$$E(R_n^2) = nD(L) + n(E(L))^2 \tag{17}$$

In uncorrelated random walks, the searching efficiency is influenced by the mean and variance of the step length. If the mean $E(L)$ is big, the searching efficiency is influenced mainly by $(E(L))^2$. If the mean $E(L)$ is same in different random walk methods, variance are the factors that influence the searching efficiency.

III. Experiments and Results

To proof whether the MSD is an effective performance measure to evaluate the searching efficiency of the random walk methods, the area-exploration missions with swarm robots are carried out in the simulation experiments. Different random walk methods are used to study how the mean and variance of the step length influence the searching efficiency. In simulation experiments, the robots use uncorrelated random walk (i.e., the step orientation is equally likely in each possible direction) and the difference between each experiment is the mean and variance of the step length. Two different types of step length are used: the first is the fixed step lengthe with the variance being zero; the second is the changing step length with different variance.

The simulation experiments were performed on Matlab and the coverage rate was used to evaluate the searching efficiency of random walk methods, where the coverage rate is the ratio of the explored area to the total area. The experimental area was a square area of $200m$ and 20 robots are used to perform the exploration mission. At the beginning of each experiment, all the robots are located in the central area. Each robot moves at a speed $v = 1m/s$ and each experiment lasts 1000s. In order to make a comparison adequately, we run the simulation experiments 20 times and then averaged the results to obtain the average coverage rate. For each random walk method, different step length are used to study the searching efficiency.

When the mean of the step length $E(L)$ is given, the variance of the fixed step length (Fix.) is $D(L) = 0$. There are no correlations between the mean and variance in normal distribution (Nor.), therefore when the step length satisfies normal distribution, different variance are used to study how the variance of the step length influence the searching efficiency. To analyze the experimental results, two indexes are used in this paper: the average coverage rate (Mean) and the standard deviation (Std.).

Equation (17) shows that, the MSD, as the performance measure, is influenced by $D(L)$ and $(E(L))^2$. Compared with the variance $D(L)$, the mean $E(L)$ has a bigger impact on the searching efficiency and only if the $D(L) > (E(L))^2$, $D(L)$ has a bigger impact on the searching efficiency.

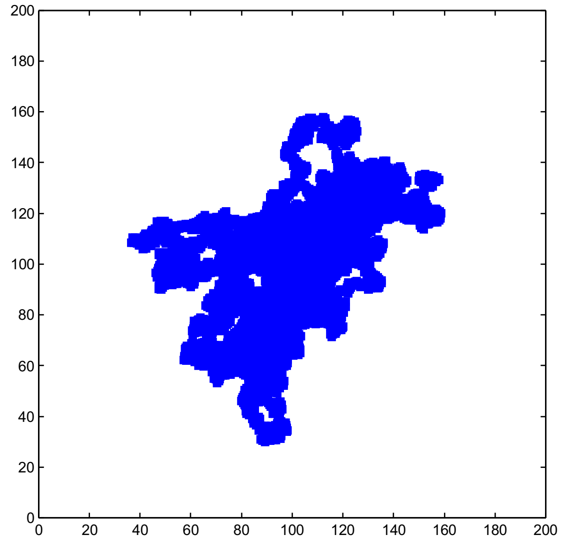

Fig. 2. The result of the exploration mission using twenty robots with the fixed step length of 1. Blue denotes explored area.

Fig. 2,3 show the results of exploration mission using 20 robots with different random walk methods. The robots with the fixed small step length can only search the surrounding area. When the step length of the robots satisfy normal distribution, the robots with bigger variance can produce larger step length; therefore, the robots can reach the far away place and the explored area also increased.

Table 1 shows the mean and standard deviation of the coverage rate for different random walk methods with the same step length of $E(L) = 1$. As shown in Table 1, the random walk method with the fixed step length $(E(L) = 1, D(L) = 0)$ has the lowest searching efficiency. In normal distribution, the coverage rate increased with the increase of the variance, which means that the searching efficiency improved. The results shown in Table 1 are consistent with the

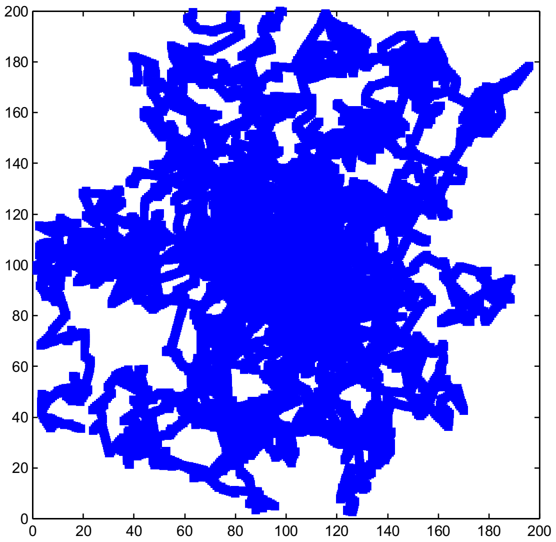

Fig. 3. The result of the exploration mission using twenty robots with the variable step-size, which satisfies normal distribution with $\mu = 1$ and $\delta = 5$.

conclusion in (17) that variance is the factor that influences the searching efficiency and when the $E(L)$ is same, the bigger $D(L)$ obtains higher searching efficiency.

TABLE I
WHEN $E(L) = 1$, THE COMPARISON AMONG THE TWO RANDOM WALK METHODS WITH DIFFERENT VARIANCE $(D(L))$.

Methods	Nor.			Fix.
$D(L)$	1	5	10	0
Mean	22.68%	41.70%	53.77%	14.50%
Std.	1.93e-02	3.08e-02	2.87e-02	1.17e-02

Table 2 shows the coverage rate for different random walk methods with the same step length of $E(L) = 5$. As shown in Table 2, the random walk method with the fixed step length $(E(L) = 5, D(L) = 0)$ has the lowest searching efficiency. In normal distribution, when the variance $(D(L))$ increases, the searching efficiency improved. Compared with the results in Table 1, the random walk method with the bigger step length has the higher searching efficiency when the variance is the same. The results shown in Table 2 are consistent with the MSD in (17).

Table 3 shows the coverage rate for different random walk methods with the same step length of $E(L) = 10$ and the random walk method with a larger variance possess the higher searching efficiency. As the variance increases, the growth rate of the searching efficiency is reduced gradually compared with Table 1 and Table 2, which means that the change of variance have less effect on the coverage rate. When the mean $(E(L))$ is small, the changes in variance have large influence on searching efficiency. When the mean $(E(L))$ is big, the changes in variance have small influence on searching efficiency. The conclusions that the step length $(E(L))$ and variance $(D(L))$ altogether affect the coverage rate are consistent with the MSD in (17).

TABLE II
WHEN $E(L) = 5$, THE COMPARISON AMONG THE TWO RANDOM WALK METHODS WITH DIFFERENT VARIANCE $(D(L))$.

Methods	Nor.			Fix.
$D(L)$	1	5	10	0
Mean	42.48%	49.50%	56.68%	39.88%
Std.	3.43e-02	2.73e-02	2.92e-02	3.28e-02

TABLE III
WHEN $E(L) = 10$, THE COMPARISON AMONG THE TWO RANDOM WALK METHODS WITH DIFFERENT VARIANCE $(D(L))$.

Methods	Nor.			Fix.
$D(L)$	1	5	10	0
Mean	54.94%	57.80%	59.58%	52.72%
Std.	2.30e-02	2.92e-02	2.16e-02	3.20e-02

Table 4, 5, 6 show the coverage rate for different random walk methods with $E(L) = 20, 30, 50$, respectively. In each table, the swarm using different random walk methods possess almost the same coverage rate, which means that the variance has little impact on searching efficiency when the mean $(E(L))$ is big. Compared with the $E(L)^2$ $(E(L)^2 = 400, 900, 2500$, respectively), the value of the variance $(D(L) = 1, 5, 10$, respectively) is too small to influence the searching efficiency. The results are consistent with the MSD in (17).

TABLE IV
WHEN $E(L) = 20$, THE COMPARISON AMONG THE TWO RANDOM WALK METHODS WITH DIFFERENT VARIANCE $(D(L))$.

Methods	Nor.				Fix.
$D(L)$	1	5	10	20	0
Mean	62.26%	63.45%	63.78%	65.43%	63.33%
Std.	2.12e-02	1.97e-02	2.21e-02	1.77e-02	2.17e-02

IV. CONCLUSION

This paper introduced mathematical theory of the random walk models to analyze and control random motion of the

TABLE V

WHEN $E(L) = 30$, THE COMPARISON AMONG THE TWO RANDOM WALK METHODS WITH DIFFERENT VARIANCE $(D(L))$.

Methods	Nor.				Fix.
$D(L)$	1	5	10	30	0
Mean	65.37%	66.62%	66.31%	68.02%	66.01%
Std.	1.73e-02	1.84e-02	1.69e-02	1.35e-02	1.64e-02

TABLE VI

WHEN $E(L) = 50$, THE COMPARISON AMONG THE TWO RANDOM WALK METHODS WITH DIFFERENT VARIANCE $(D(L))$.

Methods	Nor.				Fix.
$D(L)$	1	5	10	50	0
Mean	67.90%	67.92%	68.23%	69.03%	68.19%
Std.	1.50e-02	1.77e-02	1.22e-02	9.66e-03	1.65e-02

robots. And then the MSD was introduced as the performance measure to evaluate the searching efficiency of the random walk methods. The experimental results proved that the MSD is an effective performance measure in analyzing the searching efficiency of random walks.

Although the effectiveness of the proposed method has been verified in this paper, the practical experiments have not been performed. Therefore, as future work, we will try to verify the validity of the method in other applications and try to do the experiments with actual robots.

REFERENCES

[1] W. Rone and P. Ben-Tzvi, "Mapping, localization and motion planning in mobile multi-robotic systems," Robotica, 2013, vol. 31, no. 1, pp. 1-23.

[2] A. V. Savkin and H. Li, "A safe area search and map building algorithm for a wheeled mobile robot in complex unknown cluttered environments," Robotica, 2018, vol. 36, no. 1, pp. 96-118.

[3] S. Curtis, M. Rilee and P. Clark, "Use of swarm intelligence in spacecraft constellations for the resource exploration of the asteroid belt," In Proceedings of the Third International Workshop on Satellite Constellations and Formation Flying, 2003, pp. 24-26.

[4] S. Waharte, N. Trigoni and S. J. Julier, "Coordinated search with a swarm of uavs," IEEE Annual Communications Society Conference on Sensor, Mesh and Ad Hoc Communications and Networks Workshops, 2009, pp. 1-3.

[5] L. Pitonakova, R. Crowder and S. Bullock, "Information flow principles for plasticity in foraging robot swarms," Swarm Intelligence, 2016, vol. 10, no. 1, pp. 33-63.

[6] Y. Wei, T. Yasuda and K. Ohkura, "Autonomous task allocation for swarm robotic systems using behavioral decomposition," Intelligent and Evolutionary Systems, Cham: Springer International Publishing, 2017, pp. 469-481.

[7] D. W. Sims, A. M. Reynolds, N. E. Humphries et al., "Hierarchical random walks in trace fossils and the origin of optimal search behaviour," Proceedings of the National Academy of Sciences, vol. 111, no. 30, pp. 11073-11078, 2014.

[8] F. Ducatelle, G. D. Caro, A. F?rster et al., "Cooperative navigation in robotic swarms," Swarm Intelligence, vol. 8, no. 1, pp. 1-33, 2014.

[9] D C. imidov, G. Oriolo and V. Trianni, "Random walks in swarm robotics: an experiment with kilobots," International Conference on Swarm Intelligence. Springer, Cham, 2016: 185-196.

[10] G. Francesca, M. Brambilla, A. Brutschy, V. Trianni and M. Birattari, "AutoMoDe: A novel approach to the automatic design of control software for robot swarms," Swarm Intelligence, 2014, vol. 8, no. 2, pp. 89C112.

[11] M. Dorigo, D. Floreano, L. M. Gambardella, F. Mondada, S. Nolfi, T. Baaboura, M. Birattari, M. Bonani, M. Brambilla, A. Brutschy, D. Burnier, A. Campo, A. L. Christensen, A. Decugniere, G. Caro, F. Ducatelle, E. Ferrante, A. Forster, J. M. Gonzales and J. Guzzi, "Swarmanoid: A novel concept for the study of heterogeneous robotic swarms, "IEEE Robotics & Automation Magazine, vol. 20, no. 4, pp. 60C71, 2013.

[12] F. R. S. RobertBrown, M. R. S. E. Hon and R.I. Acad, "A brief account of microscopical observations made in the months of June, July and August 1827, on the particles contained in the pollen of plants; and on the general existence of active molecules in organic and inorganic bodies," Philosophical Magazine, 2015, vol. 4, no. 21, pp. 161-173.

[13] B. Pang, Y. Song and C. Zhang, "A Swarm Robotic Exploration Strategy Based on an Improved Random Walk Method," Journal of Robotics, 2019, 2019.

[14] E. A. Codling, M. J. Plank and S. Benhamou, "Random walk models in biology," Journal of the Royal Society Interface, 2008, vol. 5, no. 25, pp. 813-834.

[15] D. W. Sims, N. E. Humphries, R. W. Bradford, and B. D. Bruce, "Levy flight and Brownian search patterns of a free-ranging predator reflect different prey field characteristics," Journal of Animal Ecology, 2012, vol. 81, no. 2, pp. 432-442.

[16] T. Itami, "Controlling Brownianmotion applied tomacroscopic group robotswithout mutual communication," in Proceedings of the 2014 19th International Conference on Methods and Models in Automation and Robotics (MMAR), 2014, pp. 903-908.

[17] A. A. Munishkin, D. Milutinovi, and D. W. Casbeer, "Safe navigation with the collision avoidance of a brownian motion obstacle," in Proceedings of the ASME 2017 Dynamic Systems and Control Conference, American Society of Mechanical Engineers, 2017.

[18] I. A. Wagner, M. Lindenbaum, A. M. Bruckstein, "Robotic exploration, Brownian motion and electrical resistance, "Randomization and Approximation Techniques in Computer Science, 1998, pp. 116-130.

[19] F. Martinez, E. Jacinto and D. Acero, "Brownian motion as exploration strategy for autonomous swarm robots," 2012 IEEE International Conference on Robotics and Biomimetics (ROBIO). IEEE, 2012: 2375-2380.

[20] R. Fujisawa, S. Dobata, "Lvy walk enhances efficiency of group foraging in pheromone-communicating swarm robots," In System Integration (SII), 2013 IEEE/SICE International Symposium on. IEEE, 2013, pp. 808C813, 2013.

[21] G. M. Fricke, J. P. Hecker, J. L. Cannon et al., "Immune-inspired search strategies for robot swarms," Robotica, 2016, vol.34, no. 08, pp. 1791-1810.

[22] A. Schroeder, S. Ramakrishnan, M. Kumar et al., "Efficient spatial coverage by a robot swarm based on an ant foraging model and the Lvy distribution," Swarm Intelligence, 2017, vol. 11, no. 1, pp. 39C69.

[23] A. Deshpande, M. Kumar, S. Ramakrishnan, "Robot swarm for efficient area coverage inspired by ant foraging: The case of adaptive switching between Brownian motion and Lvy flight," In ASME 2017 Dynamic Systems and Control Conference, 2017, pp. V002T14A009

[24] G. Grimmett and D. Stirzaker, Probability and random process. Oxford, UK: Oxford Uiversity Press, 2001.

[25] K. V. Mardia and P. E. Jupp, Directional statistics. Chichester, UK: Wiley, 1999.

[26] R. Kareiva and N. Shigesada, "Analyzing Insect Movement as a Correlated Random Walk," Oecologia, 1983, vol. 56, no. 2, pp. 234-238.