Atmospheric Flight Dynamics A: Project report
Students: G. Borrelli, M. Rossi
Professor: F. Giulietti
Due date: 12/06/2020

1 Introduction to the project

Aim of the project is to carry out the study of a given aircraft at a chosen flight condition. Once this has been set, it is required to compare *Matlab* results of the linearized model matrices (*Alat, Alon*) and the poles and relevant zeros, with the hand-calculated ones. Finally, the construction of two Stability Augmentation systems (SAS) and two Autopilots are needed in order to improve the flight handling qualities and to track a given input signal.

The flight condition analyzed in this project is the coordinated steady turn, allowing to invert the route in exactly 60 seconds. This is observable in figure 1, where it is clearly visible how the yaw angle reaches the value of 3 *rad* and how the trajectory is completely inverted after one minute.

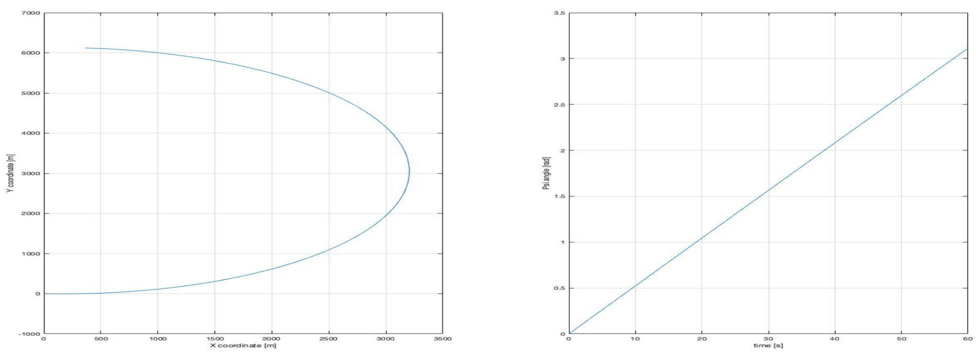


Figure 1: XY trajectory (*left*) and Yaw angle evolution (*right*) after 60 seconds.

2 Linear model of the Aircraft

2.1 The state-space structure

The model used to describe the dynamics of an Aircraft is *highly non-linear*. In order to simplify this study, it is possible to adapt a linear model, according to the *state-space structure*, which reads:

$$\begin{cases} \dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u} \\ \mathbf{y} = C\mathbf{x} + D\mathbf{u} \end{cases}$$

where $\mathbf{x}$, $\mathbf{u}$ and $\mathbf{y}$ indicate the state, input and output vectors, respectively.

2.2 Linearized model

Before starting to analyze the linearized model coming out from *Matlab*, it is useful to evaluate again which one are the terms to be taken into account during the coordinated turn.
Usually, when analyzing a symmetric maneuver (e.g. steady climb, steady level flight), a de-coupling can be observed between longitudinal and lateral-directional dynamics.
The latter won't be experienced anymore when analyzing a *non-symmetric* maneuver, as the turn one considered in this project.
This leads to the evaluation of a 8x8 matrix.

$$\begin{pmatrix} X_u & X_v + R_0 & X_w - Q_0 & X_p & X_q - W_0 & X_r & X_\varphi & X_\vartheta - \mathbf{g}_{\mathbf{x}\vartheta} \\ Y_u - R_0 & Y_v & Y_w + P_0 & Y_p + P_0 & Y_q & Y_r - U_0 & Y_\varphi + \mathbf{g}_{\mathbf{y}\varphi} & Y_\vartheta - \mathbf{g}_{\mathbf{y}\vartheta} \\ Z_u - Q_0 & Z_v - P_0 & Z_w & Z_p & Z_q + U_0 & Z_r & Z_\varphi - \mathbf{g}_{\mathbf{z}\varphi} & Z_\vartheta - \mathbf{g}_{\mathbf{z}\vartheta} \\ L'_u & L'_v & L'_w & L'_p + \mathbf{P_p} & L'_q + \mathbf{P_q} & L'_r + \mathbf{P_r} & L'_\varphi & L'_\vartheta \\ M_u & M_v & M_w & M_p + \mathbf{Q_p} & M_q & M_r + \mathbf{Q_r} & M_\varphi & M_\vartheta \\ N'_u & N'_v & N'_w & N'_p + \mathbf{R_p} & N'_q + \mathbf{R_q} & N'_r + \mathbf{R_r} & N'_\varphi & N'_\vartheta \\ 0 & 0 & 0 & 1 & \varphi_{\mathbf{q}} & \varphi_{\mathbf{r}} & \varphi_\varphi & \varphi_\vartheta \\ 0 & 0 & 0 & 0 & \vartheta_{\mathbf{q}} & \vartheta_{\mathbf{r}} & \vartheta_\varphi & 0 \end{pmatrix}$$

Where the most important stability derivatives are shown and some additional terms are highlighted. [1]

2.3 The matrices

The central finite difference scheme has been used to perform the hand calculations. This consists into applying, separately, a positive and a negative infinitesimal perturbation and to check the related outputs in the *Simulink* model (*Complete Aircraft*) as concerns the forces and the moment. To obtain the hand-calculated stability derivate, the difference is divided by the relative perturbed variable multiplied by the total mass of the aircraft or the related inertial moment.

[1]Whose formulas are reported in the appendix

Matlab reports the 4x4 matricex *Alon* and *Alat*, collecting the stability derivatives related to the Longitudinal and Lateral-Directional dynamics:

$$Alon = \begin{bmatrix} -0.0087 & 0.0667 & -13.0903 & -9.7765 \\ -0.0472 & -0.9904 & 157.2839 & -0.6171 \\ 0.0089 & -0.1642 & -1.5282 & 0 \\ 0 & 0 & 0.7616 & 0 \end{bmatrix}$$

$$Alat = \begin{bmatrix} -0.1525 & 12.9843 & -157.7847 & 7.4462 \\ -0.0319 & -1.2972 & 0.4545 & 0 \\ 0.0250 & -0.4203 & -0.2037 & 0 \\ 0 & 1.0000 & 0.0631 & 0 \end{bmatrix}$$

Where the state vectors are defined by:

$$\mathbf{x}_{lon} = [u, w, q, \vartheta]^T$$
$$\mathbf{x}_{lat} = [v, p, r, \varphi]^T$$

The comparisons with the hand-calculated matrices shows an acceptable behaviour of the error, below the 5 %.

Furthermore, here are reported the matrices related to the input vectors:

- *Blon*, describing the longitudinal inputs, in which the first column accounts for the elevator δ_e, the second for flaps δ_f, and the third for thrust δ_{th}:

$$Blon = \begin{bmatrix} 0.2583 & 0.8273 & 3.4883 \\ -12.9618 & -41.5078 & -0.1218 \\ -22.8500 & 8.2341 & 0.2859 \\ 0 & 0 & 0 \end{bmatrix}$$

- *Blat*, describing the lateral-directional inputs, in which the first column accounts for the ailerons δ_a and the second one for rudder δ_r:

$$Blat = \begin{bmatrix} -0.8895 & 3.4844 \\ -7.8973 & 1.0814 \\ -0.7958 & -5.2393 \\ 0 & 0 \end{bmatrix}$$

3 Relevant poles and zeros

One last check to be done is to carry out the hand-calculation of most relevant poles and zeros of the transfer functions deriving from the linearized model.
The approximated formula which have been used are:

- **Short Period** frequency and damping parameters:
  $\omega_{sp} = \sqrt{Z_w M_q - M_w U_0} = 5.2424$
  $\xi_{sp} = -(M_q + Z_w + M_{\dot{w}} U_0)/2\omega_{sp} = 0.3025$
- **Phugoid** frequency and damping parameters:
  $\omega_p = \sqrt{-(Z_u - M_u Z_w / M_w) g / U_0} = 0.0914$
  $\xi_p = -X_u/(2\omega_p) = 0.0477$
- **Roll** parameter:[2]
  $\frac{1}{Tr} = \frac{B^2+C}{B+C^2/D} = 1.5364$
- **Spiral** parameter:
  $\frac{1}{Ts} = Tr(-L'_r + \frac{L'_\beta}{N'_p} N'_r) g/U_0 = 0.0085$
- **Dutch Roll** frequency and damping parameters:
  $\omega_{DR} = \sqrt{\frac{D}{1/Tr}} = 2.2042$
  $\xi_{DR} = \frac{B-1/Tr}{2\omega_{DR}} = 0.0265$

The approximations are quite faithful to the real values obtained from the *Matlab* matrices, as it can been seen in the following table. Errors around 50 % and 36 % are present in the description of the phugoid damping and frequency, but the order of magnitude is still preserved.

	Damping	**Frequency** $[rad/s]$		**Damping**	**Frequency** $[rad/s]$
SP	2.40e-01	5.24e+00	R	1.00e+00	1.54e+00
P	9.37e-02	6.69e-02	S	-1.00e+00	7.71e-03
			DR	2.74e-02	2.21e+00
	Longitudinal			Lat.-Directional	

[2] Where B, C and D are the coefficients of the polynomial $s^4 + Bs^3 + Cs^2 + Ds + E$, obtained by performing the determinant of the *Alat* matrix.

Approximated formulas have been used also to calculate relevant zeros. Errors were still contained and acceptable for most of them.[3]

4 Control systems design: SAS

4.1 Flying and Handling qualities

As stated in chapter 6 of the *Automatic Flight Control System* by D. McLean, every aircraft has to satisfy qualities *'which govern the ease and precision with which a pilot is able to perform his mission'*[4]. These qualities depends on the class at which the aircraft belongs, on the flight phase and on the level of acceptability.
In the specific case of this project the aircraft considered shows the following specifications:

- **Class II**: The aircraft is of medium weight (8000 Kg) and moderate maneuvrability;
- **Phase B**: The turn maneuver is a non-terminal flight phase, not requiring for aggressive maneuvres and asking for an accurate flight path control;
- **Level 1**: The flying qualities have to be completely adequate to complete the mission.

The following table shows the Longitudinal and Lateral-Directional qualities to be satisfied, compared with the one obtained for the turn maneuver:

Flying quality	**Requirement**	**Result**
Phugoid	$\xi_{sp} \geq 0.04$	0.0937
Short Period	$0.3 \leq \xi_{sp} \leq 2.0$	**0.24**
Roll	$Tr < 1.4$	0.6496
Spiral	$Ts > 20s$	specified for an aircraft trimmed for straight and steady level flight
Dutch Roll	$\omega_{DR} > 0.5$	2.21
	$\xi_{DR} > 0.08$	**0.0274**

The highlighted terms are the one for which the requirements are not encountered, thus the design of a SAS is needed.

[3]To lighten the reading, these formulas have been reported in the appendices
[4]From a special issue of *Journal of Guidance, Control and Dynamics*

4.2 SAS Design

Two proportional controllers are introduced in the feedback path, allowing to correct the short period and dutch roll damping specifications:

- **Pitch Damper**

  The desired damping has been set to $\xi_{sp,d} = 0.5$, whilst $\xi_{sp} = 0.24$ is the one obtained.
  By using the formula, the value to assign to the controller is calculated:

  $$K_q = \frac{2\omega_{sp}(\xi_{sp,d} - \xi_{sp})}{M_{\delta_e}} = -0.1192$$

  This value has been improved by using the *Sisotool* ambient of *Matlab* on the transfer function q/δ_e, in order to reach exactly the desired damping.
  The final Kq which has been found is: $K_q = -0.13015$.
  Improvements are clearly visible by comparing the poles of the two transfer functions before and after the SAS [5]:

  $$\frac{q}{\delta_e} = \frac{-22.85s(s + 0.8934)(s + 0.01245)}{(s^2 + 0.01255s + 0.004479)(s^2 + \mathbf{2.515}s + 27.45)} \qquad \text{(before Kq)}$$

  $$\frac{q}{\delta_e} = \frac{-22.85s(s + 0.8934)(s + 0.01245)}{(s^2 + 0.01217s + 0.004084)(s^2 + \mathbf{5.489}s + 30.11)} \qquad \text{(after Kq)}$$

- **Yaw Damper**

  A similar procedure has been used to improve the damping of the dutch roll: from $\xi_{DR} = 0.0274$ to $\xi_{DR,d} = 0.2$:

  $$K_r = \frac{N'_r + Y_v + 2\sqrt{N'_b}\xi_{DR,d}}{N'_{\delta_r}} = -0.0839$$

  Still, this value has been improved with *Sisotool* and the final K_r has been set to -0.1655.
  The comparison between the two transfer functions before and after the Yaw Damper shows:

[5] The Gain has been implemented into the 'LIN_Complete' model of *Simulink*, so to have the transfer functions after the SAS implementation.

$$\frac{r}{\delta_r} = \frac{-5.2393(s + 1.221)(s^2 + 0.2984s + 0.1628)}{(s + 1.54)(s - 0.007713)(s^2 + \mathbf{0.1208}s + 4.872)} \qquad \text{(before Kr)}$$

$$\frac{r}{\delta_r} = \frac{-5.2393(s + 1.221)(s^2 + 0.2984s + 0.1628)}{(s + 1.604)(s + 0.01463)(s^2 + \mathbf{0.9014}s + 4.88)} \qquad \text{(after Kr)}$$

5 Autopilot Design and Results

By comparing the poles obtained from the 4x4 de-coupled matrices *Alon* and *Alat* with the complete 8x8 matrix, it has been noticed that the resulting frequencies and damping coefficients are quite similar.
Some differences can be appreciated in the low frequency poles. For this project, it has been chosen to carry out the design of the two Autopilots by just considering the model as if it was de-coupled, so using matrices *Alon* and *Alat*. The architecture which has been implemented on *Simulink* can be seen in figure 2

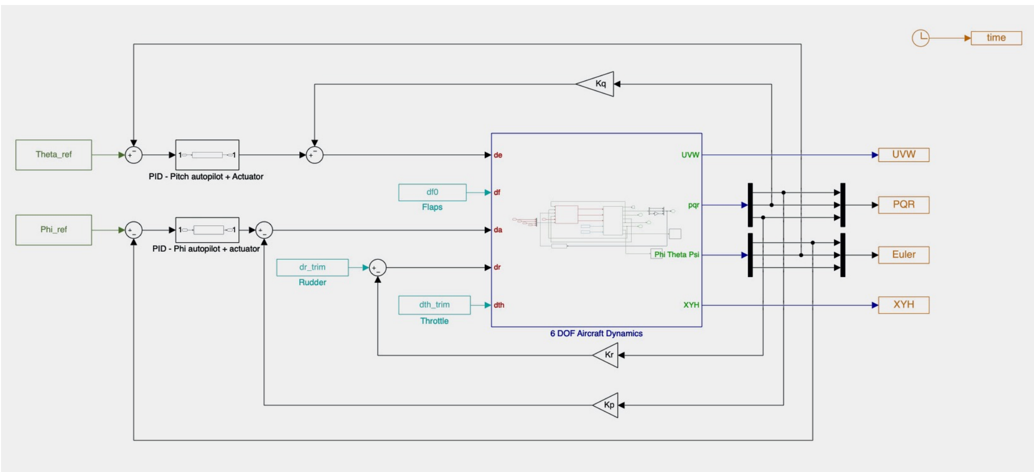


Figure 2: Scheme of the implemented SAS and Autopilots.

5.1 Pitch Autopilot

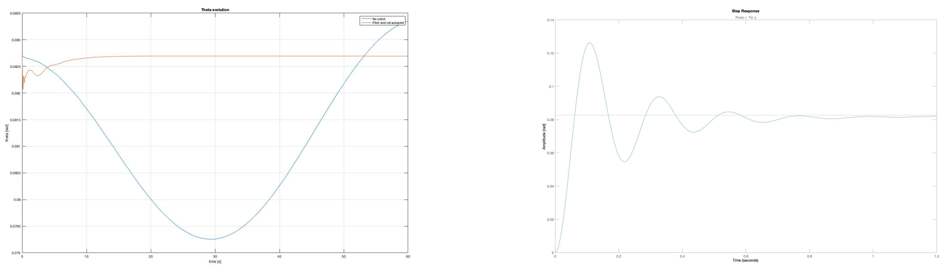
Once the handling qualities have been reached thanks to the Pitch Damper, an Autopilot is needed in order to track a desired value of pitch angle ϑ.
To better simulate the real case, an actuator has been introduced, represented by the following transfer function:

$$H(s) = \frac{15}{s + 15}$$

To obtain an ideal tracking, the best option is to implement a PID controller. Here's reported the transfer function which has been used:

$$PID_{\vartheta}(s) = \frac{-3.3(s + 8)(s + 0.3)}{s}$$

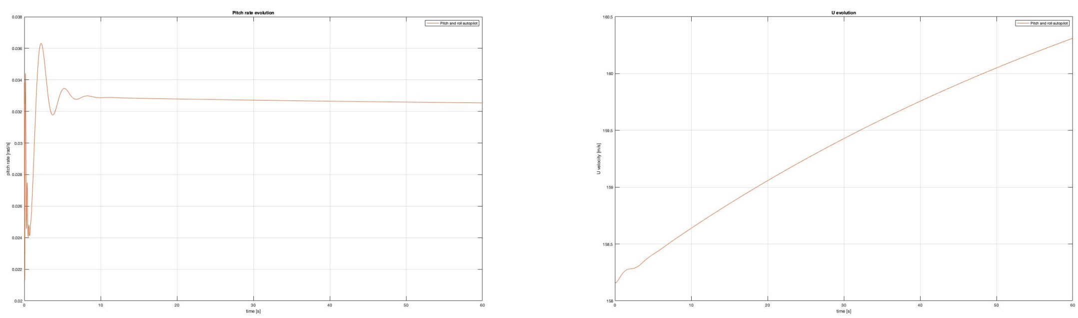
By setting these parameters, having the initial ϑ_0 as a reference value for the pitch angle, a good response is obtained in few seconds. It is also clear how the Autopilot has widely improved the tracking, showing just some small variations from the reference value which has been imposed.



(a) (b)

Figure 3: (a) Theta evolution in 60 seconds: no control (blue) and Pitch Autopilot (red). (b) Step response ($\vartheta_{ref} = 0.0827 rad$).

In figure 4 are also reported how the pitch rate and the U component of the velocity have been influenced.



(a) (b)

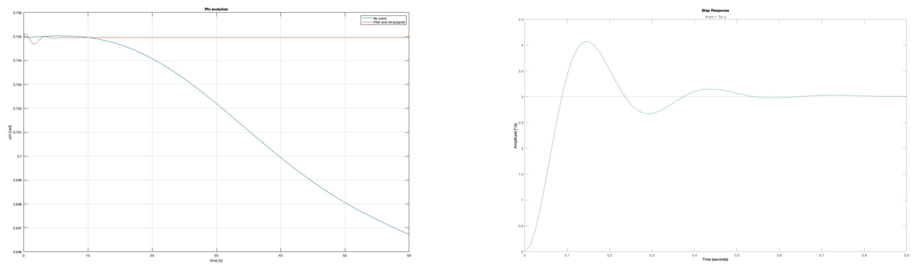
Figure 4: (a) Pitch rate and (b) U component evolution in 60 seconds.

5.2 Roll Autopilot

As for the Pitch autopilot, also in the case of the Roll Autopilot the design of a double-loop has been implemented.
It has been necessary to design the Gain of a proportional controller K_p in order to implement the inner loop. An acceptable response is obtained by setting $K_p = -0.65$ (from *Sisotool*, to obtain a time constant of about 2 seconds).
At this point, a PID has been designed with the following transfer function (still implemented together with the actuator):

$$PID_{\varphi}(s) = \frac{-4.4(s + 0.95)(s + 1.2)}{s}$$

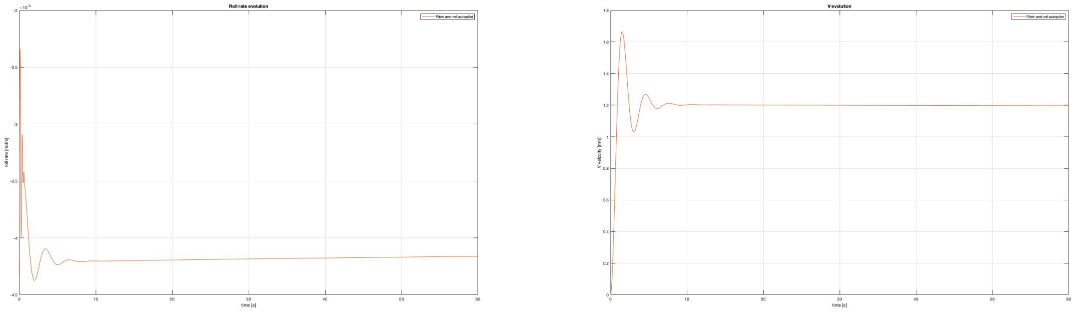
Setting the initial φ_0 as a reference value for the roll angle the behaviour of the respone is good and the tracking has much improved, as it is cleary observable in the plots of figure 5.



(a) (b)

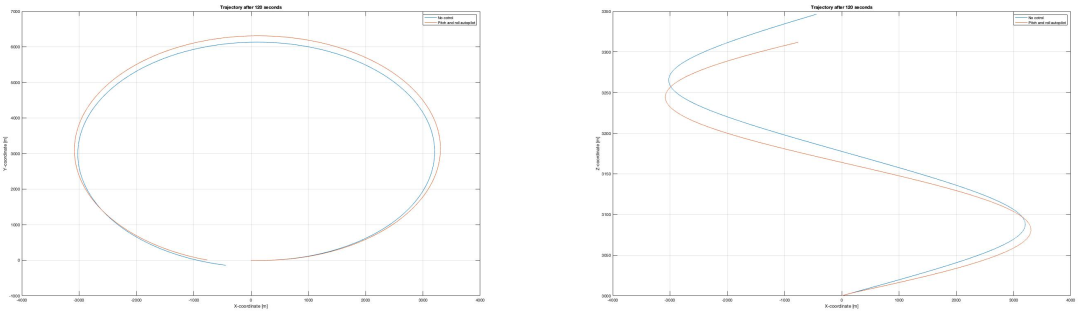
Figure 5: (a) Phi evolution in 60 seconds: no control (blue) and Pitch Autopilot (red). (b) Step response ($\varphi_{ref} = 3deg/s$).

In figure 6 are also reported how the roll rate and the V component of the velocity have been influenced.
A small side speed is introduced, but this doesn't influence the overall trajectory, as shown in figure 7.
Still the initial climb is present, but it's reduced of about 40 meters in 120 seconds.



(a) (b)

Figure 6: (a) Roll rate and (b) V component evolution in 60 seconds.



(a) (b)

Figure 7: (a) XY and (b) XZ trajectories after 120 seconds. Blue line: no control. Red line: with Pitch and Roll autopilots

References

1 D. McLean, *Automatic Flight Control Systems*, 1990, Prentice Hall

2 Class notes of *Atmospheric Flight Dynamic A* course

Appendices

8x8 matrices, added elements

$$P_p = \frac{\frac{Q_0 I_{xz}}{I_{xx}} + \frac{I_{xz}}{I_{xx}}(\frac{I_{xx}}{I_{zz}}Q_0 - \frac{I_{yy}}{I_{zz}}Q_0)}{1 - \frac{I_{xz}^2}{I_{xx}I_{zz}}}$$

$$P_q = \frac{\frac{R_0 I_{yy} + I_{xz}P_0 - I_{zz}R_0}{I_{xx}} + \frac{I_{xz}}{I_{xx}}\frac{-I_{yy}P_0 + I_{xx}P_0 - I_{xz}R_0}{I_{zz}}}{1 - \frac{I_{xz}^2}{I_{xx}I_{zz}}}$$

$$P_r = \frac{\frac{I_{yy}Q_0 - I_{zz}Q_0}{I_{xx}} - \frac{I_{xz}^2 Q_0}{I_{xx}I_{zz}}}{1 - \frac{I_{xz}^2}{I_{xx}I_{zz}}}$$

$$Q_p = \frac{-I_{xx}R_0 - 2I_{xz}P_0 + I_{zz}R_0}{I_{yy}}$$

$$Q_r = \frac{2I_{xz}R_0 + I_{zz}P_0 - I_{xx}P_0}{I_{yy}}$$

$$R_p = \frac{I_{xx}Q_0 - \frac{I_{yy}Q_0}{I_{zz}} + \frac{I_{xz}}{I_{zz}}\frac{(Q_0 I_{xz})}{I_{xx}}}{1 - \frac{I_{xz}^2}{I_{xx}I_{zz}}}$$

$$R_q = \frac{\frac{-I_{yy}P_0 + I_{xx}P_0 - I_{xz}R_0}{I_{zz}} + \frac{I_{xz}}{I_{zz}I_{xx}}(R_0 I_{yy} + I_{xz}P_0 - I_{zz}R_0)}{1 - I_{xz}^2/(I_{xx} * I_{zz})}$$

$$R_r = \frac{\frac{-I_{xz}}{I_{zz}}Q_0 + \frac{I_{xz}}{I_{zz}}(Q_0 I_{yy} - \frac{I_{zz}Q_0}{I_{xx}})}{1 - \frac{I_{xz}^2}{I_{xx}I_{zz}}}$$

$$g_{x\vartheta} = gcos(\vartheta_0)$$

$$g_{y\varphi} = gcos(\vartheta_0)cos(\varphi_0)$$

$$g_{y\vartheta} = gsin(\vartheta_0)sin(\varphi_0)$$

$$g_{z\varphi} = gcos(\vartheta_0)sin(\varphi_0)$$

$$g_{z\vartheta} = gsin(\vartheta_0)cos(\varphi_0)$$

$$\varphi_q = tan(\vartheta_0)sin(\varphi_0)$$

$$\varphi_r = tan(\vartheta_0)cos(\varphi_0)$$

$$\varphi_\varphi = -tan(\vartheta_0)(R_0 sin(\varphi_0) - Q_0 cos(\varphi_0))$$

$$\varphi_\vartheta = (tan(\vartheta_0))^2(R_0 cos(\varphi_0) + Q_0 sin(\varphi_0))$$

$$\vartheta_q = cos(\varphi_0)$$

$$\vartheta_r = -sin(\varphi_0)$$

$$\vartheta_\varphi = -(Q_0 sin(\varphi_0) + R_0 cos(\varphi_0))$$

Relevant zeros approximation

- **Elevator** input:
  $\theta_{\delta_e,sp} = -Z_w$
  $T_{1w,\delta_e} = -M_q + \frac{M_{\delta_e}}{Z_{\delta_e}} U_0$
  $Th_{2,\delta_e} = \sqrt{-Z_w U_0 M_{\delta_e}/Z_{\delta_e} + M_w U_0}$
  $Th_{3,\delta_e} = -Th_{2,\delta_e}$
  $Th_{1,\delta_e} = -X_u - (g - X_\alpha) Z_u / Z_\alpha$

- **Thrust** input:
  $T_{w,\delta_{th}} = -M_q + U_0 M_u / Z_u$
  $\theta_{\delta h} = \frac{Z_u M_w - M_u Z_w}{M_u}$

- **Aileron** input:
  $T_{2\beta,\delta_a} = -L'_p + \frac{L'_{\delta_a}}{N'_{\delta_a}} (N'_p - g/U_0)$
  $T_{1\beta,\delta_a} = \frac{g T_{2\beta,\delta_a}}{U_0 (L'_{\delta_a}/N'_{\delta_a} - L'_r)}$

- **Rudder** input:
  $T_{3\beta,\delta_r} = -(N'_r + L'_p + N'_{\delta_r}/Y_{\delta_r})$
  $T_{2\beta,\delta_r} = -L'_p + \frac{L'_{\delta_r}}{N'_{\delta_r}} (N'_p - g/U_0)$
  $T_{1\beta,\delta_r} = \frac{g T_{2\beta,\delta_r}}{U_0 (\frac{L'_{\delta_r}}{N'_{\delta_r} dr N'_r} - L'_r)}$