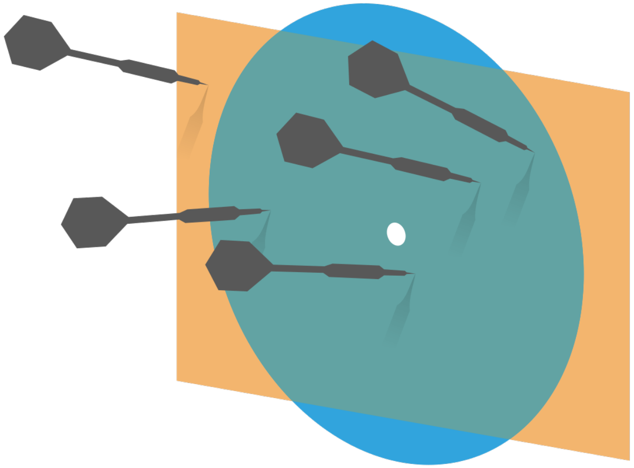
Probability and Statistics for Computer Science

HW 1



Credit: wikipedia

"Probabilistic analysis is mathematical, but intuition dominates and guides the math" – Prof. Dimitri Bertsekas

Hongye Liu, Teaching Assistant Prof, CS361, UIUC, 08.31.2023

Last time

- Median, Percentile, Mode, IQR,
- Scatter plots for relationships
- Correlation Coefficient

$\hat{y} \qquad \hat{x}$

$\hat{y}_p = r \cdot \hat{x}$

Objectives

- Probability a first look
  - Outcome and Sample Space
  - Event
  - Probability: Axioms & Properties
  - Calculating probability

Warm up (1)

10 gifts to 10 friends, each gets 1 gift.

# of arrangement = ?

1 2 3 4 5 - - 10

$10 \times 9 \times 8$ - - - - 1

$= 10!$

Warm up (11)

4 gifts to 10 friends, each gets 1 gift.

# of arrangement = ?

1st gift, 2nd gift 4th

10 9 8 7

$$10 \times 9 \times 8 \times 7 = \frac{10 \cdots 6 \cdots 1}{6 \cdots 1} = \frac{10!}{6!}$$

Warm up (III)

4 gifts to 10 friends, each gets 1 gift.

# of arrangement = ?

identical

① assume the gifts are different.

is $\frac{10!}{6!}$

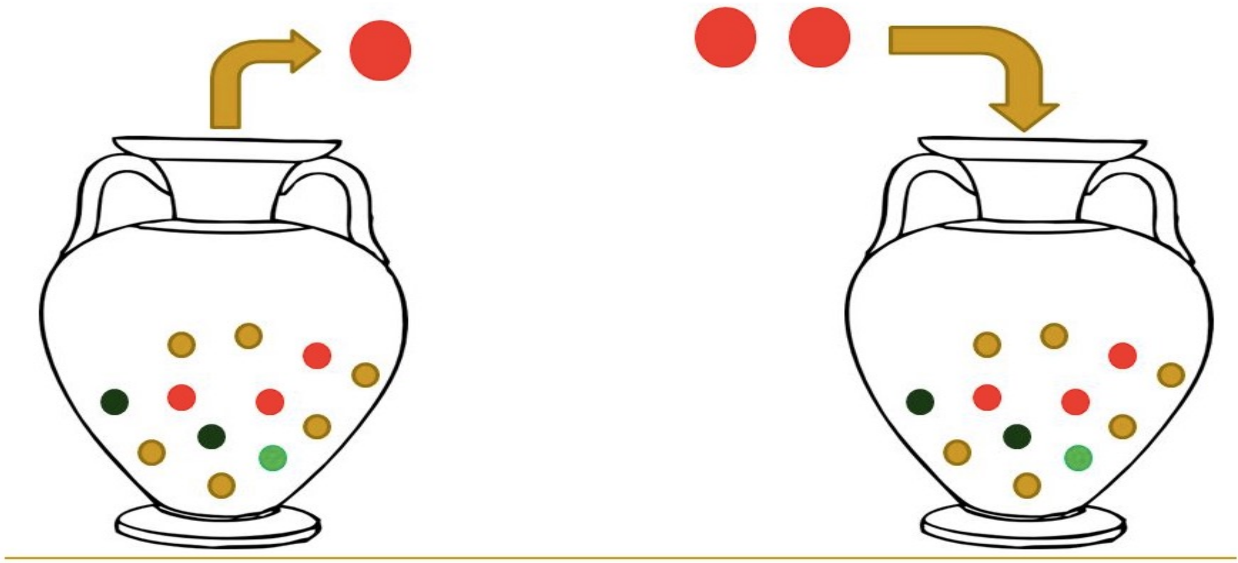
② The over count is a factor of 4! (arranging 4 different gifts)

is $\frac{10!}{6!\,4!} = \binom{10}{4} = \binom{10}{6}$

Outcome

- An outcome **A** is a possible result of a random repeatable experiment

Random: uncertain, Nondeter-ministic, …



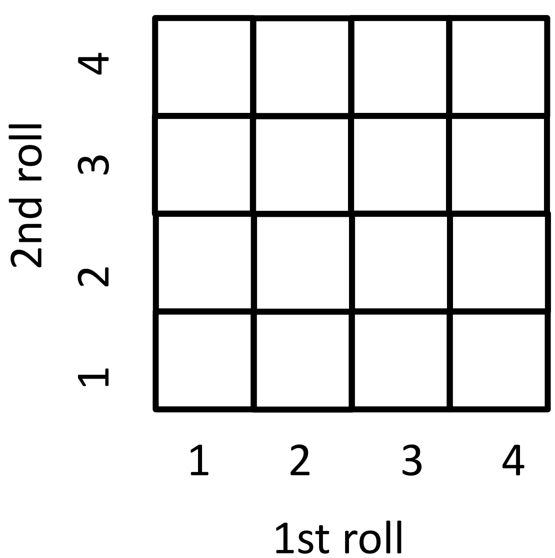
Sample space

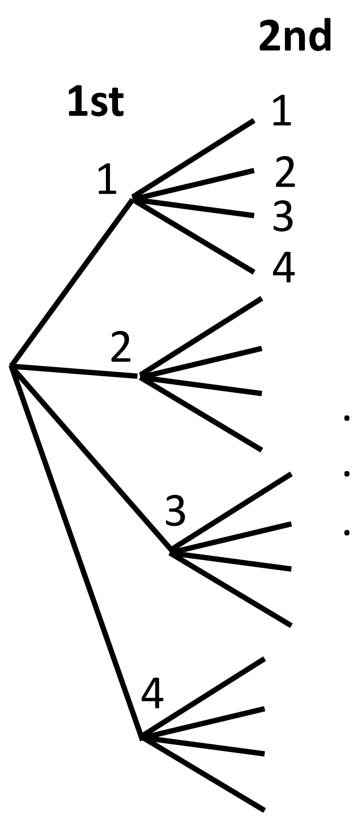
- The Sample Space, **Ω,** is the set of all possible mutually exclusive outcomes associated with the experiment
- Discrete or Continuous

Sample Space example (1)

- Experiment: we roll a 4sided-die twice
- **Discrete** Sample space:

{(1,1), (1,2)….}

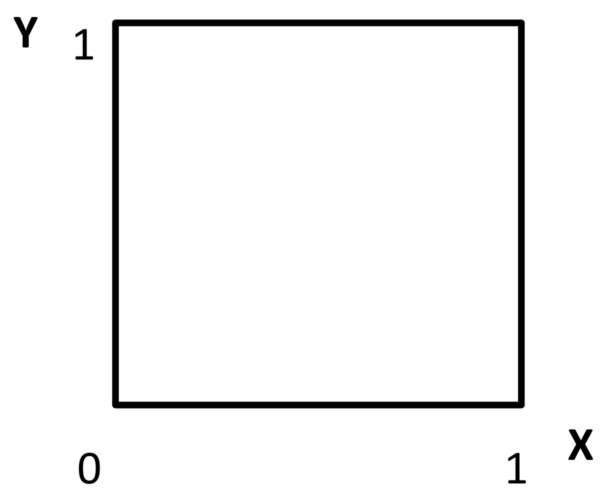




Sample Space example (2)

- Experiment: Romeo and Juliet's date
- **Continuous** Sample space:

$$\Omega = \{ (x, y) \mid 0 \leq x, y \leq 1 \}$$

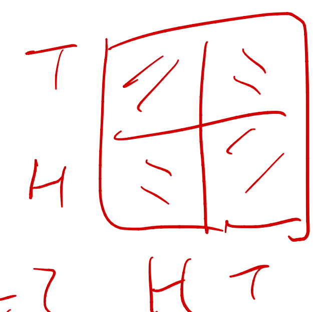


Sample Space depends on experiment (3)

- Different coin tosses
  - Toss a fair coin

    {H, T}

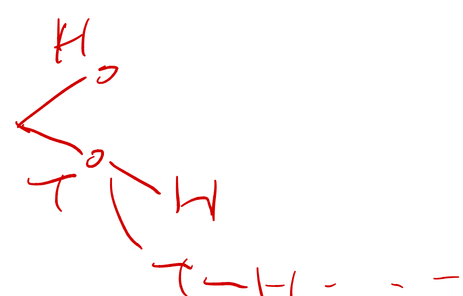
  - Toss a fair coin twice

    {HH, HT, TH, TT}

  - Toss until a head appears

    {H, TH, TTH, ...}

Sample Space depends on experiment (4)

- Drawing 2 socks one at a time from a bag containing 1 blue sock, 1 orange sock and 1 white sock **with replacement**?

$3 \times 3 = 9$ outcomes

- Drawing 2 socks one at a time from a bag containing 1 blue sock, 1 orange sock and 1 white sock **without replacement**?

$3 \times 2 = 6$

Q1. Sample Space

- Drawing 2 socks one at a time from a bag containing 1 blue sock, 1 orange sock and 1 white sock **with replacement**? What is the number of unique outcomes in the sample space?

  A. 5   B. 7   C. 9

Q2.Sample Space

- Drawing 2 socks one at a time from a bag containing 1 blue sock, 1 orange sock and 1 white sock **without replacement**? What is the number of unique outcomes in the sample space?

A. 5 B. 6 C. 9

Sample Space in real life

- Possible outrages of a power network
- Possible mutations in a gene
- A bus’ arriving time
- Possible 3-4 pl teams in CS361

Event

- An event **E** is a subset of the sample space Ω
- So, an event is a set of outcomes that is a subset of Ω, ie.
  - Zero outcome
  - One outcome
  - Several outcomes
  - All outcomes

E is ϕ, or subset, or Ω

The same experiment may have different events

- When two coins are tossed
  - Both coins come up the same?
  - At least one head comes up?

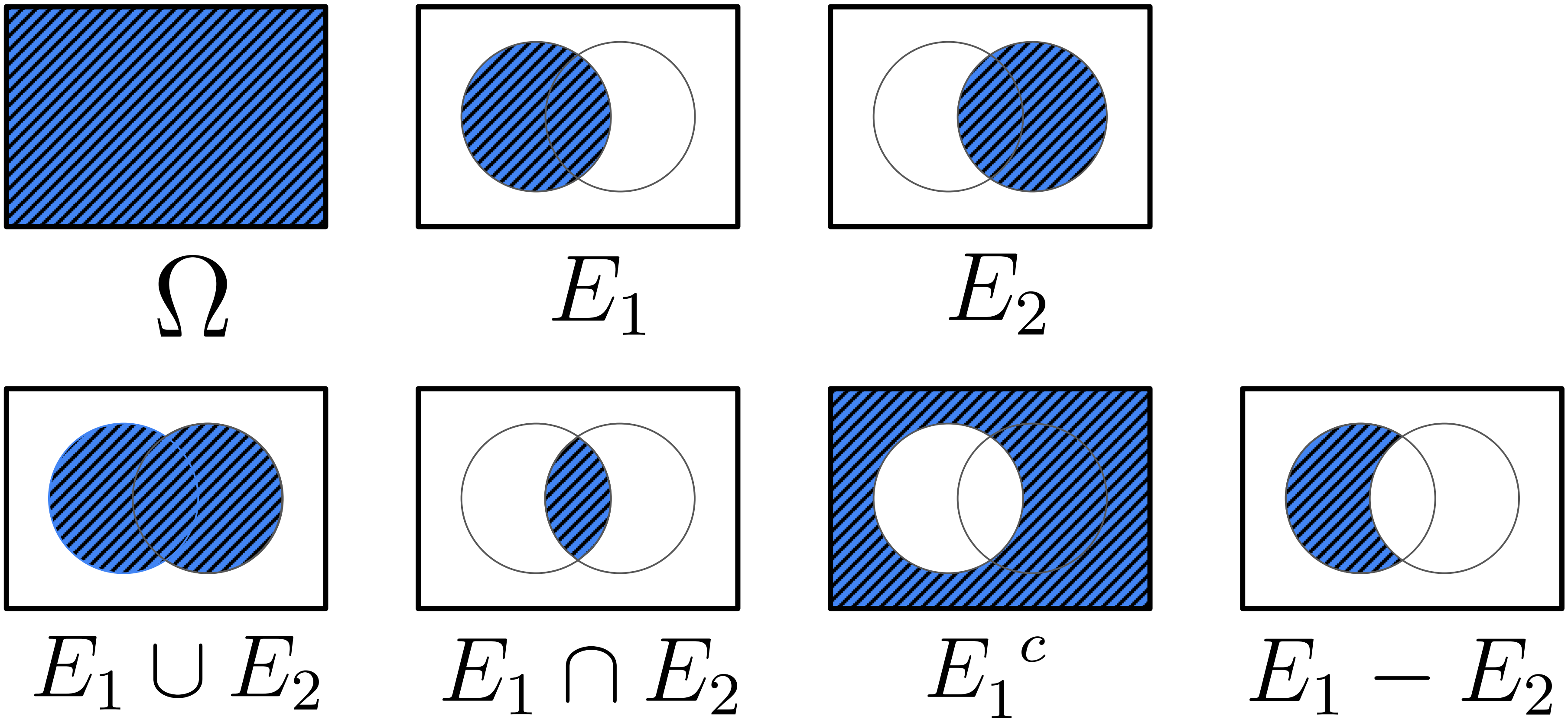
Some experiment may never end

- Experiment: Tossing a coin until a head appears

  - **E**: Coin is tossed at least 3 times

    This event includes infinite # of outcomes

Venn Diagrams of events as sets



Combining events

- Say we roll a six-sided die. Let $\Omega = ? \{1,2,3,4,5,6\}$

$$E_1 = \{1,2,5\} \ and \ E_2 = \{2,4,6\}$$

- What is $E_1 \cup E_2 = \{1,2,4,5,6\}$
- What is $E_1 \cap E_2 = \{2\}$
- What is $E_1 - E_2 = \{1,5\}$
- What is $E_1^c = \Omega - E_1 = \{3,4,6\}$

Frequency Interpretation of Probability

- Given an experiment with an outcome **A**, we can calculate the probability of **A** by repeating the experiment over and over

$$P(A) = \lim_{N->\infty} \frac{number\ of\ time\ A\ occurs}{N}$$

- So,

$$0 \leq P(A) \leq 1$$

$$\sum_{A_i \in \Omega} P(A_i) = 1$$

Axiomatic Definition of Probability

- A probability function is any function P that maps sets to real number and satisfies the following **three** axioms:

  1) Probability of any event E is non-negative

$$P(E) \geq 0$$

  2) Every experiment has an outcome &

$$P(\Omega) = 1$$

Axiomatic Definition of Probability

3) The probability of the union of disjoint events is additive

$$P(E_1 \cup E_2 \ \cup \ldots \cup \ E_N) = \sum_{i=1}^{N} P(E_i)$$

$$if \ E_i \cap E_j = \emptyset \ for \ all \ i \neq j$$

E_1 : even

E_2 : odd

Q3. Disjoint/Mutual Exclusive

- Toss a coin 3 times

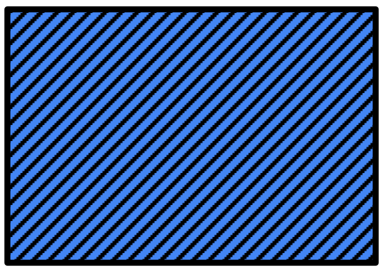
The event “exactly 2 heads appears” and “exactly 2 tails appears” are disjoint.

A. True

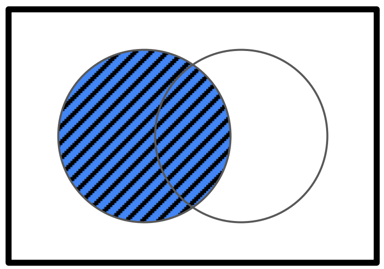
B. False

HHTT

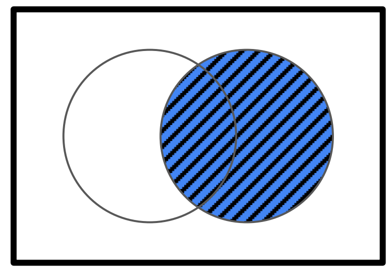
Venn Diagrams of events as sets



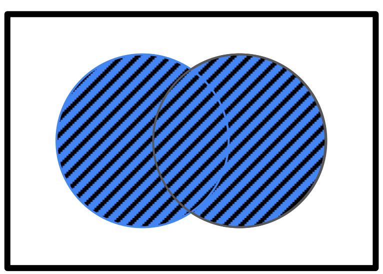
Ω



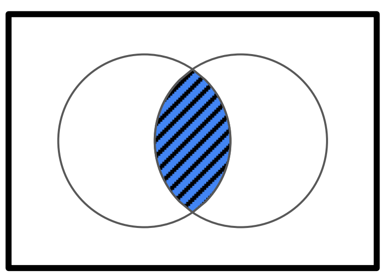
E_1



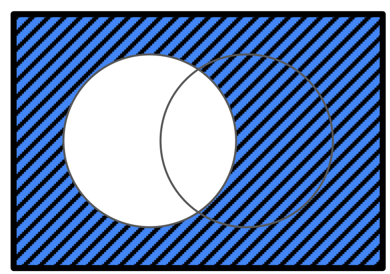
E_2



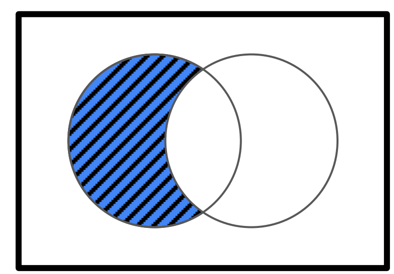
$E_1 \cup E_2$



$E_1 \cap E_2$



${E_1}^c$

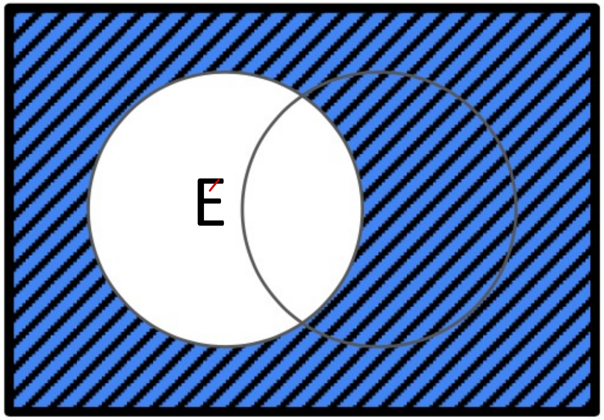


$E_1 - E_2$

Properties of probability

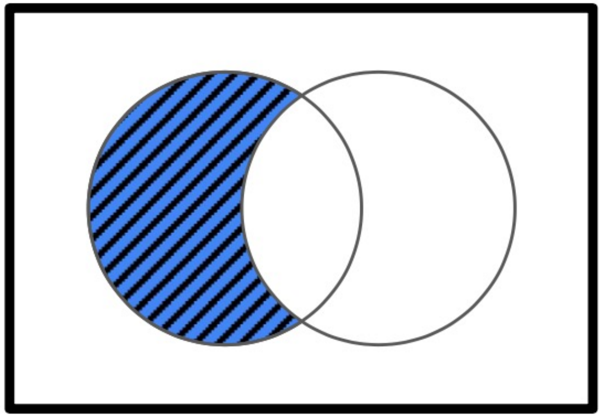
- The complement

$$P(E^c) = 1 - P(E)$$



- The difference

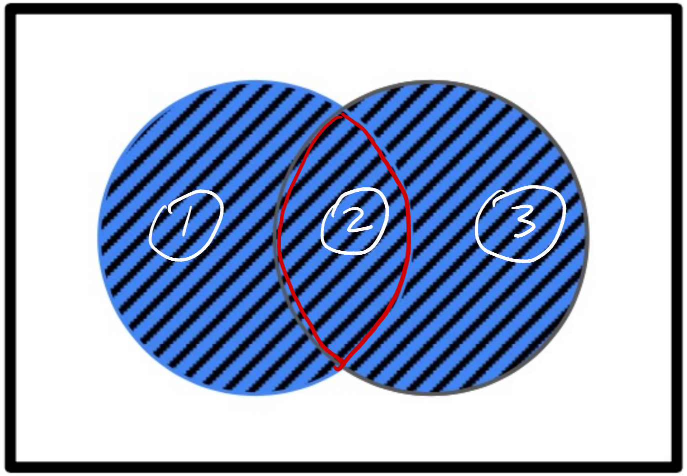
$$P(E_1 - E_2) = P(E_1) - P(E_1 \cap E_2)$$



Properties of probability

- The union

$$P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$$



- The union of multiple E

$$P(E_1 \cup E_2 \cup E_3) = P(E_1) + P(E_2) + P(E_3) - P(E_1 \cap E_2) - P(E_2 \cap E_3) - P(E_3 \cap E_1) + P(E_1 \cap E_2 \cap E_3)$$

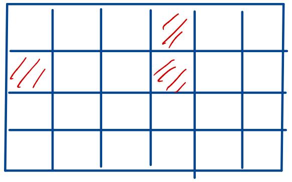
The Calculation of Probability

- Discrete countable finite event
- Discrete countable infinite event
- Continuous event

Counting to determine probability of countable finite event

- From the last axiom, the probability of event **E** is the sum of probabilities of the disjoint outcomes

$$P(E) = \sum_{A_i \in E}^{N_E = |E|} P(A_i)$$



- If the outcomes have equal probability, $\Rightarrow P(A_i) = \frac{1}{|\Omega|}$

$$P(E) = \frac{number\ of\ outcomes\ in\ E}{total\ number\ of\ outcomes\ in\ \Omega}$$

$$|E| \leftarrow \sum_{i=1}^{|E|} \frac{1}{|\Omega|}$$

Probability using counting: (1)

- Tossing a fair coin twice:
  - Prob. that it appears the same?

$$\Omega = \{HH, TT, HT, TH\}$$

$$P(E) = \frac{|E|}{|\Omega|} = \frac{2}{4} \quad E = \{HH, TT\}$$

  - Prob. that at least one head appears?

$$E = \{HT, TH, HH\}$$

$$P(E) = \frac{|E|}{|\Omega|} = \frac{3}{4}$$

Probability using counting: (2)

- 4 rolls of a 5-sided die:

  E: they all give different numbers

  - Number of outcomes in the sample space

    5^4 $5 \times 5 \times 5 \times 5$

  - Number of outcomes that make the event happen:

    $5 \times 4 \times 3 \times 2 = 5!$

  - Probability: $P(E) = \frac{|E|}{|S|} = \frac{5!}{5^4}$

Probability using counting: (2)

- What about N-1 rolls of a N-sided die?

  E: they all give different numbers

  - Number of outcomes in the sample space

    N^{N-1}

  - Number of outcomes that make the event happen: $N \cdot (N-1) \cdot (N-2) \cdots 2$

  - Probability: $\frac{N!}{N^{N-1}}$

Probability by reasoning with the complement property

- If $P(E^c)$ is easier to calculate

$$P(E) = 1 - P(E^c)$$

Probability by reasoning with the complement property

- A person is taking a test with **N** true or false questions, and the chance he/she answers any question right is 50%, what's probability the person answers **at least** one question right?

E^c : none is right

$|\Omega| : 2^N$

$|E^c| : 1$

$P(E^c) = \frac{1}{2^N}$

$P(E) = 1 - P(E^c) = 1 - \frac{1}{2^N}$

Probability by reasoning with the union property

- If E is either E1 or E2

$$P(E) = P(E_1 \cup E_2) =$$

$$P(E_1) + P(E_2) - P(E_1 \cap E_2)$$

Probability by reasoning with the properties (2)

- A person may ride a bike on any day of the year equally. What's the probability that he/she rides on a Sunday or on 15th of a month? There are two day being both Sun & 15th in 2023

E_1 : 15th of a month

E_2 : Sun. of the year

$$P(E) = \frac{12}{365} + \frac{52}{365} - \frac{2}{365}$$

365

Commutative

$$A \cap B = B \cap A$$

$$A \cup B = B \cup A$$

Associative

$$(A \cap B) \cap C = A \cap (B \cap C)$$

$$(A \cup B) \cup C = A \cup (B \cup C)$$

Distributive

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

Idempotent

$A \cap A = A$

$A \cup A = A$

Identity

$A \cup \phi = A$

$A \cap \phi = \phi$

$A \cup \Omega = \Omega$

$A \cap \Omega = A$

Complement

$A \cup A^c = \Omega$

$A \cap A^c = \phi$

$\Omega^c = \phi$

$\phi^c = \Omega$

De Morgan's

$(A \cap B)^c = A^c \cup B^c$

$(A \cup B)^c = A^c \cap B^c$

Counting may not work

- This is one important reason to use the method of reasoning with properties

What if the event has infinite outcomes

- Tossing a fair coin until head appears
  - Coin is tossed at least 3 times

    This event includes infinite # of outcomes.

    And the outcomes don’t have equal probability.

TTH, TTTH, TTTTH….

Additional References

- Charles M. Grinstead and J. Laurie Snell "Introduction to Probability”
- Morris H. Degroot and Mark J. Schervish "Probability and Statistics”

Assignments

- Work on 2nd half of Module Week2
- Quiz1 and Quiz2 due Sat. 11:59pm
- HW1 due 11:59PM Thurs. night

See you next time

See You!