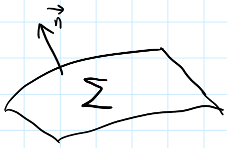
Na ćwiczeniach: całki krzywoliniowe niezorientowane
twierdzenie Greena

WYKŁAD 6:

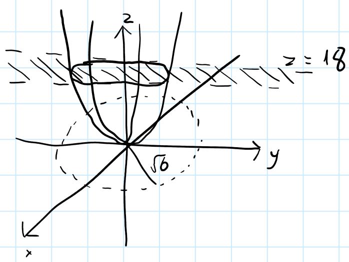


$$\iint_{\Sigma} \vec{F}(\vec{r}) \circ d\vec{S} = \iint_{\Sigma} P\,dydz + Q\,dzdx + R\,dxdy \stackrel{!}{=} \iint_{D} \left[\vec{F}(\vec{r}(u,v)) \circ (\vec{r}_u \times \vec{r}_v)\right] du\,dv$$

PRZYKŁAD

$$\iint_{\Sigma} x\,dydz + y\,dzdx + z\,dxdy = *$$

Σ - dolna strona powierzchni paraboloidy $z = 3(x^2+y^2)$
odciętej płaszczyzną $z = 18$



$x = v\cos u$ $\quad u \in \langle 0; 2\pi \rangle$
$y = v\sin u$
$z = 3v^2$ $\quad v \in \langle 0; \sqrt{6} \rangle$

$\vec{r}_u \times \vec{r}_v = [6v^2\cos u, 6v^2\sin u, -v]$ $\quad -v < 0$ ok kierunek

$\vec{F} = [x, y, z] = \vec{r}$

$\vec{r}_u = [-v\sin u, v\cos u, 0]$

$\vec{r}_v = [\cos u, \sin u, 6v]$

$$* = \iint_{D} (6v^3\cos^2 u + 6v^3\sin^2 u - 3v^3)\,du\,dv = \iint_{D} 6v^3 - 3v^3\,du\,dv = 3\iint_{D} v^3\,du\,dv =$$

$$= 3\int_0^{2\pi} du \int_0^{\sqrt{6}} v^3\,dv = 3\cdot 9\int_0^{2\pi} du = 27\cdot 2\pi = 54\pi$$

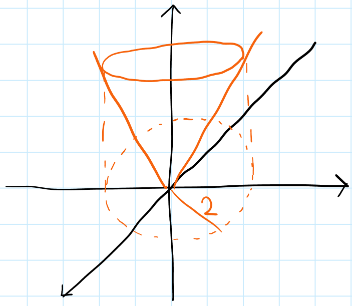
$$\left[\frac{v^4}{4}\right]_0^{\sqrt{6}} = 9$$

PRZYKŁAD:

$$\iint_{S} (y-z)\,dydz + (z-x)\,dzdx - 2y\,dxdy = *$$

Σ

Po zewnętrznej stronie stożka $z=\sqrt{x^2+y^2}$
odciętej walcem $x^2+y^2=4$



$x^2+y^2=4 \rightarrow z=2$

$$\Sigma:\begin{cases} x=v\cos u \\ y=v\sin u \\ z=v \end{cases} \qquad D:\begin{cases} u\in\langle 0;2\pi\rangle \\ v\in\langle 0;2\rangle \end{cases}$$

$$\vec{r}_u\times\vec{r}_v=[v\cos u,\ v\sin u,\ -v]$$

$$\vec{r}_u=[-v\sin u,\ v\cos u,\ 0]$$
$$\vec{r}_v=[\cos u,\ \sin u,\ 1]$$

$$\ast=\iint_D\left[v^2(\sin u-1)\cos u+v^2(1-\cos u)\sin u+2v^2\sin u\right]du\,dv=$$

$$=\iint_D v^2\left[\sin u\cos u-\cos u+\sin u-\sin u\cos u+2\sin u\right]du\,dv=$$

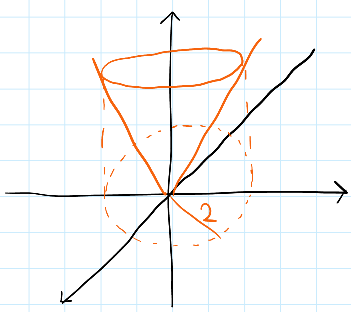
$$=\iint_D v^2(3\sin u-\cos u)\,du\,dv=\int_0^{2\pi}(3\sin u-\cos u)\,du\int_0^{2\pi}v^2\,dv=$$

$$=\frac{8}{3}\int_0^{2\pi}(3\sin u-\cos u)\,du$$

PRZYKŁAD 3

$$\iint_\Sigma (y-z)\,dy\,dz+(z-x)\,dz\,dx-2y\,dx\,dy=\ast$$

Po zewnętrznej stronie stożka $z=\sqrt{x^2+y^2}$
odciętej walcem $x^2+y^2=4$ w I oktancie



$x^2+y^2=4 \rightarrow z=2$

$$\Sigma:\begin{cases} x=v\cos u \\ y=v\sin u \\ z=v \end{cases} \qquad D:\begin{cases} u\in\langle 0;\frac{\pi}{2}\rangle \\ v\in\langle 0;2\rangle \end{cases}$$

$$\vec{r}_u\times\vec{r}_v=[v\cos u,\ v\sin u,\ -v]$$

$$\vec{r}_u=[-v\sin u,\ v\cos u,\ 0]$$
$$\vec{r}_v=[\cos u,\ \sin u,\ 1]$$

$\vec{r_u} = [-v\sin u, v\cos u, 0]$

$\vec{r_v} = [\cos u,\ \sin u,\ 1]$

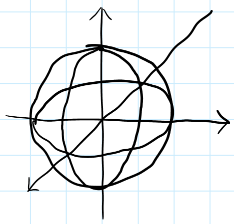
$* = \frac{-8}{3}\left[+3\cos u + \sin u\right]_0^{\frac{\pi}{2}} = \frac{16}{3}$

PRZYKŁAD 4:

Obliczyć strumień pola $\vec{F}(x,y,z) = [x,\ x+y,\ z-y]$

Σ – zewnętrzna strona sfery $x^2+y^2+z^2=4$ położona w I oktancie

$$\Phi(\vec{F}) = \iint_{\Sigma} \vec{F}(\vec{r}) \circ d\vec{S} = \iint_{\Sigma} x\,dydz + (x+y)\,dzdx + (z-y)\,dxdy$$



$$\begin{cases} x = 2\cos u\cos v \\ y = 2\sin u\cos v \\ z = 2\sin v \end{cases}$$

$$D: \begin{cases} u \in \langle 0; \frac{\pi}{2}\rangle \\ v \in \langle 0; \frac{\pi}{2}\rangle \end{cases}$$

$$\begin{array}{cccc} & i & j & k \\ \vec{r_u} = & [-2\sin u\cos v, & 2\cos u\cos v, & 0] \end{array} \qquad \vec{r_v} = [-2\cos u\sin v,\ -2\sin u\sin v,\ 2\cos v]$$

$$\vec{r_u} \times \vec{r_v} = [4\cos u\cos^2 v,\ 4\sin u\cos^2 v,\ 4\sin v\cos v]\,dv$$

$$\Phi(\vec{F}) = 8\iint_D \left(\cos^2 u\cos^3 v + \cos v(\cos u + \sin u)\sin u + (\sin v - \sin u\cos v)\sin v\cos v\right)du\,dv =$$

$$= 8\left[\iint_D \cos^3 v(1 + \cos u\sin u)\,du\,dv + \iint_D \sin^2 v\cos v\,du\,dv - \iint_D \sin u\sin v\cos^2 v\,du\,dv\right] =$$

$$= 8\left[\int_0^{\frac{\pi}{2}} du \int_0^{\frac{\pi}{2}} (1-\sin^2 v)\cos v\,dv + \int_0^{\frac{\pi}{2}} du \int_0^{\frac{\pi}{2}} \cos^3 v\cos u\sin u\,dv + \int_0^{\frac{\pi}{2}} du \int_0^{\frac{\pi}{2}} \sin^2 v\cos v\,dv - \int_0^{\frac{\pi}{2}} du\,\sin u \int_0^{\frac{\pi}{2}} \cos^2 v\sin v\,dv\right] =$$

$$= 8\left[\frac{\pi}{2}\cdot\frac{2}{3} + \int_0^{\frac{\pi}{2}} \cos u\,du \int_1^0 t^3\,dt + \frac{1}{3}\int_0^{\frac{\pi}{2}} du + \int_0^{\frac{\pi}{2}} \sin u\,du \int_1^0 t^2\,dt\right] =$$

$$= 8\left[\frac{\pi}{3} + \frac{1}{4}\cdot 1 + \frac{\pi}{6} - \frac{1}{3}\right] = 8\left[\frac{\pi}{2} - \frac{1}{12}\right] = 4\pi - \frac{2}{3}$$