Core course for *App.AI*, *Bioinfo.*, *Data Sci.&Eng.*, *Dec.Analytics*, *Q.Fin.*, *Risk Mgmt.* and *Stat.* Majors:

STAT2601A Probability and Statistics I (2023-2024 First Semester)

Chapter 9: Transformation of Multivariate Distributions

9.1 Transformation of Multivariate Distributions

Let $X_1, X_2, \ldots, X_n$ be jointly distributed continuous random variables with joint probability density function $f_{\mathbf{X}}(x_1, x_2, \ldots, x_n)$. It is sometimes necessary to obtain the joint distribution of the random variables $Y_1, Y_2, \ldots, Y_n$ which arise as functions of $X_1, X_2, \ldots, X_n$. For example, suppose $n = 2$, then

$$\begin{aligned} Y_1 &= g_1(X_1, X_2) = X_1 + X_2, \\ Y_2 &= g_2(X_1, X_2) = X_1 - X_2, \end{aligned}$$

would transform the random variables X_1, X_2 into their sum and difference. To determine the joint pdf of the transformed random variables, we may use the following theorem, which is a generalization of the one-variable transformation formula (**Theorem 6.3.**) in **Chapter 6**.

In general, let $Y_i = g_i(X_1, X_2, \ldots, X_n)$, $i = 1, 2, \ldots, n$ for some functions g_i's such that the functions g_i's satisfy the following conditions:

1. The equations $y_i = g_i(x_1, x_2, \ldots, x_n)$ can be uniquely solved for $x_1, x_2, \ldots, x_n$ in terms of $y_1, y_2, \ldots, y_n$ with solutions given by the inverse transformations, say, $x_i = h_i(y_1, y_2, \ldots, y_n)$, $i = 1, 2, \ldots, n$, i.e., the transformation from X's to Y's is one-to-one correspondence.

2. The functions g_i's have continuous partial derivatives at all points $(x_1, x_2, \ldots, x_n)$ and are such that the $n \times n$ *Jacobian determinant* is non-zero, i.e.,

$$J_0(x_1, x_2, \ldots, x_n) = \begin{vmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} & \cdots & \frac{\partial g_1}{\partial x_n} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} & \cdots & \frac{\partial g_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial g_n}{\partial x_1} & \frac{\partial g_n}{\partial x_2} & \cdots & \frac{\partial g_n}{\partial x_n} \end{vmatrix} \neq 0$$

at all points $(x_1, x_2, \ldots, x_n)$. Sometimes the calculation of the above Jacobian determinant $J_0(x_1, x_2, \ldots, x_n)$ may be complicated. The following alternative formula may be used. Consider the Jacobian determinant $J(y_1, y_2, \ldots, y_n)$ for the inverse transformation. The inverse functions h_i's should have continuous partial derivatives at all points $(y_1, y_2, \ldots, y_n)$ and the condition is again the Jacobian determinant being non-zero, i.e.,

$$J(y_1, y_2, \ldots, y_n) = \begin{vmatrix} \frac{\partial h_1}{\partial y_1} & \frac{\partial h_1}{\partial y_2} & \cdots & \frac{\partial h_1}{\partial y_n} \\ \frac{\partial h_2}{\partial y_1} & \frac{\partial h_2}{\partial y_2} & \cdots & \frac{\partial h_2}{\partial y_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial h_n}{\partial y_1} & \frac{\partial h_n}{\partial y_2} & \cdots & \frac{\partial h_n}{\partial y_n} \end{vmatrix} \neq 0$$

at all points $(y_1, y_2, \dots, y_n)$. Note that

$$J_0(x_1, x_2, \dots, x_n)^{-1} = \begin{vmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} & \cdots & \frac{\partial g_1}{\partial x_n} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} & \cdots & \frac{\partial g_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial g_n}{\partial x_1} & \frac{\partial g_n}{\partial x_2} & \cdots & \frac{\partial g_n}{\partial x_n} \end{vmatrix}^{-1} = \begin{vmatrix} \frac{\partial h_1}{\partial y_1} & \frac{\partial h_1}{\partial y_2} & \cdots & \frac{\partial h_1}{\partial y_n} \\ \frac{\partial h_2}{\partial y_1} & \frac{\partial h_2}{\partial y_2} & \cdots & \frac{\partial h_2}{\partial y_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial h_n}{\partial y_1} & \frac{\partial h_n}{\partial y_2} & \cdots & \frac{\partial h_n}{\partial y_n} \end{vmatrix} = J(y_1, y_2, \dots, y_n).$$

Under these two conditions, the joint pdf of $Y_1, Y_2, \dots, Y_n$ is given by the following formula:

$$\boxed{f_{\mathbf{Y}}(y_1, y_2, \dots, y_n) = f_{\mathbf{X}}(x_1, x_2, \dots, x_n) \times |J_0(x_1, x_2, \dots, x_n)|^{-1},}$$

or

$$\boxed{f_{\mathbf{Y}}(y_1, y_2, \dots, y_n) = f_{\mathbf{X}}(x_1, x_2, \dots, x_n) \times |J(y_1, y_2, \dots, y_n)|,}$$

where $x_i = h_i(y_1, y_2, \dots, y_n)$ for $i = 1, 2, \dots, n$.

Example 9.1.

Suppose that two random variables X_1, X_2 have a continuous joint distribution for which the joint pdf is as follows:

$$f_{\mathbf{X}}(x_1, x_2) = \begin{cases} \frac{1}{2}(x_1 + x_2)e^{-x_1 - x_2}, & x_1 > 0,\ x_2 > 0; \\ 0, & \text{otherwise.} \end{cases}$$

Obviously the transformation $Y_1 = X_1 + X_2$, $Y_2 = X_1 - X_2$ is one-to-one correspondence with the inverse transformation $X_1 = (Y_1 + Y_2)/2$, $X_2 = (Y_1 - Y_2)/2$. The Jacobian determinant is given by

$$J(y_1, y_2) = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = \frac{1}{2} \times \left(-\frac{1}{2}\right) - \frac{1}{2} \times \frac{1}{2} = -\frac{1}{2} \neq 0.$$

The joint pdf of Y_1, Y_2 is therefore given by

$$f_{\mathbf{Y}}(y_1, y_2) = f_{\mathbf{X}}(x_1, x_2) \times |J| = \frac{1}{2} y_1 e^{-y_1} \times \left|-\frac{1}{2}\right| = \frac{1}{4} y_1 e^{-y_1},$$

with support

$$x_1 > 0,\ x_2 > 0 \quad \Longleftrightarrow \quad -y_1 < y_2 < y_1,\ y_1 > 0.$$

That is,

$$f_{\mathbf{Y}}(y_1, y_2) = \begin{cases} \frac{1}{4} y_1 e^{-y_1}, & -y_1 < y_2 < y_1,\ y_1 > 0; \\ 0, & \text{otherwise.} \end{cases}$$

★

Example 9.2.

Let $X \sim \Gamma(\alpha, \lambda)$ and $Y \sim \Gamma(\beta, \lambda)$ be two independent gamma random variables. The joint pdf of X and Y is

$$f_{X,Y}(x,y) = \begin{cases} \frac{\lambda^{\alpha+\beta}}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1} y^{\beta-1} e^{-\lambda(x+y)}, & x > 0,\ y > 0; \\ 0, & \text{otherwise.} \end{cases}$$

Consider the transformation

$$U = \frac{X}{X+Y}, \qquad V = X + Y.$$

This is a one-to-one correspondence transformation with

$$X = UV, \qquad Y = V(1-U).$$

The support of the joint distribution of U, V is given by

$$\begin{cases} x > 0 \\ y > 0 \end{cases} \iff \begin{cases} uv > 0 \\ v(1-u) > 0 \end{cases} \iff \begin{cases} 0 < u < 1 \\ v > 0 \end{cases}.$$

For the Jacobian determinant:

$$\frac{\partial u}{\partial x} = \frac{y}{(x+y)^2} = \frac{1-u}{v}, \quad \frac{\partial u}{\partial y} = -\frac{x}{(x+y)^2} = -\frac{u}{v}, \quad \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = 1,$$

$$J_0(x,y) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{1-u}{v} & -\frac{u}{v} \\ 1 & 1 \end{vmatrix} = \frac{1-u}{v} \times 1 - \left(-\frac{u}{v}\right) \times 1 = \frac{1}{v} \neq 0.$$

Hence the joint pdf of U, V is given by

$$\begin{aligned} f_{U,V}(u,v) &= f_{X,Y}(x,y) \times |J_0(x,y)|^{-1} \\ &= \frac{\lambda^{\alpha+\beta}}{\Gamma(\alpha)\Gamma(\beta)} (uv)^{\alpha-1} \left[v(1-u)\right]^{\beta-1} e^{-\lambda v} \times \left|\frac{1}{v}\right|^{-1} \\ &= \frac{\lambda^{\alpha+\beta}}{\Gamma(\alpha)\Gamma(\beta)} u^{\alpha-1}(1-u)^{\beta-1} v^{\alpha+\beta-1} e^{-\lambda v}, \qquad 0 < u < 1,\ v > 0. \end{aligned}$$

From the joint pdf it is easily observed that U and V are independent. Note that the joint pdf of U, V can be written as

$$f_{U,V}(u,v) = \left[\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} u^{\alpha-1}(1-u)^{\beta-1}\right] \left[\frac{\lambda^{\alpha+\beta}}{\Gamma(\alpha+\beta)} v^{\alpha+\beta-1} e^{-\lambda v}\right], \qquad 0 < u < 1,\ v > 0,$$

and therefore,

$$U = \frac{X}{X+Y} \sim \text{Beta}(\alpha, \beta), \qquad V = X + Y \sim \Gamma(\alpha+\beta, \lambda).$$

Remark:

Some can also use the following Jacobian determinant, which may be easier to compute.

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} v & u \\ -v & 1-u \end{vmatrix} = v \times (1-u) - u \times (-v) = v = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}^{-1} = J_0^{-1}.$$

★

9.2 Some Important Transformations

We can consider the cases where two random variables are transformed into one random variable.

9.2.1 Sum of Two Random Variables

Consider

$$Z = X + Y.$$

Discrete case

$$p_Z(z) = \Pr(X + Y = z) = \sum_x p(x, z - x) = \sum_y p(z - y, y).$$

Continuous case

$$\begin{aligned}
F_Z(z) &= \Pr(Z \le z) = \Pr(X + Y \le z) \\
&= \Pr(Y \le z - X) = \int_{-\infty}^{\infty}\int_{-\infty}^{z-x} f(x, y)\mathrm{d}y\mathrm{d}x \\
&= \Pr(X \le z - Y) = \int_{-\infty}^{\infty}\int_{-\infty}^{z-y} f(x, y)\mathrm{d}x\mathrm{d}y, \\
f_Z(z) &= F'(z) = \int_{-\infty}^{\infty} f(x, z - x)\mathrm{d}x = \int_{-\infty}^{\infty} f(z - y, y)\mathrm{d}y.
\end{aligned}$$

Example 9.3.

Let X and Y be two independent geometric random variables with parameter p. What is the distribution of $Z = X + Y$?

The support of Z is $\{2, 3, 4, \ldots\}$.

$$\begin{aligned}
p_Z(z) &= \sum_x p(x, z - x) \\
&= \sum_{x=1}^{\infty} p_X(x)p_Y(z - x) \\
&= \sum_{x=1}^{z-1} \left[(1-p)^{x-1}p\right]\left[(1-p)^{z-x-1}p\right] \qquad \because p_Y(z - x) = 0 \text{ if } z - x < 1 \\
&= p^2 \sum_{x=1}^{z-1} (1-p)^{z-2} \\
&= (z-1)p^2(1-p)^{z-2} \\
&= \binom{z-1}{2-1} p^2(1-p)^{z-2}.
\end{aligned}$$

Compare with the pmf of $\mathrm{NB}(r, p)$, $p(x) = \binom{x-1}{r-1}p^r(1-p)^{x-r}$, $x = r, r+1, \ldots$, we can see that $Z \sim \mathrm{NB}(2, p)$.

★

Example 9.4.

Let (X, Y) be distributed jointly with joint pdf

$$f(x,y) = \begin{cases} e^{-y}, & 0 < x < y < \infty; \\ 0, & \text{otherwise.} \end{cases}$$

What is the distribution of $Z = X + Y$?

The support of Z is $(0, \infty)$.

$$\begin{aligned} f_Z(z) &= \int_{-\infty}^{\infty} f(z-y, y)\mathrm{d}y \\ &= \int_{z/2}^{z} e^{-y}\mathrm{d}y \qquad \because f(z-y,y) = 0 \text{ if } z-y<0 \text{ or } z-y>y \\ &= \left[e^{-y}\right]_z^{z/2} \\ &= e^{-z/2} - e^{-z}, \qquad z > 0. \end{aligned}$$

Example 9.5.

Let X and Y be two independent $\mathrm{U}(0,1)$ random variables. What is the distribution of $Z = X + Y$?

The support of Z is $(0, 2)$.

$$f_Z(z) = \int_{-\infty}^{\infty} f(z-y, y)\mathrm{d}y = \int_{-\infty}^{\infty} f_X(z-y)f_Y(y)\mathrm{d}y.$$

For the integrand to be non-zero, we must have

$$\begin{cases} 0 < x < 1 \\ 0 < y < 1 \end{cases} \iff \begin{cases} 0 < z-y < 1 \\ 0 < y < 1 \end{cases} \iff \begin{cases} z-1 < y < z \\ 0 < y < 1 \end{cases} \iff \max(z-1, 0) < y < \min(z, 1).$$

For $0 < z \le 1$,

$$f_Z(z) = \int_0^z (1)(1)\mathrm{d}y = z.$$

For $1 < z < 2$,

$$f_Z(z) = \int_{z-1}^1 (1)(1)\mathrm{d}y = 2 - z.$$

Therefore,

$$f_Z(z) = \begin{cases} z, & 0 < z \le 1; \\ 2-z, & 1 < z < 2; \\ 0, & \text{otherwise.} \end{cases}$$

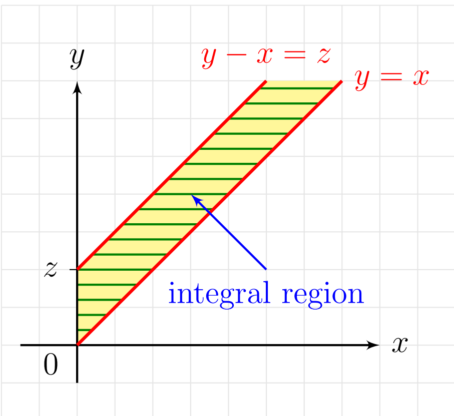
9.2.2 Difference between Two Random Variables

Consider
$$Z = X - Y.$$
Discrete case
$$p_Z(z) = \Pr(X - Y = z) = \sum_x p(x, x - z) = \sum_y p(z + y, y).$$
Continuous case
$$\begin{aligned} F_Z(z) &= \Pr(Z \le z) = \Pr(X - Y \le z) \\ &= \Pr(Y \ge X - z) = \int_{-\infty}^{\infty}\int_{x-z}^{\infty} f(x, y)\mathrm{d}y\mathrm{d}x \\ &= \Pr(X \le z + Y) = \int_{-\infty}^{\infty}\int_{-\infty}^{z+y} f(x, y)\mathrm{d}x\mathrm{d}y, \\ f_Z(z) &= F'(z) = \int_{-\infty}^{\infty} f(x, x - z)\mathrm{d}x = \int_{-\infty}^{\infty} f(z + y, y)\mathrm{d}y. \end{aligned}$$

Example 9.6.

Let (X, Y) be distributed jointly with joint pdf
$$f(x, y) = \begin{cases} e^{-y}, & 0 < x < y < \infty; \\ 0, & \text{otherwise.} \end{cases}$$
What is the distribution of $Z = Y - X$?

The support of Z is $(0, \infty)$.
$$\begin{aligned} F_Z(z) &= \Pr(Z \le z) = \Pr(Y - X \le z) \\ &= \int_0^{\infty}\int_0^{x+z} f(x, y)\mathrm{d}y\mathrm{d}x = \int_0^{\infty}\int_x^{x+z} e^{-y}\mathrm{d}y\mathrm{d}x \\ &= \int_0^{\infty}\left(e^{-x} - e^{-(x+z)}\right)\mathrm{d}x \\ &= 1 - e^{-z}, \qquad z > 0. \end{aligned}$$



$$\therefore \qquad f_Z(z) = F_Z'(z) = e^{-z}, \qquad z > 0.$$

Hence, $Z = Y - X \sim \text{Exp}(1)$.

Example 9.7.

Let X and Y be two independent $\text{U}(0,1)$ random variables. What is the distribution of $Z = X - Y$?

The support of Z is $(-1, 1)$.

$$f_Z(z) = \int_{-\infty}^{\infty} f(z+y, y)\mathrm{d}y = \int_{-\infty}^{\infty} f_X(z+y) f_Y(y)\mathrm{d}y.$$

For the integrand to be non-zero, we must have

$$\begin{cases} 0 < x < 1 \\ 0 < y < 1 \end{cases} \iff \begin{cases} 0 < z+y < 1 \\ 0 < y < 1 \end{cases} \iff \begin{cases} -z < y < 1-z \\ 0 < y < 1 \end{cases} \iff \max(-z, 0) < y < \min(1-z, 1).$$

For $-1 < z \le 0$,

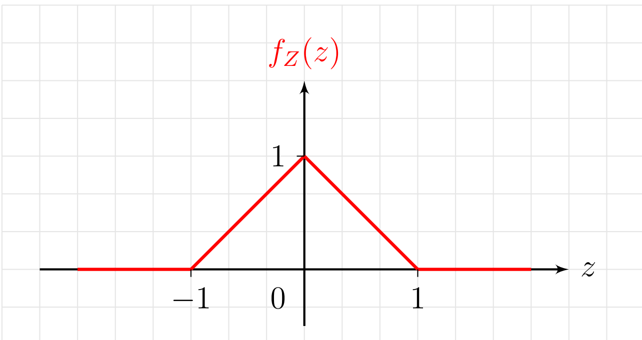
$$f_Z(z) = \int_{-z}^{1} (1)(1)\mathrm{d}y = 1 + z.$$

For $0 < z < 1$,

$$f_Z(z) = \int_{0}^{1-z} (1)(1)\mathrm{d}y = 1 - z.$$

Therefore,

$$f_Z(z) = \begin{cases} 1+z, & -1 < z \le 0; \\ 1-z, & 0 < z < 1; \\ 0, & \text{otherwise.} \end{cases}$$



9.2.3 Product of Two Random Variables

Consider

$$Z = XY.$$

Discrete case

$$p_Z(z) = \Pr(XY = z) = \sum_x p(x, z/x) = \sum_y p(z/y, y).$$

Continuous case

$$\begin{aligned}
F_Z(z) &= \Pr(Z \le z) = \Pr(XY \le z) \\
&= \Pr(Y \le z/X, X > 0) + \Pr(Y \ge z/X, X < 0) \\
&= \int_0^{\infty} \int_{-\infty}^{z/x} f(x, y) \mathrm{d}y \mathrm{d}x + \int_{-\infty}^{0} \int_{z/x}^{\infty} f(x, y) \mathrm{d}y \mathrm{d}x, \\
f_Z(z) &= F'(z) = \int_0^{\infty} f(x, z/x) \frac{1}{x} \mathrm{d}x - \int_{-\infty}^{0} f(x, z/x) \frac{1}{x} \mathrm{d}x \\
&= \int_{-\infty}^{\infty} f(x, z/x) \left| \frac{1}{x} \right| \mathrm{d}x = \int_{-\infty}^{\infty} f(z/y, y) \left| \frac{1}{y} \right| \mathrm{d}y.
\end{aligned}$$

Example 9.8.

Let X and Y be two independent $\mathrm{U}(0, 1)$ random variables. What is the distribution of $Z = XY$?

The support of Z is $(0, 1)$.

$$f_Z(z) = \int_{-\infty}^{\infty} f(x, z/x) \left| \frac{1}{x} \right| \mathrm{d}x = \int_{-\infty}^{\infty} f_X(x) f_Y(z/x) \left| \frac{1}{x} \right| \mathrm{d}x.$$

For the integrand to be non-zero, we must have

$$\begin{cases} 0 < x < 1 \\ 0 < y < 1 \end{cases} \iff \begin{cases} 0 < x < 1 \\ 0 < z/x < 1 \end{cases} \iff \begin{cases} 0 < x < 1 \\ 0 < z < x \end{cases} \iff 0 < z < x < 1.$$

Therefore,

$$f_Z(z) = \int_z^1 (1)(1) \left(\frac{1}{x} \right) \mathrm{d}x = -\ln z, \qquad 0 < z < 1.$$

9.2.4 Ratio of Two Random Variables

Consider
$$Z = \frac{X}{Y}.$$
Discrete case
$$p_Z(z) = \Pr(X/Y = z) = \sum_x p(x, x/z) = \sum_y p(zy, y).$$
Continuous case
$$\begin{aligned}
F_Z(z) &= \Pr(Z \le z) = \Pr(X/Y \le z) \\
&= \Pr(X \le zY, Y > 0) + \Pr(X \ge zY, Y < 0) \\
&= \int_0^\infty \int_{-\infty}^{zy} f(x,y)\mathrm{d}x\mathrm{d}y + \int_{-\infty}^0 \int_{zy}^\infty f(x,y)\mathrm{d}x\mathrm{d}y, \\
f_Z(z) &= F'(z) = \int_0^\infty f(zy,y)y\mathrm{d}y - \int_{-\infty}^0 f(zy,y)y\mathrm{d}y \\
&= \int_{-\infty}^\infty f(zy,y)\,|y|\,\mathrm{d}y = \int_{-\infty}^\infty f\left(x, \frac{x}{z}\right)\frac{|x|}{z^2}\mathrm{d}x.
\end{aligned}$$

Example 9.9. (Cauchy distribution)

Let X and Y be two independent $\mathrm{N}(0,1)$ random variables. What is the distribution of $Z = \dfrac{X}{Y}$?

The support of Z is $(-\infty, \infty)$.

$$\begin{aligned}
f_Z(z) &= \int_{-\infty}^\infty f(zy,y)\,|y|\,\mathrm{d}y \\
&= \int_{-\infty}^\infty f_X(zy)f_Y(y)\,|y|\,\mathrm{d}y \\
&= \int_{-\infty}^\infty \left(\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2y^2}{2}}\right)\left(\frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}\right)|y|\,\mathrm{d}y \\
&= \int_{-\infty}^\infty \frac{1}{2\pi}e^{-\frac{(1+z^2)y^2}{2}}\,|y|\,\mathrm{d}y \\
&= 2\int_0^\infty \frac{1}{2\pi}e^{-\frac{(1+z^2)y^2}{2}}y\mathrm{d}y \qquad \because \text{ the integrand is an even function} \\
&= \frac{1}{\pi}\left[\frac{1}{1+z^2}e^{-\frac{(1+z^2)y^2}{2}}\right]_\infty^0 \\
&= \frac{1}{\pi(1+z^2)}, \qquad -\infty < z < \infty.
\end{aligned}$$

This is known as the *Cauchy distribution*.

Example 9.10. (Student's t-distribution)

Let $Z \sim \mathrm{N}(0,1)$ and $W \sim \chi^2(r)$ be two independent random variables.

What is the distribution of $X = \dfrac{Z}{\sqrt{W/r}}$?

Let $Y = \sqrt{W/r}$, then, $W = rY^2$. It is a one-to-one function as W and Y are all positive.

The pdf of Y is given by

$$\begin{aligned} f_Y(y) &= f_W(ry^2) \times |2ry| = \frac{1}{\Gamma\left(\frac{r}{2}\right) 2^{\frac{r}{2}}}(ry^2)^{\frac{r}{2}-1} e^{-\frac{ry^2}{2}} \times 2ry \\ &= \frac{r^{\frac{r}{2}}}{\Gamma\left(\frac{r}{2}\right) 2^{\frac{r}{2}-1}} y^{r-1} e^{-\frac{ry^2}{2}}, \qquad y > 0. \end{aligned}$$

Therefore, using the above formula for the pdf of the ratio, the pdf of $X = \dfrac{Z}{Y}$ is given by

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f_{Z,Y}(xy, y)\,|y|\,\mathrm{d}y = \int_{-\infty}^{\infty} f_Z(xy) f_Y(y)\,|y|\,\mathrm{d}y \\ &= \int_0^{\infty} \left(\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2y^2}{2}}\right)\left(\frac{r^{\frac{r}{2}}}{\Gamma\left(\frac{r}{2}\right) 2^{\frac{r}{2}-1}} y^{r-1} e^{-\frac{ry^2}{2}}\right) y\mathrm{d}y \qquad \because f_Y(y) = 0 \text{ if } y < 0 \\ &= \frac{r^{\frac{r}{2}}}{\sqrt{\pi}\Gamma\left(\frac{r}{2}\right) 2^{\frac{r-1}{2}}} \int_0^{\infty} y^r e^{-\frac{(r+x^2)y^2}{2}} \mathrm{d}y \\ &= \frac{r^{\frac{r}{2}}}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{r}{2}\right) 2^{\frac{r-1}{2}}} \int_0^{\infty} \left(\frac{2u}{r+x^2}\right)^{\frac{r}{2}} e^{-u} \frac{1}{2}\left(\frac{2u}{r+x^2}\right)^{-\frac{1}{2}} \frac{2}{r+x^2} \mathrm{d}u \\ &\qquad\qquad \left(\text{put } u = \frac{(r+x^2)y^2}{2} \Longleftrightarrow y = \left(\frac{2u}{r+x^2}\right)^{\frac{1}{2}}\right) \\ &= \frac{r^{\frac{r}{2}}}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{r}{2}\right) 2^{\frac{r-1}{2}}} \frac{2^{\frac{r-1}{2}}}{(r+x^2)^{\frac{r+1}{2}}} \int_0^{\infty} u^{\frac{r-1}{2}} e^{-u} \mathrm{d}u \\ &= \frac{\Gamma\left(\frac{r+1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{r}{2}\right)} r^{\frac{r}{2}} (r+x^2)^{-\frac{r+1}{2}} = \frac{\Gamma\left(\frac{r+1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{r}{2}\right)} r^{-\frac{1}{2}} \left(1 + \frac{x^2}{r}\right)^{-\frac{r+1}{2}}, \qquad -\infty < x < \infty. \end{aligned}$$

It is known as the *Student's t-distribution* and is denoted as

$$X \sim t(r).$$

It plays an important role in many widely used statistical analysis methods. The derivation was first published in 1908 by a famous statistician William Sealy Gosset, under the nickname "Student", as Gosset's employer, the Guinness Brewery, forbade their staffs from publishing scientific papers.

The moment generating function is undefined while the mean and variance are

$$\mu = \begin{cases} 0, & r > 1; \\ \text{undefined}, & \text{otherwise}, \end{cases} \qquad \text{and} \qquad \sigma^2 = \begin{cases} \frac{r}{r-2}, & r > 2; \\ \text{undefined}, & \text{otherwise}. \end{cases}$$

★

Example 9.11.

Let X and Y be two independent $\mathrm{U}(0,1)$ random variables. What is the distribution of $Z = \dfrac{Y}{X}$?

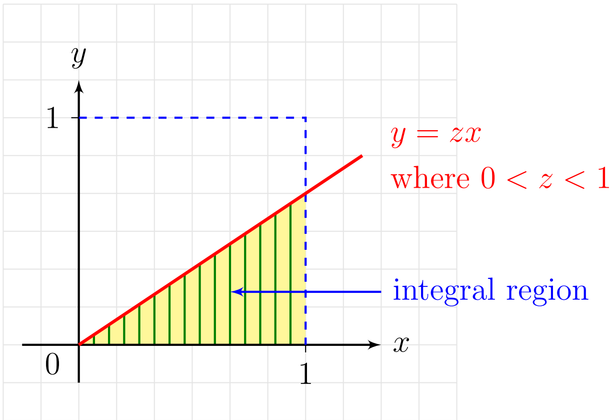
The support of Z is $(0, \infty)$.

The distribution function of Z is

$$F_Z(z) = \Pr(Z \le z) = \Pr\left(\frac{Y}{X} \le z\right) = \Pr\left(Y \le zX\right).$$

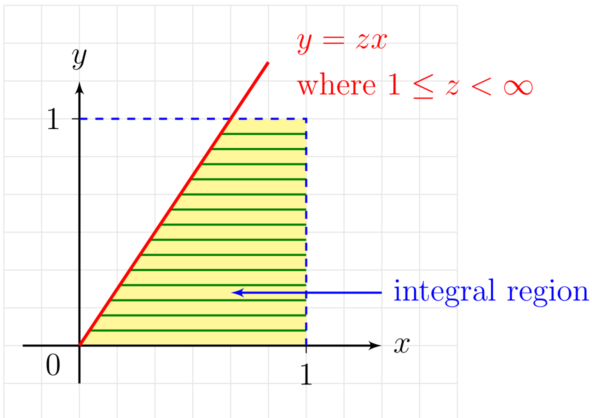
For $0 < z < 1$,

$$F_Z(z) = \int_0^1 \int_0^{zx} (1)(1)\mathrm{d}y\mathrm{d}x = \int_0^1 zx\mathrm{d}x = \frac{z}{2}.$$



For $1 \le z < \infty$,

$$F_Z(z) = \int_0^1 \int_{y/z}^1 (1)(1)\mathrm{d}x\mathrm{d}y = \int_0^1 \left(1 - \frac{y}{z}\right)\mathrm{d}y = 1 - \frac{1}{2z}.$$



Hence,

$$F_Z(z) = \begin{cases} 0, & z \le 0; \\ \frac{z}{2}, & 0 < z < 1; \\ 1 - \frac{1}{2z}, & z \ge 1, \end{cases} \quad \text{and} \quad f_Z(z) = \begin{cases} 0, & z \le 0; \\ \frac{1}{2}, & 0 < z < 1; \\ \frac{1}{2z^2}, & z \ge 1. \end{cases}$$

Remarks

To determine the distribution of the transformed random variable(s), the following three approaches are commonly used.

1. Directly apply the pdf formula if the transformation is one-to-one.
2. Determine the distribution function as in **Example 9.11.**.
3. Since the moment generating function uniquely characterizes the distribution of a random variable, one can first find the mgf of the transformed random variable and then identify its distribution.

9.2.5 Examples of the Moment Generating Function Approach

Example 9.12.

Let $X \sim \mathrm{N}(\mu_X, \sigma_X^2)$ and $Y \sim \mathrm{N}(\mu_Y, \sigma_Y^2)$ be two independent normal random variables. To find the distribution of the linear combination $W = aX + bY$, consider the mgf of W:

$$\begin{aligned} M_W(t) &= M_X(at)M_Y(bt) \\ &= \exp\left(\mu_X at + \frac{1}{2}\sigma_X^2 a^2 t^2\right) \times \exp\left(\mu_Y bt + \frac{1}{2}\sigma_Y^2 b^2 t^2\right) \\ &= \exp\left((a\mu_X + b\mu_Y)t + \frac{1}{2}(a^2\sigma_X^2 + b^2\sigma_Y^2)t^2\right) \qquad \forall t \in \mathbb{R}. \end{aligned}$$

It also takes the form of the mgf of a normal distribution. Therefore we have

$$aX + bY \sim \mathrm{N}\left(a\mu_X + b\mu_Y, a^2\sigma_X^2 + b^2\sigma_Y^2\right),$$

i.e., linear combination of independent normal random variables is also distributed as normal.

★

Note

The mgf approach is applicable only if the distribution corresponding to the resulted mgf can be easily identified.

Example 9.13.

Let X and Y be two independent random variables distributed as $\mathrm{Exp}(\lambda)$.

Let $Z = X - Y$. Then the mgf of Z can be obtained as

$$M_Z(t) = M_X(t)M_Y(-t) = \frac{\lambda}{\lambda - t} \times \frac{\lambda}{\lambda + t} = \frac{\lambda^2}{\lambda^2 - t^2}, \qquad \text{for } -\lambda < t < \lambda,$$

which does not take the form of the mgf of any distribution introduced so far. Therefore the distribution of $Z = X - Y$ is still unidentified.

(In fact, it is the mgf of the *double-exponential* distribution.)

★

~ End of Chapter 9 ~