

Ćwiczenia 15

środa, 11 kwietnia 2018

23:29

60c (walcowe): $x^2 + y^2$ $f(x, y, z) = r^2$

$$V: x^2 + y^2 \leq 4, \quad 1-x \leq z \leq 2-x$$

$$1-x \leq z \leq 2-x$$

$$\iiint_V (x^2 + y^2) dx dy dz$$

$$x = r \cos \varphi$$

$$y = r \sin \varphi$$

$$r^2 \leq 4$$

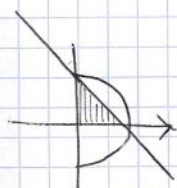
$$\varphi \in [0, 2\pi]$$

$$r \in [0, 2]$$

$$\int_0^{2\pi} d\varphi \int_0^2 r dr \int_{1-r}^{2-r} (r^2 \cos^2 \varphi + r^2 \sin^2 \varphi) r dh = \int_0^{2\pi} d\varphi \int_0^2 r^3 dr \int_{1-r}^{2-r} dh = \int_0^{2\pi} d\varphi \int_0^2 r^3 dr [h]_{1-r}^{2-r} =$$

$$= \int_0^{2\pi} d\varphi \int_0^2 r^3 (2-r \cos \varphi - (1-r \cos \varphi)) dr = \int_0^{2\pi} d\varphi \int_0^2 r^3 dr = \frac{1}{4} [r^4]_0^2 d\varphi =$$

$$= \frac{2^4}{4} \int_0^{2\pi} d\varphi = 2^2 \cdot 2\pi = 2^3 \pi = 8\pi$$



(61) d) $\iiint_V (x+y+z) dx dy dz$ $V: x^2 + y^2 \leq 1, \quad 0 \leq z \leq 2-x-y, \quad x \geq 0$

$$r^2 \leq 1, \quad 0 < r < 1, \quad \varphi \in [-\frac{\pi}{2}, \frac{\pi}{2}]$$

$$x = r \cos \varphi, \quad y = r \sin \varphi, \quad z = h, \quad 0 \leq h \leq 2-x-y$$

$$0 \leq h \leq 2-r \cos \varphi - r \sin \varphi, \quad 0 \leq 2-x-y, \quad y \leq 2-x$$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi \int_0^1 r dr \int_0^{2-r \cos \varphi - r \sin \varphi} (r^2 \cos^2 \varphi + r^2 \sin^2 \varphi + h) dh =$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi \int_0^1 r dr \int_0^{2-r \cos \varphi - r \sin \varphi} [r^2 \cos^2 \varphi + r^2 \sin^2 \varphi + \frac{h^2}{2}] dh = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi \int_0^1 r dr$$

S. 12/13:

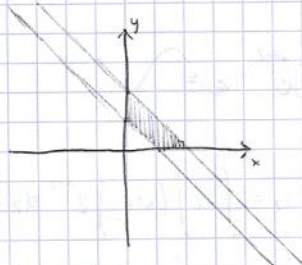
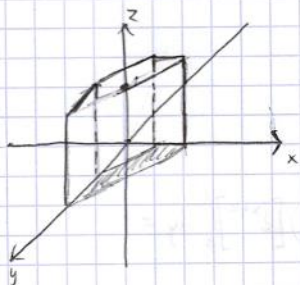
$$f(x, y, z) = z$$

$$V: x+y=1 \quad x+y=2 \quad x=0 \quad y=0 \quad z=0 \quad z=1$$

$$y=1-x$$

$$y=2-x$$

$$y=0$$



$$\iiint_V z \, dx \, dy \, dz = \int_0^1 \int_0^2 \int_0^{1-y} z \, dx \, dy \, dz = \int_0^1 \int_0^2 \frac{1}{2} (1-y)^2 \, dy \, dz = \int_0^1 \frac{1}{6} (1-y)^3 \, dy \, dz = \int_0^1 \frac{1}{24} (1-y)^4 \, dy \, dz = \frac{1}{24} \int_0^1 (1-y)^4 \, dy \, dz = \frac{1}{24} \left[-\frac{1}{5} (1-y)^5 \right]_0^1 \, dz = \frac{1}{24} \left(-\frac{1}{5} (1-1)^5 + \frac{1}{5} (1-0)^5 \right) \, dz = \frac{1}{24} \cdot \frac{1}{5} \, dz = \frac{1}{120} \, dz = \frac{1}{120} \int_0^1 dz = \frac{1}{120} \cdot 1 = \frac{1}{120}$$

$$= \int_0^1 \int_0^2 \left[\frac{1}{2} (1-y)^2 \right]_0^{1-y} \, dy \, dz = \int_0^1 \int_0^2 \frac{1}{2} (1-y)^2 \, dy \, dz = \int_0^1 \left[\frac{1}{2} \left(\frac{1}{3} (1-y)^3 \right) \right]_0^{1-y} \, dz = \int_0^1 \left[\frac{1}{6} (1-y)^3 \right]_0^{1-y} \, dz = \int_0^1 \frac{1}{6} (1-y)^3 \, dz = \frac{1}{6} \int_0^1 (1-y)^3 \, dz = \frac{1}{6} \left[-\frac{1}{4} (1-y)^4 \right]_0^1 = \frac{1}{6} \left(-\frac{1}{4} (1-1)^4 + \frac{1}{4} (1-0)^4 \right) = \frac{1}{6} \cdot \frac{1}{4} = \frac{1}{24}$$

$$\iiint_V z \, dx \, dy \, dz = \int_0^1 \int_0^{1-y} \int_0^{1-y} z \, dx \, dy \, dz = \int_0^1 \int_0^{1-y} \frac{1}{2} (1-y)^2 \, dy \, dz = \int_0^1 \frac{1}{6} (1-y)^3 \, dy \, dz = \frac{1}{6} \int_0^1 (1-y)^3 \, dz = \frac{1}{6} \left[-\frac{1}{4} (1-y)^4 \right]_0^1 = \frac{1}{6} \left(-\frac{1}{4} (1-1)^4 + \frac{1}{4} (1-0)^4 \right) = \frac{1}{6} \cdot \frac{1}{4} = \frac{1}{24}$$

$$= \int_0^1 \int_0^{1-y} \left[\frac{z^2}{2} \right]_0^{1-y} \, dy \, dz = \int_0^1 \int_0^{1-y} \frac{1}{2} (1-y)^2 \, dy \, dz = \int_0^1 \frac{1}{6} (1-y)^3 \, dy \, dz = \frac{1}{6} \int_0^1 (1-y)^3 \, dz = \frac{1}{6} \left[-\frac{1}{4} (1-y)^4 \right]_0^1 = \frac{1}{6} \left(-\frac{1}{4} (1-1)^4 + \frac{1}{4} (1-0)^4 \right) = \frac{1}{6} \cdot \frac{1}{4} = \frac{1}{24}$$

60d) $g(x, y, z) = x^2 y^2$ $U: 0 \leq x \leq y \leq z \leq 1$

$$0 \leq x \leq 1 \quad 0 \leq y \leq 1 \quad 0 \leq z \leq 1$$

$$\iiint_U g(x, y, z) \, dx \, dy \, dz = \int_0^1 \int_0^1 \int_0^1 x^2 y^2 \, dx \, dy \, dz = \int_0^1 \int_0^1 \frac{1}{3} x^3 y^2 \, dy \, dz = \int_0^1 \frac{1}{3} \left[\frac{1}{3} x^3 y^3 \right]_0^1 \, dz = \int_0^1 \frac{1}{9} x^3 \, dz = \frac{1}{9} \int_0^1 x^3 \, dz = \frac{1}{9} \left[\frac{1}{4} x^4 \right]_0^1 = \frac{1}{9} \cdot \frac{1}{4} = \frac{1}{36}$$

$$= \int_0^1 \int_0^1 \frac{1}{3} x^2 y^2 (1-y) \, dy \, dz = \int_0^1 \frac{1}{3} x^2 \int_0^1 y^2 - y^3 \, dy \, dz = \int_0^1 \frac{1}{3} x^2 \left[\frac{1}{3} y^3 - \frac{1}{4} y^4 \right]_0^1 \, dz = \int_0^1 \frac{1}{3} x^2 \left(\frac{1}{3} - \frac{1}{4} \right) \, dz = \int_0^1 \frac{1}{3} x^2 \left(\frac{1}{12} \right) \, dz = \int_0^1 \frac{1}{36} x^2 \, dz = \frac{1}{36} \int_0^1 x^2 \, dz = \frac{1}{36} \left[\frac{1}{3} x^3 \right]_0^1 = \frac{1}{36} \cdot \frac{1}{3} = \frac{1}{108}$$

$$= \int_0^1 \left(\frac{x^3}{9} - \frac{x^3}{4} - \frac{x^3}{3} + \frac{x^3}{4} \right) \, dx = \left[\frac{x^4}{36} - \frac{x^4}{12} - \frac{x^4}{18} + \frac{x^4}{28} \right]_0^1 = \frac{1}{36} - \frac{1}{12} - \frac{1}{18} + \frac{1}{28} = \frac{1}{126}$$

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$$d) \iiint_V (x+y) e^{x+z} dx dy dz \quad V = [0,1] \times [0,1] \times [0,1]$$

$$\int_0^1 \int_0^1 \int_0^1 (x+y) e^{x+z} dz dy dx =$$

$$\int_0^1 dx \int_0^1 (x+y) dy \int_0^1 e^{x+z} dz = \int_0^1 dx \int_0^1 (x+y) \int_0^1 e^{x+z} dz = \int_0^1 dx \int_0^1 (x+y) [e^{x+z}]_0^1 dy =$$

$$= \int_0^1 dx \int_0^1 (x+y) (e^{x+1} - e^x) dy = \int_0^1 (e^{x+1} - e^x) dx \int_0^1 x+y dy =$$

$$= \int_0^1 (e^{x+1} - e^x) \left[xy + \frac{y^2}{2} \right]_0^1 dx = \int_0^1 (e^{x+1} - e^x) \left(x + \frac{1}{2} \right) dx =$$

$$= \int_0^1 (e^{x+1} - e^x) \left(x + \frac{1}{2} \right) dx = \int_0^1 (e^{x+1} - e^x) \left(x + \frac{1}{2} \right) dx = *$$

$$= \left| \begin{array}{l} u = e^x \\ v = x + \frac{1}{2} \end{array} \right| = \left| \begin{array}{l} u' = e^x \\ v' = 1 \end{array} \right| =$$

$$= e^x \left(x + \frac{1}{2} \right) - \int e^x dx = e^x \left(x + \frac{1}{2} \right) - e^x = e^x \left(x + \frac{1}{2} - 1 \right) =$$

$$* = (e-1) \left(e - \frac{1}{2} - \left(1 - \frac{1}{2} \right) \right) = (e-1)^2 = e^2 - 2e + 1$$