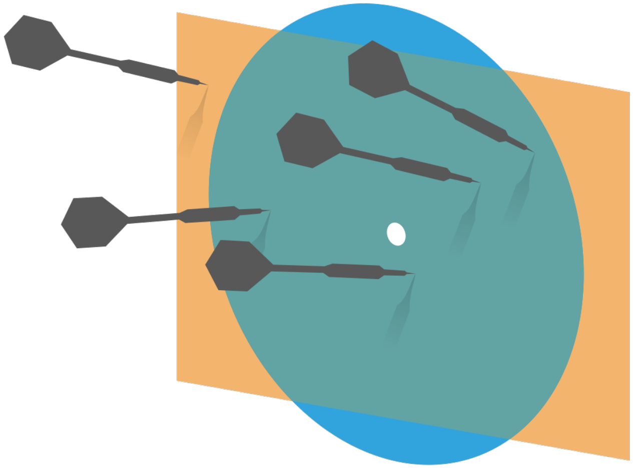
Probability and Statistics for Computer Science



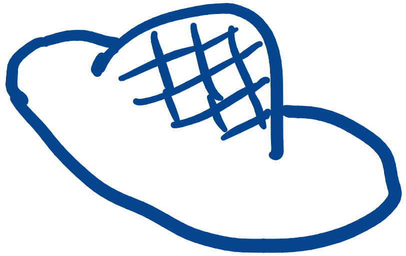
Credit: wikipedia

"A major use of probability in statistical inference is the updating of probabilities when certain events are observed" – Prof. M.H. DeGroot

Hongye Liu, Teaching Assistant Prof, CS361, UIUC, 9.7.2023

Counting: how many ways?

if we put 7 hats (indistinguishable) on 7 people out of 10 people randomly?



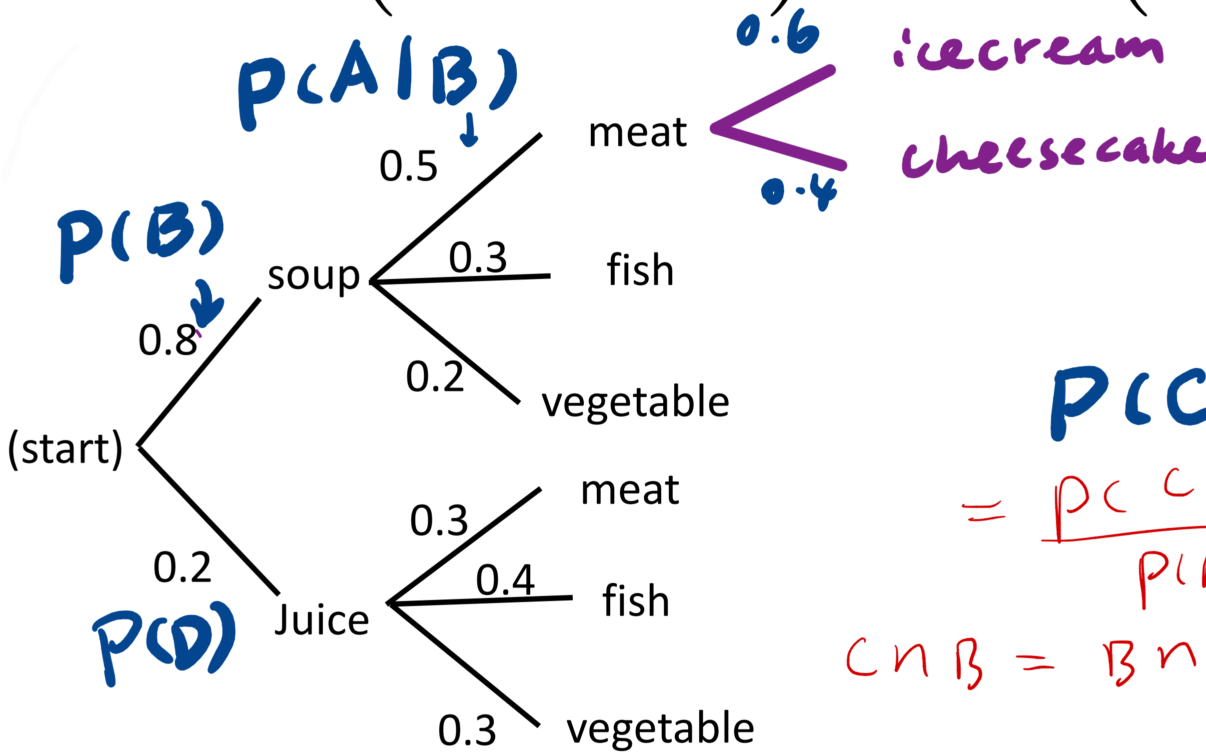
$$\binom{10}{7} = \frac{10!}{3!\,7!}$$

Last time

- More Probability Calculation
- Conditional Probability
  - Multiplication rule
  - Bayes rule

Multiplication using conditional probability

$$P(A\cap B) = P(A|B)P(B)$$



C: cheesecake

$$P(C|B) = ?$$

$$= \frac{P(C\cap B)}{P(B)} = \frac{0.8\times 0.5\times 0.4}{0.8}$$

$$C\cap B = B\cap A\cap C + \cancel{B\cap A^c\cap C} \rightarrow \emptyset$$

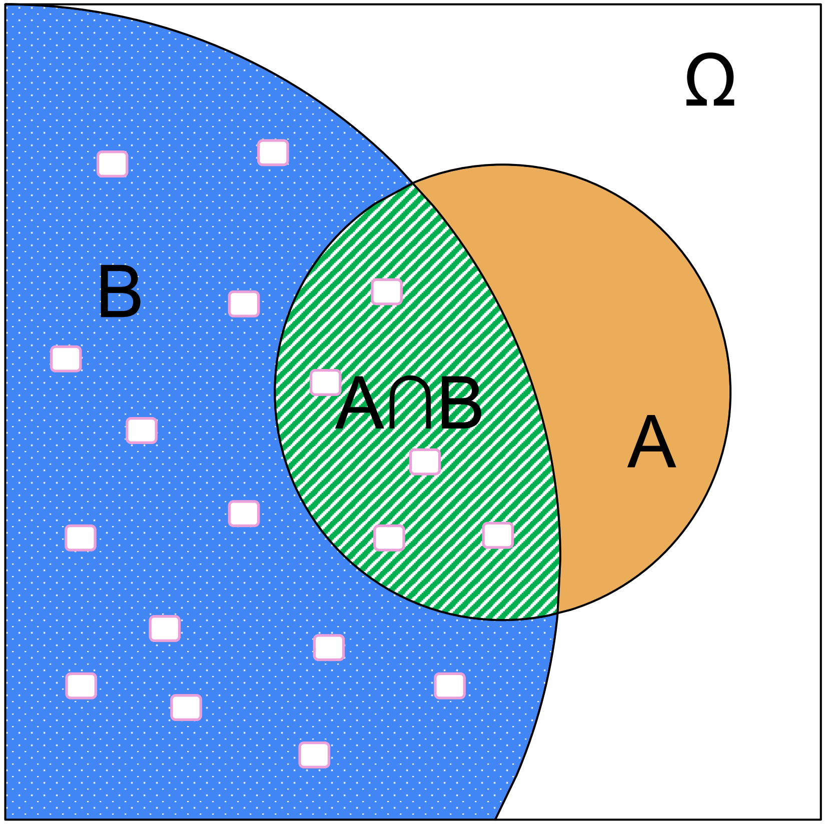
A: meat B: soup D: juice

Objectives

- Conditional Probability
  - Review
  - Bayes rule
  - Total probability
  - Independence

Conditional Probability

- The probability of ***A*** given ***B***



$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$$P(B) \neq 0$$

The area defined by the set B is the new sample space for conditional P(A|B)

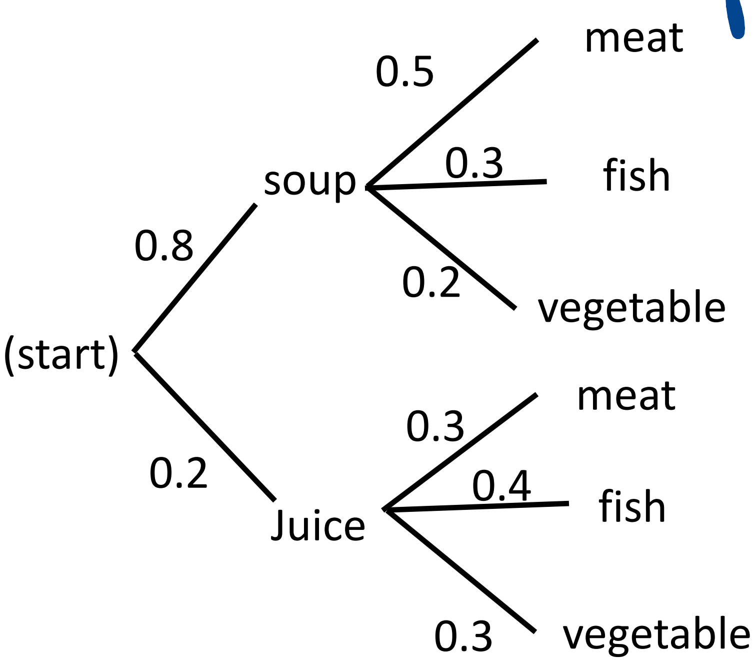
Joint Probability Calculation

$$\Rightarrow P(A \cap B) = P(A|B)P(B)$$

$P(A \cap B \cap C)$

$= P(A \cap B | C) \cdot P(C)$

$= P(A | B | C) \cdot P(B|C) \cdot P(C)$



$$P(soup \cap meat) =$$
$$P(meat|soup)P(soup)$$
$$= 0.5 \times 0.8 = 0.4$$

Symmetry of joint event in terms of conditional prob.

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \quad P(B) \neq 0$$

$$\Rightarrow P(A \cap B) = P(A|B)P(B)$$
$$\Rightarrow P(B \cap A) = P(B|A)P(A)$$

Symmetry of joint event in terms of conditional prob.

$$\because P(B \cap A) = P(A \cap B)$$

$$\Rightarrow$$

$$\boxed{P(A|B)P(B) = P(B|A)P(A)}$$

The famous **Bayes** rule

$$P(A|B)P(B) = P(B|A)P(A)$$

$$\Rightarrow \quad P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

Thomas Bayes (1701-1761)

Bayes rule: lemon cars

There are two car factories, **A** and **B**, that supply the same dealer. Factory **A** produced **1000** cars, of which **10** were lemons. Factory **B** produced **2** cars and both were lemons. You bought a car that turned out to be a lemon. What is the probability that it came from factory **B**?

E_1 : Bad car $\quad E_2$: From B

$$P(E_2|E_1) = \frac{\overset{1\times}{P(E_1|E_2)}\,P(E_2)\rightarrow\frac{2}{1002}}{P(E_1)} = \frac{\frac{2}{1002}}{\frac{12}{1002}} = \frac{1}{6}$$

Bayes rule: lemon cars

Given the above information, what is the probability that it came from factory A?

$$P(A|L) = ?$$

Bayes rule: lemon cars

Given the above information, what is the probability that it came from factory A?

$$P(A|L) = ?$$

$$P(A|L) = \frac{P(L|A)P(A)}{P(L)}$$

Or in this case

$$P(A|L) = 1 - P(B|L)$$

=

Total probability:

Sunny : 80% to workout

Rainy : 30% to workout

Snowy : 40% to workout

#Sunny / #Rainy / #Snowy = 5 : 3 : 2

P(work out) = ?   P(work out & Sunny) = P(W | Sun) P(Sun)

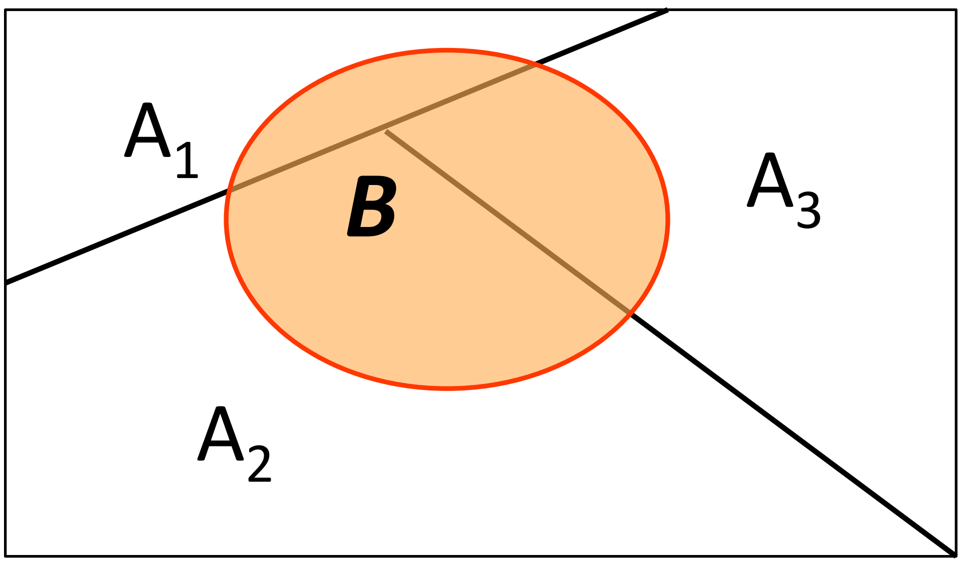
+ P(. . . & Rain) = P(W | Rain) P(Rain)

+ P(. . . & Snowy) = P(W | Snow) P(Snow)

$= 0.8 \times 0.5 + 0.3 \times 0.3 + 0.4 \times 0.2$

Total probability

$$P(B) = P(A_1 \cap B) + P(A_2 \cap B) + P(A_3 \cap B)$$
$$= P(B|A_1)P(A_1) + P(B|A_2)P(A_2)$$
$$+ P(B|A_3)P(A_3)$$



A_1, A_2, A_3 are disjoint,

$A_1 \cup A_2 \cup A_3 \supseteq B$

Total probability:

$$P(B) = P(A \cap B) + P(A^c \cap B)$$
$$= P(B|A)P(A) + P(B|A^c) \cdot P(A^c)$$

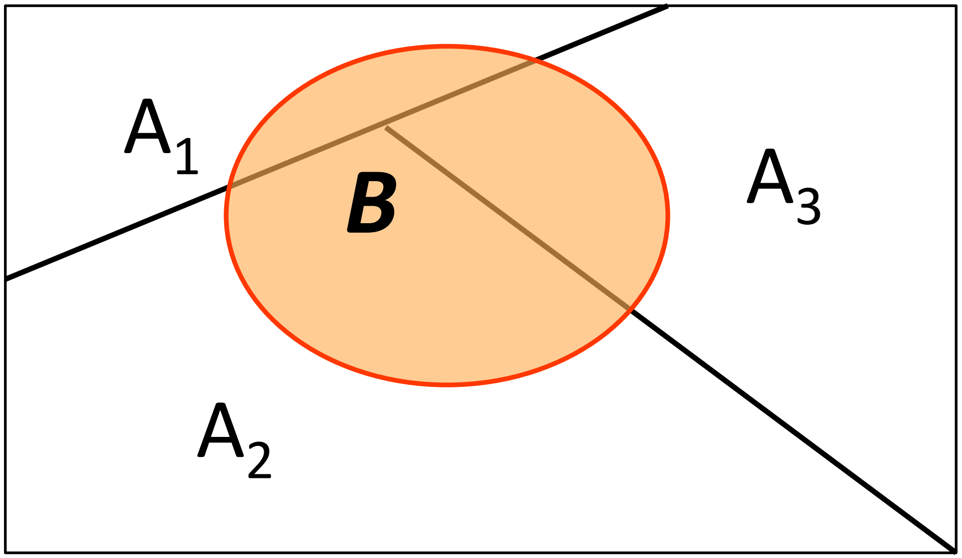
$P(B|A) =$

$P(B|A^c) =$

Total probability general form

$$P(B) = \sum_j P(B \cap A_j)$$

$$= \sum_j P(B \mid A_j) P(A_j)$$

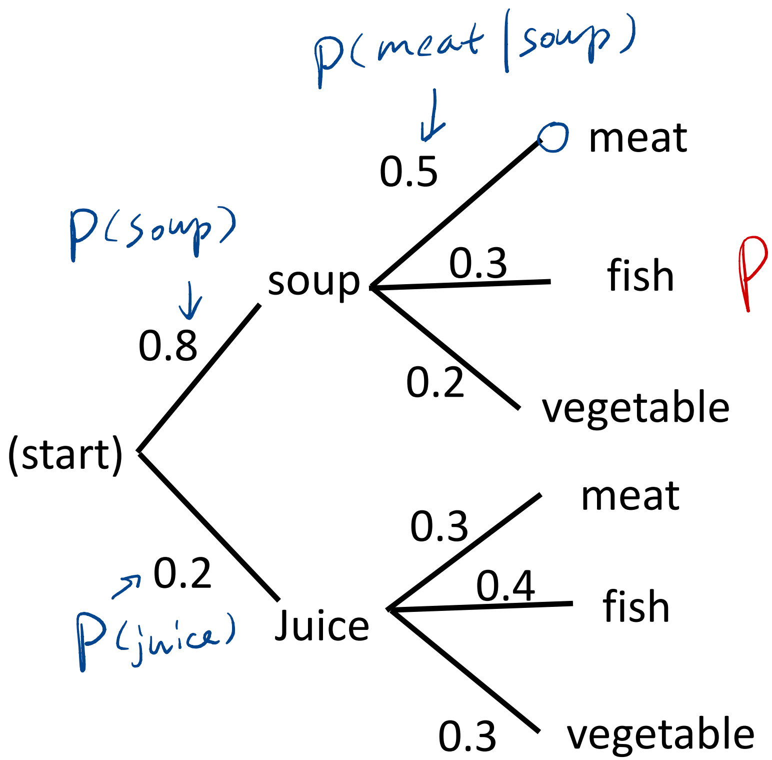


A_j, A_k are disjoint

$A_j \cap A_k = \emptyset$

$j \neq k$

Total probability:



$P(\text{meat}) = ?$ 0.46

$P(\text{soup} \cap \text{meat})$

$P(\text{soup}) \cdot P(\text{meat} \mid \text{soup})$ $+ P(\text{juice} \cap \text{meat})$

$$P(\text{soup} \mid \text{meat}) = ?$$

$$= \frac{P(\text{soup} \cap \text{meat})}{P(\text{meat})}$$

$$= \frac{0.4}{0.46}$$

Bayes rule using total prob.

$$P(A_j \mid B) = \frac{P(B \mid A_j)\, P(A_j)}{P(B)}$$

$$= \frac{P(B \mid A_j)\, P(A_j)}{\sum_j P(B \mid A_j)\, P(A_j)}$$

$A_i \cap A_j = \phi$ → disjoint

if $i \neq j$

Bayes rule: rare disease test

There is a blood test for a rare disease. The frequency of the disease is 1/100,000. If one has it, the test confirms it with probability 0.95. If one doesn't have, the test gives false positive with probability 0.001. What is $P(D|T)$, the probability of having disease given a positive test result?

$$P(D|T) = \frac{P(T|D)P(D)}{P(T)}$$

Using total prob.

0.95

$\to 1 \times 10^{-5}$

0.001

$$= \frac{P(T|D)P(D)}{P(T|D)P(D) + P(T|D^c)P(D^c)}$$

$(1-10^{-5})$

Bayes rule: rare disease test

There is a blood test for a rare disease. The frequency of the disease is **1/100,000**. If one has it, the test confirms it with probability 0.95. If one doesn't have, the test gives false positive with probability **0.001**. What is $P(D|T)$, the probability of having disease given a positive test result?

$$P(D|T) = \frac{P(T|D)P(D)}{P(T|D)P(D) + P(T|D^c)P(D^c)}$$

$$=$$

Independence

- One definition:

$$P(A|B) = P(A) \ or$$
$$P(B|A) = P(B)$$

Whether A happened doesn't change the probability of B and vice versa

Independence: example

- Suppose that we have a fair coin and it is tossed twice. let A be the event "the first toss is a head" and B the event "the two outcomes are the same."

A : 1st H

B : 1st 2nd same

HH, TT

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{1/4}{0.5} = \frac{1}{2}$$

$P(B) = \frac{1}{2}$

- These two events are

Independence

- Alternative definition

$$P(A|B) = P(A)$$

$$\Rightarrow \frac{P(A \cap B)}{P(B)} = P(A)$$

$$\Rightarrow \boxed{P(A \cap B) = P(A)P(B)}$$

Testing Independence:

- Suppose you draw one card from a standard deck of cards. E_1 is the event that the card is a King, Queen or Jack. E_2 is the event the card is a Heart. Are E_1 and E_2 independent?

$P(E_1) = \frac{3}{13}$

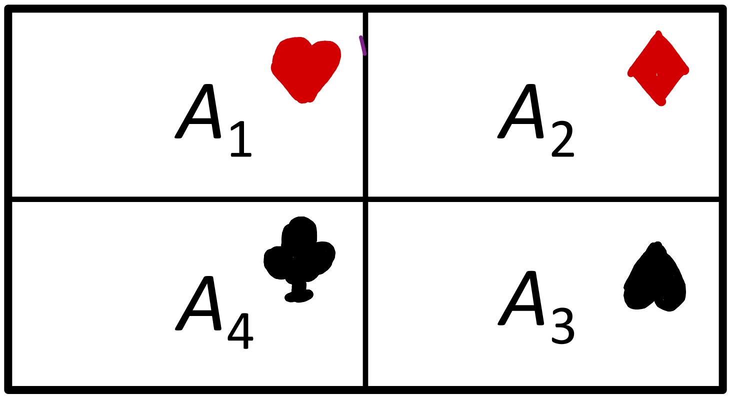
$P(E_2) = \frac{1}{4}$

$P(E_1 \cap E_2) = \frac{3}{52}$

$P(E_1 \cap E_2) = P(E_1)P(E_2)$

$\Rightarrow$ indpt!

Pairwise independence is not mutual independence in larger context



$\mathrm{P}(A_1) = \mathrm{P}(A_2) = \mathrm{P}(A_3) = \mathrm{P}(A_4) = 1/4$

$P(A) = ?\ \frac{1}{2} \quad P(B) = ?\ \frac{1}{2} \quad P(C) = ?\ \frac{1}{2}$

$$A = A_1 \cup A_2; P(A)$$

$$B = A_1 \cup A_3; P(B)$$

$$C = A_1 \cup A_4; P(C)$$

$P(A \cap B) = ?\ \overset{P(A_1)}{\frac{1}{4}} \quad P(A)P(B) = ?\ \frac{1}{4}$

$P(A \cap C) = ?\ \frac{1}{4} \quad P(A)P(C) = ?\ \frac{1}{4}$

$P(B \cap C) = ?\ \overset{P(A_1)}{\frac{1}{4}} \quad P(B)P(C) = ?\ \frac{1}{4}$

$P(A \cap B \cap C) = ? \quad P(A)P(B)P(C) = ?$

$= P(A_1) = \frac{1}{4} \qquad = \frac{1}{8}$

Mutual independence

- Mutual independence of a collection of events $A_1, A_2, A_3 ... A_n$ is :

$$P(A_i|A_j A_k ... A_p) = P(A_i)$$

$$j, k, ...p \neq i$$

- It's very strong independence!

Probability using the property of Independence: Airline overbooking (1)

- An airline has a flight with 6 seats. They always sell 7 tickets for this flight. If ticket holders show up independently with probability **p**, what is the probability that the flight is overbooked ?

$P(7 \text{ pl show up})$

$= P(1\text{st} \cap 2\text{nd} \cap 3\text{rd} \cdots \cap 7\text{th})$

$= P(1\text{st})\, P(2\text{nd}) \cdots P(7\text{th})$

$= p^7$

Probability using the property of Independence: Airline overbooking (2)

- An airline has a flight with 6 seats. They always sell 8 tickets for this flight. If ticket holders show up independently with probability **p**, what is the probability that exactly 6 people showed up?

P(6 people showed up) = $\binom{8}{6} \underline{p^6 (1-p)^2}$

Probability using the property of Independence: Airline overbooking (3)

- An airline has a flight with 6 seats. They always sell 8 tickets for this flight. If ticket holders show up independently with probability **p**, what is the probability that the flight is overbooked ?

P(overbooked) = $P(7\ \text{ppl}) + P(8\ \text{ppl})$

$= \binom{8}{7} p^7 (1-p) + p^8$

Assignments

- Module week3 on Canvas
- Next time: Random variable

Additional References

- Charles M. Grinstead and J. Laurie Snell "Introduction to Probability”
- Morris H. Degroot and Mark J. Schervish "Probability and Statistics”

See you next time

See You!