39 GRADIENTY

a) $f(x,y) = x^3 + xy^2 + 2 \qquad (1,-2)$

$\frac{\partial f}{\partial x} = 3x^2 + y^2 \Big|_{(1,-2)} = 3+4=7$

$\frac{\partial f}{\partial y} = 2yx = -4$

$\nabla f = (7,-4)$

b) $f(x,y) = \ln(x+\ln y) \qquad (e,1)$

$\frac{\partial f}{\partial x} = \frac{1}{x+\ln y} \Big|_{e,1} = \frac{1}{e} = \frac{1}{e}$

$\frac{\partial f}{\partial y} = \frac{1}{x+\ln y} \cdot \left(0 + \frac{1}{y}\right) = \frac{1}{(x+\ln y)y} = \frac{1}{e}$

$\text{grad} f = \left(\frac{1}{e}, \frac{1}{e}\right)$

c) $f(x,y) = (1+xy)^y \qquad (0,0)$

$\frac{\partial f}{\partial x} = y(1+xy)^{y-1} \cdot (0+y) = y^2(1+xy)^{y-1} \Big|_{0,0} = 0$

$\frac{\partial f}{\partial y} = y(1+xy)^{y-1} \cdot (0+x) = xy(1+xy)^{y-1} = 0$

d) $g(x,y,z) = x\sqrt{y} - e^z \ln y \qquad (1,1,0)$

$\frac{\partial g}{\partial x} = \sqrt{y} - 0 = \sqrt{y} \Big|_{1,1,0} = 1$

$\frac{\partial g}{\partial y} = \frac{1}{2}x \cdot \frac{1}{\sqrt{y}} - e^z \cdot \frac{1}{y} = \frac{x}{2\sqrt{y}} - \frac{e^z}{y} \Big|_{1,1,0} = \frac{1}{2} - \frac{1}{1} = -\frac{1}{2}$

$\frac{\partial g}{\partial z} = 0 - \ln y\, e^z \cdot 1 = -\ln y \cdot e^z = \Big|_{1,1,0} = 0$

$\text{grad}\, g(x,y,z) = \left[1, -\frac{1}{2}, 0\right]$

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b) $f(x,y) = x - \frac{y}{x^2} + y \qquad (1,1) \qquad \overline{v} = \left[\frac{3}{5}; -\frac{4}{5}\right]$

$\nabla f(x,y) = [3,0] \qquad \frac{\partial f}{\partial v} = \frac{9}{5}$

c) $g(x,y,z) = e^{x^2y-z} \qquad (-1,1,-1)$

$\frac{\partial f}{\partial x} = e^{x^2y-z} \cdot (2xy) \Big|_{(-1,1,-1)} = -2e^2$

$$\frac{\partial f}{\partial x} = e^{x^2y-z} \cdot (2xy)\Big|_{(-1,1,1)} = -2e^2$$

$$\frac{\partial f}{\partial y} = e^{x^2y-z} \cdot (x^2)\Big|_{(-1,1,-1)} = e^2$$

$$\frac{\partial f}{\partial z} = e^{x^2y-z} \cdot 1 \Big|_{(-1,1,-1)} = -e^2$$

$$\frac{\partial f}{\partial \bar{v}} = \bar{v} \circ (-2e^2, e^2, -e^2) =$$

$$= \left[\tfrac{2}{3}, -\tfrac{2}{3}, \tfrac{1}{3}\right] \circ \left[-2e^2, e^2, -e^2\right] = -\tfrac{4}{3}e^2 + \left(-\tfrac{2}{3}e^2\right) - \frac{e^2}{3} = -2\tfrac{1}{3}e^2$$

(38) Znaleźć wektor $\bar{v}$ dla którego pochodna kierunkowa

$f(x,y) = y - x^2 + 2\ln(xy)$ w punkcie $\left(-\frac{1}{2}, -1\right)$

wynosi 0

$$\frac{\partial f}{\partial x} = 0 - 2x + 2 \cdot \frac{1}{x\cancel{y}} \cdot \cancel{y} = \frac{2}{x} - 2x \Big|_{(-\frac{1}{2},-1)} = -4 + 1 = -3$$

$$\frac{\partial f}{\partial y} = 1 - 0 + \frac{2}{\cancel{x}y} \cdot \cancel{x} = 1 - \frac{2}{y} \Big|_{(-\frac{1}{2},-1)} = -1$$

$$\nabla f(x,y) = [-3, -1]$$

$$-3x_0 + (-1)y_0 = 0$$

$$\begin{cases} -3x = 1y \\ x^2 + y^2 = 1 \end{cases}$$

$$9x^2 + x^2 = 1 \qquad x^2 = \frac{1}{10} \qquad x = \sqrt{\frac{1}{10}}$$

$$y = \sqrt{1 - x^2} = \sqrt{1 - \frac{1}{10}} = \sqrt{\frac{9}{10}}$$

$$V_{x_1} = \frac{1}{\sqrt{10}} \qquad V_{x_2} = \frac{-1}{\sqrt{10}}$$

$$V_{y_1} = \frac{3}{\sqrt{10}} \qquad V_{y_2} = \frac{-3}{\sqrt{10}}$$

(40) a)

$$f(x,y) = x^3 + 3xy^2 - 51x - 24y$$

$$\frac{df}{\partial x} = 3x^2 + 3y^2 - 51$$

$$\frac{\partial f}{\partial y} = 6xy - 24$$

$$\begin{cases} 3x^2 + 3y^2 - 51 = 0 \\ 6xy - 24 = 0 \end{cases} \rightarrow \quad 3x^2 + 3 \cdot \frac{16}{x^2} - 51 = 0$$

$$x^2 + \frac{16}{x^2} - 17 = 0$$

$$xy = 4 \qquad y = \frac{4}{x}$$

$$x^2 + \frac{16}{x^2} - 17 = 0 \qquad x_1 = 1 \qquad x_2 = -1$$

$x^4 + 16 - 17x^2 = 0 \quad x_3 = 4 \quad x_4 = -4$

$\Delta = 225 \quad \sqrt{\Delta} = 15$

$x = \frac{17 \pm 15}{2} = 16$

$xy = 4 \rightarrow A(1,4)$
$B(-1,-4)$
$C(4,1)$
$D(-4,-1)$

$\frac{\partial f}{\partial x} = 3x^2 + 3y^2 - 51$

$\frac{\partial f}{\partial x \partial x} = 6x$

$\frac{\partial f}{\partial y} = 6xy - 24$

$\frac{\partial f}{\partial y \partial y} = 6x$

$\frac{\partial f}{\partial x \partial y} = 6y$

$H = \begin{vmatrix} 6x & 6y \\ 6y & 6x \end{vmatrix}$

$H_1(1,4) = \begin{vmatrix} 6 & 24 \\ 24 & 6 \end{vmatrix} = -540$

$H_2(4,1) = \begin{vmatrix} 24 & 6 \\ 6 & 24 \end{vmatrix} = 540$ MAKSIMUM $z = 152$

$H_2(-4,-1) = \begin{vmatrix} -24 & -6 \\ -6 & -24 \end{vmatrix} = 540$ MINIMUM $z = -152$

$H_4(-1,-4) = -540$

d) $f(x,y) = y\sqrt{x} - y^2 - x + 6y$

max : $f(4,4) = 12$

g) $f(x,y) = xy + \ln y + x^2$

$\frac{\partial f}{\partial x} = y + 2x$

$\frac{\partial f}{\partial y} = x + \frac{1}{y}$

$\begin{cases} 2x + y = 0 \\ x + \frac{1}{y} = 0 \end{cases}$

$\frac{-2}{y} + y = 0 \quad y^2 - 2 = 0 \quad y^2 = 2$

~~$y_1 = -\sqrt{2}$~~ ~~$x_1 = \frac{1}{\sqrt{2}}$~~

$y_2 = \sqrt{2}$ $x_2 = \frac{-1}{\sqrt{2}}$

$f_{xx} = 2$

$f_{xy} = 1$

$f_{yy} = \frac{-1}{y^2}$

$H = \begin{vmatrix} 2 & 1 \\ 1 & \frac{-1}{y^2} \end{vmatrix} = \frac{-2}{y^2} - 2$