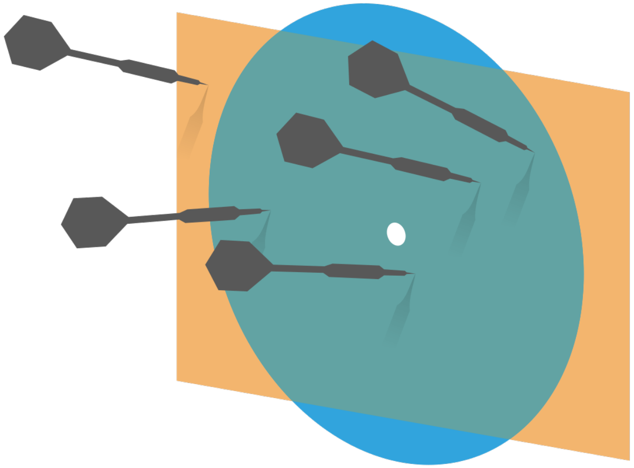
# Probability and Statistics for Computer Science



Credit: wikipedia

"A major use of probability in statistical inference is the updating of probabilities when certain events are observed" – Prof. M.H. DeGroot

Hongye Liu, Teaching Assistant Prof, CS361, UIUC,9.5.2023

# Warm up

1) 10 people queue

$10 \times 9 \times 8 \times \cdots \times 1 = 10!$

2) 4 out of 10 queue

$10 \times 9 \times 8 \times 7 = \frac{10!}{6!}$

3) 6 out of 10 group

$$\frac{\frac{10!}{4!}}{6!}$$

→ redundancy factor is 6!

# Warm up (II)   work at home

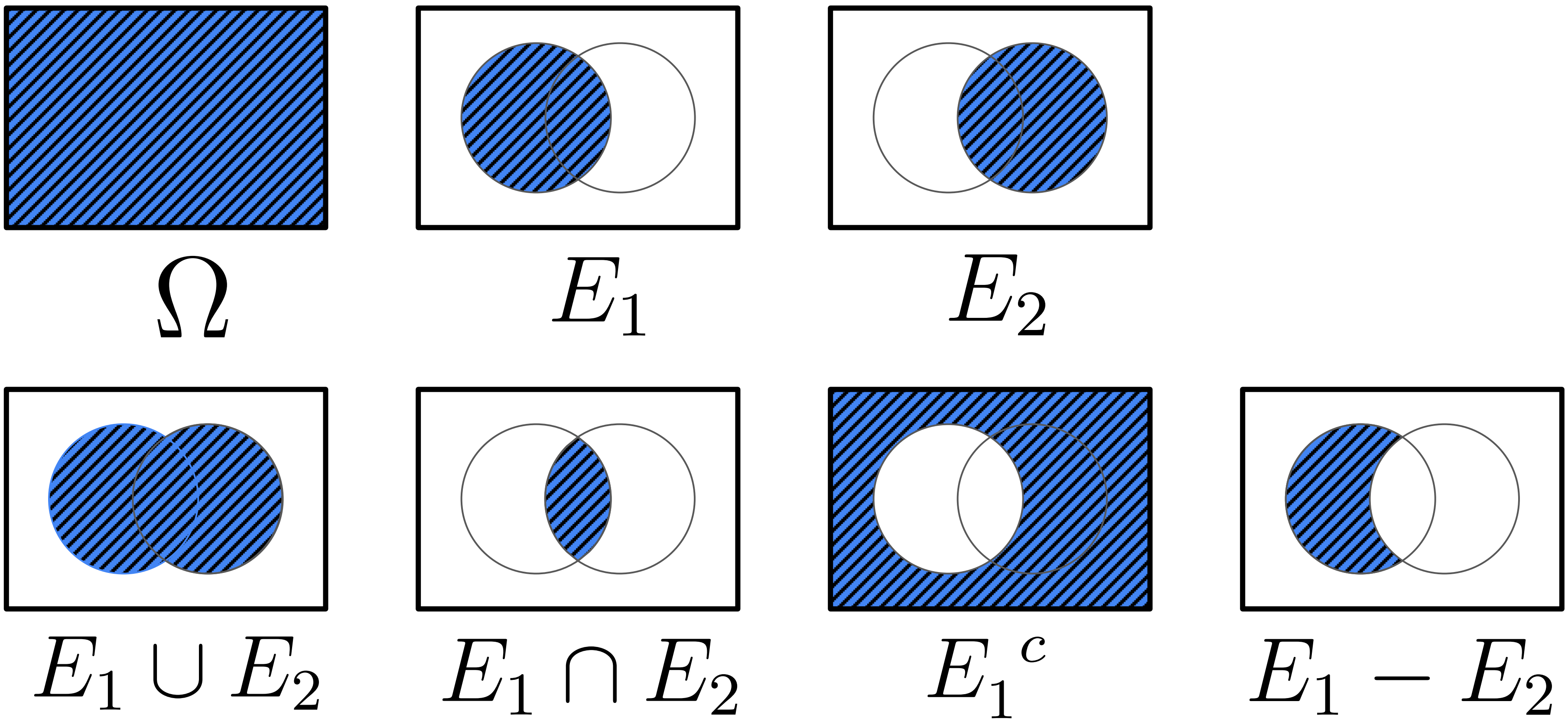
- Fill the blanks:

"I am an avid vegetarian and I enjoy eating all day long, people admire my appetite and like to watch me eat." I am a _ _ _ _ _. How many ways are there to rearrange these 5 letters? _______. If you draw 2 letters from them, how many outcomes (order matters) are there that are without "a"?

# Objectives

- Probability
  - Probability calculation
  - Conditional Probability
    - Bayes rule
    - Independence

# Venn Diagrams of events as sets



# Senate Committee problem

The United States Senate contains **two** senators from each of the **50** states. If a committee of eight senators is selected at random, what is the probability that it will contain at least one of the two senators from **IL**?

$E^c$ : none of 2 IL got in

$|\Omega| = \binom{100}{8}$

$|E|$ $\quad P(E) = 1 - P(E^c) = 1 - \frac{|E^c|}{|\Omega|}$

$1 - \frac{\binom{98}{8}}{\binom{100}{8}}$

# Probability: Birthday problem

- Among 30 people, what is the probability that at least 2 of them celebrate their birthday on the same day? Assume that there is no February 29 and each day of the year is equally likely to be a birthday.

$E^c$: none of them share

$|\Omega| : 365 \times 365 \cdots = 365^{30}$

$|E^c| : \underbrace{365 \times 364 \times 363 \cdots}_{30} = \frac{365!}{334!}$

# Counting may not work

- This is one important reason to use the method of reasoning with properties

# What if the event has infinite outcomes

- Tossing a fair coin until head appears
  - Coin is tossed at least 3 times

    This event includes infinite # of outcomes.

    And the outcomes don't have equal probability.

E: {1st is T & 2nd is T & 3rd is H}

TTH, TTTH, TTTTH….

↳ a joint event

# Conditional Probability

- Motivation of conditional probability

# Conditional Probability

- Example:

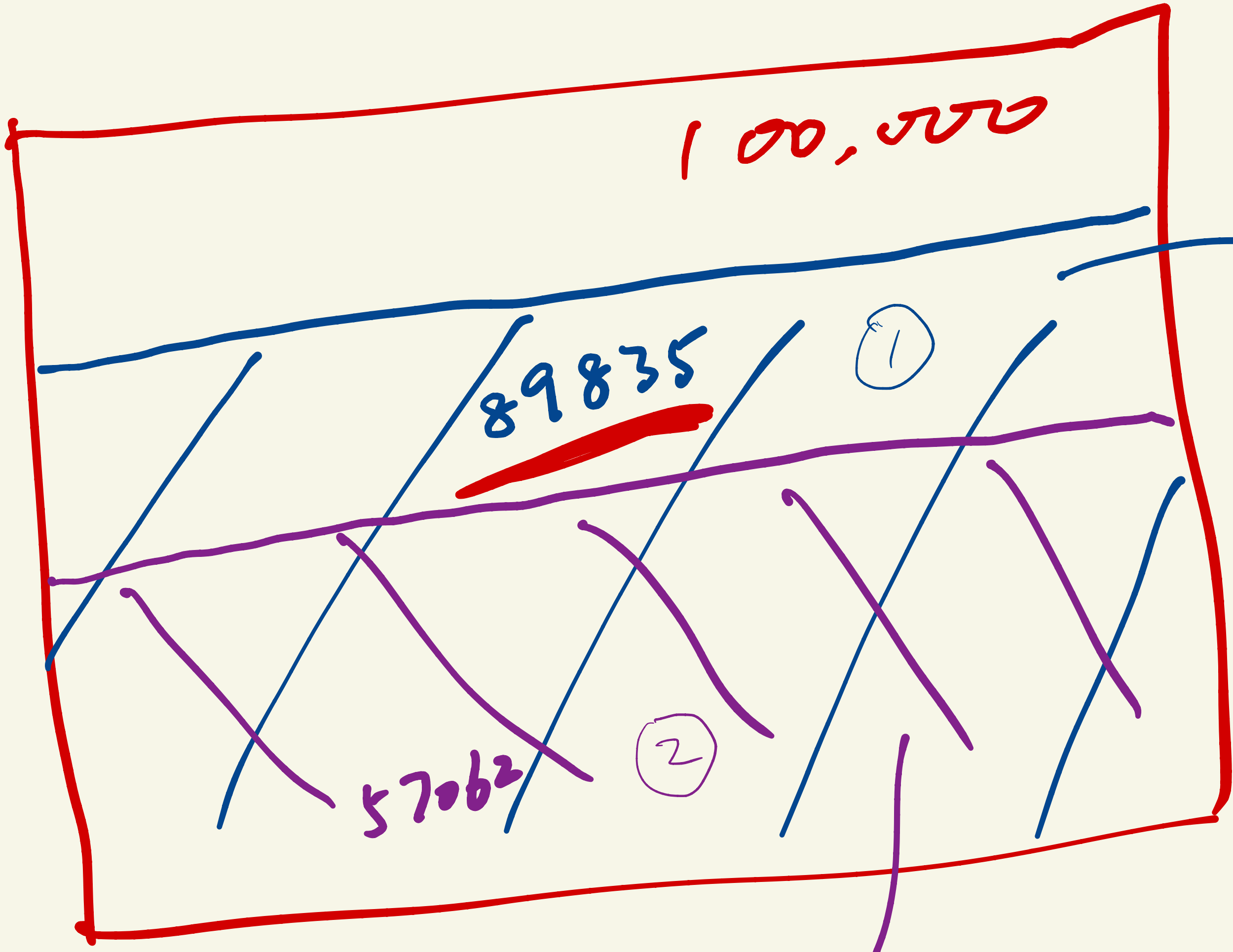
An insurance company knows in a population of 100 thousands females, 89.835% expect to live to age 60, while 57.062% can expect to live to 80. Given a woman at the age of 60, what is the probability that she lives to 80?

# Conditional Probability

- Given the condition she is 60 already, the size of the sample space for the outcomes has changed to

  **89,835** instead of 100,000

$|\Omega| = 100{,}000$




live to 60

$|E_{60}| =$

$|E_{80}| =$ 57062

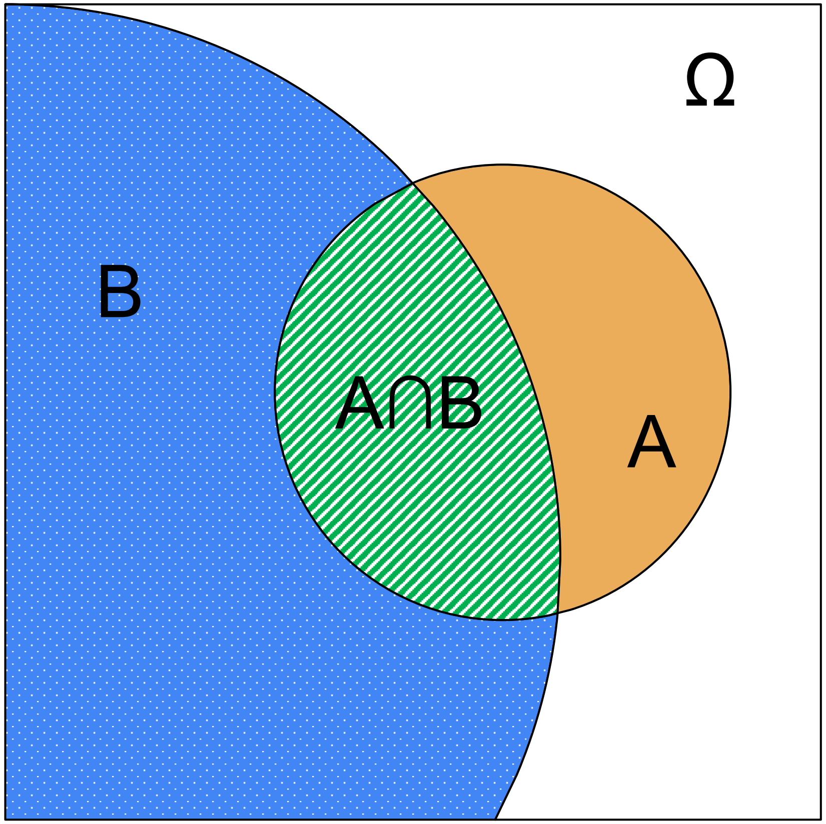
live to 80

$$P(E_{80} \mid E_{60}) = \frac{|E_{80}|}{|\Omega_n|} =$$

$|\Omega_n| = ?$ 89835

# Conditional Probability

- The probability of ***A*** given ***B***




$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$$P(B) \neq 0$$

The "Size" analogy

# Conditional Probability

$A$ : a woman lives to 80

$$P(A|B) = \frac{57,062}{89,835} = 0.6352$$

$B$ : a woman is at 60 now

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{57,062}{1521}}{\frac{89,835}{1521}}$$

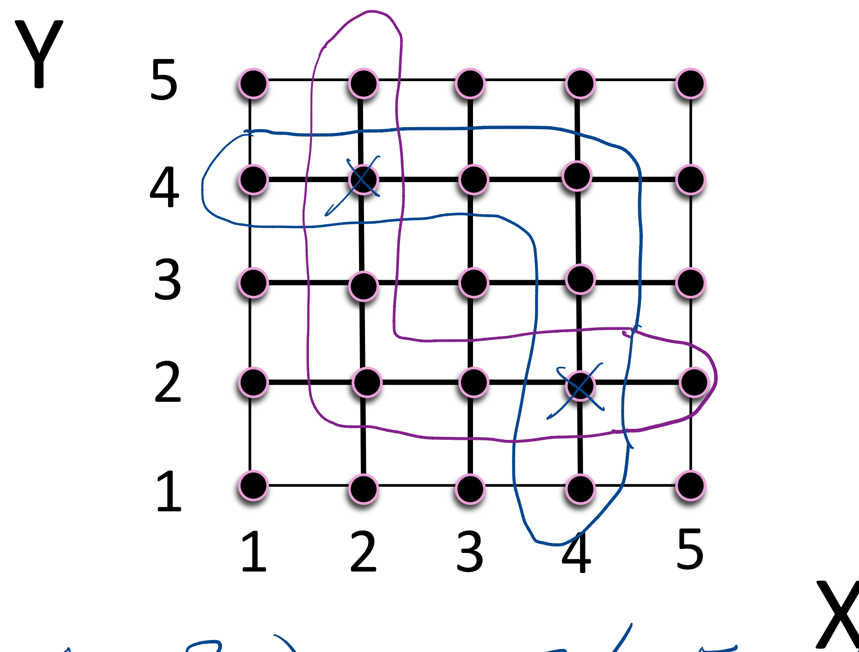
$$= 0.6352$$

While $P(A) = \frac{57,062}{100,000} = 0.57062$

# Conditional Probability: die example

Throw 5-sided fair die twice.

$$A : max(X, Y) = 4$$
$$B : min(X, Y) = 2$$




$$P(A|B) =? \; \frac{P(A \cap B)}{P(B)} = \frac{2/25}{7/25} = \frac{2}{7}$$

# Conditional probability, that is?

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \quad P(B) \neq 0$$

Still a probability! It satisfies the three axioms

$P(\Omega) = 1$

$P(A|B) = \frac{P(A \cap B)}{P(B)}$

$P(A^c|B) = \frac{P(A^c \cap B)}{P(B)}$

$P(A|B) + P(A^c|B) = ?$

$P(A_1 \cup A_2|B) = ?$ if $A_1 \cap A_2 = \phi$

$= P(A_1|B) + P(A_2|B)$

# Multiplication rule using conditional probability

- Joint event

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \quad P(B) \neq 0$$

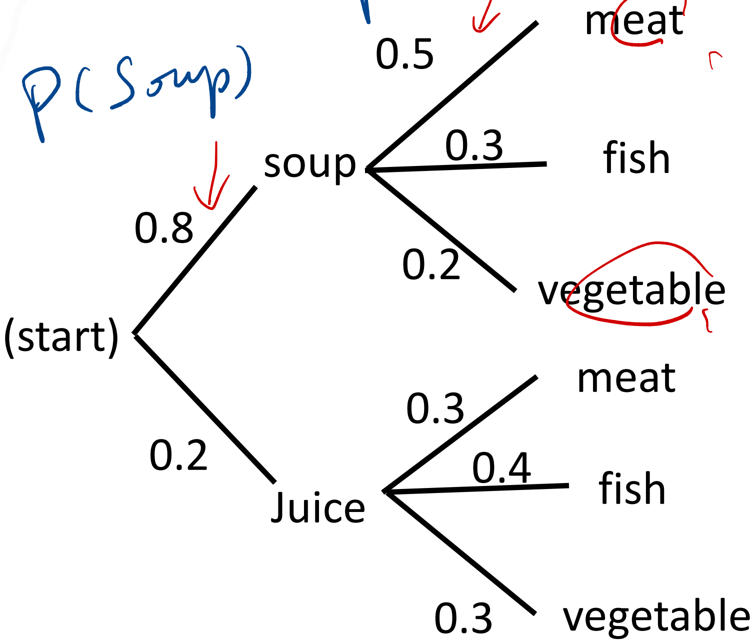
$$\Rightarrow \boxed{P(A \cap B) = \underbrace{P(A|B)}_{\text{conditional}} \underbrace{P(B)}_{\text{prior}}}$$

# Multiplication using conditional probability

$$P(A \cap B) = P(A|B)P(B)$$

$P(\text{meat} \mid \text{soup})$

$P(\text{Soup})$




$P(\text{soup} \cap \text{meat})$ ?

$= P(\text{soup}) \cdot P(\text{Meat} \mid \text{soup})$

$= 0.8 \times 0.5 = 0.4$

$P(\text{soup} \cap \overline{\text{fish}})$

$= ?\ P(\text{soup} \cap \text{meat}) + P(\text{soup} \cap \text{veg})$

$= 0.8 \times 0.5 + 0.8 \times 0.2$

# Multiplication using conditional probability

$$P(A \cap B) = P(A|B)\underline{P(B)}$$

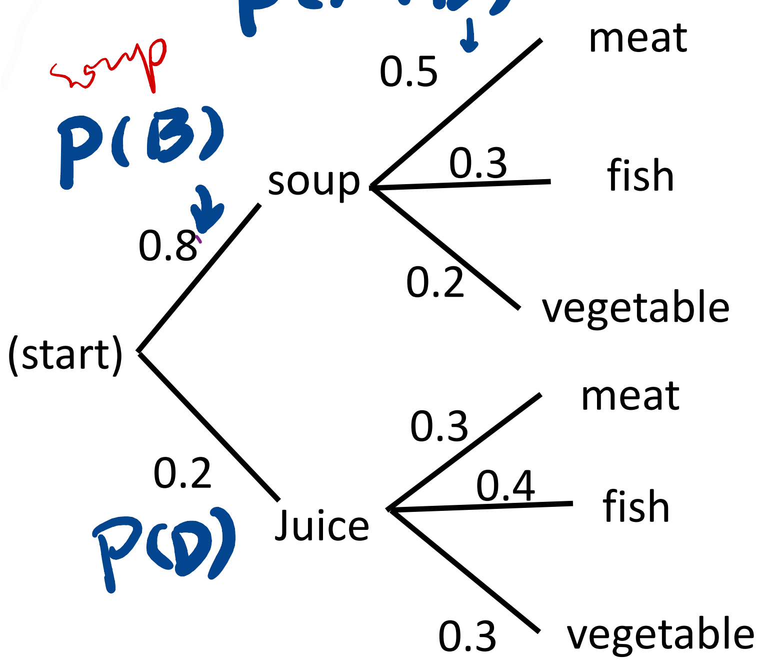
meat

prior

$P(A|B)$

soup

$P(B)$




$P(D)$

$\underline{P(A^c|B) = ?}$

$P(A^c \cap D) = ?$

$P(A^c|B) = \dfrac{P(A^c \cap B)}{P(B)}$

$P(A^c|B) = 1 - P(A|B)$

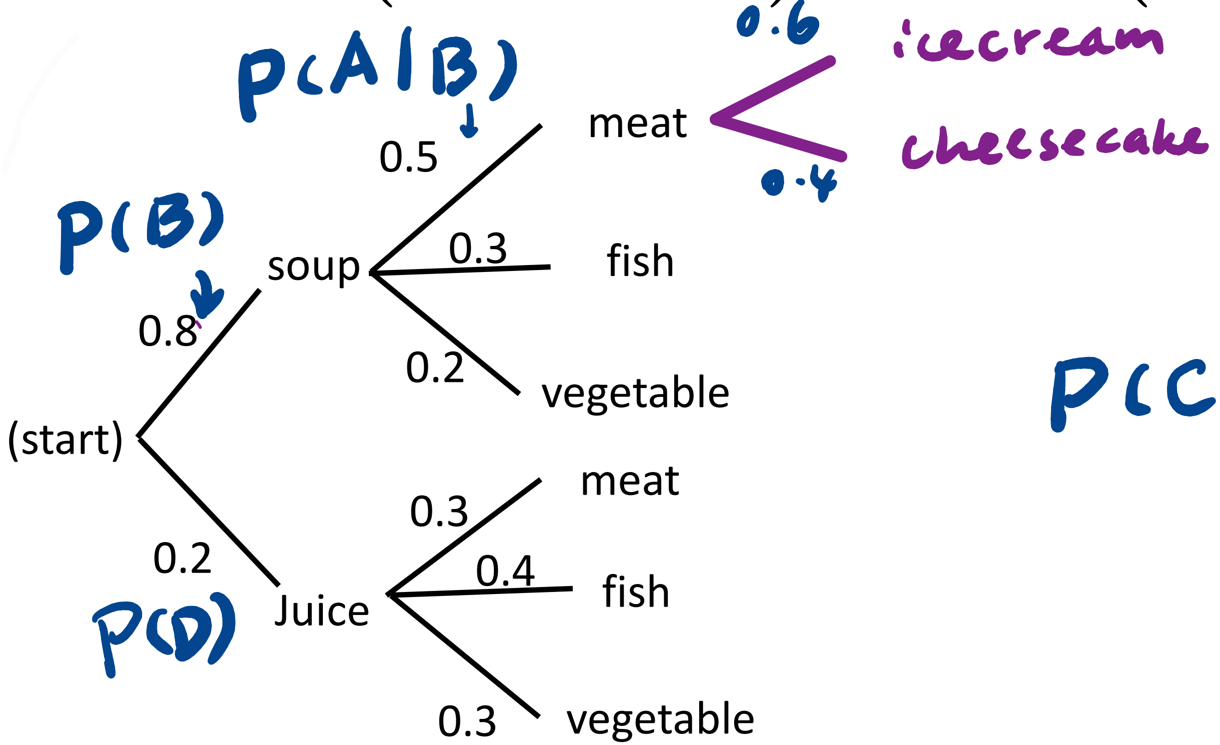
A : meat

B : soup

D : juice

# Multiplication using conditional probability

$$P(A \cap B) = P(A|B)P(B)$$




C: cheesecake

$P(C|B) = ?$

A: meat B: soup D: juice

# Symmetry of joint event in terms of conditional prob.

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \quad P(B) \neq 0$$

$$P(B|A) = \frac{P(B \cap A)}{P(A)}$$

$$\Rightarrow P(A \cap B) = P(A|B)P(B)$$

$$\Rightarrow P(B \cap A) = P(B|A)P(A)$$

# Symmetry of joint event in terms of conditional prob.

$$\because P(B \cap A) = P(A \cap B)$$

$$\Rightarrow$$

$$\boxed{P(A|B)P(B) = P(B|A)P(A)}$$

# The famous **Bayes** rule

$$P(A|B)P(B) = P(B|A)P(A)$$

$$\Rightarrow \quad P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

$$P(A|D) = \frac{P(D|A) \cdot P(A)}{P(D)}$$

Thomas Bayes (1701-1761)

# Bayes rule: lemon cars

There are two car factories, **A** and **B**, that supply the same dealer. Factory **A** produced **1000** cars, of which **10** were lemons. Factory **B** produced **2** cars and both were lemons. You bought a car that turned out to be a lemon. What is the probability that it came from factory **B**?

$$P(B|L) = \frac{P(L|B) \cdot P(B)}{P(L)} = \frac{1 \times \frac{2}{1002}}{\frac{12}{1002}}$$

# Simulation of Conditional Probability

http://www.randomservices.org/random/apps/ConditionalProbabilityExperiment.html

# Additional References

- Charles M. Grinstead and J. Laurie Snell "Introduction to Probability"
- Morris H. Degroot and Mark J. Schervish "Probability and Statistics"

# Assignments

- Work on Module Week 3 on Canvas
- Next time: More on independence and conditional probability

# See you next time

*See You!*