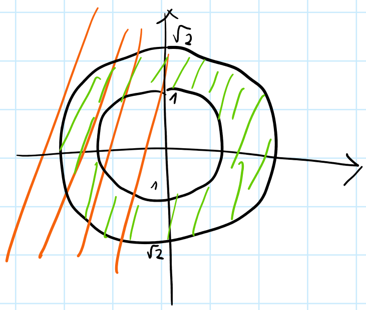
Ćwiczenia 14

niedziela, 25 marca 2018 12:49

(51) b) $\iint_D xy^2\,dx\,dy \quad D:\ x \geq 0\ ;\ 1 \leq x^2+y^2 \leq 2$



$1 \leq r^2 \leq 2$

$1 \leq r \leq \sqrt{2}$

$$\iint_D xy^2\,dx\,dy = \iint_\Delta r^3 \cos\varphi \sin^2\varphi\, r\, d\varphi\, dr =$$

$$= \iint_\Delta r^4 \cos\varphi \sin^2\varphi\, dr\, d\varphi = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi \int_1^{\sqrt{2}} r^4 \cos\varphi \sin^2\varphi\, dr =$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\varphi \sin^2\varphi\, d\varphi \left[\frac{r^5}{5}\right]_1^{\sqrt{2}} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\varphi \sin^2\varphi\, d\varphi \cdot \left(\frac{2^{\frac{5}{2}}}{5} - \frac{2}{5}\right) =$$

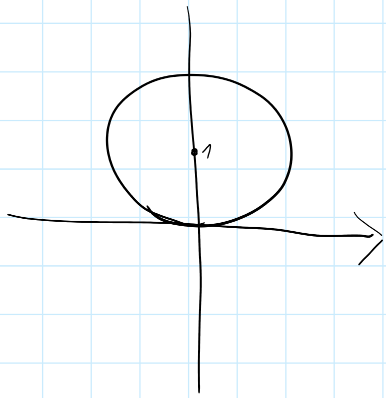
$$= \underbrace{\left(\frac{2^{\frac{5}{2}} - 2}{5}\right)}_{Z} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\varphi \sin^2\varphi\, d\varphi = \left| \begin{array}{l} t = \sin\varphi \\ dt = \cos\varphi\, d\varphi \\ \begin{array}{c|c|c} t & -1 & 1 \\ \hline \sin & -\frac{\pi}{2} & \frac{\pi}{2} \end{array} \end{array} \right| = Z \int_{-1}^{1} t^2\, dt =$$

$$= Z \cdot \frac{t^3}{3}\Big|_{-1}^{1} = Z \cdot \left(\frac{1}{3} + \frac{1}{3}\right) = \frac{2}{3} Z = \frac{2}{3}\left(\frac{2^{\frac{2}{5}} - 1}{5}\right)$$

d) $\iint_D x^2\,dx\,dy \qquad D:\ x^2+y^2 \leq 2y$

$D:\ x^2 + (y^2 - 2y + 1) \leq 1$

$x^2 + (y-1)^2 \leq 1$



$r^2 \leq 2r\sin\varphi$

$$r^2 \leqslant 2r\sin\varphi$$

$$0 \leq r \leqslant 2\sin\varphi$$

$$\iint_{\Delta} r^2\cos^2\varphi\, r\,dr\,d\varphi$$

$$\int_0^{\pi} d\varphi \int_0^{2\sin\varphi} r^3\cos^2\varphi\, d\varphi = \int_0^{\pi} d\varphi\cos^2\varphi \int_0^{2\sin\varphi} r^3\,dr =$$

$$= \int_0^{\pi} d\varphi\cos^2\varphi \cdot \left[\frac{r^4}{4}\right]_0^{2\sin\varphi} = \int_0^{\pi} d\varphi\cos^2\varphi \cdot \left(\frac{2^4\sin^4\varphi}{4}\right) =$$

$$= 4\int_0^{\pi} \cos^2\varphi\sin^4\varphi\, d\varphi = 4\int_0^{\pi} \cos^2\varphi\left(1-\cos^2\varphi\right)^2 d\varphi =$$

$$= 4\int_0^{\pi} \frac{2\cos2\varphi+1}{2} \cdot \left(1-\frac{2\cos2\varphi+1}{2}\right)^2 d\varphi = \frac{1}{2}\int_0^{\pi}(2\cos2\varphi+1)\left(1-\frac{2\cos2\varphi+1}{2}\right)^2 d\varphi =$$

$$= \frac{1}{2}\int_0^{\pi}(2\cos2\varphi+1)(1-2\cos2\varphi)^2\,d\varphi =$$

$$= \frac{1}{2}\int_0^{\pi}(2\cos2\varphi+1)\,(1-4\cos2\varphi+4\cos^2 2\varphi)\,d\varphi =$$

$$= \frac{1}{2}\int_0^{\pi}(2\cos2\varphi+8\cos^3 2\varphi+8\cos^4 2\varphi+1-4\cos2\varphi+4\cos^2 2\varphi) =$$

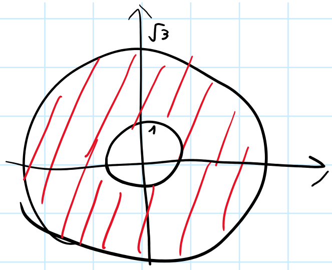
$$= \frac{1}{2}\int_0^{\pi}(4\cos^2 2\varphi-2\cos2\varphi+1)\,d\varphi =$$

$$= \frac{1}{2}\int_0^{\pi}(2\cdot 2\cos4\varphi+1)-\cos2\varphi+1)\,d\varphi =$$

$$= \int_0^{\pi} 2\cos4\varphi-\cos2\varphi+\frac{3}{2})\,d\varphi = \left[2\cdot\frac{\sin4\pi\varphi}{4}-\frac{\sin2\varphi}{2}+\frac{3}{2}\varphi\right]_0^{\pi} =$$

$$= \frac{3}{2}\pi$$

h) $\iint_D \ln(1+x^2+y^2)\,dx\,dy \qquad D:\ 1 \leq x^2+y^2 \leqslant 9$



$1 \leq r^2 \leq 9$

$1 \leq r \leq 3 \qquad 0 \leq \varphi \leq 2\pi$

$$\int_0^{2\pi} d\varphi \int_1^3 \ln(1+r^2\cos^2\varphi + r^2\sin^2\varphi)\, r\, dr = \int_0^{2\pi} d\varphi \int_1^3 \ln(1+r^2)\, r\, dr =$$

$$= \int_0^{2\pi} d\varphi \int_1^3 (r)' \ln(1+r^2)\, r\, dr = \int_0^{2\pi} d\varphi \int_1^3 r \ln(1+r^2)\, dr =$$

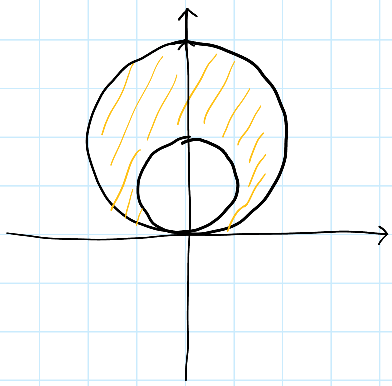
$$= \left| \begin{array}{l} t = 1+r^2 \\ dt = 2r \end{array} \right| = \frac{1}{2}\Big[t \ln t\Big]_1^3 \int_1^3 \frac{t}{t}\, dt = \frac{1}{2}(3\ln 3 - \ln 1)$$

$$= \frac{1}{2}(10\ln 10 - 2\ln 2)(10-2) = \frac{1}{2}(10\ln 10 - 2\ln 2)(-8) =$$

$$= \cancel{\tfrac{1}{2}}\, 4 \ln \frac{10^{10}}{2^2} = \ln \frac{10^5}{2}$$

52 b) D: $x^2 + y^2 - 2y = 0 \qquad x^2 + y^2 - 4y = 0$

$x^2 + (y-1)^2 = 1 \qquad x^2 + (y-2)^2 = 4$



$1 \leq r \leq 2$

$\varphi \in \langle 0; \pi \rangle$

$$\int_0^{\pi} d\varphi \int_1^2$$

$x_1 = r\cos\varphi$

$y_1 = r\sin\varphi$

$$2r\sin\varphi \leqslant r^2 \leqslant 4r\sin\varphi$$

$$2\sin\varphi \leqslant r \leqslant 4r\sin\varphi$$

$$\int_0^{\pi} d\varphi \int_{2\sin\varphi}^{4\sin\varphi} r\,dr \Rightarrow \int_0^{\pi} d\varphi \left[\frac{r^2}{2}\right]_{2\sin\varphi}^{4\sin\varphi} = \int_0^{\pi} d\varphi \left[\frac{16\sin^2\varphi}{2} - \frac{4\sin^2\varphi}{2}\right] =$$

$$= 6\int_0^{\pi} \sin^2\varphi\, d\varphi = 6\int_0^{\pi} \frac{1-\cos\varphi}{2}\, d\varphi = 3\int_0^{\pi}(1-\cos 2\varphi)\,d\varphi =$$

$$= 3\left[\varphi - \frac{\sin 2\varphi}{2}\right]_0^{\pi} = 3^3\left[\pi - 0 - 0 + 0\right] = 3\pi$$