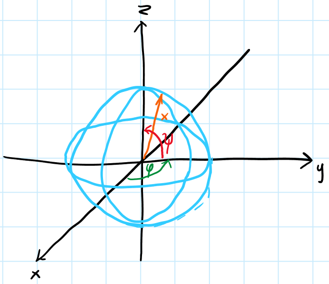
Wykład 15 - współrzędne sferyczne

niedziela, 15 kwietnia 2018 12:48

WSPÓŁRZĘDNE SFERYCZNE



$x = r\cos\varphi\cos\psi$

$y = r\sin\varphi\cos\psi$

$z = r\sin\psi$

$r \geq 0$

$\varphi \in \langle 0;2\pi)$

$\psi \in \langle -\frac{\pi}{2};\frac{\pi}{2}\rangle$

$x^2+y^2+z^2=r^2$

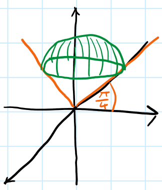
$dx\,dy\,dz \rightarrow r^2\cos\psi\, d\varphi\, dr\, d\psi$

PRZYKŁAD:

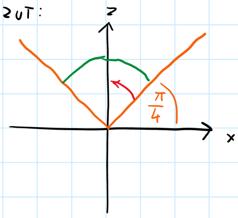
$\iiint_V (x^2+y^2)\,dx\,dy\,dz$

stożek   górna część sfery

$\sqrt{x^2+y^2} \leq z \leq \sqrt{1-x^2-y^2}$



RZUT:



$0 \leq \varphi \leq 2\pi$

$0 \leq r \leq 1$

$\frac{\pi}{4} \leq \psi \leq \frac{\pi}{2}$

$$\iiint_V (x^2+y^2)\,dx\,dy\,dz = \int_0^{2\pi} d\varphi \int_0^1 dr \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} r^2\cos^2\psi\, r^2\cos\psi\, d\psi =$$

$$= \left(\int_0^{2\pi} d\varphi\right)\left(\int_0^1 r^4\,dr\right)\left(\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos^3\psi\, d\psi\right) = \left[\varphi\right]_0^{2\pi}\left[\frac{r^5}{5}\right]_0^1\left[\sin^3\psi - \sin\psi\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}} =$$

$$= 2\pi \cdot \frac{1}{5} \cdot \frac{\sqrt{2}}{4} = \frac{\sqrt{2}\pi}{10}$$

OBJĘTOŚĆ BRYŁY PRZEZ CAŁKĘ POTRÓJNA

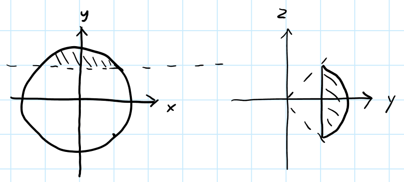
$|V| = \iiint_V 1\,dx\,dy\,dz$

PRZYKŁAD:

63d Obszar ograniczony powierzchniami

$x^2+y^2+z^2=2$ $y \geq 1$

$x^2+y^2+z^2=2 \qquad y \geqslant 1$



$$\frac{1}{\sin\varphi\cos\psi} \leqslant r \leqslant \sqrt{2}$$

$$\tfrac{1}{4}\pi \leqslant \varphi \leqslant \tfrac{3}{4}\pi$$

$$-\frac{\pi}{4} \leqslant \psi \leqslant \frac{\pi}{4}$$

$$y = 1 = r\sin\varphi\cos\psi$$

$$r = \frac{1}{\sin\varphi\cos\psi}$$

$$|V| = \iiint_V dx\,dy\,dz = \int_{\frac{\pi}{4}}^{\frac{3}{4}\pi} d\varphi \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} d\psi \int_{\frac{1}{\sin\varphi\cos\psi}}^{\sqrt{2}} r^2\cos\psi\, dr =$$

$$\int_{\frac{\pi}{4}}^{\frac{3}{4}\pi} d\varphi \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos\psi\, d\psi \int_{\frac{1}{\sin\varphi\cos\psi}}^{\sqrt{2}} = \int_{\frac{\pi}{4}}^{\frac{3}{4}\pi} d\varphi \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos\psi\, d\psi \left[\frac{r^3}{3}\right]_{\frac{1}{\sin\varphi\cos\psi}}^{\sqrt{2}} =$$

$$= \frac{1}{3}\int_{\frac{\pi}{4}}^{\frac{3}{4}\pi} d\varphi \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos\psi\, d\psi \left(2\sqrt{2} - \frac{1}{\sin^3\varphi\cos^3\psi}\right) d\psi =$$

$$= \frac{1}{3}\int_{\frac{\pi}{4}}^{\frac{3}{4}\pi} d\varphi \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left(2\sqrt{2}\cos\psi - \frac{1}{\sin^3\varphi\cos^2\psi}\right) d\psi =$$

$$= \frac{1}{3}\int_{\frac{\pi}{4}}^{\frac{3}{4}\pi} d\varphi \left(2\sqrt{2}\sin\psi - \frac{1}{\sin^3\varphi}\cdot \operatorname{tg}\psi\right)\Big|_{-\frac{\pi}{4}}^{\frac{\pi}{4}} =$$

$$= \frac{1}{3}\int_{\frac{\pi}{4}}^{\frac{3}{4}\pi} \left(4 - \frac{2}{\sin^3\varphi}\right) d\varphi =$$

$$\int \frac{\sin\varphi}{\sin^4\varphi}\, d\varphi = \int \frac{\sin\varphi\, d\varphi}{(1-\cos^2\varphi)^2} = -\int \frac{dt}{(1-t^2)^2}$$

$$\frac{1}{(1-t)^2(1+t)^2} = \frac{-A}{t-1} + \frac{B}{(t-1)^2} + \frac{C}{t+1} + \frac{D}{(1+t)^2} =$$

$$= A(1-t)(t+1)^2 + B(t+1)^2 + C(t+1)(t-1)^2 + D(t-1)^2$$

<po obliczeniu> : $A = \frac{1}{4} \quad B = \frac{1}{4} \quad C = \frac{1}{4} \quad D = \frac{1}{4}$

$$= -\frac{1}{4}\int \left(\frac{1}{1-t} + \frac{1}{(1-t)^2} + \frac{1}{1+t} + \frac{1}{(1+t)^2}\right) dt =$$

$$= -\frac{1}{4}\left[-\ln|1-t| + \ln|1+t| + \frac{1}{1-t} + \frac{-1}{1+t}\right]_{-\frac{\sqrt{2}}{2}}^{\frac{\sqrt{2}}{2}}$$

$$= \left[\ln\left|\frac{1+t}{1-t}\right| + \frac{2t}{1-t^2}\right]_{-\frac{\sqrt{2}}{2}}^{\frac{\sqrt{2}}{2}} = \ln\left|\frac{1+\frac{\sqrt{2}}{2}}{1-\frac{\sqrt{2}}{2}}\right| - \ln\left|\frac{1-\frac{\sqrt{2}}{2}}{1+\frac{\sqrt{2}}{2}}\right| + \frac{\sqrt{2}}{\frac{1}{2}} - \frac{-\sqrt{2}}{\frac{1}{2}} =$$

$$= \ln\left|\frac{(1+\frac{\sqrt{2}}{2})^2}{(1-\frac{\sqrt{2}}{2})^2}\right| + 4\sqrt{2}$$