Core course for *App.AI*, *Bioinfo.*, *Data Sci.&Eng.*, *Dec.Analytics*, *Q.Fin.*, *Risk Mgmt.* and *Stat.* Majors:

STAT2601A Probability and Statistics I (2023-2024 First Semester)

Chapter 2: Counting

The branch of mathematics which studies the finite or countable discrete structures, such as the number of different ways of arranging things is called *combinatorial analysis.*

This chapter applies the principles in *combinatorics* to the counting procedures and methods. We can then compute the *combinatorial probability* which is consistent to the "probability of occurrence of an event" defined as in the classical school introduced in the last chapter.

2.1 Counting Rules

Theorem 2.1. (The Product Rule)
Suppose that a procedure can be broken down into a sequence of two tasks. If there are n_1 ways to do the first task and for each of these ways of doing the first task, there are n_2 ways to do the second task, then there are $n_1 n_2$ ways to do the procedure.

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Corollary 2.1. (Multiplication Principle)
If an operation consists of a sequence of k separate steps of which the first can be performed in n_1 ways, followed by the second n_2 ways, and so on until the k-th can be performed in n_k ways, then the operation consisting of k steps can be performed by $n_1 \times n_2 \times \cdots \times n_k = \prod_{i=1}^{k} n_i$ ways.

▶

Example 2.1.

How many different ordered outcomes are possible when a red die, a blue die and a white die are rolled?

Solution:

1st step:	rolling a red die	6 ways
2nd step:	rolling a blue die	6 ways
3rd step:	rolling a white die	6 ways

Total number of possible outcomes $= 6 \times 6 \times 6 = 216$.

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Example 2.2.

How many ways can four people be seated in a row of four chairs?

Solution:

1st step:	people in the 1st chair	4 ways
2nd step:	people in the 2nd chair	3 ways
3rd step:	people in the 3rd chair	2 ways
4th step:	people in the 4th chair	1 way

Total number of possible outcomes $= 4 \times 3 \times 2 \times 1 = 24$.

Example 2.3.

How many different 6-place license plates are possible if the first 2 places are for letters and the other 4 for numbers?

Solution: $26 \times 26 \times 10 \times 10 \times 10 \times 10 = 6760000$.

Theorem 2.2. (The Sum Rule)
If a task can be done either in one of n_1 ways or in one of n_2 ways, where none of the set of n_1 ways is the same as any of the set of n_2 ways, then there are $n_1 + n_2$ ways to do the task.

Example 2.4.

Suppose that either a member of the Science Faculty or a student who is a statistics major is chosen as a representative to a university committee. How many different choices are there for this representative if there are 37 members of the Science Faculty and 83 statistics majors and no one is both a faculty member and a student?

Solution: $37 + 83 = 120$.

Theorem 2.3. (The Subtraction Rule)
If a task can be done in either n_1 ways or n_2 ways, then the number of ways to do the task is $n_1 + n_2$ minus the number of ways to do the task that are common to the two different ways.

The subtraction rule is a simple case of the *inclusion-exclusion principle* (to be introduced in the next chapter), when it is used to count the number of elements in the union of two sets.

Example 2.5.

A computer company receives 350 applications from computer graduates for a job planning a line of new web servers. Suppose that 220 of these applicants majored in computer science, 147 majored in business, and 51 majored both in computer science and in business. How many of these applicants majored neither in computer science nor in business?

Solution:

To find the number of these applicants who majored neither in computer science nor in business, we can subtract the number of students who majored either in computer science or in business (or both) from the total number of applicants.

The number of students who majored either in computer science or in business (or both) is $220 + 147 - 51 = 316$.

Therefore, we conclude that $350 - 316 = 34$ of the applicants majored neither in computer science nor in business.

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Theorem 2.4. (The Division Rule)
There are n/d ways to do a task if it can be done using a procedure that can be carried out in n ways, and for every way w, exactly d of the n ways correspond to way w.

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Example 2.6.

How many different ways are there to seat four people around a circular table, where two seating arrangements are considered the same when each person has the same left neighbor and the same right neighbor?

Solution:

We arbitrarily select a seat at the table and label it seat 1. We number the rest of the seats in numerical order, proceeding clockwise around the table. Note that are four ways to select the person for seat 1, three ways to select the person for seat 2, two ways to select the person for seat 3, and one way to select the person for seat 4. Thus, there are $4! = 24$ ways to order the given four people for these seats. However, each of the four choices for seat 1 leads to the same arrangement, as we distinguish two arrangements only when one of the people has a different immediate left or immediate right neighbor. Because there are four ways to choose the person for seat 1, by the division rule there are $24/4 = 6$ different seating arrangements of four people around the circular table.

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2.2 Permutation

Definition 2.1.
A permutation of a number of objects is any arrangement of these objects in a definite order.

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Number of permutations of n **distinguishable** objects

$$
\begin{aligned}
&= (\text{number of possible choice of the 1st object}) \\
&\qquad \times (\text{number of possible choice of the 2nd object}) \\
&\qquad\qquad \times (\text{number of possible choice of the 3rd object}) \\
&\qquad\qquad\qquad \times \cdots \\
&\qquad\qquad\qquad\qquad \times (\text{number of possible choice of the } n\text{-th object}) \\
&= n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1 \\
&= n!
\end{aligned}
$$

Therefore the number of permutations of a set of n **distinguishable** objects, taken all together, is $n!$. (The notation $n!$ is called the *factorial* of n. By convention, $0!$ is defined as 1.)

Example 2.7.

(a) In how many ways can 6 people sit in a row?

(b) In how many ways can 6 people sit around a round table?

(c) In how many ways can 6 distinct beads form a single ring-shaped bracelet?

Solution:

(a) $6! = 720$.

(b) $(6-1)! = 120$.

(c) $(6-1)!/2 = 60$.

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Definition 2.2.
An arrangement of r objects taken from a set of n **distinguishable** objects is called a permutation of n objects taken r at a time. The total number of such ordered subsets are symbolized ${}_nP_r$ $(r \le n)$ and can be calculated by

$$ {}_nP_r = n(n-1)\cdots(n-r+1) = \frac{n!}{(n-r)!}. $$

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For convention, we define ${}_nP_r = 0$ for $r > n$.

Example 2.8.

How many possible Tierce outcomes are there in a horse race with 14 horses?

Solution: $\quad {}_{14}P_3 = \dfrac{14!}{11!} = 2184.$

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Example 2.9.

How many different letter arrangements can be formed using the letters LETTERS?

Solution: $\dfrac{7!}{1!1!1!2!2!} = 1260 \qquad$ (1 L, 1 R, 1 S, 2 E's, 2 T's).

Theorem 2.5. (Permutation of indistinguishable objects)
In general, if n_1 objects are of type 1, n_2 objects are of type 2, ..., n_k objects are of type k; then the number of permutations of these n objects $(n = n_1 + n_2 + \cdots + n_k)$ is

$$\frac{n!}{n_1!n_2!\cdots n_k!}.$$

2.3 Combination

Definition 2.3.
A subset of r objects selected without regard to order from a set of n **distinguishable** objects is called a combination of n objects taken r at a time. The total number of such combination is symbolized by

$$_nC_r \qquad \text{or} \qquad \binom{n}{r}.$$

Example 2.10.

What is the number of combinations of three letters chosen from four letters A, B, C and D?

Combination	Permutation
{A, B, C}	(A,B,C), (A,C,B), (B,A,C), (B,C,A), (C,A,B), (C,B,A)
{A, B, D}	(A,B,D), (A,D,B), (D,A,B), (D,B,A), (B,A,D), (B,D,A)
{A, C, D}	(A,C,D), (A,D,C), (C,A,D), (C,D,A), (D,A,C), (D,C,A)
{B, C, D}	(B,C,D), (B,D,C), (C,B,D), (C,D,B), (D,B,C), (D,C,B)

$$\text{Number of permutations} = {}_4P_3 = \frac{4!}{(4-3)!} = \frac{24}{1} = 24.$$

$$\text{Number of combinations} = {}_4C_3 = \frac{\text{no. of permutations of 3 objects from 4 objects}}{\text{no. of permutations identified as the same collection}} = \frac{24}{3!} = 4.$$

In general, we have

$$_nC_r = \begin{cases} \frac{n!}{r!(n-r)!}, & \text{for } r \le n; \\ 0, & \text{for } r > n. \end{cases}$$

Note: $_nC_r = {}_nC_{n-r}$ or $\binom{a+b}{a} = \binom{a+b}{b}$.

Example 2.11.

What is the number of combinations of six numbers chosen from $\{1, 2, \ldots, 49\}$?

Solution: $${}_{49}C_6 = \frac{49!}{6!43!} = 13983816.$$

Example 2.12.

In order to play a game of basketball, 10 children at a playground divide themselves into two teams of 5 each. How many different divisions are possible?

Solution: $$\frac{{}_{10}C_5}{2} = 126 \qquad \text{(Order of the two teams is irrelevant).}$$

Example 2.13.

If 6 products are randomly chosen from a lot containing 20 good and 5 defective products, what is the chance that there will be exactly 4 good products in the selected sample?

Solution:

Total number of possible samples is ${}_{25}C_6$.

Number of samples with 4 good and 2 defective products is ${}_{20}C_4 \times {}_5C_2$.

$$\Pr(\text{4 good products in the sample}) = \frac{{}_{20}C_4 \times {}_5C_2}{{}_{25}C_6} = \frac{48450}{177100} = 0.2736.$$

Example 2.14.

What is the probability of getting a hand of full house if we randomly draw five cards from a deck of 52 cards?

Solution:

Total number of possible outcomes $= {}_{52}C_5 = 2598960$.

Total number of possible full houses $= {}_{13}C_2 \times 2 \times {}_4C_3 \times {}_4C_2 = 3744$.

$$\Pr(\text{full house}) = \frac{3744}{2598960} = 0.00144.$$

2.4 Selection of Distinguishable Objects

Both the formulae ${}_nP_r$ and ${}_nC_r$ described in previous sections can be used to count the number of subsets of size r from a set of n distinguishable objects, with ${}_nP_r$ as the number of *ordered* subsets and ${}_nC_r$ as the number of *unordered* subsets. One point to note is that in both cases, the subsets are selected **without replacement**, i.e., repeated objects are not allowed in the selection.

In some problems we may need to evaluate the number of subsets selected **with replacement**.

Example 2.15.

The pass code of a number lock consists of 4 digits, each digits may be set to 0,1,2,...,9. How many possible pass codes are there?

It is a typical example of counting the number of *ordered* subsets (4 digits) from a set of distinguishable objects $(0-9)$ **with replacement** as there can be repeated digits in the pass code (e.g., 9901, 1535, etc.) and different orderings of the digits lead to different pass code (e.g., 1234 is different from 2314, 4478 is different from 7484, etc.). The number of possible pass codes can be easily computed as

$$10 \times 10 \times 10 \times 10 = 10000.$$

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Theorem 2.6. (Ordered selection with replacement)
In general, the number of ordered selections of r out of n distinguishable objects can be computed by n^r.

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Example 2.16.

The casino game sic-bo (also known as tai-sai, big-small) is a casino game played with three dice. How many unordered outcomes are there?

It is a typical example of counting the number of *unordered* subsets (3 numbers) from a set of distinguishable objects (faces $1-6$) with replacement as there can be more than one die landing on the same number (e.g., $\{1, 1, 3\}$, $\{6, 6, 6\}$, etc.) and different orderings of the numbers are treated as the same outcome (e.g., $\{1, 2, 3\}$ is the same as $\{2, 1, 3\}$, $\{4, 5, 5\}$ is the same as $\{5, 4, 5\}$, etc.).

To count the number of possible selections, we may consider an equivalent problem: allocating 3 indistinguishable balls into 6 distinguishable cells (separated by $6-1=5$ sticks).

$$\begin{array}{cl} \circ\circ \mid \mid \circ \mid \mid \mid & \text{is equivalent to outcome } \{1, 1, 3\} \\ \mid \mid \mid \mid \mid \circ\circ\circ & \text{is equivalent to outcome } \{6, 6, 6\} \\ \mid \circ \mid \mid \circ \mid \circ \mid & \text{is equivalent to outcome } \{2, 4, 5\} \\ \vdots & \qquad\qquad \vdots \end{array}$$

Therefore, number of unordered selections of 3 out of 6 numbers with replacement

$$\begin{aligned} &= \text{number of different arrangements of the 8 symbols } (\circ\circ\circ \mid\mid\mid\mid\mid) \\ &= \frac{8!}{3!5!} = \binom{6+3-1}{3} = 56. \end{aligned}$$

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Theorem 2.7. (Unordered selection with replacement)
In general, the number of unordered selections of r out of n distinguishable objects can be computed by

$${}_nH_r = \binom{n+r-1}{r}.$$

▲

The following table summarizes the four different scenarios:

	With replacement	Without replacement
Ordered	n^r	${}_nP_r = \dfrac{n!}{(n-r)!}$
Unordered	${}_nH_r = \dbinom{n+r-1}{r}$	${}_nC_r = \dbinom{n}{r} = \dfrac{n!}{r!(n-r)!}$

2.5 Partition of Objects

Theorem 2.8. (Partition of distinguishable objects)
If a set of n **distinct** objects is to be partitioned into r **distinct** groups of respective size $n_1, n_2, \ldots, n_r$ (all being nonnegative integers) where $n = n_1 + n_2 + \cdots + n_r$, then the number of different partitions is

$$\binom{n}{n_1, n_2, \ldots, n_r} = \frac{n!}{n_1! n_2! \cdots n_r!}.$$

It is known as the multinomial coefficient.

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Example 2.17.

A box contains four books on history and twelve books on mathematics. If the books are distributed equally at random among four students, find the probability that each student will get a book on history.

Solution:

Number of ways that each student will get a book on history (and thus 3 books in mathematics) is

$$\frac{4!}{1!1!1!1!} \times \frac{12!}{3!3!3!3!} = 8870400.$$

Number of ways of distributing the 16 books to the four students is

$$\frac{16!}{4!4!4!4!} = 63063000.$$

Hence, the required probability is

$$\frac{8870400}{63063000} = 0.1407.$$

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Theorem 2.9. (Partition of indistinguishable objects)
If a set of n **indistinguishable** objects is to be partitioned into r **distinguishable** groups (where some groups can be empty), then the number of different partitions is

$$_rH_n = \binom{r+n-1}{n}.$$

▲

Example 2.18.

20 red balls are distributed into 4 different urns. How many different partitions are possible?

Solution:

$$\binom{4+20-1}{20} = 1771.$$

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Example 2.19.

How many non-negative integer-valued solutions are there satisfying $x_1 + x_2 + x_3 + x_4 + x_5 = 12$?

Solution:

$$\binom{5+12-1}{12} = 1820.$$

Thought Question 1:

How many positive integer-valued solutions are there satisfying $x_1 + x_2 + x_3 + x_4 + x_5 = 12$?

Thought Question 2:

How many non-negative integer-valued solutions are there satisfying $x_1 + x_2 + x_3 + x_4 + x_5 < 12$?

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Remarks

The counting formulae we have seen so far can be found as the coefficients in the following identities:

1. Binomial Theorem

$$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^r y^{n-r}$$

2. Multinomial Theorem

$$(x_1 + x_2 + \cdots + x_r)^n = \sum_{\substack{(n_1,\ldots,n_r) \\ n_1+\cdots+n_r=n}} \frac{n!}{n_1!n_2!\cdots n_r!} x_1^{n_1} x_2^{n_2} \cdots x_r^{n_r}$$

3. Negative Binomial Series

$$\frac{1}{(1-x)^n} = \sum_{r=0}^{\infty} \binom{n+r-1}{r} x^r \qquad \text{for } |x| < 1$$

2.6 Some Harder Examples

In **Theorem 2.8.** and **Theorem 2.9.**, we are dealing with cases where distinguishable or indistinguishable objects are placed into distinguishable boxes. However, when the boxes are indistinguishable, there is no simple closed-formed formula for the number of ways of distribution.

Example 2.20. (Distinguishable objects and indistinguishable boxes)

How many ways are there to put four different employees into three indistinguishable offices, when each office can contain any number of employees?

We represent the four employees by A, B, C, and D. First, we can distribute employees so that all four are put into one office, three are put into one office and a fourth is put into a second office, two employees are put into one office and two put into a second office, and finally, two are put into one office, and one each put into the other two offices. Each way to distribute these employees to these offices can be represented by a way to partition the elements A, B, C, and D into disjoint subsets.

We can put all four employees into one office in exactly one way, represented by {{A,B,C,D}}. We can put three employees into one office and the fourth employee into a different office in exactly four ways, represented by {{A,B,C}, {D}}, {{A,B,D}, {C}}, {{A,C,D}, {B}}, and {{B,C,D}, {A}}. We can put two employees into one office and two into a second office in exactly three ways, represented by {{A,B}, {C,D}}, {{A,C}, {B,D}}, and {{A,D}, {B,C}}. Finally, we can put two employees into one office, and one each into each of the remaining two offices in six ways, represented by {{A,B}, {C}, {D}}, {{A,C}, {B}, {D}}, {{A,D}, {B}, {C}}, {{B,C}, {A}, {D}}, {{B,D}}, {A}, {C}}, and {{C,D}, {A}, {B}}.

Counting all the possibilities, we find that there are 14 ways to put four different employees into three indistinguishable offices.

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Example 2.21. (Indistinguishable objects and indistinguishable boxes)

How many ways are there to pack six copies of the same book into four identical boxes, where a box can contain as many as six books?

Solution:

We will enumerate all ways to pack the books. For each way to pack the books, we will list the number of books in the box with the largest number of books, followed by the numbers of books in each box containing at least one book, in order of decreasing number of books in a box. The ways we can pack the books are

$$\{6\},\ \{5,\ 1\},\ \{4,\ 2\},\ \{4,\ 1,\ 1\},\ \{3,\ 3\},\ \{3,\ 2,\ 1\},\ \{3,\ 1,\ 1,\ 1\},\ \{2,\ 2,\ 2\},\ \{2,\ 2,\ 1,\ 1\}.$$

For example, $\{4,\ 1,\ 1\}$ indicates that one box contains four books, a second box contains a single book, and a third box contains a single book (and the fourth box is empty). We conclude that there are nine allowable ways to pack the books, because we have listed them all.

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Thought Question:
20 balls are divided into 4 groups, both balls and groups are indistinguishable. Any group can be empty. How many possibilities are there?

Solution:
Denote $q(n,k)$ as the number of different ways to allocate n identical balls to k indistinguishable boxes. Then obviously, $q(n,1) = q(0,k) = q(1,k) = 1$. Also if there are more boxes than balls ($k > n$), we can always remove the extra empty boxes as there cannot be more non-empty boxes than balls. Therefore, $q(n,k) = q(n,n)$ if $k > n$.

Now suppose that $k \le n$. We can consider the following two scenarios: there is at least one empty box; or all the boxes are non-empty.

Case 1: there is at least one empty box
Under this scenario, we can remove one empty box and the number of possibilities is the same as the number of ways to allocate n balls to $k-1$ boxes, i.e., $q(n,k-1)$.

Case 2: every box is non-empty
Under this scenario, we can remove one ball from each of the boxes. The number of possibilities is then the same as the number of ways to allocate $n-k$ balls to k boxes, i.e., $q(n-k,k)$.

Combining the two scenarios yields the following recursive formula:

$$q(n,k) = q(n-k,k) + q(n,k-1)$$

Using the recursive formula, we can start with evaluating

$$q(n,2) = q(n-2,2) + q(n,1) = q(n-2,2) + 1$$

for $n = 1, 2, \ldots, 20$, and then evaluating

$$q(n,3) = q(n-3,3) + q(n,2)$$

for $n = 1, 2, \ldots, 20$, and finally evaluating

$$q(n,4) = q(n-4,4) + q(n,3)$$

for $n = 1, 2, \ldots, 20$. (to be continued on the next page...)

Example 2.22.

A medical student has to work in a hospital for five days in January. However, he is not allowed to work two consecutive days in the hospital. In how many different ways can he choose the five days he will work in the hospital?

Solution:

The difficulty here is to make sure that we do not choose two consecutive days. This can be assured by the following trick. Let a_1, a_2, a_3, a_4, a_5 be the dates of the five days of January that the student will spend in the hospital, in increasing order. Note that the requirement that there are no two consecutive numbers among the a_i, and $1 \le a_i \le 31$ for all i is equivalent to the requirement that $1 \le a_1 < a_2 - 1 < a_3 - 2 < a_4 - 3 < a_5 - 4 \le 27$.

Therefore, the selection requirement is equivalent to choosing 5 distinct numbers a_1, $a_2 - 1$, $a_3 - 2$, $a_4 - 3$, and $a_5 - 4$ from the set $\{1, 2, \ldots, 27\}$.

The answer is then $\binom{27}{5} = 80730$.

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The following table shows the values of $q(n, k)$ for $k = 1, 2, 3, 4$ and $n = 1, 2, \ldots, 20$.

	k			
n	1	2	3	4
0	1	1	1	1
1	1	1	1	1
2	1	2	2	2
3	1	2	3	3
4	1	3	4	5
5	1	3	5	6
6	1	4	7	9
7	1	4	8	11
8	1	5	10	15
9	1	5	12	18
10	1	6	14	23
11	1	6	16	27
12	1	7	19	34
13	1	7	21	39
14	1	8	24	47
15	1	8	27	54
16	1	9	30	64
17	1	9	33	72
18	1	10	37	84
19	1	10	40	94
20	1	11	44	108

Therefore the number of different ways to allocate 20 identical balls to 4 indistinguishable boxes is $q(20, 4) = 108$.

2.7 Generating Function

Definition 2.4.
A generating function is a power series whose coefficients give a specific sequence of numbers, i.e., the generating function of a sequence $\{a_0, a_1, a_2, \ldots\}$ is given by

$$\begin{aligned} G(x) &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots . \end{aligned}$$

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Note that the term a_k can be obtained by

$$a_k = \frac{1}{k!} G^{(k)}(0) = \frac{1}{k!} \frac{\mathrm{d}^k}{\mathrm{d}x^k} G(x) \bigg|_{x=0} .$$

Example 2.23.

1. $(1+x)^n = \displaystyle\sum_{r=0}^{n} \binom{n}{r} x^r$

   is the generating function of $\displaystyle\binom{n}{r}$, $r = 0, 1, 2, \ldots, n$.

2. $(1-x)^{-n} = \displaystyle\sum_{r=0}^{\infty} \binom{n+r-1}{r} x^r$ for $|x| < 1$

   is the generating function of $\displaystyle\binom{n+r-1}{r}$, $r = 0, 1, 2, \ldots$.

3. $e^x = \displaystyle\sum_{r=0}^{\infty} \frac{x^r}{r!}$

   is the generating function of $\dfrac{1}{r!}$, $r = 0, 1, 2, \ldots$.

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The concept of generating functions provides a useful tool for counting problems. Roughly speaking, generating functions transform problems about sequences into problems about functions. We can carry out all sorts of manipulations on sequences by performing mathematical operations on their corresponding generating functions, as illustrated by the following examples.

Example 2.24.

In how many ways are there to buy 15 fruits from a store which has 2 apples, 3 bananas, 6 coconuts, and many pears? (Suppose that fruits of the same type are identical.)

Solution:

The generating function corresponding to the selection of apples is $A(x) = 1+x+x^2$, where x^0, x^1, x^2 respectively represent the selection of 0, 1, 2 apples. Similarly, the generation functions corresponding to the selections of the other fruits are given by:

$$\begin{array}{ll} \text{Bananas} & B(x) = 1 + x + x^2 + x^3 \\ \text{Coconuts} & C(x) = 1 + x + x^2 + x^3 + x^4 + x^5 + x^6 \\ \text{Pears} & P(x) = 1 + x + x^2 + \cdots \end{array}$$

The product of these generating functions will give the generating function for the number of possible collections of n fruits:

$$\begin{aligned} G(x) &= A(x)B(x)C(x)P(x) \\ &= (1+x+x^2)(1+x+x^2+x^3) \\ &\qquad \times(1+x+x^2+x^3+x^4+x^5+x^6)(1+x+x^2+\cdots) \\ &= \frac{1-x^3}{1-x}\frac{1-x^4}{1-x}\frac{1-x^7}{1-x}\frac{1}{1-x} \\ &= (1-x^3)(1-x^4)(1-x^7)(1-x)^{-4} \\ &= (1-x^3-x^4+x^{10}+x^{11}-x^{14})\sum_{r=0}^{\infty}\binom{r+3}{r}x^r. \end{aligned}$$

In particular, the number of ways to buy 15 fruits can be obtained as the coefficient of x^{15}, which is given by

$$\begin{aligned} & \binom{15+3}{15} - \binom{12+3}{12} - \binom{11+3}{11} + \binom{5+3}{5} + \binom{4+3}{4} - \binom{1+3}{1} \\ =& \binom{18}{15} - \binom{15}{12} - \binom{14}{11} + \binom{8}{5} + \binom{7}{4} - \binom{4}{1} \\ =& \binom{18}{3} - \binom{15}{3} - \binom{14}{3} + \binom{8}{3} + \binom{7}{3} - \binom{4}{1} \\ =& 816 - 455 - 364 + 56 + 35 - 4 \\ =& 84. \end{aligned}$$

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Example 2.25.

Suppose four fair dice are rolled. What is the probability that the sum of their faces is divisible by 7?

Solution:

The total number of possible outcomes of four rolled dice is $6^4 = 1296$.

To count the number of possible outcomes with sum divisible by 7, we can make use of the generating functions. The generating function of the outcome of each particular die is

$$\begin{aligned} D(x) &= x + x^2 + x^3 + x^4 + x^5 + x^6 \\ &= \frac{x(1-x^6)}{1-x}. \end{aligned}$$

Hence the generating function of the number of possible outcomes on four dice with specific sums is given by

$$\begin{aligned} G(x) &= [D(x)]^4 \\ &= x^4(1-x^6)^4(1-x)^{-4} \\ &= (x^4 - 4x^{10} + 6x^{16} - 4x^{22} + x^{28}) \sum_{r=0}^{\infty} \binom{r+3}{r} x^r. \end{aligned}$$

Since the sum on the outcomes of four dice cannot exceed 24, the possible sums that are divisible by 7 are 7, 14, 21. The number of possible outcomes on the four dice such that the sum is divisible by 7 can be obtained as the sum of the coefficients of x^7, x^{14}, x^{21} in the generating function $G(x)$, which is given by

$$\begin{aligned} & \binom{3+3}{3} + \left[\binom{10+3}{10} - 4\binom{4+3}{4}\right] + \left[\binom{17+3}{17} - 4\binom{11+3}{11} + 6\binom{5+3}{5}\right] \\ = & \binom{6}{3} + \left[\binom{13}{10} - 4\binom{7}{4}\right] + \left[\binom{20}{17} - 4\binom{14}{11} + 6\binom{8}{5}\right] \\ = & \binom{6}{3} + \left[\binom{13}{3} - 4\binom{7}{3}\right] + \left[\binom{20}{3} - 4\binom{14}{3} + 6\binom{8}{3}\right] \\ = & 20 + (286 - 4 \times 35) + (1140 - 4 \times 364 + 6 \times 56) \\ = & 20 + 146 + 20 \\ = & 186. \end{aligned}$$

Therefore the probability that the sum is divisible by 7 is

$$\Pr(\text{sum of 4 rolled dice is divisible by 7}) = \frac{186}{1296} = \frac{31}{216}.$$

Example 2.26.

How many integer solutions to the equation $a + b + c = 6$ satisfy $-1 \le a \le 2$, $1 \le b \le 4$ and $1 \le c \le 4$?

Solution:

Since $-1 \le a \le 2$, the variable a contributes a term $x^{-1} + x^0 + x + x^2$ to the generating function. Similarly, each of b and c contributes a term $x + x^2 + x^3 + x^4$. Hence the generating function is

$$\begin{aligned}
G(x) &= (x^{-1} + x^0 + x + x^2)(x + x^2 + x^3 + x^4)^2 \\
&= x(1 + x + x^2 + x^3)^3 \\
&= x\left(\frac{1 - x^4}{1 - x}\right)^3 \\
&= x(1 - 3x^4 + 3x^8 - x^{12})(1 - x)^{-3} \\
&= (x - 3x^5 + 3x^9 - x^{13})\sum_{r=0}^{\infty}\binom{r+2}{r}x^r.
\end{aligned}$$

The answer is the coefficient of x^6 in $G(x)$, which is given by

$$\begin{aligned}
& \binom{5+2}{5} - 3\binom{1+2}{1} \\
=\ & \binom{7}{5} - 3\binom{3}{1} \\
=\ & 21 - 9 \\
=\ & 12.
\end{aligned}$$

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Extra Example

Superman has super-strength and can carry any number of boulders, but insists on carrying an odd number. Batman can carry up to 40 boulders. Mighty Mouse can only carry up to 2 boulders. Batman and Mighty Mouse might go empty-handed. How many ways can the three distribute exactly (a) 37 and (b) 87 boulders to carry?

∼ End of Chapter 2 ∼