The Geometry of Donaldson-Witten Theory

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Declaration

The work in this thesis is my own except where otherwise stated.

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Abstract

Donaldson-Witten theory is Witten's supersymmetric topological quantum field theory that realises the Donaldson invariants of smooth 4-manifolds as correlation functions of certain operators. This thesis culminates with a detailed exposition of the differential geometric interpretation of this formulation by Atiyah and Jeffrey. They show that the action of the theory can be understood as an infinite dimensional analogue of the Chern-Gauss-Bonnet formula for the Euler characteristic of a vector bundle. The bulk of this thesis introduces the background necessary to understand this. The framework of equivariant cohomology is used to construct the Mathai-Quillen representative of a universal Thom form. We then demonstrate its application to localisation of integrals, starting with a proof of the Poincaré-Hopf theorem and proceeding to arbitrary vector bundles. Finally, the construction of Donaldson invariants is sketched before expounding the Atiyah-Jeffrey interpretation.

Contents

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Notation and terminology

Notation

$\Gamma(M, E)$	Sections of a vector bundle $E \to M$, also denoted $\Gamma(E)$
$\mathcal{X}(M)$	Vector fields on a manifold M, i.e. sections of its tangent space
$\Omega(M)$	Differential forms on a manifold M
$\Omega_{cv}(E)$	Differential forms on a vector bundle $E \xrightarrow{\pi} M$ which are compactly supported along the fiber
$\Omega_{rd}(E)$	Differential forms on a vector bundle $E \xrightarrow{\pi} M$ which are rapidly decreasing along the fiber
$\Omega_G(M)$	A more compact notation for the Cartan model $(S(\mathfrak{g}^*) \otimes \Omega(M))^G$
$W(\mathfrak{g})$	The Weil algebra for a Lie algebra $\mathfrak{g}$, defined as $\bigwedge(\mathfrak{g}^*) \otimes S(\mathfrak{g}^*)$
$H_G(M)$	The equivariant cohomology ring of a G-manifold M
$\chi(E \to M)$ or $\chi(E)$	A representative of the Euler class of a vector bundle $E \to M$
V_pP, H_pP	The vertical and horizontal tangent subspaces respectively of a principal bundle at the point $p \in P$
b_i	The i^{th} Betti number of a topological space M, defined as the rank of $H_i(M, \mathbb{Z})$
$\mathcal{A}(E)$	The affine space of connections on a vector bundle E
$\mathrm{Ad}\, P$	The fiber bundle $P \times_{\mathrm{Ad}} G$
$\mathrm{ad}(P)$	The associated vector bundle $P \times_{\mathrm{Ad}} \mathfrak{g}$
$\int^B$	The Berezin integral

$\mathrm{Pf}(A)$	The Pfaffian polynomial of a skew-symmetric matrix A

Terminology

SD/ASD	self-dual/anti-self-dual; meaning contained in the 1/-1 eigenspace of the Hodge star operator on $\Omega^2(M)$ for a Riemannian 4-manifold M
characteristic homomorphism	another name for the Weil homomorphism $w : W(\mathfrak{g}) \to \Omega(P)$ associated to a connection on a principal bundle
$\mathfrak{g}$-dga	A $\mathfrak{g}$-differential graded algebra
TQFT/TFT	Topological (quantum) field theory
SUSY	Supersymmetric
YM	Yang-Mills
equivariant form	An invariant element of the Cartan model for a given G-manifold M

Mathematics	Physics
local trivialisation (or section)	gauge
choice of local trivialisation	gauge fixing
change of local trivialisation	local gauge symmetry/transformation
bundle automorphism	global symmetry/transformation
connection	gauge field/potential
curvature of connection	gauge field strength

Introduction

Brief history of gauge theory

In order to provide the context for the main topic of this thesis, we begin with a brief history of gauge theory. There is a long-standing tradition of rich areas of mathematics originating from physics, of which gauge theory and Donaldson theory is no exception.

In 1918, Weyl introduced the concepts of gauge transformation and gauge invariance, while seeking to unify electromagnetism with general relativity. After the development of quantum mechanics, an idea was to make the global phase symmetry into a local gauge symmetry by replacing the momentum operator with a covariant derivative in the Schrodinger equation, given by $\widehat{p} = \frac{\hbar}{i}(\nabla + iA)$. The result of this U(1) gauge symmetry turns out to be profound, since it introduces eletromagnetic interactions with a charged quantum mechanical particle. This was the first widely accepted gauge theory, and popularised by Pauli (1941).[28]

Gauge theory was mostly limited to electromagnetism and general relativity until the paper of Yang and Mills (1954). They extended the concept of gauge theory to non-abelian groups to understand the strong interaction. This idea later found application to the model of the electroweak interaction which incorporated the Higgs mechanism by Weinberg and Salam (1967). Yang-Mills theory also lead to the development of a strong force gauge theory, which is now known as quantum chromodynamics. The Standard Model unifies the description of electromagnetism, weak interactions and strong interactions in the language of quantized gauge theory.

Donaldson-Witten theory

The development of the study of solutions to the Yang-Mills equations has proven to be fruitful for differential topology. In 1983, Simon Donaldson built on his doctoral advisor Atiyah's work on Yang-Mills instantons to introduce his famous polynomial invariants, and prove Donaldson's theorem.[13] In particular, he constructed invariants of smooth four dimensional manifolds from moduli spaces of anti-self-dual connections on principal SU(2)-bundles. These invariants are sensitive to differentiable structures, whereas typical invariants in topology are stable under homeomorphisms. Furthermore, Freedman used his work to exhibit the existence of manifolds that are homeomorphic but not diffeomorphic to Euclidean $\mathbb{R}^4$.

However, Donaldson's work did not indicate any relation to physical ideas which Yang-Mills theory is based on. As a result, topologists with little or no knowledge of physics were able to learn and make important contributions to Donaldson theory. However, if physics had lead to such a profound impact, perhaps there was more to be gained from a physical formulation of the theory. Indeed this was the case, as we will see next.

The origin of TQFT can be traced to the work of Albert Schwarz and Edward Witten. Schwarz (1978) showed that the Ray-Singer torsion, a particular topological invarint, could be represented as the partition function of a certain quantum field theory. Unrelated to this observation, Witten gave a framework in supersymmetric (SUSY) quantum mechanics that lead to a generalisation of Morse theory (1982). It turns out that Donaldson invariants on a cylinder $Y^3 \times \mathbb{R}$ is formally a Morse theory for a certain action functional. Thus, Witten [33] (1988) applied his idea to Donaldson invariants on a $Y^3 \times \mathbb{R}$ to obtain a $N = 2$ SUSY quantum theory. But since this Morse theory is on a set of gauge fields, we get a QFT. In the same paper, he came up with a relativistically covariant twisted version of QFT by generalising the existing fields and adding new fields, in which the Donaldson polynomials appeared as correlation functions of certain observables. This became known as Donaldson-Witten theory.

This construction is a remarkable achievement because it showed a profound connection between SUSY TQFT and Donaldson theory that one might expect. Topologists were eager to understand the insights that gave rise to the radically new view of smooth four-manifold invariants. However, this area of theoretical physics was accessible to relatively few mathematicians. Fortunately, Atiyah and Jeffrey showed how to arrive at the very same action by purely geometric means. Starting with the universal Mathai-Quillen formula, they made a series of manipulations which resulted in the same action of Donaldson-Witten theory, but which could now be interpreted as an Euler form on an infinite-dimensional vector bundle.

Donaldson-Witten theory caught the attention of many, but this approach did not lead many mathematicians to gain new insight about Donaldson invariants, for a few reasons. Aside from the time and effort required to learn the relevant physics, the mathematics required to make the path integrals rigourous was not (and is still not) available. But in 1994, Witten's SUSY TQFT revealed its practical significance in a dramatic way. Seiberg and Witten discovered new dualities in SUSY theories, which illustrates features such as quark confinement. These ideas led to the unification of string theories into M-theory, and other developments in SUSY theories. They showed that the dual theory to Donaldson-Witten theory also has SUSY and the same twist obtains a new TQFT. The corresponding correlation functions, called the Seiberg-Witten invariants, revolutionised the study of 4-manifolds once again.

The new invariants are technically far easier to compute, and also leads to new simpler proofs in Donaldson theory. Moreover, the duality predicts particular formulas that relate Donaldson invariants to Seiberg-Witten invariants, although a rigorous proof has still not yet been found.

Structure of the thesis

If there is anything to take away from the history presented, it is that topologists can benefit from a deeper understanding of Donaldson-Witten theory. The goal for this thesis is to provide a thorough explanation of Atiyah and Jeffrey's interpretation of Witten's correlation functions and Lagrangian in terms of mathematically familiar concepts. Although the main breakthrough has been made in in the SUSY TQFT formulation, this provides a useful framework for mathematicians to think about the relation of Donaldson invariants with physics. The structure of the thesis is as follows.

In Chapter 1, we motivate equivariant cohomology and proceed to describe the Borel construction. This construction is topological in nature, and we formulate an equivalent algebraic model called the Weil model, analogous to the de Rham isomorphism. We also describe the simpler Cartan model, and its properties.

In Chapter 2, we introduce the Thom isomorphism for vector bundles, and then construct the Mathai-Quillen formula for the Thom form. Subsequently, we consider the generalisation of this formula to the Cartan model, called the universal Thom form.

In Chapter 3, we demonstrate the appliation of the Mathai-Quillen formula, and more generally the Thom form to localisation of integrals on manifolds. We prove a theorem stated in [6] by utilising some results in intersection theory.

In Chapter 4, we sketch the construction of Donaldson invariants. This requires first describing the Yang-Mills equations, then explaining the structure of the anti-self-dual moduli space. The theory required to do this is highly technical, so we do not go into the details of the analytic arguments, but include proofs of some results needed in the next chapter. Finally the Donaldson map is defined.

In Chapter 5, we finally give an exposition of Atiyah and Jeffrey's paper.[2] We first need to construct an analogue of the Thom form for a principal bundle. This is used in one of the series of manipulations of the Mathai-Quillen formula for the universal Thom form. The result is then applied to Donaldson-Witten theory to show that the action functional is reproduced, which allows us to interpret it as an infinite dimensional analogue of an Euler class.

Assumed knowledge

The reader is required to be familiar with the theory in introductory texts on smooth manifolds, such as *An Introduction to Manifolds* by Loring Tu [30], or Introduction to Smooth Manifolds by John Lee [21]. Topics in particular include submanifolds, Lie groups, differential forms, integration on manifolds, de Rham cohomology, Riemannian metrics and the exponential map.

Furthermore we assume the reader is familiar with the theory of connections, curvature and Chern-Weil theory on vector bundles and principal bundles. An excellent textbook for this is *Differential Geometry - Connections, Curvature, and Characteristic Classes* by Loring Tu [31].

The reader is recommended to have a basic background in algebraic topology, which includes knowledge of simplicial homology and cohomology groups, and the cup product.

Bibliographical notes

- A overview of the main ideas in the two formulations of Donaldson-Witten theory is given in Labastida and Marino [19], focusing more on TQFT.
- The article "What Do Topologists Want From Seiberg-Witten Theory?" [18] presents a more thorough review of the history of the development of Donaldson-Witten theory and Seiberg-Witten theory, with brief explanations of the ideas.

Chapter 1

Equivariant Cohomology

1.1 Preliminaries

Equivariant differential topology extends the results of differential topology to manifolds/topological spaces with a group action, called a G-space. A G-space is a topological space X with a continuous action $X \times G \to X$ such that $x \cdot e = x$ for e the identity in G and $(x \cdot g) \cdot h = x \cdot (gh)$ for $g, h \in G$.

We will often be dealing with both left and right group actions, but these are equivalent. If we have a right G action on X, this can be turned into a left action by defining $g \cdot x = x \cdot g^{-1}$, so that $h \cdot (g \cdot x) = (x \cdot g^{-1}) \cdot h^{-1} = x \cdot (g^{-1} h^{-1}) = (hg) \cdot x$. The same argument works in the other direction.

There are two main formulations of equivariant cohomology: the Borel construction which uses classifying spaces, and the Cartan model. These have been proven to be equivalent in the case of compact, connected Lie groups by Cartan (Theorem 1.28 and 1.29). First let us recall some basic definitions in algebraic topology. All maps considered are continuous.

Definition 1.1. A map $f : X \to Y$ is a weak homotopy equivalence if it induces an isomorphism of homotopy groups $f_* : \pi_q(X) \to \pi_q(Y)$ for all $q \geq 0$. A space X is weakly contractible if $\pi_q(X) = 0$ for all $q \geq 0$.

Theorem 1.2 (Whitehead's theorem [17, Thm 4.5])**.** *If a continuous map f : $X \to Y$ of CW complexes is a weak homotopy equivalence, then f is a homotopy equivalence.*

In particular, this means that a weakly contractible CW complex is contractible, using the inclusion map $x_0 \to X$.

Theorem 1.3 ([17, Prop 4.21])**.** *A weak homotopy equivalence $f : X \to Y$ induces isomorphisms $f^* : H^n(Y; R) \to H^n(X; R)$ in cohomology for all n.*

Definition 1.4. Given manifolds F, E and B, a fiber bundle with fiber F is a surjection $\pi : E \to B$ that is locally homeomorphic to a product $U \times F$; i.e. every $b \in B$ has a neighborhood $U \subset B$ such that there is a fiber-preserving homeomorphism $\phi_U : \pi^{-1}(U) \to U \times F$.

A useful tool for computing homotopy groups is the homotopy exact sequence of a fiber bundle.

Theorem 1.5 ([17, Thm 4.41])**.** *Suppose $(E, x_0) \xrightarrow{\pi} (B, b_0)$ is a fiber bundle with fiber $F = \pi^{-1}(b_0)$ and path-connected base space B. Let x_0 also be the basepoint of F, and $i : (F, x_0) \to (E, x_0)$ the inclusion map. Then there exists a long exact sequence*

$$\ldots \to \pi_n(F, x_0) \xrightarrow{i_*} \pi_n(E, x_0) \xrightarrow{\pi_*} \pi_n(B, b_0) \to \pi_{n-1}(F, x_0) \to \ldots \to \pi_0(E, x_0) \to 0$$

All maps are group homomorphisms except the last three maps which are set maps.

Proposition 1.6. *If $E \xrightarrow{\pi} B$ is a fiber bundle with weakly contractible fiber F and path-connected base B, then π is a weak homotopy equivalence. Furthermore if B and F are CW complexes, then π is a homotopy equivalence.*

***Proof*.** By Theorem 1.5, the sequence of induced maps

$$\ldots \to \pi_n(F) \to \pi_n(E) \to \pi_n(B) \to \pi_{n-1}(F) \to \ldots$$

is exact. So the induced maps $\pi_* : \pi_n(E) \to \pi_n(B)$ on homotopy groups are isomorphisms. Hence π is a weak homotopy equivalence.

Now suppose in addition that B and F are CW complexes. Since each cell of B is contractible, the fiber bundle restricted to a cell is trivial and is homeomorphic to cell$\times F$. Since F also has a CW structure, the product is also a CW complex, and gluing these pieces together shows that E is a CW complex. Hence, by Whitehead's theorem, π is a homotopy equivalence. $\square$

Let G be a topological group. If P is a right G-space and M is a left G-space, the Cartan mixing space of P and M is the quotient $P \times_G M := (P \times M)/\sim$ by the equivalence relation $(p, m) \sim (pg, g^{-1}m)$ for all $g \in G$. Equivalently, this is the orbit space $(P \times M)/G$ under the diagonal action $(p, m) \cdot g = (pg, g^{-1}m)$.

If $P \xrightarrow{\pi} B$ is a principal G-bundle, also define the projection $\tau_1 : P \times_G M \to B$ by $\tau_1([p, m]) = \pi(p)$. This is well defined because π preserves the fiber.

Proposition 1.7. *If $P \xrightarrow{\pi} B$ is a principal G-bundle and M is a left G-space, then $\tau_1 : P \times_G M \to B$ is a fiber bundle with fiber M.*

***Proof*.** Suppose $\pi^{-1}(U) \simeq U \times G$. It suffices to show $\tau_1^{-1}(U) \simeq U \times M$.

$$\begin{aligned}\tau_1^{-1}(U) &= \{[p,m] \in P \times_G M \mid \pi(p) \in U\} \\ &= \pi^{-1}(U) \times_G M \\ &\simeq (U \times G) \times_G M\end{aligned}$$

To show this is homeomorphic to $U \times M$, we define $\varphi : (U \times G) \times_G M \to U \times M$ by $[(x,g),m] \mapsto (x, gm)$. It has inverse $(x,m) \mapsto [(x,1),m]$. $\square$

1.2 Borel construction

The motivation of equivariant cohomology is to study the cohomology of the quotient space of a G-space M by the group action. But if the G-action is not free, the space M/G doesn't capture information from non-trivial stabilisers. For instance, if S^1 acts on S^2 by rotation about the vertical axis, the quotient is a segment which has trivial cohomology. Furthermore, if the action is not free and we have a G-equivariant homotopy equivalence $f : M \to N$, then the induced map $M/G \to N/G$ is not necessarily a homotopy equivalence. As an example, consider $M = \mathbb{R}$ with $\mathbb{Z}$-action given by translation, and a point $N = *$ with trivial $\mathbb{Z}$-action. Then $\mathbb{R}/\mathbb{Z} \simeq S^1$ but $N/\mathbb{Z}$ is a point.

To address these limitations, the idea behind the Borel constuction is to force the action to be free by replacing M with $E \times M$ where E is a G-space with a free action, and then studying the quotient $(E \times M)/G$. But we need an appropriate choice of E such that the cohomology does not depend on it.

Definition 1.8. Given a topological group G, let $EG \to BG$ be a principal G-bundle with weakly contractible total space EG and any base space BG. Define the homotopy quotient of a G-space M by $M_G := EG \times_G M$.

The equivariant cohomology of M is defined to be the singular cohomology of the homotopy quotient: $H^*_G(M; R) := H^*(M_G; R)$.

Of course, for this definition to make sense we need to show that it is independent of the choice of weakly contractible EG and find out when EG exists, which we do in this section. But first, let us show that when the action is free, equivariant cohomology is the cohomology of the orbit space.

Proposition 1.9. *If M has a free G-action, then $H^*(M_G) \simeq H^*(M/G)$.*

Proof. Let $EG \to BG$ be a principal G-bundle with weakly contractible total space. Since the G-action is free, $M \to M/G$ is a principal bundle. By Proposition 1.7, $(EG \times M)/G \to M/G$ is a fiber bundle with fibre homeomorphic to EG. Then by Proposition 1.6, the result follows. □

Lemma 1.10. *If E is a weakly contractible G-space, and $P \to P/G$ is a principal G-bundle, there is a weak homotopy equivalence $(E \times P)/G \to P/G$.*

Proof. Consider $E \times_G P = (E \times P)/G$ as the orbit space under the diagonal action $(e, p)g = (g^{-1}e, pg)$. By Proposition 1.7, $(E \times P)/G \to P/G$ is a fiber bundle with fiber E. Then the result follows by Proposition 1.6. □

The next proposition shows that the definition of equivariant cohomology is independent of the choice of E.

Proposition 1.11. *Suppose M is a left G-space. If $E \to B$ and $E' \to B'$ are two principal G-bundles with weakly contractible total spaces, then $H^*(E \times_G M) \simeq H^*(E' \times_G M)$.*

Proof. Since the G-action on $E' \times M$ is also free, $E' \times M \to (E' \times M)/G$ is a principal bundle. Then by the lemma above, there is a weak homotopy equivalence $(E \times E' \times M)/G \to (E' \times M)/G$. By Theorem 1.3, it induces isomorphisms in cohomology $H^n((E \times E' \times M)/G) \simeq H^n((E' \times M)/G)$ for all $n \geq 0$.

By symmetry of E and E', we also conclude $H^n((E' \times E \times M)/G) \simeq H^n((E \times M)/G)$ for all $n \geq 0$. The canonical homeomorphism $(E' \times E \times M)/G \to (E \times E' \times M)/G$ induces an isomorphism on cohomology, and thus $H^n((E \times M)/G) \simeq H^n((E' \times M)/G)$. □

The remaining question in our definition of equivariant cohomology of a G-space M is the existence of a weakly contractible principal G-bundle EG. It turns out that for CW complexes, a weakly contractible G-bundle is equivalent to a universal G-bundle, as defined below.

Definition 1.12. A principal G-bundle $\pi : EG \to BG$ is a universal G-bundle if the following two conditions hold:

(i) for any principal G-bundle P over a CW complex X, there exists a continuous map $h : X \to BG$ such that $P \simeq h^*EG$

(ii) If $h_0, h_1 : X \to BG$ and $h_0^*EG \simeq h_1^*EG$ over a CW complex X, then h_0 and h_1 are homotopic

The base space BG of a universal G-bundle is called a classifying space for G.

Theorem 1.13 (Steenrod [29, p. 102]). *Let $E \to B$ be a principal G-bundle. If E is weakly contractible, then $E \to B$ is a universal bundle. Conversely, if $E \to B$ is a universal bundle and B is a CW complex, then E is weakly contractible.*

Theorem 1.14 (Milnor's construction [22]). *A universal G-bundle exists for any topological group G.*

Theorem 1.15. *If a CW classifying space exists for a topological group G, it is unique up to homotopy equivalence.*

Proof. Suppose $E \to B$ and $E' \to B'$ are universal bundles, where B and B' are CW complexes. Since E' is universal there is a map $f : B \to B'$ such that $E' \simeq h^*E$. Similarly there is a map $h : B' \to B$ such that $E \simeq h^*E'$. Therefore, $E \simeq f^*h^*E = (h \circ f)^*E$.

But this means $(h \circ f)^*E = \mathrm{id}_B^* E$, so by condition (ii) of a universal bundle, $h \circ f \simeq \mathrm{id}_B$. Similarly, $f \circ h \simeq \mathrm{id}_{B'}$. Therefore B and B' are homotopy equivalent. □

1.3 Weil model

The Weil model of equivariant cohomology, introduced by Cartan based on unpublished work of André Weil, is an algebraic alternative to the Borel construction. It is akin to the de Rham chain complex of differential forms as a model for singular cohomology. To motivate the definition, we first recall some basic facts about differential forms on principal bundles and G-manifolds.

1.3.1 Results on principal bundles

Definition 1.16. An antiderivation of a graded algebra $A = \bigoplus_{k=0}^{\infty} A^k$ is a linear map $D : A \to A$ such that

$$D(ab) = (Da)b + (-1)^{\deg a} a(Db)$$

If there is an integer m such that $DA^k \subset A^{k+m}$ for all k, then we call it an antidervation of degree m.

Definition 1.17. Let $\omega \in \Omega(P)$ be a differential form on a principal G-bundle $P \xrightarrow{\pi} M$. Then ω is a basic form if $\omega = \pi^*\eta$ where $\eta \in \Omega(M)$. It is G-invariant if $R_g^*\omega = \omega$ for all $g \in G$, where $R_g : P \to P$ is right multiplication by g. It is horizontal if at each $p \in P$, ω vanishes whenever one of its arguments is vertical: $\iota_Y \omega_p = 0$ for all $Y \in V_p$.

The definition of G-invariant also extends to forms on a G-manifold. We can also identify each $X \in \mathfrak{g}$ with a vertical vector field via $X_p = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} p \exp(tX)$.

Theorem 1.18 ([32, Theorem 12.2]). *For a connected Lie group G, a form $\omega \in \Omega(M)$ on a G-manifold is G-invariant if and only if $\mathcal{L}_A\omega = 0$ for all $A \in \mathfrak{g}$ in the Lie algebra.*

Definition 1.19. If V is a vector space, the interior multiplication (or contraction) operator $\iota(v) : \Lambda V^* \to \Lambda V^*$ for $v \in V$ is the unique antidervation of -1 such that $\iota(v)\alpha = \alpha(v)$ if $\alpha \in V^*$.

In particular, we have interior multiplication with a vector field $\iota(X) : \Omega^*(M) \to \Omega^{*-1}(M)$. The definition similarly extends to vector valued forms.

Theorem 1.20 ([32, Theorem 12.5]). *A form $\omega \in \Omega^k(P)$ on a principal G-bundle is basic if and only if it is G-invariant and horizontal.*

Theorem 1.21 (Cartan's homotopy formula [30, Theorem 20.10]). *Let $X \in \mathcal{X}(M)$ be a vector field on a manifold M, and $\alpha \in \Omega(M)$. Then*

$$\mathcal{L}(X) = d\iota(X) + \iota(X)d$$

Theorem 1.22 ([31, Theorem 30.4]). *If a principal bundle $P \to M$ has connection $\omega \in \Omega^1(P, \mathfrak{g})$, its curvature form F satisfies the following properties:*

(i) (Horizontality) For $p \in P$ and $X_p, Y_p \in T_pP$, $F_p(X_p, Y_p) = (d\omega)_p(hX_p, hY_p)$ where hX_p is the horizontal projection. Consequently, F is horizontal, i.e. $\iota_X F = 0$ for any vertical vector $X \in T_pP$

(ii) (G-equivariance) $R_g^* F = \mathrm{Ad}_{g^{-1}} \circ F$

1.3.2 Differential graded algebras

We would like to introduce the concept of equivariant differential forms on a G-manifold M, which should play the role of differential forms on M_G. Our starting point is to abstract some of the properties above to a more general algebra.

Definition 1.23. Let $\mathfrak{g}$ be a Lie algebra. A $\mathfrak{g}$-differential graded algebra is a graded-commutative algebra $\Omega = \bigoplus_{k\geq 0} \Omega^k$ with

- an antiderivation $d : \Omega \to \Omega$ of degree 1 such that $d \circ d = 0$
- two actions of $\mathfrak{g}$: $\iota : \mathfrak{g} \times \Omega \to \Omega$ and $\mathcal{L} : \mathfrak{g} \times \Omega \to \Omega$, where for $X \in \mathfrak{g}$, ι_X and $\mathcal{L}_X$ are $\mathbb{R}$-linear in X, ι_X acts on Ω as an antiderivation of degree -1, $\iota_X^2 = 0$, and $\mathcal{L}_X$ acts as a derivation of degree 0.

Furthermore, the operators satisfy Cartan's homotopy formula: $\mathcal{L}_X = d\iota_X + \iota_X d$.

Note that graded-commutative algebra means that if $a \in \Omega^k, b \in \Omega^l$ then $ba = (-1)^{kl}ab$. From the previous subsection, it is clear that differential forms $\Omega(N)$ on a G-manifold N is an example of a $\mathfrak{g}$-dga.

Definition 1.24. An element $\alpha \in \Omega$ of a $\mathfrak{g}$-dga is horizontal if $\iota_X\alpha = 0$ for all $X \in \mathfrak{g}$. It is invariant if $\mathcal{L}_X\alpha = 0$ for all $X \in \mathfrak{g}$. It is basic if it is both horizontal and invariant.

Definition 1.25. A morphism $\phi : \Omega' \to \Omega$ of $\mathfrak{g}$-differential graded algebras is a graded algebra homomorphism that commutes with d, ι_X and $\mathcal{L}_X$ for all $X \in \mathfrak{g}$.

It follows directly that morphism $\phi : \Omega' \to \Omega$ of $\mathfrak{g}$-differential graded algebras maps basic elements to basic elements.

If (A, d_A) and (B, d_B) are differential graded algebras, then their tensor product has multiplication given by

$$(a \otimes B)(a' \otimes b') = (-1)^{(\deg b)(\deg a')} aa' \otimes bb' \tag{1.1}$$

which respects the grading $(A \otimes B)^k = \bigoplus_{i+j=k} A^i \otimes B^j$. The differential, interior product and Lie derivative on $A \otimes B$ are defined by

$$\begin{aligned} d(a \otimes b) &= (d_A a) \otimes b + (-1)^{\deg a} a \otimes d_B b \\ \iota_X(a \otimes b) &= (\iota_X a) \otimes b + (-1)^{\deg a} a \otimes \iota_X b \\ \mathcal{L}_X(a \otimes b) &= (\mathcal{L}_X a) \otimes b + a \otimes \mathcal{L}_X b \end{aligned} \tag{1.2}$$

Proposition 1.26 (Operations on DGAs)**.**

(i) If (A, d_A) and (B, d_B) are $\mathfrak{g}$-differential graded algebras, then $(A \otimes B, d)$ is a $\mathfrak{g}$-differential graded algebra

(ii) If Ω is a $\mathfrak{g}$-differential graded algebra, the vector subspace of basic elements Ω_{bas} is a $\mathfrak{g}$-differential graded algebra.

***Proof** (sketch).* (i) It is required to check that d and ι are antiderivations, $\mathcal{L}_X$ is a derivation, $d \circ d = 0$, $\iota_X^2 = 0$, and Cartan's homotopy formula. These are all straightforward calculations.

(ii) It is required to show that if α and β are basic, then so are $\alpha + \beta$, $\alpha\beta$, $d\alpha$, $\iota_X\alpha$ and $\mathcal{L}_X\alpha$.

For $\alpha + \beta$, it follows from linearity of ι and $\mathcal{L}_X$. For $\alpha\beta$, it follows from the (anti)derivation property of ι_X and $\mathcal{L}_X$. For $d\alpha$, we have

$$\begin{aligned} \iota_X(d\alpha) &= (\mathcal{L}_X - d\iota_X)\alpha = 0 && \text{(by Cartan's homotopy formula)} \\ \mathcal{L}_X(d\alpha) &= d\mathcal{L}_X\alpha = 0 && \text{(again by Cartan's homotopy formula)} \end{aligned}$$

The basicness of $\iota_X \alpha$ and $\mathcal{L}_X \alpha$ also follow from application of Cartan's homotopy formula. $\square$

By contrast, the subalgebra of horizontal elements is not d-invariant.

1.3.3 Weil model

Let $P \xrightarrow{\pi} M$ be a principal G-bundle. With a choice of connection $\omega \in \Omega^1(P, \mathfrak{g})$ and associated curvature $\Omega \in \Omega^2(P, \mathfrak{g})$, the Weil homomorphism defines a homomorphism $\bigwedge(\mathfrak{g}^*) \otimes S(\mathfrak{g}^*) \to \Omega(P)$. To construct this, define the linear map

$$f_1 : \mathfrak{g}^* \to \Omega^1(P) \text{ given by } f_1(\alpha) = \alpha \circ \omega$$

which we can extend to a unique algebra homomorphism $\bigwedge(\mathfrak{g}^*) \to \Omega(P)$ given by $f_1(\beta_1 \wedge \ldots \wedge \beta_k) = f_1(\beta_1) \wedge \ldots \wedge f_1(\beta_k)$ for $\beta_i \in \mathfrak{g}^*$. This is well defined because 1-forms anticommute, thus we can define an alternating k-linear map and use the universal property of $\bigwedge^k$. Similarly, define the linear map

$$f_2 : \mathfrak{g}^* \to \Omega^2(P) \text{ given by } f_2(\alpha) = \alpha \circ \Omega$$

Again, we can extend f_2 to a unique algebra homomorphism $S(\mathfrak{g}^*) \to \Omega(P)$ given by $f_2(\beta_1 \cdots \beta^k) = f_2(\beta_1) \wedge \ldots \wedge f_2(\beta_k)$ for $\beta_i \in \mathfrak{g}^*$, because 2-forms commute.

Combining f_1 and f_2 gives a bilinear map $f_1 \times f_2 : \bigwedge(\mathfrak{g}^*) \times S(\mathfrak{g}^*) \to \Omega(P)$ given by $(\alpha, \beta) \mapsto f_1(\alpha) \wedge f_2(\beta)$, and hence a linear map on the tensor product

$$w : \bigwedge(\mathfrak{g}^*) \otimes S(\mathfrak{g}^*) \to \Omega(P)$$

such that $w(\alpha \otimes \beta) = f_1(\alpha) \wedge f_2(\beta)$. We call w the <u>Weil homomorphism</u> (also called characteristic homomorphism), and $W(\mathfrak{g}) = \bigwedge(\mathfrak{g}^*) \otimes S(\mathfrak{g}^*)$ is called the <u>Weil algebra</u>. Moreover, w is a graded-algebra homomorphism if we assign degree 1 to elements of $\mathfrak{g}^*$ in $\bigwedge(\mathfrak{g}^*)$ and degree 2 to elements of $\mathfrak{g}^*$ in $S(\mathfrak{g}^*)$.

We want to make the Weil algebra into a $\mathfrak{g}$-differential graded algebra, so we will define a differential operator. Let $X_1, \ldots, X_n$ be a basis for $\mathfrak{g}$, with dual basis $\alpha^1, \ldots, \alpha^n$. So the Weil algebra is generated by

$$\theta_i = \alpha^i \otimes 1, \quad u_i = 1 \otimes \alpha^i \quad \in \bigwedge(\mathfrak{g}^*) \otimes S(\mathfrak{g}^*) \tag{1.3}$$

We can write the connection $\omega \in \Omega^1(P, \mathfrak{g})$ as a linear combination $\omega = \sum \omega^k X_k$ and the curvature as $\Omega = \sum \Omega^k X_k$, where ω^k and Ω^k are $\mathbb{R}$-valued forms. Then the Weil homomorphism acts by

$$w(\theta_k) = \theta_k \circ \omega = \omega^k \qquad w(u_k) = u_k \circ \Omega = \Omega^k \tag{1.4}$$

We want to define a differential operator δ such that $w \circ \delta = d \circ w$, so that w is a $\mathfrak{g}$-dga homomorphism. By the structural equation $\Omega = d\omega + \frac{1}{2}[\omega, \omega]$ and Bianchi identity $d\Omega = [\Omega, \omega]$, we have

$$d\omega^k = \Omega^k - \frac{1}{2}\sum c^k_{ij}\omega^i \wedge \omega^j, \qquad d\Omega^k = \sum c^k_{ij}\Omega^i \wedge \omega^j$$

where $[X_i, X_j] = \sum c^k_{ij} X_k$ are the structure constants of the Lie algebra. Hence, we define δ by

$$\delta\theta_k = u_k - \frac{1}{2}\sum c^k_{ij}\theta_i\theta_j, \qquad \delta u_k = \sum c^k_{ij} u_i \theta_j \tag{1.5}$$

and extend δ to $W(\mathfrak{g})$ as an antiderivation as in Definition 1.16. Similarly, we will define interior multiplication. On $\Omega(P)$, we have for $A \in \mathfrak{g}$

$$\iota_A \omega^k = \omega^k(\underline{A}) = \alpha^k(A), \qquad \iota_A \Omega^k = 0$$

the first property follows from $\omega(\underline{A}) = A$ for a connection form, and the second property follows from Ω being a horizontal form (see Theorem 1.22). Hence in order for the Weil homomorphism to preserve interior multiplication, ι_A should be defined by

$$\iota_A \theta_k = \alpha^k(A), \qquad \iota_A u_k = 0 \tag{1.6}$$

and extend ι_A to $W(\mathfrak{g})$ as an antiderivation. Finally, the Lie derivative is defined by Cartan's homotopy formula. Applying equations (1.5) and (1.6), we obtain

$$\mathcal{L}_{X_j}\theta_k = \sum c^k_{ij}\theta_i, \qquad \mathcal{L}_{X_j} u_k = \sum c^k_{ij} u_i \tag{1.7}$$

Since w commutes with δ and ι_A, it will commute with $\mathcal{L}_A$. The final property to check for $W(\mathfrak{g})$ to be a $\mathfrak{g}$-dga is that $\delta^2 = 0$.

Theorem 1.27 ([32, Theorem 19.1])**.** *On the Weil algebra $W(\mathfrak{g})$, $\delta^2 = 0$*

Proof *(sketch).* Since δ is an antiderivation, δ^2 is a derivation. So it suffices to check $\delta^2 = 0$ on a set of algebra generators of $W(\mathfrak{g})$. One such set is $\{\theta_1, \ldots, \theta_n, \delta\theta_1, \ldots, \delta\theta_n\}$, noting that u_k is generated by this set via equation (1.5). Thus it suffices to show $\delta^2\theta_k = 0$, which can be computed directly using equation (1.5). $\square$

Finally, we are ready to state the relation between the Weil model and the Borel construction of equivariant cohomology.

Theorem 1.28 (Equivariant de Rham theorem [32, Theorem 19.4]). *For a compact connected Lie group G with Lie algebra $\mathfrak{g}$, and G-manifold M, there is a graded-algebra isomorphism*

$$H_G^*(M) \simeq H^*((W(\mathfrak{g}) \otimes \Omega(M))_{bas}, \delta)$$

The complex $(W(\mathfrak{g})\otimes\Omega(M))_{bas}$ with the Weil differential is called the Weil model. It is the basic subcomplex of the tensor product of two $\mathfrak{g}$-dgas which is well defined by Proposition 1.26.

1.4 Cartan model

1.4.1 Weil-Cartan isomorphism

The Cartan model arises from algebraically solving for the horizontal forms of the Weil model $W(\mathfrak{g}) \otimes \Omega(M)$. This leads to a $\mathfrak{g}$-dga that is equivalent to the Weil model and simpler to work with. We consider the same generating elements for $W(\mathfrak{g})$ as in equation (1.3). Then for a $\mathfrak{g}$-dga $\mathcal{A}$, we can write

$$W(\mathfrak{g}) \otimes \mathcal{A} = \bigwedge(\mathfrak{g}^*) \otimes S(\mathfrak{g}^*) \otimes \mathcal{A} = \bigwedge(\theta_1, \dots, \theta_n) \otimes \mathcal{A}[u_1, \dots, u_n]$$

Thus, an element $\alpha \in W(\mathfrak{g}) \otimes \mathcal{A}$ can be written as

$$\alpha = a + \sum \theta_I a_I, \qquad a_I \in \mathcal{A}[u_1, \dots, u_n]$$

Theorem 1.29 (Weil-Cartan isomorphism). *Let G be a connected Lie group, and $\mathcal{A}$ be a $\mathfrak{g}$-dga. There is a graded-algebra isomorphism*

$$F : (W(\mathfrak{g}) \otimes \mathcal{A})_{hor} \to S(\mathfrak{g}^*) \otimes \mathcal{A} \qquad a + \sum \theta_I a_I \mapsto a$$

which induces a graded-algebra isomorphism on the basic subalgebras $F : (W(\mathfrak{g})\otimes \mathcal{A})_{bas} \to (S(\mathfrak{g}^*) \otimes \mathcal{A})^G$.

Typically, we are interested in $\mathcal{A} = \Omega(M)$ for a G-manifold M. The complex $\Omega_G(M) := (S(g^*) \otimes \Omega(M))^G$, called the Cartan model, is the subalgebra of invariants elements, where the Lie derivative acts the same way as in the Weil model. Elements of the Cartan model are called equivariant forms. Note that the Cartan model is not a $\mathfrak{g}$-dga, but a description of the basic Weil subcomplex.

Proof *(sketch).* Firstly, we verify that F commutes with $\mathcal{L}_X$ for all $X \in \mathfrak{g}$, so that the restriction to basic subalgebras is well defined. It can be shown that $\mathcal{L}_i\theta_k = -\sum_j c_{ij}^k\theta_j$, so $(F \circ \mathcal{L}_i)(a + \sum \theta_I a_I) = \mathcal{L}_i a = \mathcal{L}_i \circ F(a + \sum \theta_I a_I)$.

We will prove the theorem by constructing a projection H onto the horizontal subalgebra of $W(\mathfrak{g})\otimes\mathcal{A}$. Denote $E = S(\mathfrak{g}^*)\otimes\mathcal{A}$, more generally this will work for a graded algebra that admits a interior multiplication operator ι_X. Let $\theta_1, \dots, \theta_n$ be a basis for $\mathfrak{g}^*$ in $\bigwedge(\mathfrak{g}^*)$. Define

$$H_i := 1 - \theta_i\iota_i : \bigwedge(\mathfrak{g}^*) \otimes E \to \bigwedge(\mathfrak{g}^*) \otimes E, \qquad H := \prod H_i$$

where ι_i acts by the diagonal action on $\bigwedge(\mathfrak{g}^*) \otimes E$ (see equation (1.2)). Let

$$J := \bigcap_i \ker \iota_i = (\bigwedge(\mathfrak{g}^*) \otimes E)_{hor}$$

We can show $H|_E : E \to J$ is a graded algebra isomorphism, and in fact the horizontal projection of the Weil model, by the following steps:

(1) H_i is a ring map, and hence H is a ring map
(2) $H_iH_j = H_jH_i$
(3) $\iota_iH_i = 0$, hence $\mathrm{Im}H_i \subset \ker \iota_i$ and $\mathrm{Im}H \subset J$
(4) $H|_J = 1_J$, thus $\mathrm{Im}H = J$
(5) $H_i(\theta_i) = 0$, thus H is supported on E, and $H|_E$ is surjective
(6) $H|_E$ is injective

The calculations above are all straightforward. The proof is concluded by noting that F is a left inverse for H, and hence the unique inverse for H. □

Remark 1.30. A potential point of confusion is that $H : W(\mathfrak{g}) \to W(\mathfrak{g})$ is the projection on to the horizontal component, so in this case $S(\mathfrak{g}^*) = W(\mathfrak{g})_{hor}$. However as an operator on $W(\mathfrak{g}) \otimes \mathcal{A}$, this is no longer true because elements in $S(\mathfrak{g}^*)\otimes\mathcal{A}$ are no longer horizontal. This means that $E \neq J$ in the proof, and neither is contained within the other. Consequently, the Cartan model $(S(\mathfrak{g}^*)\otimes\mathcal{A})^G$ consists of invariant but not necessarily horizontal elements, but is isomorphic to the algebra of basic forms.

1.4.2 Cartan differential

The Weil-Cartan isomorphism carries the Weil differential δ_W to a differential δ_C on the Cartan model, i.e. defined by the commutative diagram

$$\begin{tikzcd}
(W(\mathfrak{g}) \otimes \Omega(M))_{\mathrm{bas}} \arrow[d, "\delta_W"'] & (S(\mathfrak{g}^*) \otimes \Omega(M))^G \arrow[l, "H"', "\simeq"] \arrow[d, dashed, "\delta_C"] \\
(W(\mathfrak{g}) \otimes \Omega(M))_{\mathrm{bas}} \arrow[r, "F", "\simeq"'] & (S(\mathfrak{g}^*) \otimes \Omega(M))^G
\end{tikzcd}$$

This ensures that the Cartan model is isomorphic to the basic subcomplex as a differential graded algebra, so becomes another model of equivariant cohomology. As expected, we have $\delta_C F = F\delta_W HF = F\delta_W$. To find an explicit description of δ_C, let $a \in (S(\mathfrak{g}^*) \otimes \Omega(M))^G$. We have

$$H(a) = \left(\prod(1-\theta_i\iota_i)\right) a = a - \sum_i \theta_i\iota_i a + \sum_{i,j}(\theta_i\iota_i)(\theta_j\iota_j)a - (\cdots)$$

where $(\cdots)$ are terms which contain some θ_i as a factor. Then

$$\delta_W H(a) = \delta_W a - \sum_i \left(u_i - \frac{1}{2}\sum c^i_{kl}\theta_j\theta_l\right)\iota_i a + (\cdots)$$

where we have used the antiderivation property of δ_W and the definition of $\delta_W\theta_i$. If we write $a = \sum u^I\eta_I$ where $\eta_I \in \Omega(M)$, then

$$\delta_W a = \sum(\delta_W u^I)\eta_I + \sum u^I d\eta_I$$

Note that $\deg u^I$ is always even, and all terms in $\delta_W u^I$ contain a factor of some θ_i. Recall that F simply drops all terms containing θ_i, so it leaves

$$\delta_C a := F\delta H(a) = \sum u^I d\eta_I - \sum_i u_i\iota_i a$$

If we define d on the Cartan model by $d\left(\sum u^I\eta_I\right) = \sum u^I d\eta_I$, then

$$\delta_C a = \left(d - \sum u^i\iota_i\right) a \tag{1.8}$$

Note that $\delta_W u_i = \sum c^i_{kl}u_k\theta_l$, while $\delta_C u_i = 0$ in the Cartan model.

It may be unsatisfying to give a definition which is based on the choice of a basis for $\mathfrak{g}$, so we now give an intrinsic, basis-free description. An element $\alpha = \sum u_I\alpha_I \in S(\mathfrak{g}^*) \otimes \Omega(M)$, where $\alpha_I \in \Omega(N)$, can be interpreted as a function $\mathfrak{g} \to \Omega(M)$ given by

$$\alpha(X) = \sum u_{i_1}(X)\cdots u_{i_n}(X)\alpha_I$$

where I is a multi-index of arbitrary length allowing for repeated indices. Thus, $(S(\mathfrak{g}^*) \otimes \Omega(M))^G$ are $\Omega(M)$-valued polymomials on $\mathfrak{g}$.

Theorem 1.31. *The Cartan differential $\delta_C : (S(\mathfrak{g}^*) \otimes \Omega(M))^G \to (S(\mathfrak{g}^*) \otimes \Omega(M))^G$ is given by*

$$(\delta_C\alpha)(X) = (d - \iota_X)(\alpha(X)), \quad \textit{for } X \in \mathfrak{g}.$$

Proof. Since both sides are linear, it suffices to assume $\alpha = u^I \beta$ where $\beta \in \Omega(M)$. Then

$$(\delta_C \alpha)(X) = u_I(X) d\beta - \sum_i u_i(X) \iota_i u^I(X) \beta$$

In the second term of this expression, the operator can be written

$$\sum_i u_i(X) \iota_{X_i} = \iota_{\sum u_i(X) X_i} = \iota_X$$

using linearity of ι, and that u_i is the dual basis, completing the proof. □

1.4.3 Cartan map

The notation for the Cartan model alludes to a natural action of G on the complex $W(\mathfrak{g})$. The action is induced by the coadjoint representation $\mathrm{Ad}^* : G \to \mathrm{Aut}(\mathfrak{g}^*)$ of G on $\mathfrak{g}^*$. It is the unique action such that $\langle \mathrm{Ad}_g^* \alpha, \mathrm{Ad}_g X \rangle = \langle \alpha, X \rangle$ for $\alpha \in \mathfrak{g}^*, X \in \mathfrak{g}, g \in G$ where $\langle \alpha, X \rangle = \alpha(X)$ denotes the value of the linear functional, given by

$$\langle \mathrm{Ad}_g^* \alpha, Y \rangle := \langle \alpha, \mathrm{Ad}_{g^{-1}} Y \rangle \tag{1.9}$$

The induced representation $\mathrm{ad}^* : \mathfrak{g} \to \mathrm{End}(\mathfrak{g}^*)$ on the Lie algebra is found by differentiating Ad^* as usual, which gives

$$\langle \mathrm{ad}_X^* \alpha, Y \rangle = \langle \alpha, -\mathrm{ad}_X Y \rangle = -\langle \alpha, [X, Y] \rangle \tag{1.10}$$

This is precisely the same as the Lie derivative $\mathcal{L}_X$ action on the Weil model. To see this, the action in the basis $X_1, \ldots, X_n$ of $\mathfrak{g}$ with dual basis $\alpha_1, \ldots, \alpha_k$ of $\mathfrak{g}^*$ is

$$\begin{aligned}
\langle \mathrm{ad}_{X_i}^* \alpha_k, Y \rangle &= \Big\langle \mathrm{ad}_{X_i}^* \alpha_k, \sum_j b_j X_j \Big\rangle \\
&= -\Big\langle \alpha_k, \sum_j b_j [X_i, X_j] \Big\rangle \\
&= -\Big\langle \alpha_k, \sum_{j,l} b_j c_{ij}^l X_l \Big\rangle \\
&= -\sum_j b_j c_{ij}^k = -\sum_j c_{ij}^k \alpha^j(Y)
\end{aligned}$$

Comparing with equation (1.7), this proves the claim, since the action of $\mathcal{L}_{X_i}$ is the same on both the even and odd generators. In this view, the Lie derivative on $W(\mathfrak{g})$ is the coadjoint representation of $\mathfrak{g}$ on $\mathfrak{g}^*$. This means that $\alpha \in W(\mathfrak{g})$ is invariant, i.e. satisfies $\mathcal{L}_X \alpha = 0$ for all $X \in \mathfrak{g}$ if and only if $\mathrm{Ad}_g^* \alpha = \alpha$ for all $g \in G$, assuming G is connected. This can be proved by writing any $g \in G$ in the form $A = e^{Y_1} e^{Y_2} \cdots e^{Y_m}$ for some $Y_i \in \mathfrak{g}$.[16, Corollary 3.47]

On the other hand, we know from Theorem 1.18 that on a G-manifold M, $\mathcal{L}_X\omega = 0$ if and only if $R_g^*\omega = \omega$. Therefore, elements in $W(\mathfrak{g}) \otimes \Omega(M)$ are invariant if and only if they are invariant under the product group action $(\alpha \otimes \omega) \cdot g = \mathrm{Ad}_g^* \alpha \otimes R_g^*\omega$.

Viewed as polymomials, $S(\mathfrak{g}^*)^G$ is the subalgebra of $\mathrm{Ad}(G)$-invariant polynomials. From Theorem 1.29, we have $W(\mathfrak{g})_{bas} = S(\mathfrak{g}^*)^G$. Thus, a Weil homomorphism $w : W(\mathfrak{g}) \to \Omega(P)$ induced by a connection on a principal bundle $P \to M$ descends to the Chern-Weil homomorphism on the basic subcomplexes $w : S(\mathfrak{g}^*)^G \to \Omega(P)_{bas}$, given by

$$\alpha = \sum a_I u_I \mapsto \sum a_I \Omega^{i_1} \wedge \cdots \wedge \Omega^{i_k}$$

Denote the image of the Weil map by $\alpha(\Omega)$, which indicates "evaluating the polynomial at the curvature". We wish to generalise the Chern-Weil map to $S(\mathfrak{g}^*) \otimes \Omega(P) \to \Omega(P)$, by defining $\alpha \otimes \eta \mapsto \alpha(\Omega) \wedge \eta$. However, the problem is that elements in the Cartan model are invariant but not horizontal as mentioned in Remark 1.30, and hence $\alpha(\Omega) \wedge \eta$ will be invariant but not necessarily horizontal. We can remedy this by projecting onto the horizontal subspace of $\mathrm{Hor}_\omega : \Omega(P) \to \Omega(P)_{hor}$. Thus, we have motivated the Cartan map

$$\mathrm{Car}_\omega : (S(\mathfrak{g}^*) \otimes \Omega(P))^G \to \Omega(P)_{bas}, \qquad \alpha \mapsto \mathrm{Hor}_\omega(\alpha(\Omega)) \tag{1.11}$$

As Proposition 1.9 shows, the equivariant cohomology of a G-manifold P with a free action is isomorphic to the singular cohomology of the quotient $H_G(P) \simeq H^*(P/G)$. The Cartan map can be used to establish an algebraic counterpart of this same result.

Theorem 1.32 (Cartan's theorem)**.** *Suppose $P \xrightarrow{\pi} M$ is a principal G-bundle with a connection ω, where G is a connected Lie group. Then the associated Cartan map defines a quasi-isomorphism*

$$(S(\mathfrak{g}^*) \otimes \Omega(P))^G \underset{i}{\overset{\mathrm{Car}_\omega}{\rightleftarrows}} \Omega(P)_{bas} \simeq \Omega(M)$$

where $i : \omega \mapsto 1 \otimes \omega$ is the inclusion. Here a quasi-isomorphism means a chain map that induces isomorphisms $H_G^*(P) \simeq H^*(M)$ on all cohomology groups.

Cartan [8] constructs an explicit homotopy operator to show that the inclusion i is the homotopy inverse. The original paper can be found in appendix A of the text by Guillemin and Sternberg [15], and also summarised in chapter 5 of the same book. A more detailed proof is given in Appendix A of the book by Tu

[32]. The Cartan map will turn out be very useful in the next chapter for the Mathai-Quillen construction.

The broader significance of Cartan's theorem is that it can be used to prove the equivariant de Rham theorem (Theorem 1.28), by showing that $H^*(M_G) \simeq H^*(\Omega_G(M))$. However, the total space EG of the universal bundle of a Lie group G is not a finite-dimensional manifold, which is necessary in order to apply Cartan's theorem. If G is compact and connected, the idea is to approximate EG by finite-dimensional manifolds, which reproduce the cohomology of EG. These details are all also given in Appendix A of [32].

Bibliographical notes

- An excellent book on equivariant cohomology which this chapter is largely based on is *Introductory Lectures on Equivariant Cohomology* by Tu [32].
- A more comprehensive treatment of equivariant cohomology is given in Guillemin and Sternberg [15], which covers topics not included in Tu's book such as equivariant symplectic forms and the Mathai-Quillen construction. However, Tu's writing is more pedagogical.
- The main results of this chapter stem from Cartan's seminal paper [8] where he introduces the Weil model and proves that it computes the cohomology of the base of a principal bundle.

Chapter 2

Mathai-Quillen formula

The Mathai-Quillen formula is an explicit differential form representative of the Thom class of a vector bundle. The significance of the Thom class is that its pullback by a section to the base manifold gives a representative of the Euler class, thus it can be used to calculate the Euler number of the vector bundle. Recall that there are two quite different approaches for calculating the Euler number $\chi(X) = \chi(TX)$ of an oriented even dimensional manifold X. This first is topological, and counts the signed isolated zeros of a vector field on X, via the Poincaré-Hopf theorem. The second is differential geometric and represents $\chi(X)$ as the integral over X of a density constructed from the curvature of a connection on X, via the Gauss-Bonnet theorem. The Mathai-Quillen representative of the class depends on both a section s and connection ∇ on the vector bundle, and can be regarded as a formula which interpolates between the two approaches.

Our interest in the formula is further motivated by its application in providing a geometric interpretation of the action principle of Witten's TQFT, where he characterised Donaldson invariants as correlation functions of observables, as we will see later in Chapter 5.

2.1 Integration along the fiber

Let $\pi : E \to B$ be an oriented fiber bundle over a manifold B with oriented fiber F. Suppose $\dim F = m$ and $\dim B = n$.

Definition 2.1. Integration along the fiber is a map $\pi_* : \Omega^*(E) \to \Omega^{*-m}(B)$, defined as follows. Let $\alpha \in \Omega^k(E)$ and $v_1, \ldots, v_{k-m} \in T_b B$, then π_* is given by

$$(\pi_*\alpha)_b(v_1, \ldots, v_{k-m}) = \int_{\pi^{-1}(b)} \alpha(\widetilde{v}_1, \ldots, \widetilde{v}_{k-m}, -)$$

where $\widetilde{v}_i$ is any lift of v_i to TE. The integral is computed by pulling back the form via a trivialisation $\phi : F \xrightarrow{\simeq} \pi^{-1}(b)$.

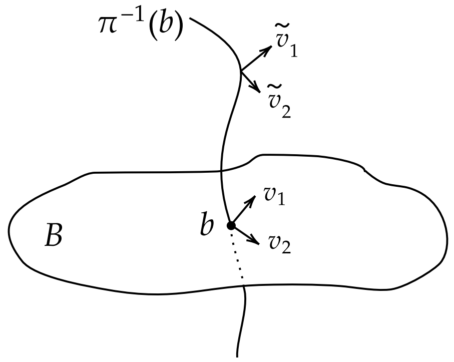


Figure 2.1: Lift of tangent vectors along $\pi^{-1}(b)$

To show that this is well defined, we need to check

- the integral is independent of choice of trivialisation
- definition is independent of the choices of lifts
- integrand is a smooth differential form on $\pi^{-1}(b)$
- $\pi_*\alpha$ is a smooth differential form on B

Denote $\beta = \alpha(\widetilde{v}_1, \ldots, \widetilde{v}_{k-m}, -)$.
(1) Choice of trivialisation. This comes from the fact that the trivialisation is oriented: if $\phi_1, \phi_2 : F \to \pi^{-1}(b)$ are two trivialisations, then $\psi := \phi_1^{-1}\phi_2 : F \to F$ is an orientation preserving diffeomorphism. Hence, on an open subset $V_\alpha \subset F$ the integral is written in coordinates as

$$\int_{V_\alpha} \phi_1^*\beta \,\mathrm{d}f_1 \ldots \mathrm{d}f_m = \int_{V_\alpha} \psi^*\phi_1^*\beta \,\mathrm{d}g_1 \ldots \mathrm{d}g_m$$

where $g_i = \psi(f_i)$ are the transformed coordinate functions. Since $\det(D\psi) = 1$ by assumption, the two integrals are equal by the change of coordinates theorem.

Moreover, for vector bundles and principal bundles, the definition is also independent of the choice of oriented trivialisation with the same orientation. Even if ϕ_1, ϕ_2 are taken from different trivialisations, the transition function $\phi_1^{-1}\phi_2 : F \to F$ will be an orientation preserving diffeomorphism due to the extra structure on ϕ_1 and ϕ_2, i.e. linear isomorphism or G-equivariant map.

(2) Choice of lifts. Fix $a \in \pi^{-1}(b)$, and let $\widetilde{v}_1'$ be another lift of $v_1 \in T_bB$. That is, $D\pi|_a(\widetilde{v}_1) = D\pi|_a(\widetilde{v}_1') = v_1$. So $\widetilde{v}_1' - \widetilde{v}_1 \in \ker D\pi|_a = T_a(p^{-1}(b))$.

Since the fiber dimension is m, $m+1$ vectors $w_1, \ldots, w_m, \widetilde{v}_1' - \widetilde{v}_1 \in T_a(\pi^{-1}(b))$ are linearly dependent, and therefore

$$\alpha(\widetilde{v}_1' - \widetilde{v}_1, \widetilde{v}_2, \ldots, \widetilde{v}_{k-m}, w_1, \ldots, w_m) = 0$$

which shows the integrand is independent of the choice of lifts.

(3) Integrand is smooth on $\pi^{-1}(b)$. Since α is a smooth differential form, we only need to check that its dependence on the lifted vectors is smooth. Given a local trivialisation $\phi : U \times F \xrightarrow{\simeq} \pi^{-1}(U)$, at $a \in \pi^{-1}(b)$ we can choose the lift $\widetilde{v}_1 = D\phi|_{\phi^{-1}(a)}(v_1)$. This is a smooth choice, and hence α is smooth by part (2).

(4) $\pi_*\alpha$ is a smooth differential form on B. To see this, we can write it out in coordinates. Let $dx_1, \ldots, dx_n, dt_1, \ldots, dt_m \in T^*E|_U$ be a basis, where t_i are coordinate functions for F and x_i are coordinate functions for B. Assume α is a product of 1-forms and has a factor of $dt_1 \wedge \cdots \wedge dt_m$, since otherwise β will not be a top form on $\pi^{-1}(b)$. Then integration along the fiber is locally described by

$$\alpha = \pi^*\eta \wedge f(x,t)\,\mathrm{d}t_1 \cdots \mathrm{d}t_m \mapsto \eta \wedge \int_{\pi^{-1}(b)} f(x,t)\,\mathrm{d}t_1 \cdots \mathrm{d}t_m$$

for some $\eta \in \Omega(B)$. It is now apparent that the dependence on x, i.e. the coordinates for B, is smooth.

Remark 2.2. Note that not all differential forms in $\Omega^*(E)$ are integrable, so we typically restrict the domain of π_* to some space of integrable forms. For now, assume this has been done.

In this chapter, we are interested in integration along the fiber for oriented vector bundles in particular, however the general definition will be needed later when we integrate over G-orbits of principal bundles. The case of vector bundles is slightly simpler, because the fiber has global coordinates.

Proposition 2.3. *Integration along the fiber π_* commutes with the exterior derivative d.*

***Proof*.** Since π_* and d are linear, it suffices to prove this proposition on the restriction $E|_U \simeq U \times F$ to a subset in the trivialising open cover. If $\omega \in \Omega^*(E)$, to write it in coordinates we need to further choose a subset $U \times V$ which is diffeomorphic to an open neighbourhood of $\mathbb{R}^{n+m}$. Then we can write it locally as $\omega = \pi^*\eta \wedge f(x,t)dt_1 \cdots dt_n$ for some $\eta \in \Omega^*(B)$. Then

$$\begin{aligned} d\pi_*\omega &= d\left(\eta \wedge \int f(x,t)\,\mathrm{d}t_1 \cdots \mathrm{d}t_m\right) \\ &= d\eta \wedge \int f(x,t)\,\mathrm{d}t_1 \cdots \mathrm{d}t_m + \eta \wedge \sum_i dx_i \int \partial_{x_i} f(x,t)\,\mathrm{d}t_1 \cdots \mathrm{d}t_m \end{aligned}$$

and

$$\begin{aligned} \pi_* d\omega &= \pi_*\Big(\pi^* d\eta \wedge f(x,t)dt_1 \cdots dt_n + \pi^*\eta \wedge \sum_i \partial_{x_i} f(x,t) dx_i dt_1 \cdots dt_n\Big) \\ &= d\eta \wedge \int f(x,t)\,\mathrm{d}t_1 \cdots \mathrm{d}t_m + \eta \wedge \sum_i dx_i \int \partial_{x_i} f(x,t)\,\mathrm{d}t_1 \cdots \mathrm{d}t_m \end{aligned}$$

where we have used $d\pi^*\eta = \pi^* d\eta$ in the first line. The proof is concluded by summing over the partition of unities subordinate to trivialising covers of F and E. $\square$

Therefore, π_* descends to a well defined map on de Rham cohomology. Integration along the fiber can also be defined by the property (b) in the following proposition, where the existence of $\pi_*\alpha$ can be seen as a generalisation of Fubini's theorem. Here $\Omega_c^*(E)$ denotes the space of forms with compact support, while $\Omega_{cv}^*(E)$ denotes the space of forms with compact support along the fiber: ω is in $\Omega_{cv}^*(E)$ if for every compact set $K \subset B$, $\pi^{-1}(K) \cap \operatorname{supp}(\omega)$ is compact in E.

Proposition 2.4 (Projection formula)**.** *Let $\pi : E \to B$ be a fiber bundle with m-dimensional fiber F, and both B and E are oriented.*
(a) For $\alpha \in \Omega_{cv}(E)$ and $\beta \in \Omega(B)$,

$$\beta \wedge \pi_*\alpha = \pi_*(\pi^*\beta \wedge \alpha)$$

(b) If $\alpha \in \Omega_{cv}(E)$ and $\beta \in \Omega_c(B)$,

$$\int_B \beta \wedge \pi_*\alpha = \int_E \pi^*\beta \wedge \alpha$$

Proof. (a) Write $\alpha = \pi^*\eta \wedge f(x,t)dt_1 \wedge \cdots \wedge dt_m$ in terms of local coordinates on $E|_U$. Then

$$\pi_*(\pi^*\beta \wedge \alpha) = \pi^*(\beta \wedge \eta) \wedge \int_{\pi^{-1}(b)} f(x,t)dt_1 \wedge \cdots \wedge dt_m = \beta \wedge \pi_*(\alpha)$$

Note that there is an implicit sum over a partition of unity for F.
(b) Again, it suffices to prove this on the restriction $E|_U \simeq U \times F$ to a subset in the trivialising cover for E. Furthermore, let $U \times V_\alpha$ be a trivialising open cover of $U \times F$, where U and each $V_\alpha \subset F$ are diffeomorphic to an open subset of $\mathbb{R}^d$. After summing over the partition of unity subordinate to an open cover of E, the result will still hold. On $E|_U$, the integral can be written

$$\begin{aligned}\int_{E|_U} \pi^*\beta \wedge \alpha &= \sum_\alpha \int_{U\times V_\alpha} \pi^*\beta \wedge \alpha \\ &= \sum_\alpha \int_U \int_{V_\alpha} \pi^*\beta \wedge \alpha = \int_U \int_{\pi^{-1}(b)} \pi^*\beta \wedge \alpha = \int_U \beta \wedge \pi_*\alpha\end{aligned}$$

where we have applied Fubini's theorem in the second line, and the result of part (a) in the last equality. $\square$

2.2 Thom isomorphism

Theorem 2.5 (Poincaré duality [6, p.44])**.** *If M is an oriented manifold of dimension n, then the pairing*

$$\int_M : H^q(M) \otimes H_c^{n-q}(M) \to \mathbb{R}$$

is non-degenerate. Therefore, $H^q(M) \simeq (H_c^{n-q}(M))^*$.

Consider a vector bundle $E \to M$ of rank n, over a manifold of dimension m. Since the zero section embeds M in E, M is a deformation retract of E. Since homotopic maps induce the same map in de Rham cohomology, it follows that

$$H^*(E) \simeq H^*(M)$$

If we further assume E and M are orientable manifolds, then a similar statement also holds for compact cohomology using Poincaré duality

$$H_c^*(E) \simeq (H^{n+m-*}(E))^* \simeq (H^{n+m-*}(M))^* \simeq H_c^{*-n}(M)$$

The Thom isomorphism is concerned with compact vertical cohomology. From the previous section, integration along the fiber descends to a map on compact vertical cohomology $\pi_* : H_{cv}^*(E) \to H^*(M)$. In fact, we can show that it induces an isomorphism, called the Thom isomorphism. We shall prove this in the trivial bundle case $M \times \mathbb{R}^n \to M$.

Proposition 2.6. *Integration along the fiber defines an isomorphism*

$$\pi_* : H_{cv}^*(M \times \mathbb{R}^n) \to H^{*-n}(M)$$

***Proof*.** Let $\pi_*^n : H_{cv}^*(M \times \mathbb{R}^n) \to H_{cv}^{*-1}(M \times \mathbb{R}^{n-1})$ be integration along the last component of $\mathbb{R}^n$, and for ease of notation denote $H_{cv}^*(M \times \mathbb{R}^0) := H^*(M)$. We will prove that π_*^n defines an isomorphism, because then the proposition follows by taking the composition $\pi_* = \pi_*^1 \circ \cdots \circ \pi_*^n$ (using Fubini's theorem).

Let $dt := dt_1 \wedge \cdots \wedge dt_n \in \Omega^n(\mathbb{R}^n)$ be the top form corresponding to orthonormal coordinates on $\mathbb{R}^n$. To produce a map in the reverse direction to π_*^n, let $e(t)dt \in \Omega_c^1(\mathbb{R})$ be a compactly supported 1-form with integral 1, and define

$$e_* : \Omega_{cv}^*(M \times \mathbb{R}^{n-1}) \to \Omega_{cv}^{*+1}(M \times \mathbb{R}^n), \qquad \omega \mapsto \omega \wedge e(t_n)\, dt_n$$

It is clear that e_* commutes with d, so induces a map on cohomology. Moreover, $\pi_*^n \circ e_* = 1$ on $\Omega_{cv}^*(M \times \mathbb{R}^{n-1})$. While $e_* \circ \pi_*^n \neq 1$ on the levels of forms, our strategy is to construct a homotopy operator $K : \Omega^*(M \times \mathbb{R}^n) \to \Omega^{*-1}(M \times \mathbb{R}^n)$ such that

$$1 - e_*\pi_*^n = (-1)^{q-1}(dK - Kd), \qquad \text{on } \Omega^q(M \times \mathbb{R}^n) \tag{2.1}$$

This will imply that for any closed $\omega \in \Omega^*(M \times \mathbb{R}^n)$, $(1 - e_*\pi_*^n)\omega$ is proportional to $dK\omega$, which is exact. Thus $e_*\pi_*^n = 1$ on cohomology. Let $\omega \in \Omega^*(M \times \mathbb{R}^n)$. If ω does not contain a factor of dt_n, define $K(\omega) = 0$. Otherwise define

$$\omega = (\pi^*\phi)f(x,t)dt_I dt_n \mapsto \pi^*\phi dt_I \Big(\int_{-\infty}^t f \,\mathrm{d}t_n - \int_{-\infty}^t e \,\mathrm{d}t_n \int_{\mathbb{R}} f \,\mathrm{d}t_n \Big)$$

where $I \subset \{1, \dots, n-1\}$. It is now straightforward to prove equation (2.1): compute and compare $(dK - Kd)\omega$ with $(1 - e_*\pi_*^n)\omega$ for the cases where ω contains a factor of dt_n or not. $\square$

The proof above shows that the inverse map in cohomology is given by the wedge product with a compactly supported form $\Phi \in \Omega^n(\mathbb{R}^n)$ whose integral along each component is 1. This can be generalised to arbitrary vector bundles $E \to M$, known as the Thom isomorphism.

Theorem 2.7 (Thom Isomorphism [6, Theorem 6.17])**.** *For a vector bundle E over compact manifold M, integration along the fiber induces an isomorphism $H^*_{cv}(E) \simeq H^{*-n}(M)$*

Definition 2.8. Under the Thom isomorphism $H^*(M) \to H^{*+n}_{cv}(E)$, the image of $1 \in H^0(M)$ is called the Thom class $\Phi \in H^n_{cv}(E)$ of the vector bundle.

In other words, the representatives of the Thom class are precisely the forms which are compactly supported along the fiber and have integral 1 along the fibers. As a direct consequence of Proposition 2.4, the Thom class $\Phi \in H^n_{cv}(E)$ satisfies for all $\beta \in \Omega^*_c(M)$

$$\int_M \beta = \int_E \pi^*\beta \wedge \Phi \tag{2.2}$$

2.3 Mathai-Quillen construction of Thom form

The Mathai-Quillen construction uses a slight variation of the Thom isomorphism. Differential forms with compact support in each fibre are replaced by forms which are rapidly decreasing in the fibre directions.

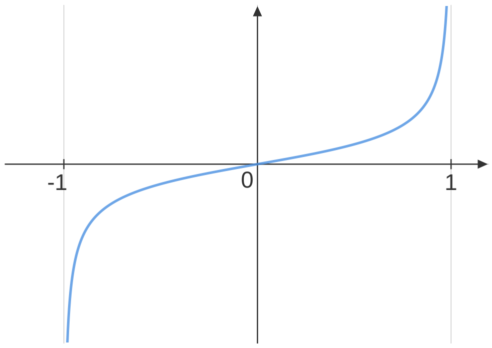
Definition 2.9. Let $E \to M$ be a vector bundle with fiber V of rank n. A differential form $\omega \in \Omega(E)$ is rapidly decreasing along the fiber if the restriction to the fiber $\omega_{\pi^{-1}(x)} : V \to \mathbb{R}$ is a Schwartz function for all $x \in M$, i.e.

$$\forall\, a, b \in \mathbb{N}^n, \|\omega_{\pi^{-1}(x)}\|_{a,b} < \infty$$

where $\|f\|_{a,b} := \sup_{x\in\mathbb{R}^n} |x^a(D^b f)(x)|$. The space of such forms is denoted $\Omega_{rd}(E)$.

Using the diffeomorphism $h : E \to E$ mapping the unit ball to $\mathbb{R}^n$

$$h(y) = \frac{y}{\sqrt{1 - \|y^2\|}}$$



we can pull back a rapidly decreasing form $\omega \in \Omega^*_{rd}(E)$ to $\Omega^*_{cv}(E)$ which is supported on the unit ball bundle. Note that the Schwartz property guarantees that $h^*\omega$ is smooth, i.e. all derivatives approach zero as $y \to 1$ since ω decays faster than any polynomial.

Let us first look at the construction on an oriented Euclidean vector bundle $E \xrightarrow{\pi} M$ of rank n over a point $M = \{x_0\}$. Given coordinates $x^1, \ldots, x^n$ on E from the local trivialisation, a Thom form $U \in \Omega^n_{rd}(E)$ can be defined as

$$U(x) = (2\pi)^{-n/2} e^{-|x|^2/2} dx^1 \wedge \ldots \wedge dx^n$$

Given the basis $e_1, \ldots, e_n$ of V corresponding to the coordinates $x^1, \ldots, x^n$, denote the map $dx = dx^k \otimes e_k \in \Omega^1(E, V)$. Then $e^{-idx} \in \Omega(E, \bigwedge V)$ by taking a formal sum of wedge products. We can express this form in terms of the Berezin integral, considered as a map $\int^B := \int \mathrm{d}e_1 \cdots \mathrm{d}e_n : \Omega^*(E, \bigwedge V) \to \Omega^*(E)$. The reader is advised at this stage to refer to Appendix A for the definition and properties of the Berezin integral, which will be utilised extensively in this chapter.

Lemma 2.10. *The form $U \in \Omega^n_{cv}(E)$ equals*

$$U(x) = (2\pi)^{-n/2} e^{-|x|^2/2} \epsilon(n) \int^B e^{-idx}$$

where $\epsilon(n) = 1$ if n is even, otherwise i.

Proof. The Berezin integral kills any forms of order less than n, so only multiples of $e_1 \wedge \ldots \wedge e_n$ remain. Hence,

$$\begin{aligned}
\int^B e^{-idx} &= (-i)^n \frac{1}{n!} \int^B (dx^k \otimes e_k)^n \\
&= (-i)^n \int^B (dx^1 \otimes e_1) \ldots (dx^n \otimes e_n) \\
&= (-i)^n (-1)^{n(n-1)/2} \int^B dx^1 \wedge \ldots \wedge dx^n \otimes e_1 \wedge \ldots \wedge e_n \\
&= (-i)^n (-1)^{n(n-1)/2} dx^1 \wedge \ldots \wedge dx^n
\end{aligned}$$

Multiplication of elements of $\Omega(E, \bigwedge V)$ occur in the graded tensor product, so that this is a graded algebra. In the second line, we have permuted all $n!$ terms into standard order, and terms of degree two commute. In the third line, dx^i are separated from the e_i, which anti-commute due to $\Omega(E, \bigwedge V)$ being a graded algebra. The lemma now follows, since $(-i)^n(-1)^{n(n-1)/2}$ equals 1 if n is even, and otherwise $-i$. □

Our goal is to now define the Thom form on a more general manifold by modification of the above lemma. We define the tautological section $x \in \Omega^0(E, \pi^* E)$ by $e \in E_p \mapsto e \in (\pi^* E)_e = E_p$. Then choosing a Euclidean connection ∇^E on E, we have an induced covariant derivative $\nabla^{\pi^* E}$ on $\pi^* E \to E$, which we denote by just ∇. Our idea is to replace dx by $\nabla^{\pi^* E} x \in \Omega^1(E, \pi^* E)$. In the rest of this section, denote $V = \pi^* E$ to simplify notation.

Next, recall that the curvature is a form in $\Omega^2(E, \mathrm{End}(V))$ which is skew-symmetric valued. We can identify $\mathfrak{so}(V)$ with $\bigwedge^2 V$ in the following way:

$$A \in \mathfrak{so}(V) \mapsto \sum_{i<j} \langle Ae_i, e_j \rangle \, e_i \wedge e_j \tag{2.3}$$

given a frame of V. Hence, we denote the curvature form as $F \in \Omega^2(E, \bigwedge^2 V)$.

We now define the Mathai-Quillen Thom form (corresponding to $t = 1$)

$$U_t = (2\pi)^{-n/2} \epsilon(n) \int^B e^{-\frac{1}{2} t^2 |x|^2 - it \nabla x - F} \in \Omega^n(E) \tag{2.4}$$

To justify why $U_t \in \Omega^n(E)$, note that $\omega_t \in \bigoplus_{i=0}^2 \Omega^i(E, \bigwedge^i V)$, and hence $e^{-\omega_t} \in \bigoplus_{i=0}^n \Omega^i(E, \bigwedge^i V)$. The Berezin integral then only retains the top degree part.

To prove that U is indeed a Thom form, we first need to define a contraction operator $\iota(s) : \Omega^i(E, \bigwedge^j V) \to \Omega^i(E, \bigwedge^{j-1} V)$ for $s \in \Gamma(V)$ in a similar fashion to definition 1.19, by the following properties:

(1) If $w \in \Omega^0(E, V)$, then $\iota(s) w = \langle s, w \rangle$

(2) If $\alpha \in \Omega^i(E, \bigwedge^j V), \beta \in \Omega^k(E, \bigwedge^l V)$, then

$$\iota(x)(\alpha \wedge \beta) = (\iota(x)\alpha) \wedge \beta + (-1)^{i+j} \alpha \wedge (\iota(x)\beta)$$

Note that this uniquely defines $\iota(s)$ because the formulas can be applied to a basis of the graded tensor product $\bigwedge^i T^*E \otimes \bigwedge^j V$.

Proposition 2.11. *If ∇ is a metric connection on π^*E, this induces a covariant derivative on $\bigwedge \pi^*E$. Then for any $\alpha \in \Omega(E, \bigwedge V)$ and $s \in \Gamma(E, V)$, we have*

$$d \int^B \alpha = \int^B \nabla\alpha = \int^B (\nabla - it\iota(s))\alpha$$

Proof. Given a non-vanishing section $\nu \in \Omega^1(E, \bigwedge^n V)$, the Berezin integral is given by the induced metric on $\bigwedge V$ with ν. From section A.2.1, the induced covariant derivative ∇ on $\bigwedge V$ is compatible with this metric, and

$$d \int^B \omega = d \langle \nu, \omega \rangle = \langle \nabla\nu, \omega \rangle + \langle \nu, \nabla\omega \rangle = \int^B \nabla\omega$$

where we have used the fact that $\nabla\nu = 0$, by definition of the Berezin integral on an oriented vector bundle with a metric connection. We can extend this to $d \int^B \alpha = \int^B (\nabla - it\iota(s))\alpha$ for any section of π^*E because $\iota(s)\alpha$ has no component in the top exterior power. □

Proposition 2.12. *Let $x \in \Gamma(E, V)$ be the tautological section on E, and ∇ the induced connection on $\bigwedge V$. Let $\omega_t = \frac{1}{2}t^2 |x|^2 + it\nabla x + F \in \Omega(E, \bigwedge V)$. Then*

$$(\nabla - it\iota(x))\omega_t = 0$$

Proof. By metric compatibility, $\nabla |x|^2 = 2 \langle \nabla x, x \rangle = -2\iota(x)\nabla x$. Next, we have $\nabla(\nabla x) = \iota(x)F$. Finally by Bianchi's identity, $\nabla F = 0$. Combining this,

$$\nabla\omega_t = -t^2\iota(x)\nabla x + it\iota(x)F$$

On the other hand, $\iota(x) |x|^2 = 0$ by definition of $\iota(x)$, so

$$it\iota(x)\omega_t = -t^2\iota(x)\nabla x + it\iota(x)F$$

from which it follows that $(\nabla - it\iota(x))\omega_t = 0$. □

Theorem 2.13. *The Mathai-Quillen form $U \in \Omega^n_{rd}(E)$ is a Thom form.*

Proof. We need to show that U is closed, and integration along the fiber gives $1 \in \Omega^0(E)$. From equation (A.2), the exponential of $-\omega_t$ can be written

$$e^{-\omega_t} = \sum_{k=0}^{n} \frac{e^{-t^2|x|^2/2}}{k!}(-it\nabla x - F)^k = \sum_{k=0}^{\infty} \frac{1}{k!}(-\omega_t)^k \tag{2.5}$$

Since $\nabla - it\iota(x)$ is an antiderivation, it follows from Proposition 2.12 that $(\nabla - it\iota(x))e^{-\omega_t} = 0$. And therefore by Proposition 2.11 it follows that $U_t \propto \int^B e^{-\omega_t}$ is closed.

From equation (2.5), we can write the Thom form as

$$U_t = (2\pi)^{-n/2}\epsilon(N)e^{-\frac{1}{2}|x|^2} \int^B e^{-i\nabla x - F}$$

To integrate $U_t \in \Omega(E)$ along the fiber of $V = \pi^* E$, we extract from the Berezin integral the part which is a n-form on the fiber of V,

$$\int_E (2\pi)^{-n/2}\epsilon(n)e^{-\frac{1}{2}|x|^2} \int^B e^{-i\nabla x - F} = (2\pi)^{-n/2}\epsilon(n) \int_E e^{-\frac{1}{2}|x|^2} \int^B e^{-idx} = 1$$

we are only left with the dx term because the induced connection and curvature forms on $\pi^* E$ do not have any components along the fiber. We have evaluated the integral using Lemma 2.10. □

Next, we will show one of the key properties of the Thom form: its pullback to M by any section $s : M \to E$ is a representative of the Euler class of the vector bundle. The Mathai-Quillen formula gives us a direct way to prove this, via the following transgression formula, which is the reason we defined the parameter t.

Lemma 2.14. *The form $U_t \in \Omega^n(E)$ satisfies the transgression formula*

$$\frac{\mathrm{d}}{\mathrm{d}t}U_t = -i(2\pi)^{-n/2}\epsilon(n)d \int^B xe^{-\omega_t}$$

Proof. With $\omega_t = \frac{1}{2}t^2 |x|^2 + it\nabla x + F \in \Omega(E, \bigwedge V)$ as before, we have

$$\frac{\mathrm{d}\omega_t}{\mathrm{d}t} = t|x|^2 + i\nabla x = i(\nabla - it\iota(x))x$$

Since $(\nabla - it\iota(x))\omega_t = 0$ by Proposition 2.12, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}e^{-\omega_t} = -\frac{\mathrm{d}\omega_t}{\mathrm{d}t}e^{-\omega_t} = -i((\nabla - it\iota(x))x)e^{-\omega_t} = -i(\nabla - it\iota(x))(xe^{-\omega_t})$$

and hence by application of Proposition 2.11,

$$\begin{aligned}\frac{\mathrm{d}U_t}{\mathrm{d}t} &= -i(2\pi)^{-n/2}\epsilon(n) \int^B (\nabla - it\iota(x))(xe^{-\omega_t}) \\ &= -i(2\pi)^{-n/2}\epsilon(n)d \int^B xe^{-\omega_t}\end{aligned}$$

□

Proposition 2.15. *Let $s \in \Gamma(M, E)$ be a section of E. The cohomology class of the pullback $s^*U \in \Omega^n(M)$ is independent of the section $s \in \Gamma(E)$, and is a representative of the Euler class for even n.*

***Proof*.** First observe that $s^*U_0 = (2\pi)^{-n/2}\epsilon(n)\int^B e^{-s^*F} \in \Omega(M)$. From equation (2.3), $F \in \Omega^2(E, \bigwedge^2 V)$ is given in a frame of V by

$$F = \sum_{i<j} F_{ji} e_i \wedge e_j = -\frac{1}{2}\sum_{i,j} F_{ij} e_i \wedge e_j$$

Since $F = \pi^*\Omega$ is the pullback of the curvature Ω on $E \to M$, the pullback of F by any section s gives the original curvature form, since $s^*\pi^* = (\pi \circ s)^*$. Therefore, s^*U_0 does not depend on s, and is equal to $s^*U_0 = (2\pi)^{-n/2}\epsilon(n)\,\mathrm{Pf}(\Omega)$ by Lemma A.5, which is a representative of the Euler class from standard Chern-Weil theory.

To show this also holds for s^*U, we can integrate the transgression formula above from 0 to 1,

$$U_1 - U_0 = -i(2\pi)^{-n/2}\epsilon(n) d\int_0^1 \int^B x e^{-t^2|x|^2/2 - it\nabla x - F}\,\mathrm{d}t$$

Then taking the pullback by s on both sides,

$$s^*U - s^*U_0 = -i(2\pi)^{-n/2}\epsilon(n) d\int_0^1 \int^B s \wedge e^{-t^2|s|^2/2 - it\nabla s - \Omega}\,\mathrm{d}t$$

we find that s^*U is cohomologous to the Euler class for any section s. □

2.4 Universal Mathai-Quillen formula

There is a slight generalisation of the Mathai-Quillen formula in the framework of equivariant cohomology. This is useful because we are often interested in the Thom form of a vector bundle associated to a principal bundle, where equivariant differential forms may be easier to work with. Furthermore, the universal Mathai-Quillen formula does not depend on the choice of a connection.

Let $E \to M$ be an oriented vector bunde or rank n with a metric and compatible connection, and fiber V. Then E can be identified as the associated bundle $\mathrm{Fr}_{\mathrm{SO}}(E) \times_{\mathrm{SO}} V$ with the induced connection using the following proposition.

Proposition 2.16. *Let $E \to M$ be an oriented vector bundle with fiber $V = \mathbb{R}^n$ and $P = \mathrm{Fr}_{\mathrm{SO}}(E)$ be the principal $\mathrm{SO}(n)$-bundle of orthonormal oriented frames on E. Then $P \times_{\mathrm{SO}(n)} V$ is canonically isomorphic to E as a vector bundle.*

Proof. The action of $A \in \mathrm{SO}(n)$ on $P \times V$ is $([v_1 \cdots v_n], v) \cdot A = ([v_1 \cdots v_n]A, A^{-1}v)$. Then the canonical isomorphism $\psi : P \times_{\mathrm{SO}(n)} V \to E$ is defined by

$$[[v_1 \cdots v_n], v] \mapsto [v_1 \cdots v_n]v$$

It is clear that this is well defined, linear, smooth and preserves the fiber. The map is injective because $v_1, \ldots, v_n$ must be linearly independent, and surjective because $v_1, \ldots, v_n$ is a basis for E_x. □

This shows that all vector bundles are associated to some principal bundle. So more generally, consider a connected Lie group G and principal G-bundle $P \to M$ over a manifold of dimension n with a connection ω. Let $\rho : G \to \mathrm{SO}(V)$ be a representation of rank n (same as $\dim M$). This determines the associated vector bundle $E := P \times_\rho V \to M$ with typical fiber V.

Let p_1, p_2 be the projection maps from $P \times V$ to P and V respectively. Note that both maps are G-equivariant, where V is considered as a G-manifold with action $v \cdot g = \rho(g)^{-1}v$. The principal bundle $P \times V \to P \times_\rho V$ has the connection $p_1^*\omega$ (the reader should check this is a valid connection). The Cartan map associated to this connection gives the homomorphism

$$\Omega_G(V) \xrightarrow{p_2^*} \Omega_G(P \times V) \xrightarrow{\mathrm{Car}_\omega} \Omega(P \times V)_{bas} \simeq \Omega(E)$$

given by $\alpha \mapsto \mathrm{Hor}_\omega((p_2^*\alpha)(\Omega))$. Since $P \times V \to E$ is a principal bundle, $\Omega^*(E) \simeq \Omega^*(P \times V)_{bas}$ by definition of basic forms on a principal bundle.

Definition 2.17. An element $U \in (S(\mathfrak{g}^*) \otimes \Omega(V))^G$ is a <u>universal Thom form</u> if for any principal G-bundle $P \to M$ with a connection ω, a representation $\rho : G \to \mathrm{SO}(V)$ and oriented associated bundle $P \times_\rho V$, the Cartan map $\mathrm{Car}_\omega : \Omega_G(V) \to \Omega(P \times_\rho V)$ carries U to a form representing the Thom class of $P \times_\rho V$.

The integral over V for equivariant forms is a map $\int_V : S(\mathfrak{g}^*) \otimes \Omega(V) \to S(\mathfrak{g}^*)$ defined by $\left(\int_V \alpha\right)(X) = \int_V \alpha(X)$, where elements of $S(\mathfrak{g}^*)$ are viewed as polynomials on $\mathfrak{g}$.

Theorem 2.18. *A differential form $U \in (S(\mathfrak{g}^*) \otimes \Omega(V))^G$ is a universal Thom form if and only if U is closed and $\int_V U = 1$.*

Proof. Assume we have the same objects as above. When we pull back the form on $\Omega(V)$ to $\Omega(P \times V)$, it will still be a basic element because the projection p_2 is G-equivariant. Next, the image of $w : (S(\mathfrak{g})^* \otimes \Omega(P \times V))^G \to \Omega(P \times V)_{inv}$ will be closed because the Weil homomorphism commutes with the differential operators. Finally, its horizontal projection is also closed. The same argument in reverse proves that U must be closed if $\mathrm{Car}_\omega(p_2^*U))$ is closed.

It remains to check that integration along the fiber of $\tau := \mathrm{Car}_\omega(p_2^*U) \in \Omega(E)$ equals the integral over V of the universal Thom form. Fix a point $x_0 \in M$, let $P_0 = \pi^{-1}(x_0)$ be the fiber over x_0 and E_0 be the fiber of E over x_0. The fiber integral of τ only depends on the value of τ at the tangent space of the fiber submanifold, i.e. in the vertical direction (see figure 2.1). So to evaluate the fiber integral at any point in E_0, it suffices to consider the restriction to the principal G-bundle $P_0 \times V \to E_0$. The tangent space at $(p, v) \in P_0 \times V$ is $V_pP \oplus V$. But the

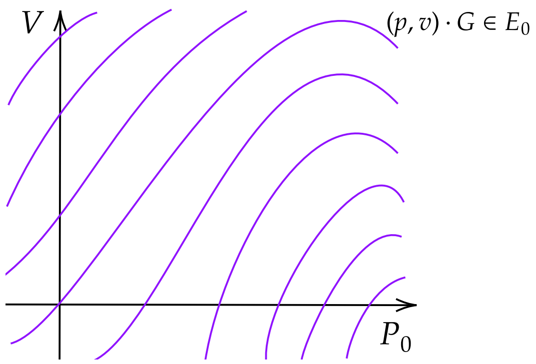


Figure 2.2: Elements of the fiber E_0 viewed as orbits of $P_0 \times V$. The diagram reflects G being a free action on P_0, but not necessarily on V

connection on $P \times V$ is the pullback $p_1^*\omega$, which is constant in the V direction, and the curvature form vanishes on $V_pP \oplus V$ since it is a horizontal form. If $\alpha_1, \ldots, \alpha_d$ is a dual basis for $\mathfrak{g}^*$, the element p_2^*U is a sum of terms of the form $\beta_I \alpha_I$, where $\beta_I \in \Omega(P \times V)$ are invariant forms. After we replace the α_i by the curvature forms Ω_i, as a form on $P_0 \times V$ we are only left with the term β_0. Then taking the horizontal component amounts to only evaluating tangent vectors in V at each point $(p, v) \in P_0 \times V$. Subsequently, the fiber integral is the same as the integral of β_0 over V, which equals 1 because $\int_V U = 1$. □

The above theorem is proved in chapter 10 of [15] in the more general context of a K-manifold $E = P \times_\rho V$, and a principal G-bundle $P \times V \to P \times_\rho V$, with associated Cartan map $\mathrm{Car}_\omega : \Omega_{G\times K}(P \times V) \to \Omega_K(E)$. The Thom form is obtained from the map $\mathrm{Car}_\omega(p_2^*U)$ where $U \in \Omega_{G\times K}(V)$.

We shall now construct the Mathai-Quillen universal Thom form. Let V be a real vector space with standard inner product, and $\rho : G \to \mathrm{SO}(V)$ be a rep of the Lie group G, with induced rep $\rho_* : \mathfrak{g} \to \mathfrak{so}(V)$. We will apply the Berezin integral to a form in $S(\mathfrak{g}^*)\otimes\Omega(V)\otimes\bigwedge V$. Fix an oriented orthonormal basis $\{e_1, \ldots, e_n\}$ for V, which allows us to define the Berezin integral, with corresponding coordinate functions $x_1, \ldots, x_n$. Let $\{X_1, \ldots, X_m\}$ be a basis of $\mathfrak{so}(V)$, with corresponding dual basis $\{u_1, \ldots, u_m\}$ where $m = \frac{1}{2}n(n-1)$. Recall that we can identify $\mathfrak{so}(V)$ with $\bigwedge^2 V$ via equation (2.3), restated here

$$X_a \in \mathfrak{so}(V) \mapsto \sum_{i<j} \langle X_a e_i, e_j \rangle \, e_i \wedge e_j = \frac{1}{2} \sum_i e_i \wedge X_a e_i$$

Denote $dx := \sum_i dx_i \otimes e_i \in \Omega^1(V) \otimes V$. Define the element

$$\sigma := -\frac{1}{2} |x|^2 - idx - \sum_a u_a \rho_* \otimes X_a \quad \in S(\mathfrak{g}^*) \otimes \Omega(V) \otimes \bigwedge V \tag{2.6}$$

which is analogous to ω_1 as defined in Prop 2.12. Note that $u_a \rho_* \in \mathfrak{g}^*$. In components, if $M^a = [\langle X_a e_i, e_j \rangle]_{ji}$, we can write the same element as

$$\sigma = -\frac{1}{2} |x|^2 - i \sum_i dx_i \otimes e_i - \frac{1}{2} \sum_{i,j} (\rho_*)_{ji} e_i \wedge e_j \tag{2.7}$$

since $u_a X_a$ is the identity on $\mathfrak{so}(V)$, and $(\rho_*)_{ji} \in \mathfrak{g}^*$. Consider its Berezin integral

$$U := \epsilon(n)(2\pi)^{-n/2} \int^B \exp(\sigma) \quad \in S(\mathfrak{g}^*) \otimes \Omega(V) \tag{2.8}$$

where $\epsilon(n) = 1$ if n is even and otherwise $\epsilon(n) = i$.

Theorem 2.19. *The element $U \in S(\mathfrak{g}^*) \otimes \Omega(V)$ defined above is a universal Thom form.*

Proof. We need to show that it is invariant and equivariantly closed in the Cartan model, and integration along the fiber gives 1.

(1) Closed. We can consider the Cartan differential $\delta_C = d - \sum u_k \iota_k$ as an operator on $S(\mathfrak{so}(V)^*) \otimes \Omega(V) \otimes \bigwedge V$ by acting trivially on the last factor, so that it commutes with the Berezin integral. It suffices to show U is closed as a form on $S(\mathfrak{so}(V)^*) \otimes \Omega(V)$, because the map $\rho^* : S(\mathfrak{so}(V)^*) \to S(\mathfrak{g}^*)$ given by $u_i \mapsto u_i \circ \rho_*$ commutes with the Cartan differential. This can be seen most easily by viewing the elements as polynomials as in Theorem 1.31, which shows that $(\delta_C \rho_* \alpha)(X) = (\rho_* \delta_C \alpha)(\rho(X)) = (d - \iota_{\rho(X)})\alpha(\rho(X))$, noting that ι_X and $\iota_{\rho(X)}$ act the same way on $\Omega(V)$.

For the first term, only d acts to give $\delta_C |x|^2 = 2x_i dx_i$. On the second term, $d^2 = 0$ and we need to evaluate $u_a \iota_a dx_i$. Since x_i projects to the ith component of a vector in V, and the action of e^{tX_k} is matrix multiplication, for $v \in V$ in the tangent space

$$\iota_a dx_i(v) = dx_i \left(\left. \frac{\mathrm{d}}{\mathrm{d}t} \right|_{t=0} e^{-tX_a} v \right) = -x_i (X_a v) \tag{2.9}$$

Hence, $u_a \iota_a dx_i = -u_a \otimes x_i \circ X_a$. The third term is a multiple of u_i and constant on $\Omega(V)$, hence maps to zero by δ_C. Thus,

$$\begin{aligned}(d - u_a\iota_a)\sigma &= -x_i dx_i - u_a \otimes ix_i X_a \otimes e_i \\ &= -x_i dx_i - u_a \otimes ix_j \otimes X_a e_j \end{aligned} \tag{2.10}$$

which is justified by x_j being the coordinate functions of e_j:

$$x_i X_a \otimes e_i = \langle X_a e_j, e_i \rangle \, x_j \otimes e_i = x_j \otimes \langle X_a e_j, e_i \rangle \, e_i = x_j \otimes X_a e_j \tag{2.11}$$

The key observation is that δ_C acts by lowering the degree of the $\bigwedge V$ part. More precisely, consider the action of the Berezin derivative/integral

$$\begin{aligned}\frac{\partial}{\partial e_k}\left(-\sum_{i<j} u_a \otimes e_i \wedge M^a_{ji} e_j\right) &= -\sum_{i>k} u_a \otimes M^a_{ik} e_i + \sum_{i<k} u_a \otimes M^a_{ki} e_i \\ &= -\sum_i u_a \otimes M^a_{ik} e_i = -u_a \otimes X_a e_k\end{aligned}$$

since $M^a_{ij} = -M^a_{ji}$. Applying $\frac{\partial}{\partial e_k}$ to the other term,

$$\frac{\partial}{\partial e_k}(-idx_i \otimes e_i) = idx_k$$

because e_i anti-commutes with dx_i. We have shown that

$$\delta_C \sigma = \left(\sum_k ix_k \frac{\partial}{\partial e_k}\right)\sigma \tag{2.12}$$

It follows that this also holds for $\exp(\sigma)$, because δ_C is an anti-derivation. Therefore,

$$\delta_C \int^B \exp(\sigma) = \int^B \delta_C \exp(\sigma) = \int^B ix_k \frac{\partial}{\partial e_k} \exp(\sigma) = 0$$

as the Berezin derivative kills any top degree terms in $\bigwedge V$.

(2) Integral along the fiber. We need to extract the coefficient of $dx_1 \wedge \ldots \wedge dx_n$. Therefore, the last term in σ does not contribute, and

$$\int_V U = \epsilon(n)(2\pi)^{-n/2} \int_V e^{-|x|^2} \int^B \exp(-idx_k \otimes e_k)$$

But we know $\int^B e^{-idx_k \otimes e_k}$ evaluates to $(-i)^n(-1)^{n(n-1)/2} dx_1 \wedge \ldots \wedge dx_n$ by Lemma 2.10. Hence, the integral of U along the fiber V is 1.

(3) Invariant. It suffices to show the terms in σ are invariant, because the Lie derivative is a derivation. We can extend the action of G to the graded algebra

$S(\mathfrak{g}^*) \otimes \Omega(V) \otimes \bigwedge V$, acting on the $\bigwedge V$ component by multiplication by $\rho(g)^{-1}$. Note that the Berezin integral is invariant under the action on $\bigwedge V$. This gives an associated Lie derivative, corresponding to multiplication by $\rho_*(X)$ for $X \in \mathfrak{g}$.

It is clear that $|x|^2$ is invariant, because $\rho(g)$ has determinant 1. Next, we show $dx_i \otimes e_i$ is invariant by applying $\mathcal{L}_X$,

$$\mathcal{L}_X(dx_i \otimes e_i) = \iota_X(dx_i) \otimes e_i + dx_i \otimes \rho_*(X)e_i = 0$$

by equations (2.9) and (2.11). Finally, to apply $\mathcal{L}_X$, recall that the action on $S(\mathfrak{g}^*)$ is the coadjoint representation $\mathcal{L}_X\alpha(Y) = -\alpha([X,Y])$. The action on V is $\mathcal{L}_X v = \rho_*(X)v$. We can denote the third term in σ as $\frac{1}{2}e_i \wedge (\rho_*)_{ji}e_j \in \mathfrak{g}^* \otimes \bigwedge^2 V$, since $u_a X_a$ is just the identity. For $X, Y \in \mathfrak{g}$, with $\rho_*(X) = A, \rho_*(Y) = B$,

$$\begin{aligned} &(\mathcal{L}_X(e_i \wedge (\rho_*)_{ji}e_j))(Y) \\ &= (Ae_i \wedge \rho_* e_i)(Y) - e_i \wedge \rho_*([X,Y])e_i + (e_i \wedge \rho_* Ae_i)(Y) \\ &= Ae_i \wedge Be_i - e_i \wedge (AB - BA)e_i + e_i \wedge ABe_i && (\star) \\ &= -e_i \wedge BAe_i - e_i \wedge (AB - BA)e_i + e_i \wedge ABe_i = 0 && (\star\star) \end{aligned}$$

where we begin by using the derivation property of $\mathcal{L}_X$, and explain the next two lines below.

Proof of $(\star)$: This step initially looks wrong, but recall that the term is to be interpreted as

$$\sum_{i,j} e_i \wedge (\rho_*(Y))_{ji}(Ae_j) = \sum_{i,j} e_i \wedge \sum_k B_{ji}A_{kj}e_k = \sum_i e_i \wedge \sum_{k,j} A_{kj}B_{ji}e_k = \sum_i e_i \wedge ABe_i$$

Proof of $(\star\star)$: This is just an application of the more general property

$$Ae_i \wedge Be_i = A_{ji}e_j \wedge B_{ki}e_k = e_j \wedge B_{ki}A^{\intercal}_{ij}e_k = e_j \wedge BA^{\intercal}e_j$$

Finally, U is invariant because $\mathcal{L}_X$ commutes with the Berezin integral. □

The proof of (1) closure is due to Guillemin and Sternberg [15] and part of (3) invariance is a corrected version of the proof in Constantinescu [9, p.27]. In particular, the steps $(\star)$ and $(\star\star)$ are corrected, as well as the action of $\mathcal{L}_X$ on $\mathfrak{g}^*$.

Remark 2.20. The minus sign in e^{-tX_a} in the fundamental vector field over V in equation (2.9) arises due to the fact that the SO(V)-action on V is a right action. In general, this is so that the map $\mathfrak{g} \to \Gamma(TM)$ is a Lie algebra homomorphism.

To relate our formula to other versions in literature, let $\chi = (e_1, \ldots, e_n)$ be the Grassman variables for V. In more compact notation, the third term of σ can be written as $\frac{1}{2}e_i \wedge (\rho_*)_{ij}e_j = \frac{1}{2}\chi^{\intercal}(\rho_*)\chi$. Then the Mathai-Quillen Thom form $U \in S(\mathfrak{g}^*) \otimes \Omega(V)$ can be alternatively written as

$$U = \epsilon(n)(2\pi)^{-n/2}e^{-|x|^2/2}\int^B \exp\left(\frac{1}{2}\chi^{\intercal}(\rho_*)\chi - idx^{\intercal}\chi\right)\mathrm{d}\chi \tag{2.13}$$

If $\rho_*(X) \in \mathfrak{so}(V)$ is always invertible, we can apply Proposition A.8, to get

$$U = \epsilon(n)(2\pi)^{-n/2}\operatorname{Pf}(\rho_*)\exp(-|x|^2/2 - dx^{\intercal}(\rho_*)^{-1}dx)$$

We can also obtain a universal Thom form in the Weil model, by applying the Weil-Cartan isomorphism $(S(\mathfrak{g}^*) \otimes \Omega(V))^G \xrightarrow{\cong} (W(\mathfrak{g}) \otimes \Omega(V))_{bas}$.

Bibliographical notes

- A more comprehensive treatment of the Thom isomorphism can be found in Bott and Tu [6].
- The Mathai-Quillen formula described in section 2.3 is based on section 1.6 in [5].
- The textbook by Guillemin and Sternberg [15] gives an exposition of the Mathai-Quillen formula in the equivariant case

Chapter 3

Localisation

3.1 Poincaré-Hopf theorem

One of the primitive applications of the Mathai-Quillen formula is to prove the Poincaré-Hopf theorem. Let $v \in \mathcal{X}(M)$ be a vector field on M. At a zero $p \in M$ of v, the Lie bracket of vector fields defines an endomorphism $\mathcal{L}_p(v) : T_pM \to T_pM$ by $X \mapsto [X, v]$. Here X is interpreted locally as a vector field with constant coefficients. In a coordinate system in which $v = v^i \partial_i$, $\mathcal{L}_p(v)$ is given by

$$\mathcal{L}_p(v)\partial_i = \sum_{j=1}^{n} [\partial_i, v^j(p)\partial_j] = \sum_{j=1}^{n} (\partial_i v^j(p))\partial_j$$

since ∂_i is a constant vector field. A zero $p \in M$ of v is called non-degenerate if $\mathcal{L}_p(v)$ is invertible. In this case, denote $\nu(p, v) = \operatorname{sgn}\det(\mathcal{L}_p(v)) \in \{\pm 1\}$. If all of the zeros of v are non-degenerate, we call v non-degenerate.

Lemma 3.1. *If v is a non-degenerate vector field on a compact manifold M, then there are finitely many zeros.*

Proof. Choose a chart $\phi_p : U_p \to \mathbb{R}^n$ in a neighbourhood U_p of each zero $p \in M$ of v, which gives coordinates on the tangent bundle $d\phi_p : TU_p \to \mathbb{R}^n$. The vector field v defines a smooth map $d\phi_p \circ v : U_p \to \mathbb{R}^n$ with $v(p) = 0$, and we claim that its derivative is $d\phi_p \circ \mathcal{L}_p(v)$. To see this, write $v = v^j \partial_j$ in the coordinates of TU_p, i.e. $d\phi_p(\partial_i) = e_i$. Then

$$\partial_i(d\phi_p \circ v) = \partial_i(v^j e_j) = (\partial_i v^j)e_j = d\phi_p \circ \mathcal{L}_p(v)$$

As $\mathcal{L}_p(v)$ is invertible, the inverse function theorem tells us $d\phi_p \circ v$ is a local diffeomorphism between open neighbourhoods $V_p \subset U_p$ and $B \subset \mathbb{R}^n$.

Now suppose for a contradiction that are infinitely many zeros, in which case we can find a convergent sequence of distinct zeros $p_k \to p \in M$. Then $v(p) = 0$ as well by continuity. So $d\phi_p \circ v : V_p \to B \subset \mathbb{R}^n$ is a diffeomorphism for some $V_p \subset M$, with $d\phi_p \circ v(p) = 0$. But by definition of convergent sequence, there exists a distinct zero $p_k \in V_p$ which means $d\phi_p \circ v(p_k) = 0$, contradicting $d\phi_p \circ v$ being a diffeomorphism. □

Theorem 3.2 (Poincare-Hopf). *If v is a non-degenerate vector field on an oriented compact manifold M of dimension n, then*

$$\int_M \chi(TM) = \sum_{\{p|v(p)=0\}} \nu(p, v)$$

where $\chi(TM) \in H^n(M)$ denotes the Euler class of the vector bundle.

Proof. (adapted from [5, Theorem 1.56]) Define $\phi_p : V_p \to \mathbb{R}^n$ as in the proof of the previous lemma, chosen such that $\phi_p(p) = 0$. The orientation on V_p induced by the oriented chart ϕ_p and by the local diffeomorphism $d\phi_p \circ v$ differ by the sign $\nu(p, v)$. We may assume V_p are disjoint for each p. Choose a Riemannian metric on M which agrees with the metric on each V_p induced by the diffeomorphism into $\mathbb{R}^n$. Since M is compact, $\exists\, \epsilon > 0$ such that $\|v\| \geq \epsilon$ on the compact set $M \setminus \bigcup_p V_p$, because if $\|v\|$ can get arbitrarily close to 0, we would find a convergent subsequence in $M \setminus \bigcup_p V_p$ whose limit is a zero of v.

Let $U \in \Omega^n(TM)$ be the Mathai-Quillen Thom form of TM with respect to the Riemannian metric and associated Levi-Civita connection. Let $f : \mathbb{R}_+ \to [0, 1]$ be a smooth bump function such that $f(s) = 1$ if $s < \epsilon^2/4$ and $f(s) = 0$ if $s > \epsilon^2$. Then for all $t > 0$

$$\int_M \chi(TM) = \int_M v_t^* U = \int_M (1 - f(\|v\|^2)) v_t^* U + \int_M f(\|v\|^2) v_t^* U \tag{3.1}$$

where $v_t := tv$. From equation (2.5), we see that $v_t^* U$ is of the form

$$v_t^* U = e^{-t^2\|v\|^2/2} \sum_{k=0}^{n} t^k \alpha_k$$

where $\alpha_k \in \Omega(M)$. Therefore, the first integral in equation (3.1) is rapidly decreasing in t since $\|v\|^2 > \epsilon^2/4$, and approaches zero as $t \to \infty$.

On each V_p, the metric and connection are trivial since they are induced by v. If we write $v = \sum_{i=1}^n x^i \partial_i$ in the coordinates of $d\phi_p$,

$$\begin{aligned} v_t^* U|_{V_p} &= (2\pi)^{-n/2} e^{-t^2\|x\|^2/2} \int^B e^{-itdv} \\ &= (2\pi)^{-n/2} \nu(p, v) e^{-t^2\|x\|^2/2} t^n dx^1 \wedge \ldots \wedge dx^n \qquad \text{(using Lemma 2.10)} \end{aligned}$$

The sign $\nu(p, v)$ comes from the Berezin integral using the orientation of the manifold, while $x^1, \ldots, x^n$ are coordinates of the vector field. Therefore,

$$\begin{aligned}\lim_{t\to\infty}\int_{V_p} f(\|v\|^2)v_t^*U &= (2\pi)^{-n/2}\nu(p,v)\lim_{t\to\infty}\int_{\mathbb{R}^n} f(\|x\|^2)e^{-t^2\|x\|^2/2}t^n dx^1\wedge\ldots\wedge dx^n \\ &= (2\pi)^{-n/2}\nu(p,v)\lim_{t\to\infty}\int_{\mathbb{R}^n} f(\|t^{-1}y\|^2)e^{-\|y\|^2/2}dy^1\wedge\ldots\wedge dy^n \\ &= \nu(p,v)\end{aligned}$$

where we have made the change of variables $x = t^{-1}y$. Summing over each of the neighbourhoods V_p of zeros, we get the result. □

The Mathai-Quillen formula also gives us intuition on why the Euler number should only depend on the zeroes of a section. The $e^{-|\epsilon s(x)|^2/2}$ term in the formula tells us that as $\epsilon \to 0$, the integral of the Euler form rapidly decays to zero at any point outside $s^{-1}(0)$.

There is a generalisation of this result, where we assume that the zeros of v are isolated instead of non-degenerate.[5, Theorem 1.58] It is desirable to generalise this further to relate the Euler number of arbitrary vector bundles to zeros of a section, where the zeros may not be isolated, but the Poincaré-Hopf theorem relies on the Lie bracket of vector fields to define the orientation of each zero. In order to describe such a generalisation we first need to introduce some intersection theory.

3.2 Poincaré dual as a Thom class

Let M be an oriented manifold and $i : S \hookrightarrow M$ be a compact, oriented submanifold with $\dim S = k, \dim M = n$. Poincaré duality (Theorem 2.5) can be used to associate S to a unique cohomology class $[\eta_S] \in H_c^{n-k}(M)$ called its Poincare dual as follows. Define the linear functional

$$H^k(M) \to \mathbb{R}, \qquad \omega \mapsto \int_S i^*\omega$$

Here $\int_S i^*\omega$ is defined because S is compact. It follows by Poincare duality that integration over S corresponds to a unique cohomology class $[\eta_S] \in H_c^{n-k}(M)$, with the property that for any $\omega \in H^k(M)$

$$\int_S i^*\omega = \int_M \omega \wedge \eta_S \tag{3.2}$$

Remark 3.3. The same argument can be applied to define the Poincaré dual of an oriented submanifold $S \subset M$ that is not necessarily compact. This is associated to a linear functional $H_c^k(M) \to \mathbb{R}$ by integrating over S. This is well defined because for any compact $K \subset M$, $K \cap S$ is compact in S. This corresponds to a unique cohomology class $[\eta_S] \in H^{n-k}(M)$, satisfying equation (3.2) for $\omega \in H_c^k(M)$.

Notice that the Poincaré dual has a remarkably similar property to the Thom class (c.f. equation (2.2)). To relate the two ideas, this requires constructing a vector bundle over the submanifold S, which leads to the concept of a tubular neighbourhood.

Definition 3.4. The normal bundle of S in M is the quotient vector bundle $N_S \to S$ defined by the exact sequence

$$0 \longrightarrow TS \longrightarrow TM|_S \longrightarrow N_S \longrightarrow 0$$

Let $S \subset M$ be a submanifold. A tubular neighbourhood of S in M is an open neighbourhood of S in M diffeomorphic to the normal bundle of S in M, such that S is diffeomorphic to the zero section.

For a Riemannian manifold, one can identify N_S with the orthogonal complement $(TS)^\perp \subset TM$. If we further assume S and M are oriented, then there is an induced orientation on the normal bundle via

$$\mathrm{or}(TM|_S) = \mathrm{or}(TS) \wedge \mathrm{or}(N_S) \tag{3.3}$$

where $\mathrm{or}(N_S)$ denotes the orientation form on the vector bundle.

Theorem 3.5 (Tubular neighbourhood theorem [20, Thm 5.25])**.** *Every submanifold S of a Riemannian manifold M has a tubular neighbourhood $T \subset M$. If S is compact, the tubular neighbourhood can be chosen to have constant radius.*

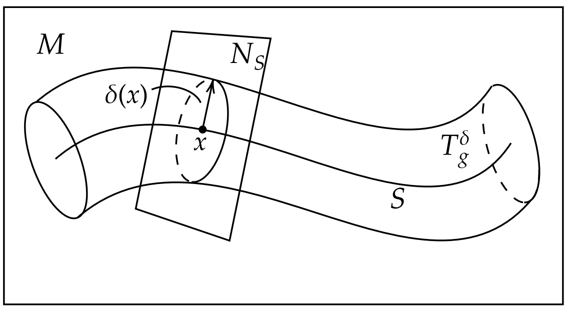


Figure 3.1: A tubular neighbourhood

The proof of the theorem shows that the tubular neighbourhood is the diffeomorphic image under the exponential map of a subset of the form

$$V_g^\delta = \{(x,v) \in N_S \mid |v|_g < \delta(x)\} \;\mapsto\; T_g^\delta = \{x \in M \mid \mathrm{dist}_g(x,S) < \delta(x)\} \tag{3.4}$$

for some positive smooth function $\delta : S \to \mathbb{R}$, called the radius, which can be chosen as small as desired. Note that V_g^δ is diffeomorphic to N_S. In this construction, the fiber over $x \in S$ is

$$T_g^\delta(x) = \{z \in M \mid \mathrm{dist}_g(z, S) = d_g(z, x) < \delta(x)\} \tag{3.5}$$

We can now explain the relation between the Poincaré dual and Thom class. Let S be an oriented submanifold of an oriented manifold M. If $j : T \hookrightarrow M$ is the inclusion of a tubular neighborhood of S, we have a Thom class $U \in H^{n-k}_{cv}(T)$ by identifying T with the normal bundle of S. We can define a map $j_* : H^*_{cv}(T) \to H^*(M)$ which extends by zero, because forms that are compactly supported along the fiber go to zero near the boundary of T.

Theorem 3.6 (Poincaré dual as a Thom class)**.** *The Poincaré dual η_S of S is equal to the Thom class $U \in H^{n-k}_{cv}(T)$ of the tubular neighbourhood of S. That is,*

$$\int_M \omega \wedge j_* U = \int_S i^*\omega \qquad \text{for all } \omega \in H^k(M)$$

where U is defined by identifying T with the normal bundle.

***Proof*.** Let $\omega \in H^k(M)$. Consider ω as a form on T, since we are not concerned with its values outside this region, so we may view the inclusion $i : S \to T$ as the zero section. Note that $\pi : T \to S$ induces a deformation retraction of T onto S, so π and i are inverse isomorphisms in cohomology. This means ω differs from $\pi^* i^* \omega$ by an exact form: $\omega = \pi^* i^* \omega + d\tau$.

$$\begin{aligned}
\int_M \omega \wedge j_* U = \int_T \omega \wedge U &= \int_T (\pi^* i^* \omega + d\tau) \wedge U \\
&= \int_T (\pi^* i^* \omega) \wedge U && \text{(by Stokes' theorem)} \\
&= \int_S i^* \omega \wedge \pi_* U && \text{(by Proposition 2.4)} \\
&= \int_S i^* \omega && \text{(since } \pi_* U = 1)
\end{aligned}$$

□

In particular, this means that the Poincaré dual of the image of the zero section $M_0 \subset E$ of a vector bundle $E \to M$ is the equal to the Thom class of E. This is because the exact sequence

$$0 \longrightarrow TM_0 \longrightarrow TE|_{M_0} \longrightarrow E \longrightarrow 0$$

shows that the normal bundle of the zero section is E itself.

Remark 3.7 (Localisation principle)**.** Another observation is that the support of the Poincaré dual of S can be shrunk into any arbitrarily small tubular neighbourhood of S, by simply pulling back the Thom class of the normal bundle into the tubular neighbourhood.

To find the Poincaré dual of the zero set of an arbitrary section, or more generally the inverse image of a submanifold, we first need a result in intersection theory. It will be useful at this stage to review transversality in Appendix B before continuing.

3.3 Poincaré dual of submanifolds

Definition 3.8. Let S and L be oriented submanifolds of the oriented manifold M such that $\dim S + \dim L = \dim M$. Suppose S and L intersect transversely. Then for each $p \in L \cap S$, define $\epsilon(S, L, p) \in \{\pm 1\}$ via

$$\mathrm{or}(T_pS) \wedge \mathrm{or}(T_pL) = \epsilon(S, L, p)\,\mathrm{or}(T_pM)$$

If $S \cap L$ is finite, we can define the <u>intersection number</u> of L and S to be the integer

$$[S] \cdot [L] = \sum_{p \in S \cap L} \epsilon(S, L, p)$$

Remark 3.9. One way to guarantee that $S \cap L$ is finite is when $S \cap L$ is compact. Theorem B.4 applied to the inclusion $i : S \to M$ and L tells us that $S \cap L$ is a zero dimensional submanifold. If $S \cap L$ is not finite, there is a sequence $(x_n)_{n \in \mathbb{N}} \subset S \cap L$ such that $x_n \to x \in S \cap L$. So there is no neighbourhood of x in M whose restriction to $S \cap L$ is a point, contradicting that $S \cap L$ is a zero dimensional submanifold.

Theorem 3.10. *Suppose S and L are oriented submanifolds of the oriented manifold M, and S or L is compact. Assume $S \pitchfork L$ and $\dim L + \dim S = \dim M$. Then*

$$[S] \cdot [L] = \int_M \eta_S \wedge \eta_L = \int_L \eta_S$$

***Proof*.** Set $s = \dim S$, $l = \dim L$ and $n = s + l$. For any $p \in S \cap L$, there are submersions $f : U_p \to \mathbb{R}^l$ and $g : U_p \to \mathbb{R}^s$ from a neighbourhood of p such that $f^{-1}(0) = S \cap U_p$ and $g^{-1}(0) = L \cap U$. Then the joint function $\psi = (g, f) : U_p \to \mathbb{R}^n$ satisfies

$$S \cap U_p = \{x_{s+1} = \cdots = x_{s+l} = 0\}, \quad L \cap U_p = \{x_1 = \cdots = x_s = 0\},$$

Since $\dim M = l + s$ and $S \pitchfork L$, the tangent space is a direct sum $T_pM = T_pS \oplus T_pL$. Since f and g are also submersions, it follows that $D\psi|_p : T_pS \oplus T_pL \to \mathbb{R}^{s+l}$ is bijective. Hence, by the inverse function theorem, we can assume ψ is a diffeomorphism to its image.

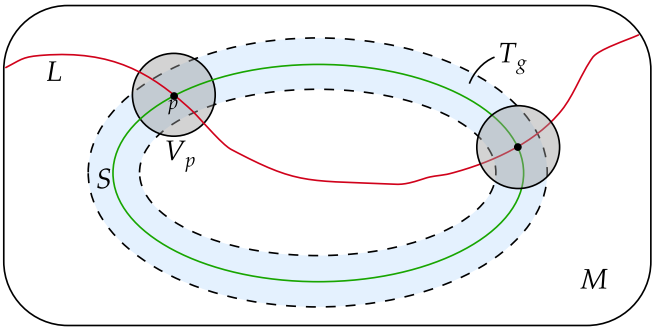


Figure 3.2: The fibers of the tubular neighbourhood of S at points of transverse intersection coincides with L

Our key idea is that we can choose neighbourhoods $V_p \subset U_p$ and a Riemannian metric g on M such that on V_p the metric is the Euclidean norm $(dx_1)^2 + \ldots + (dx_n)^2$. This means that we can choose a tubular neighbourhood $T_g \subset M$ of S such that the fibers coincide with the submanifold L, i.e. (see Figure 3.2)

$$T_g(p) = (L \cap V_p) \cap T_g$$

where we must choose T_g small enough so that the fiber of p is contained in V_p. Note we can assume V_p are disjoint, so that $L \cap T_g = \bigcup_{p \in S \cap L} T_g(p)$. Let $\eta_S \in \Omega^l(M)$ be a representative of the Poincaré dual of S with support on T_g.

The fiber $T_g(p)$ is equipped with the co-orientation of S in M (equation (3.3)). Denote $L_p := T_g(p)$ to be the same fiber but equipped with the orientation induced by L. Then we have the relation

$$\operatorname{or}(L_p) = \epsilon(S, L, p) \operatorname{or}(T_g(p))$$

Now observe that

$$\int_M \eta_S \wedge \eta_L = \int_L \eta_S = \sum_{p \in S \cap L} \int_{L_p} \eta_S = \sum_{p \in S \cap L} \epsilon(S, L, p) \int_{T_g(p)} \eta_S = \sum_{p \in S \cap L} \epsilon(S, L, p)$$

where $\int_{T_g(p)} \eta_S = 1$ by Theorem 3.6. $\square$

This gives a new perspective on the Euler number as the self intersection number of the diagonal $\Delta \subset M \times M$, i.e. $\int_\Delta \eta_\Delta = \int_M \chi(M)$. This follows from η_Δ being equal to the Thom class of N_Δ. Its integral on Δ is viewed as the integral on the zero section of N_Δ, and hence computes the Euler number of the normal bundle. It can be shown that the normal bundle N_Δ is isomorphic to the tangent bundle of Δ,[6, Lemma 11.23] and hence equals the Euler number of Δ. Note that Δ and M are diffeomorphic.

We are now ready to generalise Theorem 3.6 by finding the Poincaré dual of the zero set of an arbitrary section, or more generally the inverse image of a submanifold.

Suppose M and N are oriented manifolds of dimensions m and n, $f : N \to M$ is a smooth map, and $S \subset M$ is a submanifold of dimension k that is transverse to f. By Theorem B.4, $f^{-1}S \subset N$ is a submanifold of dimension $n - m + k$, assuming of course that $n + k \geq m$. The following lemma gives an orientation on the normal bundle $N_{f^{-1}(S)}$, which in turn gives a natural orientation on $f^{-1}(S)$ via equation (3.3).

Lemma 3.11. *There is a bundle isomorphism $Df : N_{f^{-1}(S)} \to f^* N_S$.*

***Proof*.** First observe that $f \pitchfork S$ if and only if $Df|_x : T_x N \to T_{f(x)} M$ maps surjectively to the quotient vector space $T_{f(x)}M/T_{f(x)}S = (N_S)_{f(x)}$ for all $x \in f^{-1}(S)$. Note that $Df|_x$ maps $T_x f^{-1}(S)$ into $T_{f(x)}S$ (this is true in general), and hence the kernel of the map $Df|_x : T_x N \to (N_S)_{f(x)}$ is $T_x f^{-1}(S)$. The kernel cannot be any larger due to dimension constraints from surjectivity. Therefore, by the first isomorphism theorem $Df|_x : (N_{f^{-1}(S)})_x \to (N_S)_{f(x)}$ is a linear isomorphism. Consequently, $Df : N_{f^{-1}(S)} \to f^* N_S$ is a bundle isomorphism. □

Theorem 3.12. *Let $f : N \to M$ and $S \subset M$ as above such that $f^{-1}(S)$ is compact. Then $f^*\eta_S = \eta_{f^{-1}(S)}$, i.e.*

$$\int_N \omega \wedge f^*\eta_S = \int_{f^{-1}(S)} i^*\omega \qquad \textit{for all } \omega \in H^{n-m+k}(N)$$

where $i : f^{-1}(S) \to N$ is the inclusion map.

***Proof*.** Fix Riemannian metrics g on N and h on M. Let $T_g^\epsilon \subset N$ and $T_h^\delta \subset M$ be tubular neighbourhoods of $f^{-1}(S)$ in N and S in M, of the form in equation (3.4). Since $f^{-1}(S)$ is compact, assume ϵ is constant. Denote by $T_g^\epsilon(x)$ the fiber over $x \in f^{-1}(S)$, defined in equation (3.5). We claim that there exists $\epsilon > 0$ such that the restriction of f to $T_g^\epsilon(x)$ is diffeomorphism to its image for all $x \in f^{-1}(S)$.

> *Proof of claim.* By Lemma 3.11, $Df|_x : (N_{f^{-1}(S)})_x \to (N_S)_{f(x)}$ is an isomorphism. By identifying the fiber $(N_{f^{-1}(S)})_x$ as the tangent space to the fiber of the tubular neighbourhood, we can write $Df|_x : T_x T_g^\epsilon(x) \to T_{f(x)} T_h^\delta(f(x))$. This can be interpreted as the differential of the restriction to $T_g^\epsilon(x)$. So

by the inverse function theorem, for each $x \in f^{-1}(S)$, there exists a neighbourhood $U_x \subset N$ of x and $\epsilon(x) > 0$ such that the restriction to $T_g^{\epsilon(x)}(x)$ is a diffeomorphism to its image. Since $f^{-1}(S)$ is compact, there is a finite subcover $U_{x_1}, \ldots, U_{x_r}$ and define $\epsilon = \min\{\epsilon(x_1), \ldots, \epsilon(x_r)\}$.

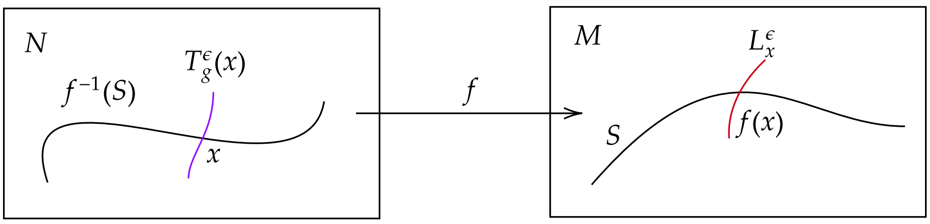


Figure 3.3: The diffeomorphism $f : T_g^\epsilon(x) \to L_x^\epsilon$ mapping the normal fiber

For such an ϵ, denote the image of the embedding $L_x^\epsilon := f(T_g^\epsilon(x))$.

Choose a form η_S representing the Poincaré dual with support in T_h^δ. The integral of $f^*\eta_S$ along the fiber $T_g^\epsilon(x)$ is

$$\int_{T_g^\epsilon(x)} f^*\eta_S = \int_{L_x^\epsilon} \eta_S = [S] \cdot [L_x^\epsilon] = 1$$

The first equality applies via the diffeomorphism $f : T_g^\epsilon(x) \to L_x^\delta$. The second equality follows from Theorem 3.10, since L_x^ϵ is compact.

Therefore, $f^*\eta_S$ is the pullback of Thom class of the normal bundle over $f^{-1}(S)$, so it is equal to the Poincaré dual of $f^{-1}(S)$ by Theorem 3.6. □

Corollary 3.13. *Let $E \xrightarrow{\pi} M$ be an oriented vector bundle of rank k over an oriented manifold of dimension n, and $s : M \to E$ be a section which is transverse to the zero section. If $n \geq k$ and $s^{-1}(0)$ is compact, then $s^*\Phi(E) = \eta_{s^{-1}(0)}$, i.e.*

$$\int_M \omega \wedge s^*\Phi(E) = \int_{s^{-1}(0)} i^*\omega \qquad \textit{for all } \omega \in H^{n-k}(M)$$

Proof. Denote $M_0 \subset E$ as the image of the zero section. By direct application of the previous theorem, we have $s^*\eta_{M_0} = \eta_{s^{-1}(0)}$. But we know from Theorem 3.6 that $\eta_{M_0} = \Phi(E)$, and the result follows. □

Note that $s^*\Phi(E) = \chi(E)$ is the Euler class of the vector bundle for any choice of section, as we have proved. This is remarkable, because the left side is independent of the section, but can be computed as the integral over the zero set of any transverse section.

The result above can be viewed as the generalisation of the Poincaré-Hopf theorem that we alluded to, when we take the vector bundle to be the tangent space and $\omega = 1$, in which case $s^{-1}(0)$ is zero dimensional. The orientation of each zero is based on whether the isomorphism $Ds|_x : T_xM \to E_x$ (see Lemma 3.11) reverses orientation or not.

There is an extension of the previous result for a section s which is not transverse to the zero section. Set $s = \dim s^{-1}(0)$. Choose a connection on E, which defines a linear map $\nabla_x s : T_xM \to E_x$ for each $x \in M$. Recall that when $s \pitchfork M_0$, there is a surjection $Ds : T_xM \to T_{s(x)}E/T_{s(x)}M_0$ from which we deduce that $n = k + s$. But now suppose instead that $s > n - k$, so the section is not transverse to M_0, but such that $s^{-1}(0)$ is a submanifold and the fibers of $\operatorname{coker}(\nabla s)$ defined by

$$0 \to \operatorname{Im}(\nabla_x s) \to E_x \to \operatorname{coker}(\nabla_x s) \to 0$$

have constant rank so that $\operatorname{coker} \nabla s$ is a vector bundle over $s^{-1}(0)$. The rank of $\operatorname{coker}(\nabla s)$ is $r := k - (n - s)$. Notice that $\nabla_x s = Ds|_x$ for all $x \in s^{-1}(0)$. Given orientations of M and E, $\operatorname{coker}(\nabla s)$ is canonically oriented.

Theorem 3.14. *Let $E \xrightarrow{\pi} M$ be an oriented vector bundle over an oriented manifold, and $s : M \to E$ be a section with the properties above such that $s^{-1}(0)$ is compact. Then for all $\omega \in H^{s-r}(M)$*

$$\int_M \omega \wedge \chi(E \to M) = \int_{s^{-1}(0)} i^*\omega \wedge \chi(\operatorname{coker}(\nabla s) \to s^{-1}(0))$$

Proof *(sketch).* In the following, E is understood to be a vector bundle over M, while $\operatorname{coker}(\nabla s)$ is a vector bundle over $s^{-1}(0)$. To uncover the relation between $\chi(E)$ and $\chi(\operatorname{coker}(\nabla s))$, pick a generic section $u : s^{-1}(0) \to \operatorname{coker}(\nabla s)$. Then by Corollary 3.13, the zero set $u^{-1}(0) \subset s^{-1}(0)$ is Poincaré dual to $\chi(\operatorname{coker}(\nabla s))$, i.e.

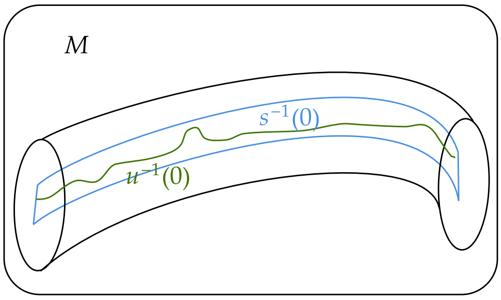
$$\int_{s^{-1}(0)} \omega \wedge \chi(\operatorname{coker}(\nabla s)) = \int_{u^{-1}(0)} \omega, \qquad \text{for all } \omega \in H^{s-r}(s^{-1}(0))$$

Since $u^{-1}(0) \subset s^{-1}(0) \subset M$, $u^{-1}(0)$ also determines a cohomology class of M. We claim that the Poincaré dual of $u^{-1}(0)$ in M is $\chi(E)$, i.e.

$$\int_M \omega \wedge \chi(E) = \int_{u^{-1}(0)} \omega, \qquad \text{for all } \omega \in H^{s-r}(M)$$

from which the theorem follows directly. To prove this, lift and extend u to $\overline{u} : M \to E$ such that $\overline{u}$ vanishes outside some tubular neighbourhood of $s^{-1}(0)$. The lift can be done by identifying $\operatorname{coker}(\nabla s) = (\operatorname{Im}\nabla s)^{\perp}$ via a metric on E.

The main idea is that the section $s + \epsilon\overline{u}$ is now transverse to $M_0 \subset E$ for all $\epsilon > 0$ at points $x \in u^{-1}(0)$. This is because $\nabla_x s : T_x M / T_x s^{-1}(0) \to \mathrm{Im}(\nabla_x s)$ is surjective, while $\nabla_x \overline{u} : T_x s^{-1}(0) \to \mathrm{Im}(\nabla_x s)^{\perp}$ is surjective. Hence, $\nabla_x(s + \epsilon\overline{u}) : T_x M \to E_x$ is surjective. Moreover, we extend u to the neighbourhood of $s^{-1}(0)$ in such a way that $s + \epsilon\overline{u}$ is transverse to the zero section for all $\epsilon > 0$. Although



$$\begin{tikzcd} & \mathrm{coker}(\nabla s) \arrow[d, shift right] & E \arrow[d, shift right] \\ u^{-1}(0) \arrow[r, hook] & s^{-1}(0) \arrow[r, hook] \arrow[u, shift right, "u"'] & M \arrow[u, shift right, "s"'] \end{tikzcd}$$

$s + \epsilon\overline{u}$ may have zeros outside of $u^{-1}(0)$ where the two functions cancel out, these values must become non-zero as $\epsilon \to 0$. This is because outside $s^{-1}(0)$, $\epsilon\overline{u} \to 0$ while s is non-zero, and on $s^{-1}(0) \setminus u^{-1}(0)$, $s = 0$ while $\epsilon\overline{u}$ is non-zero. Hence as $\epsilon \to 0$, the zero set approaches a neighbourhood of $u^{-1}(0)$.

Therefore, we have

$$\int_M \omega \wedge \chi(E) = \int_{(s+\epsilon\overline{u})^{-1}(0)} \omega \xrightarrow{\epsilon \to 0} \int_{u^{-1}(0)} \omega, \qquad \text{for all } \omega \in H^{s-r}(M)$$

The limit is defined because $u^{-1}(0)$ is compact. Also note $(s + \epsilon\overline{u})^{-1}(0)$ has dimension $n - k$, while $u^{-1}(0)$ has dimension $s - r$, which are equal because $r = k - (n - s)$. $\square$

Note that if s is a generic section then $\mathrm{coker}(\nabla s) = \{0\}$, which reduces to Corollary 3.13. This generalisation is due to Witten [34, Sec 3.3], which is the only paper I found where any justification of this is given, and the proof is my best attempt to make it precise.

Bibliographical notes

- The intersection theory in this chapter draws heavily from the notes by Nicolaescu [27], whose writing is very detailed and precise. The notes are intended to provide a complete proof of Theorem 3.12, which is claimed at the end of page 69 in Bott and Tu [6].

Chapter 4

Donaldson Invariants

In order to discuss the Atiyah and Jeffrey's interpretation of Witten's TQFT using the Mathai-Quillen form, we first need to take a brief detour to introduce the Donaldson polynomial invariants of smooth 4-manifolds. The details of the construction of the invariants are highly technical, which this chapter will not be concerned with, as that would be beyond the scope of the thesis.

4.1 Differential forms as normed spaces

We must first describe how to consider the space of differential forms as a normed space, because this will be central to the concepts of this chapter. Let $E \to M$ be vector bundle with a metric connection over an oriented Riemannian manifold. Denote $\nabla : \Omega^k(M, E) \to \Omega^{k+1}(M, E)$ to be the exterior covariant derivative. Using the metric on TM and E, this induces a metric on $\bigwedge^k(T^*M) \otimes E$. We denote the induced norm as $\|\cdot\|_g$ below, but later just as $|\cdot|$. The metric g on M also gives rise to the unique Riemannian volume form $\mathrm{d}V_g$ (see [20, Prop 2.41]).

Definition 4.1. For $1 \leq p < \infty$, and $s \in \Omega^l(M, E)$, define the $\underline{L^p \text{ norm}}$

$$\|s\|_{L^p} = \left(\int_M \|s(x)\|_g^p \,\mathrm{d}V_g\right)^{1/p}$$

and the <u>Sobolev norm</u>

$$\|s\|_{L^p_k} = \left(\sum_{j=0}^{k} \|\nabla^j s\|_{L^p}^p\right)^{1/p} = \left(\int_M \|s(x)\|_g^p + \|\nabla s(x)\|_g^p + \cdots + \|\nabla^k s(x)\|_g^p \,\mathrm{d}V_g\right)^{1/p}$$

In this definition the Sobolev metric depends on the metric on $T^*M^{\otimes l} \otimes E$ as well as the connection on E. It can be shown that the Sobolev norm is up to equivalence, independent of the choices of metrics and connections [11, Lemma 11.22].

Definition 4.2. Let $1 \leq p < \infty$ and $k \in \mathbb{Z}_{\geq 0}$. The Sobolev space $\Omega^l_{L^p_k}(M, E)$ of L^p_k-sections is the completion of $\Omega^l(M, E)$ in the Sobolev norm $\|\cdot\|_{L^p_k}$.

We are often interested in Lie algebra valued differential forms $\Omega(P, \mathfrak{g})$ on a principal bundle , so we need to define a metric on both P and $\mathfrak{g}$. The metric on T_pP can be obtained by identifying the horizontal subspace induced by the connection with $T_{\pi(p)}M$, and the vertical subspace with $\mathfrak{g}$. Assuming G is compact, we can always construct a metric on $\mathfrak{g}$ as follows.

Theorem 4.3. *Let G be a compact Lie group, with a representation $\rho : G \to \mathrm{GL}(V)$. Then there exits an invariant inner product on V, i.e. $\langle \rho(g)v, \rho(g)w \rangle = \langle v, w \rangle$ for all $g \in G$. Consequently, ρ is an orthogonal rep, and the induced rep ρ_* on $\mathfrak{g}$ is skew-symmetric valued.*

The construction is based on choosing an arbitrary inner product on V, and defining $\langle v, w \rangle := \int_G \langle \rho(A)w, \rho(A)w \rangle_V \, \mathrm{d}vol(A)$, where $\mathrm{d}vol(A)$ is a right-invariant differential form on G (we can also construct the integral based on the right-invariant Haar measure on G). In particular, this gives an Ad-invariant metric on $\mathfrak{g}$. This also gives an induced metric on $\mathrm{ad}\, P = P \times_{\mathrm{Ad}} \mathfrak{g}$, which is well defined because the metric on both P and $\mathfrak{g}$ are invariant under the action of G.

4.2 Yang-Mills theory

Donaldson's polynomial invariants is based on the study of the moduli space of gauge equivalent solutions to the Yang-Mills equations on a 4-dimensional manifold. The solutions can be easily described using the Hodge star operator, which we define below.

Definition 4.4. Let (M, g) be an oriented Riemannian n-manifold. The Hodge star operator $\star : \Omega^k(M) \to \Omega^{n-k}(M)$ is the unique smooth bundle homomorphism satisfying

$$\omega \wedge \star\eta = \langle \omega, \eta \rangle_g \, dV_g \tag{4.1}$$

where $\langle \cdot, \cdot \rangle_g$ is the induced metric on $\bigwedge^k(T^*M)$. In orthonormal coordinates, this is given by

$$\star(dx_{i_1} \wedge \cdots \wedge dx_{i_k}) = \mathrm{sgn}(I) dx_{i_{k+1}} \wedge \cdots \wedge dx_{i_n}$$

where $dx_1 \wedge \cdots \wedge dx_n = dV_g$ on some open subset.

The Hodge star extends to vector valued forms by only acting on the $\bigwedge^k(T^*M)$ part. We are interested in the Hodge star acting on 2-forms on a 4-manifolds in particular, since $\star : \Omega^2(M, \mathfrak{g}) \to \Omega^2(M, \mathfrak{g})$ is a linear operator with $\star^2 = 1$. The two eigenvalues 1 and -1 allow us to decompose into its eigenspaces $\Omega^2(M, \mathfrak{g}) = \Omega^{2,+}(M, \mathfrak{g}) \oplus \Omega^{2,-}(M, \mathfrak{g})$, called self-dual (SD) and anti-self-dual (ASD) two-forms respectively.

4.2.1 General definition

Let $P \to M$ be a principal G-bundle over an n-dimensional Riemannian manifold M, with connection A and associated curvature F. Assume G is a compact Lie group, though we will typically choose $G = \mathrm{SU}(2)$ or $G = \mathrm{SO}(3)$. Since the curvature form of a principal bundle is horizontal and Ad-equivariant, we can interpret it as an ad(P)-valued form on the base $F \in \Omega(M, \operatorname{ad} P)$. The Yang-Mills functional is defined by the L^2 norm squared of the curvature

$$S_{YM}(A) := \|F\|_{L^2}^2 = \int_M |F|^2 \, dV_g$$

where $|F|^2$ comes from the metric on $\bigwedge^2 T^*M \otimes \operatorname{ad} P$. The significance in physics is that this represents the action of a free field, generalising electromagnetism for non-abelian gauge groups.[3, p. 277]

The principal of least action in physics dictates that the classical solutions are the connections that satisfy the Euler-Lagrange equations of this action functional, that is, locally minimise S_{YM}. Given a perturbation $A+ta$ of a connection A in the affine space $\Omega^1(M, \operatorname{ad} P)$, the curvature is related by

$$F_{A+ta} = d_A(A+ta) + (A+ta) \wedge (A+ta) = F_A + t d_A a + t^2 a \wedge a$$

Therefore,

$$\begin{aligned}
&\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} S_{YM}(A+ta) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \int_M \left|F_A + t d_A a + t^2 a \wedge a\right|^2 \\
&= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \int_M \left(|F_A|^2 + 2t \langle F_A, d_A a\rangle + 2t^2 \langle F_A, a \wedge a\rangle + t^4 |a \wedge a|^2\right) dV_g \\
&= 2 \int_M \langle F_A, d_A a\rangle \, dV_g = 2 \int_M \langle d_A^* F_A, a\rangle \, dV_g
\end{aligned}$$

where $d_A^* : \Omega^k(M, \operatorname{ad} P) \to \Omega^{k-1}(M, \operatorname{ad} P)$ is the formal adjoint of d_A using the L^2 inner product on $\Omega^k(M, \operatorname{ad} P)$, defined by $\langle d_A s, t\rangle_{L^2} = \langle s, d_A^* t\rangle_{L^2}$. It can be shown that its explicit form is $d_A^* = (-1)^{n(k+1)-1} \star d_A \star$. The equation above shows that the connection A is a critical point if and only if $d_A^* F_A = 0$, or $d_A \star F_A = 0$. A connection satisfying this condition is called a Yang-Mills connection. If $\star F_A = \pm F_A$, then by the Bianchi identity it is a YM connection.

4.2.2 Orthogonal or unitary structure group

In this subsection we assume $G = \mathrm{SO}(N)$ or $G = \mathrm{SU}(N)$, which allows us to make a number of simplifications. We define the trace wedge for matrix Lie algebra valued forms $\omega, \eta \in \Omega(M, \mathfrak{g})$ by

$$\operatorname{Tr}(\omega \wedge \eta) = \sum_{j,k} \omega_j \wedge \eta_k \operatorname{Tr}(X_j X_k) \tag{4.2}$$

For an arbitrary Lie algebra valued form, equation (4.1) will not hold. But for the case where $G = \mathrm{SO}(N)$ or $\mathrm{SU}(N)$, we can choose the Ad-invariant Hilbert-Schmidt metric on $\mathfrak{g}$ defined by $\langle X, Y \rangle = \operatorname{Tr}(X^* Y)$ where X^* is the adjoint. In this case, if $\alpha, \beta \in \Omega^k(M, \mathfrak{g})$, we have

$$-\operatorname{Tr}(\alpha \wedge \star\beta) = \langle \alpha, \beta \rangle \, dV_g \tag{4.3}$$

which follows from expanding using equation (4.2), applying the property property $\operatorname{Tr}(X^* Y) = -\operatorname{Tr}(XY)$ for $X, Y \in \mathfrak{g}$, and finally using equation (4.1). Hence the Yang-Mills functional can be rewritten using equation (4.3) as

$$S_{YM}(A) = -\int_M \operatorname{Tr}(F \wedge \star F)$$

Recall from Chern-Weil theory that there are two important classes of $\operatorname{Ad} \mathrm{GL}(r, \mathbb{C})$ invariant polynomials on $\mathfrak{gl}(r, \mathbb{C})$. The first is the coefficients $f_k(X)$ of λ^{r-k} in the characteristic polynomial $\det(\lambda I + X)$. Another class consists of the trace polynomials $\operatorname{Tr}(X^k)$. The Chern classes of P are defined by $c_k(P) = [f_k\left(\frac{i}{2\pi} F\right)] \in H^{2k}(M)$ and is independent of the choice of connection. Their importance is due to the following theorem:

Theorem 4.5 ([14, Theorem E.5]). *The second Chern class $c_2(E) \in H^4(M, \mathbb{Z})$ classifies up to isomorphism $\mathrm{SU}(2)$-bundles over any compact connected oriented four-manifold M.*

From Newton's identity for symmetric polynomials with $k = 2$ (see [31, Theorem B.2]), we have $\operatorname{Tr}(X^2) - f_1(X)\operatorname{Tr}(X) + 2f_2(X) = 0$, giving the following relation for any complex vector bundle

$$\left[\frac{1}{8\pi^2} \operatorname{Tr}(F^2)\right] = c_2(E) - \frac{1}{2} c_1(E)^2 \in H^4(M)$$

where we have used the fact that $f_1(X) = \operatorname{Tr}(X)$. In the case where the structure group is $\mathrm{SU}(n)$ or $\mathrm{SO}(n)$, the trace of the curvature is zero, so $c_1(E) = 0$ is trivial and we can define

$$k := c_2(E)[M] = \frac{1}{8\pi^2}\int_M \operatorname{Tr}(F^2) \tag{4.4}$$

Now, we can decompose the field strength $F = F^+ + F^-$ into SD and ASD parts. Using the fact that $\Omega^{2,+}$ and $\Omega^{2,-}$ are orthogonal, $\operatorname{Tr}(X^*Y) = -\operatorname{Tr}(XY)$ and the properties of the hodge star, this gives

$$\begin{aligned} k = \frac{1}{8\pi^2}\int_M \operatorname{Tr}(F^2) &= \frac{1}{8\pi^2}\int_M (\operatorname{Tr}(F^+\wedge F^+) + \operatorname{Tr}(F^-\wedge F^-)) \\ &= \frac{1}{8\pi^2}\int_M (-|F^+|^2 + |F^-|^2) dV_g \end{aligned} \tag{4.5}$$

Comparing with the Yang-Mills action,

$$S_{YM}(A) = \int_M (|F^+|^2 + |F^-|^2) dV_g = \begin{cases} 8\pi^2 k + 2\int_M |F^+|^2 dV_g \\ -8\pi^2 k + 2\int_M |F^-|^2 dV_g \end{cases}$$

We see that for $k > 0$, the action is bounded below by $S(A) \geq 8\pi^2 k$ and the ASD connections $F^+ = 0$ are minimisers. For $k < 0$ the action is bounded below by $S(A) \geq -8\pi^2 k$, and minimised by SD connections $F^- = 0$. Thus the classical equations of motion are equivalent to $\star F = \pm F$, whose solutions are called (anti-)instantons.

4.3 Space of connections

Proposition 4.6. *Let $E_1, \ldots, E_k$ and F be vector bundles over a manifold M. There is a bijection*

$$\left\{ \begin{matrix} C^\infty\text{-multilinear maps} \\ T : \Gamma(E_1)\times\cdots\times\Gamma(E_k) \to \Gamma(F) \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \text{sections} \\ \Gamma(M, E_1^*\otimes\cdots\otimes E_k^*\otimes F) \end{matrix} \right\}$$

Proposition 4.7. *If $\nabla^A, \nabla^B : \mathcal{X}(M)\times\Gamma(E) \to \Gamma(E)$ are two connections on a vector bundle $E \to M$, then*

$$\nabla^A - \nabla^B \in \Omega^1(X, \operatorname{End} E)$$

Conversely, given $a \in \Omega^1(X, \operatorname{End} E) \simeq \operatorname{Hom}(T^*M \otimes E, E)$, then $\nabla^A + a$ interpreted as

$$(\nabla^A + a)_X(s) = \nabla^A_X s + a(X\otimes s)$$

is again a connection on E. This proposition is summarised by the statement: the space of connections is an affine space modelled on $\Omega^1(X, \operatorname{End} E)$.

In one direction, it is easy to show $\nabla^A - \nabla^B$ is tensorial (meaning $C^\infty(M)$ linear in both components), and apply the previous proposition. In the other direction, we only need to verify the Leibniz rule. Thus, a choice of reference connection on E defines a bijection between the space of all connections and $\Omega^1(M, \operatorname{End} E)$. There is a similar result for principal bundles.

Proposition 4.8. *The space of connections $\mathcal{A}$ on a principal G-bundle $P \to M$ is an affine space modelled on the vector space $\Omega^1(M, \operatorname{ad} P) \simeq \Omega^1(M, P \times_{\mathrm{Ad}} \mathfrak{g})$.*

This follows from the fact that the difference of two connections $\theta_1 - \theta_2 \in \Omega^1(P, \mathfrak{g})$ is a horizontal and Ad-equivariant form. Finally we identify this space as $\Omega^1(M, P \times_{\mathrm{Ad}} \mathfrak{g})$, by the isomorphism between these spaces (see [31, Theorem 31.9]). Conversely, the sum of θ_1 with a horizontal and Ad-equivariant form is again a connection on P.

Given a real or complex vector bundle E of rank n, we can always construct its frame bundle with structure group $\mathrm{GL}(n)$ (or $\mathrm{GL}(n, \mathbb{C})$). Additional algebraic structure on E yields a principal bundle with smaller structure group. For example, if E is a complex vector bundle with a Hermitian metric, we get a principal $\mathrm{U}(n)$-bundle of orthonormal frames in E. In these cases, we can identify the space of connections on E as $\Omega^1(M, \mathfrak{g}_E)$, where the restriction to $\mathfrak{g}_E \subset \operatorname{End} E$ ensures that the new connection is compatible with the structure group of E. If the structure group is $\mathrm{SU}(2)$, for example, then $\mathfrak{g}_E$ consists of skew-adjoint, trace-free endomorphisms of the rank two vector bundle E. The point is, we may view connections on principal bundles as a generalisation of the vector bundle case, so we work with the former.

Proposition 4.9. *To every connection $\omega \in \Omega^1(M, \operatorname{ad} P)$ on a principal G-bundle $P \to M$, there is an associated vector bundle connection d_ω on $\operatorname{ad} P \to M$, where $\operatorname{ad} P$ acts on itself by the fiberwise Lie bracket. Assume G is a matrix Lie group.*

***Proof*.** Let $v \in \mathcal{X}(M)$ and $\lambda \otimes X \in \Gamma(\operatorname{ad} P)$, where $\lambda \in C^\infty(M), X \in \operatorname{ad} P$. We define the connection by $d_\omega(v, \lambda \otimes X) = D\lambda(v) \otimes X + \lambda[\omega(v), X]_{\operatorname{ad} P}$. Here the Lie bracket on elements of $\operatorname{ad} P$ is defined by

$$[[p, A], [p, B]]_{\operatorname{ad} P} := [p, [A, B]_{\mathfrak{g}}]$$

Then d_ω is C^∞-linear in v, and linear in λ. The fiberwise Lie bracket is well defined because

$$[p \cdot g, [g^{-1}Ag, g^{-1}Bg]] = [p \cdot g, g^{-1}[A, B]g] = [p, gg^{-1}[A, B]gg^{-1}] = [p, [A, B]]$$

Finally, d_ω satisfies the Leibniz rule by construction. □

Definition 4.10. The gauge group $\mathcal{G}$ of a vector bundle $E \to M$ is the group of all vector bundle automorphisms $\operatorname{Aut}(E)$. Similarly, the gauge group of a principal bundle $P \to M$ is the group of principal bundle automorphisms $\operatorname{Aut}(P)$.

Proposition 4.11. *The group of gauge transformations $\mathcal{G} = \operatorname{Aut}(P)$ is isomorphic to sections $\Omega^0_{\operatorname{Ad}}(P, G) \simeq \Omega^0(M, \operatorname{Ad} P)$ under fiber-wise multiplication*

***Proof*.** A gauge transformation $f : P \to P$ preserves the fibers of P and satisfies $f(p \cdot g) = f(p) \cdot g$. For a fixed $p \in P$, we have $f(p) = p \cdot \psi(p)$ for some $\psi(p) \in G$ because the action is transitive. Then f acts by multiplication by $\psi(p)$ on the whole fiber. Hence, $f(p) = p \cdot \psi(p)$ for a smooth function $\psi : P \to G$.

Substituting this into $f(p \cdot g) = f(p) \cdot g$, we find $\psi(p \cdot g) = g^{-1}\psi(p)g$ since the action is free. So $R_g^*\psi = \operatorname{Ad}_{g^{-1}} \psi$. Hence, $\psi \in \Omega^0_{\operatorname{Ad}}(P, G) \simeq \Omega^0(M, P \times_{\operatorname{Ad}} G)$. This isomorphism can be proved in the same way as the associated vector bundle case. Finally, note that composition of automorphisms corresponds to multiplication of ψ in the fibers of groups. $\square$

The action of $f \in \operatorname{Aut}(P)$ on a connection $\omega \in \Omega^1(P, \mathfrak{g})$ is $f^*\omega \in \Omega^1(P, \mathfrak{g})$. Consequently, the action of $\Omega^0(M, \operatorname{Ad} P)$ on $\Omega^1(M, \operatorname{ad} P)$ is still denoted the same way.

The purpose of Propositions 4.11 and 4.8 is two-fold: this description allows us to define Sobolev completions of these spaces; the other purpose is to identify $\mathcal{G}$ as an infinite dimensional Lie group with Lie algebra $\Omega^0(M, \operatorname{ad} P)$. The gauge group $\mathcal{G}$ has the fiberwise exponential map $\exp : \Omega^0(M, \operatorname{ad} P) \to \Omega^0(M, \operatorname{Ad} P)$ which assigns to any section $\sigma \in \Gamma(\operatorname{ad} P)$ the section $s(x) = \exp(\sigma(x))$. To see that this gives a well defined map, we use the fact that $\exp : \mathfrak{g} \to G$ satisfies $\exp(g^{-1}Ag) = g^{-1}\exp(A)g$. The Lie bracket on $\Omega^0(M, \operatorname{ad} P)$ is defined by the fiberwise bracket, which we have shown is well defined.

4.4 Moduli space

Before defining the Donaldson invariants, we study the structure of the moduli space, which will allow us to understand the invariants better. However, the details rely on elliptic theory and is quite involved, so we only sketch the arguments.

The first step is to form the quotient space $\mathcal{A}/\mathcal{G}$ of gauge equivalent connections. Singularities in this quotient space are characterised by reducible connections, as defined below.

Definition 4.12. A connection $\omega \in \Omega(P, \mathfrak{g})$ on a principal G-bundle $P \to M$ is reducible if the gauge group $\mathcal{G}$ modulo its center does not act freely on the connection ω.

For the group $G = \mathrm{SU}(2)$, the center of $\mathcal{G} \simeq \Gamma(P \times_{\mathrm{Ad}} \mathrm{SU}(2))$ is $Z := \Gamma(P \times_{\mathrm{Ad}} \mathbb{Z}_2)$, since $\mathbb{Z}_2$ is the center of $\mathrm{SU}(2)$.

Theorem 4.13 ([14, Theorem 3.1]). *Suppose ω is a connection on the principal $\mathrm{SU}(2)$-bundle $P \to M$ which is not flat. Then the following are equivalent:*

(a) *ω is reducible, i.e. $\mathcal{G}_\omega / Z \neq 1$, where $\mathcal{G}_\omega$ is the stabiliser of ω*
(b) $\mathcal{G}_\omega / Z \simeq U(1)$
(c) *$\nabla : \Omega^0(\mathrm{ad}\, P) \to \Omega^1(\mathrm{ad}\, P)$ has a nonzero kernel, where ∇ is the induced covariant derivative*
(d) *For any associated bundle $\eta = P \times_G V$ and induced connection D, the bundle $\eta = \eta_1 \oplus \eta_2$ and the connection $D = d_1 \oplus d_2$ both split.*

Let us denote by $\mathcal{A}^*$ the space of irreducible connections on a principal bundle, and $\mathcal{B}^* := \mathcal{A}^*/\mathcal{G}$ the topological quotient by the gauge group. The classical equation of motion $\star F = -F$ is a non-linear differential equation for non-abelian gauge groups, and defines a subspace of the infinite dimensional space of connections $\mathcal{A}$. The key property is that ASD connections are preserved by the action of the gauge group. Therefore, ASD connections are a subspace of $\mathcal{B}^*$, which we call the moduli space $\mathcal{M}^*$.

Remark 4.14. If we reverse the orientation of M, then this swaps the SD and ASD forms in $\Omega^2(M, \mathrm{ad}\, P)$. Since the two theories are completely equivalent, we could work with SD connections. However, there is an important class of 4-manifolds which have a natural orientation - complex manifolds. Over this orientation, it turns out that ASD connections are associated to holomorphic objects, while SD connections to anti-holomorphic objects.[24, p.95] For this reason, we choose to work in terms of ASD connections by default.

By working with Sobolev completion spaces, $\mathcal{M}^*$ can be shown to have a manifold structure by application of the slice theorem. Denote

- the vector space of L^2_l $\mathrm{ad}\, P$-valued forms by $\Omega^*_l(M, \mathrm{ad}\, P)$.
- the space of L^2_l-connections by $\mathcal{A}_l(P)$ which is modelled on $\Omega^1_l(M, \mathrm{ad}\, P)$.
- the group of L^2_l-gauge transformations by $\mathcal{G}_l(P) = \Omega^0_l(M, \mathrm{ad}\, P)$

The Sobolev spaces of sections are Banach manifolds, and the L^2_k spaces are in fact Hilbert manifolds with the inner product defined similarly to the L^2 norm.

Theorem 4.15 ([24, Section 4.5])**.** *The space of gauge equivalent irreducible connections on a principal* $\mathrm{SU}(2)$*-bundle* $\mathcal{B}_2^*(P) := \mathcal{A}_2^*(P)/\mathcal{G}_3(P)$ *is a Hilbert manifold.*

To show that the moduli space has a manifold structure, there is more work to do. Let us clarify one point straight away: the reason we can analyse the ASD subspace of $\mathcal{B}_2^*$ in place of smooth connections is that for any L_l^2 connection A, there is a L_{l+1}^2 gauge transformation u such that $u(A)$ is a smooth connection.[12, Prop 4.2.16] Therefore, the moduli spaces $\mathcal{M}^*$ and $\mathcal{M}_l^*$ are actually the same. Since we will not be dealing with the analysis related to the properties of the Sobolev spaces, from this point forward we will drop the subscripts. We now establish a couple of basic properties of the curvature operator and gauge group actions, that will be important in the next chapter.

Proposition 4.16 ([14, p.54])**.** *Fix a reference connection* ω_0 *to identify* $\mathcal{A}$ *with the affine space* $\Omega^1(M, \operatorname{ad} P)$*. The curvature operator* $F : \Omega^1(M, \operatorname{ad} P) \to \Omega^2(M, \operatorname{ad} P)$ *is smooth, and its differential at* ω *is the covariant derivative*

$$d_\omega : \Omega^1(M, \operatorname{ad} P) \to \Omega^2(M, \operatorname{ad} P)$$

This is not hard to see, because at $\omega + tA$,

$$F(\omega + tA) = F(\omega) + t d_\omega A + t^2 A \wedge A$$

since the exterior derivative d on $\Omega(P, \mathfrak{g})$ corresponds to the covariant derivative d_ω on $\Omega(M, P \times_{\mathrm{Ad}} \mathfrak{g})$.

Proposition 4.17. *Fix* $\omega \in \mathcal{A}$ *as a reference connection. The derivative of the gauge group action* $L_\omega : \mathcal{G} \to \mathcal{A}$*, where* $L_\omega(f) = f^*\omega$*, is given by*

$$DL_\omega|_{\mathrm{id}}(\phi) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} L_\omega(\exp(t\phi)) = d_\omega \phi$$

where $d_\omega : \Omega^0(M, \operatorname{ad} P) \to \Omega^1(M, \operatorname{ad} P)$ *is the covariant derivative.*

***Proof*.** Viewing ω as a connection on $\operatorname{ad} P \to M$, consider the connection 1-form $A_\alpha \in \Omega^1(U_\alpha, \mathfrak{gl}(\mathfrak{g}))$ in a local trivialisation for $\operatorname{ad} P$. Any $u \in \Omega^0(M, \operatorname{Ad} P)$ acts as a gauge transformation of $\operatorname{ad} P$, and the connection 1-form transforms as

$$u_\alpha^* A_\alpha = (du_\alpha)u_\alpha^{-1} + u_\alpha^{-1} A_\alpha u_\alpha$$

where $u_\alpha : U_\alpha \to \mathrm{GL}(\mathfrak{g})$ is a matrix valued function under the trivialisation. Therefore, the derivative of the gauge action at $\phi \in \Omega^0(M, \operatorname{ad} P)$ is

$$\begin{aligned}
\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (\exp(t\phi)^* A_\alpha) &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (d(\exp(t\phi)) \exp(-t\phi) + \exp(-t\phi) A_\alpha \exp(t\phi)) \\
&= d\phi - \phi A_\alpha + A_\alpha \phi \\
&= d\phi + [A_\alpha, \phi]
\end{aligned}$$

which is precisely the covariant derivative $d_\omega \phi$ under the trivialisation. □

The strategy to work out the local structure of $\mathcal{M}^*$ is to apply an infinite dimensional version of the implicit function theorem. Roughly, it states that the zero set of a smooth Fredholm map between Hilbert manifolds is locally isomorphic to a subset defined by a smooth map between finite dimensional manifolds (see [24, Lemma 5.2.1] for the precise statement).

If d_ω^* is the formal adjoint of $d_\omega : \Omega^k(M, \operatorname{ad} P) \to \Omega^{k+1}(M, \operatorname{ad} P)$, then it can be shown that $d_\omega^* \circ d_\omega$ is an elliptic operator. The structure of the moduli space is unraveled by studying what is called the fundamental elliptic complex $\mathcal{E}_\omega$ associated with an ASD connection ω:

$$0 \longrightarrow \Omega^0(M, \operatorname{ad} P) \xrightarrow{d_\omega} \Omega^1(M, \operatorname{ad} P) \xrightarrow{d_\omega^+} \Omega^{2,+}(M, \operatorname{ad} P)$$

where d_ω^+ denotes the covariant derivative folowed by projection to the self-dual subspace. The generalised Hodge decomposition guarantees that the cohomology groups of this complex $H^*(\mathcal{E}_\omega)$ are finite dimensional. The first linear operator is the differential of the action of $\mathcal{G}$ on $\mathcal{A}$ by Proposition 4.17, thus its image is the tangent space to the $\mathcal{G}$-orbit through ω. The second operator is the differential of the self dual curvature map at ω by Proposition 4.16, thus its kernel is the formal tangent space of the ASD subspace of $\mathcal{A}$ at ω. Hence, the first cohomology $H^1(\mathcal{E}_\omega)$ is the formal tangent space to $\mathcal{M}$ at $[\omega]$.

The Hodge decomposition theorem also allows us to deduce that d_ω^+ restricted to $\ker d_\omega^*$ is a Fredholm operator, i.e. its kernel and cokernel are finite dimensional. Let $\omega \in \mathcal{A}^*$ be an irreducible ASD connection. Then by application of the implicit function theorem, one can show there is a mapping

$$\Psi : H^1(\mathcal{E}_\omega) \to H^2(\mathcal{E}_\omega)$$

such that $\Psi^{-1}(0)$ is identified with a neighbourhood of $[\omega] \in \mathcal{M}^*$. In the case where $H^2(\mathcal{E}_\omega)$ is zero, the moduli space is a smooth manifold near $[\omega]$ of dimension equal to $\dim H^1(\mathcal{E}_\omega)$. This can be computed via the Atiyah-Singer Index Theorem, which tells us that the dimension is $8c_2(P) - 3(b_0(M) - b_1(M) + b_2^+(M))$, where $b_i(M) = \dim H^i(M, \mathbb{R})$ and $b_2^+(M) = \dim H_+^2(M, \mathbb{R})$, i.e. the dimension of the maximal positive definite subspace of $H^2(M, \mathbb{R})$ under the cup product.

Unfortunately, the moduli space is not compact, so Donaldson constructed what is called the Uhlenbeck compactification $\overline{\mathcal{M}}(P)$ of the moduli space. Donaldson showed that sequences of ASD connections can have curvatures which blow up near a finite number of points, but elsewhere converge to an ASD connection. The compactification is constructed by adjoining these limit connections paired with the blowup points to the moduli space. After defining the appropriate topology, this becomes compact.

4.5 Donaldson invariants

Finally, our goal is to sketch the definition of the Donaldson polynomials invariants. In this section, assume M is a closed, oriented, simply connected smooth 4-manifold with a fixed orientation of $H_+^2(M; \mathbb{R})$ (which induces one of the moduli space), and $P \to M$ is a principal SU(2)-bundle with $c_2(P) > 0$. Define

$$d = 4c_2(P) - \frac{3}{2}(1 + b_2^+(M))$$

which is half of the formal dimension of the moduli space $\mathcal{M}^*(P)$. The Donaldson invariant associated to P is a symmetric multilinear function of degree d on $H_2(M; \mathbb{Z})$

$$D_{M,d} : H_2(M; \mathbb{Z})^{\otimes d} \to \mathbb{Q}$$

Very roughly, the idea behind the definition is as follows: The Donaldson μ map $\mu : H_2(M, \mathbb{Z}) \to H^2(\mathcal{M}^*(P), \mathbb{Z})$ associates homology classes in M to cohomology classes in the moduli space. This map can then be extended to take values in $H^2(\overline{\mathcal{M}}(P), \mathbb{Z})$, which we call $\overline{\mu}$. For $\gamma_1, \ldots, \gamma_d \in H_2(M, \mathbb{Z})$, we define the invariant to be the integral over the compactified moduli space

$$D_{M,d}(\gamma_1, \ldots, \gamma_d) = \int_{\overline{\mathcal{M}}(P)} \overline{\mu}(\gamma_1) \wedge \cdots \wedge \overline{\mu}(\gamma_d)$$

Hence the Donaldson invariant can be interpreted as an element in the polynomial algebra of $H^2(M, \mathbb{Z})$, i.e. dual to $H_2(M, \mathbb{Z})$. To be more precise, $\overline{\mathcal{M}}(P)$ is generally not a manifold, and the above refers to the pairing of the cup products of $\mu(\gamma_i)$ with the fundamental homology class $[\overline{\mathcal{M}}(P)]$. One finds that $\overline{\mathcal{M}}(P)$ admits a fundamental class only if k is in what is called the stable range of M: $k > \frac{3}{4}(1 + b_2^*(M))$. Refer to equation 4.5 for the definition of k.

The definition of the μ map relys on a certain SO(3)-bundle, which we now sketch. Consider the principal $\mathcal{G}$-bundle

$$\mathcal{A}^* \times P \to \mathcal{A}^* \times_{\mathcal{G}} P$$

where the action is given by $(\omega, p) \cdot f = (f^*\omega, f^{-1}(p))$. This action is free because the elements of $\mathcal{A}^*$ are irreducible: $(f^*\omega, f^{-1}(p)) = (\omega, p)$ implies f is a stabiliser of ω, and $f^{-1}(p) = p$ leaves only the possibility that $f = \mathrm{id}$.

There is a natural projection $\mathcal{A}^* \times_{\mathcal{G}} P \to \mathcal{B}^* \times M$ given by $[\omega, p] \mapsto ([\omega], \pi(p))$. The SU(2) action on $\mathcal{A}^* \times P$ given by multiplication on P descends to the quotient $\mathcal{A}^* \times_{\mathcal{G}} P$ because it commutes with the $\mathcal{G}$ action. But this action $[\omega, p] \cdot g = [\omega, p \cdot g]$ is not free because g is a stabiliser if and only if $g = \pm\,\mathrm{id} \in \mathrm{SU}(2)$. Thus, there is a free $\mathrm{SU}(2)/\{\pm\,\mathrm{id}\} \simeq \mathrm{SO}(3)$ action and we have a principal SO(3)-bundle

$$\xi := \mathcal{A}^* \times_{\mathcal{G}} P \to \mathcal{B}^* \times M$$

As an SO(3)-bundle, it can be shown that the first Pontryagin class $p_1(\xi) \in H^4(\mathcal{B}^* \times M, \mathbb{Z})$ is divisible by 4.[24, Lemma 7.2.1] Recall that the slant product in algebraic topology is a map

$$H^{p+q}(X \times Y) \times H_p(Y) \to H^q(X), \qquad (x, \alpha) \mapsto x/\alpha$$

Then we can define the map $H_2(M, \mathbb{Z}) \to H^2(\mathcal{B}^*, \mathbb{Z})$ given by $\gamma \mapsto -\frac{1}{4}p_1(\xi)/\gamma$. The reason for the $-1/4$ is that when an SO(3)-bundle $\mathcal{P}$ lifts to an SU(2)-bundle $\mathcal{P}'$, we have the relation $c_2(\mathcal{P}') = -\frac{1}{4}p_1(\mathcal{P})$. Finally, the μ map is defined as the restriction of $\mathcal{B}^*$ to the moduli space:

$$\mu : H_2(M, \mathbb{Z}) \to H^2(\mathcal{M}^*, \mathbb{Z}), \qquad \mu(\gamma) = -\frac{1}{4}p_1(\xi)/\gamma$$

The extension of this map to the Uhlenbeck compactification $\overline{\mathcal{M}}(P)$ requires the Taubes gluing procedure, for details refer to [24] or [12].

Bibliographical notes

- The local model for $\mathcal{M}$ obtained from the elliptic complex was first studied by Atiyah, Hitchin, and Singer [1].
- The geometry of the the moduli space, and the theory leading up to the Donaldson polynomial invariants are discussed in more detail in Morgan [24] and Donaldson and Kronheimer [12].
- A succint review of Donaldson theory and the invariants can be found in the conference paper by Naber [25].

Chapter 5

Donaldson-Witten theory

In finite dimensions, the Mathai-Quillen formula gives an explicit differential form representative of the Euler class. It was shown by Atiyah and Jeffrey [2] that not only can the zero dimensional Donaldson invariant can be identified with the Euler number of a vector bundle over $\mathcal{A}^*/\mathcal{G}$, but the Donaldson invariants in general can be written as an integral of a Mathai-Quillen type form over the gauge equivalence classes of irreducible connections, analogous to Corollary 3.13. Moreover, they have been able to reproduce Witten's action functional from twisted SUSY TQFT term by term from purely geometric considerations. Conversely, this formalism makes it possible to construct a TQFT starting from a moduli problem.

5.1 Thom form analogue for principal bundles

Before we construct the Atiyah-Jeffrey formula, we first need to develop an analogue of the Thom form for a principal G-bundle $P \to M$. Assume G is a connected compact Lie group with $\dim G = d, \dim M = n$. We wish to construct a differential form $W \in \Omega^d(P)$ such that its integral along the fiber equals 1, so that by Proposition 2.4, for all $\beta \in \Omega_c^B(M)$

$$\int_M \beta = \int_P \pi^*\beta \wedge W \tag{5.1}$$

Of course, we will need an orientation on M, G and P.

Denote by $\omega \in \Omega^1(P, \mathfrak{g})$ the connection on P induced by a metric on P so that the horizontal and vertical subspaces are orthogonal complements. Let $\eta_1, \ldots, \eta_d$ be an orthonormal basis for $\mathfrak{g}$ consistent with the orientation on G. We will show that

$$W := \int^B \exp(\omega) d\eta = \int^B \exp(\omega_a \eta_a) d\eta = \omega_1 \wedge \cdots \wedge \omega_d \ \in \Omega^d(P) \tag{5.2}$$

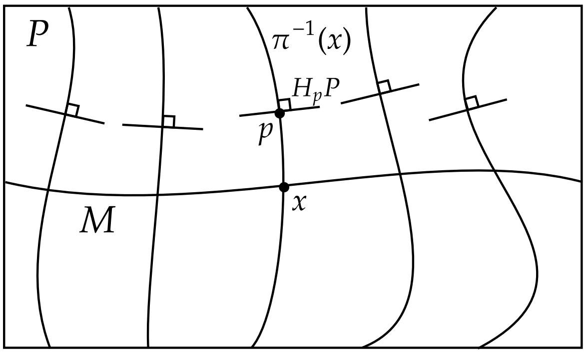


Figure 5.1: Horizontal subspaces defined by orthogonal complement

is the desired vertical volume form. Multiplication inside the Berezin integral occurs in the graded tensor product $\Omega(P) \otimes \mathfrak{g}$, so $\omega_a \eta_a$ all have degree two.

The integral along the G-orbit $\pi_* : \Omega^*(P \times V) \to \Omega^{*-d}(P \times_G V)$ is defined as in section 2.1, where the integral defined by pulling back to a form on G via a trivialisation $\psi : G \xrightarrow{\simeq} \pi^{-1}(x)$.

We will now turn G into Riemannian manifold by extending a metric on $\mathfrak{g}$ to T_gG via left translations. That is, define $\langle u, v \rangle_g = \langle DL_g^{-1}(u), DL_g^{-1}(v) \rangle_e$ where $u, v \in T_gG, L_g(h) = gh$. By construction, this is a left-invariant metric. In fact, because G is connected and compact we can turn it into a bi-invariant metric by Haar integrating the metric: $\langle u, v \rangle := \int_G \langle DR_g u, DR_g v \rangle$.

Proposition 5.1. *For all $p \in P$, $L_p^* W \in \Omega^d(G)$ is equal to the unique Riemannian volume form induced by the bi-invariant metric on G.*

Proof. Let $\epsilon_1, \ldots, \epsilon_d \in T_e^* G$ be the orthonormal coframe dual to $\eta_1, \ldots, \eta_d$. At a point $g \in G$, the tangent vectors are transported to $DL_g \eta_i$ while the coframe is transported to $\epsilon_i \circ DL_g^{-1} \in T_g^* G$, where L_g is left multiplication by g. Then the unique Riemannian volume form is given by

$$\epsilon_1 \circ DL_g^{-1} \wedge \cdots \wedge \epsilon_d \circ DL_g^{-1}$$

The covectors are orthonormal in the induced metric on T^*G because the metric is left invariant. For $X \in \mathfrak{g}$, the pullback by $L_p : G \to P_{\pi(p)}$ of $\theta := \omega_a \in \Omega^1(P)$ is

$$\begin{aligned}
(L_p^* \theta)_g (DL_g X) &= \theta_{p \cdot g}(DL_p \circ DL_g(X)) \\
&= \theta_{p \cdot g}(DL_{p \cdot g}(X)) \\
&= \theta_{p \cdot g}\left(\left. \frac{\mathrm{d}}{\mathrm{d}t} \right|_{t=0} (p \cdot g \exp(tX)) \right) \\
&= \epsilon_a(X) = (\epsilon_a DL_g^{-1}) DL_g(X)
\end{aligned}$$

where the second and third lines are applications of the chain rule, and the last line uses the property $\omega(\underline{X}) = X$ of a connection 1-form. This shows that $L_p^*\omega_a = \epsilon_a \circ DL_g^{-1}$, so

$$L_p^*(\omega_1 \wedge \cdots \omega_d) = \epsilon_1 \circ DL_g^{-1} \wedge \cdots \wedge \epsilon_d \circ DL_g^{-1}$$

as required. □

Assume that the metric on G is normalised so that the volume of G is 1. In turn, we can show that the integral along the G-orbit $\pi_* W = 1$.

Corollary 5.2. *The integral along the fiber of W is $\pi_* W = 1 \in \Omega^0(G)$*

***Proof*.** For $b \in M$, by definition we have $\pi_* W = \int_{\pi^{-1}(b)} W$ which is is computed by pulling back W by a local trivialisation $\psi : G \to \pi^{-1}(b)$. Let $p = \psi(e)$, then $\psi(g) = \psi(e) \cdot g = p \cdot g = L_p(g)$ because ψ is a G equivariant map. Then by the previous proposition, the pullback form $L_p^* W$ equals same volume form on G at every point in the fiber. Therefore, $\pi_* W = 1$ is the constant function on G. □

The next proposition explains why W is also called the projection form in [10].

Proposition 5.3. *Suppose $\eta \in \Omega^k(P)$ is an invariant differential form, then $\pi_*(\eta \wedge W) \in \Omega^k(M)$ is the horizontal projection of η onto M.*

***Proof*.** Let $b \in M, p \in \pi^{-1}(b)$. We know W is a top form on the vertical tangent space, since ω_a are all linearly independent in $V_p^* P$. Let $dx_1, \ldots, dx_n \in H_p^* P$ be an orthonormal basis for the horizontal cotangent space.

We may assume η is a product of 1-forms. If η contains any factor of ω_a, then its evaluation on all horizontal vectors will be zero, which agrees with $\pi_*(\eta \wedge W) = 0$. Now suppose $\eta = dx_I$, for an ordered multi-index $I \subset \{1, \ldots, n\}$. By definition of the fiber integral, for $v_1, \ldots, v_k \in T_b M$

$$\pi_*(\eta \wedge W)_b(v_1, \ldots, v_k) = \int_{\pi^{-1}(b)} (\eta \wedge W)(\widetilde{v}_1, \ldots, \widetilde{v}_k, -)$$

where $\widetilde{v}_i \in TP$ is any lift of v_i. In particular, at a point $p \in \pi^{-1}(b)$, we can choose them to be horizontal lifts, then extend to the whole fiber by right translation R_g. Note that if $v \in H_p P$, is horizontal, then $DR_g(v) \in H_{p \cdot g} P$ is still horizontal because the connection form ω satisfies $\mathrm{Ad}_g\, R_g^* \omega = \omega$.

$$\begin{aligned}\pi_*(\eta \wedge W)_b(v_1, \ldots, v_k) &= \int_{\pi^{-1}(b)} \eta(\widetilde{v}_1, \ldots, \widetilde{v}_k) \wedge W \\ &= \eta_p(\widetilde{v}_1, \ldots, \widetilde{v}_k) \int_{\pi^{-1}(b)} W = \eta_p(\widetilde{v}_1, \ldots, \widetilde{v}_k)\end{aligned}$$

the second line follows from η being an invariant form: $R_g^* \eta = \eta$ and that is how we've chosen the lifted horizontal vectors. □

5.2 Atiyah-Jeffrey formula

This section is a much more thorough explanation of section 2 of the paper by Atiyah and Jeffrey [2], where a formula for the Euler number is obtained starting from the Mathai-Quillen Thom form. The purpose is to formally apply it to the Donaldson-Witten context in the next section. We consider the following setup:

- G is a compact connected Lie group of dimension d
- $P \to M$ is a principal G-bundle
- Choose an Ad-invariant metric on $\mathfrak{g}$ using Theorem 4.3
- Similarly, construct a metric on P which is invariant under the action of G, by averaging any Riemannian metric on P with respect to a Haar measure
- At each point $p \in P$, this Riemannian metric defines an orthogonal complement to the vertical tangent space. Since these subspaces are invariant under the action of G, this determines a connection ω on P. Moving forward, we will solely use this connection.
- V is a real vector space, with $\dim V = 2m$ and a rep $\rho : G \to \mathrm{SO}(V)$
- $E := P \times_G V$ is the associated vector bundle.

Our starting point is the universal Mathai-Quillen formula (equation 2.13), an element in the Cartan model $S(\mathfrak{g}^*) \otimes \Omega(V)$, restated here

$$U = (2\pi)^{-m} e^{-|x|^2/2} \int^B \exp\left(\frac{1}{2}\chi^\intercal \rho_* \chi - idx^\intercal \chi\right) \mathrm{d}\chi \tag{5.3}$$

where $\chi = (e_1, \ldots, e_{2m})$ is a basis for V.

Manipulation 1: Replace Ω with $d\omega$

Recall that we can obtain a Thom form on E via the Cartan map, which is given by $\alpha \to \mathrm{Hor}(p_2^*\alpha(\Omega)) \in \Omega(P \times V)_{bas}$, where $p_2 : P \times V \to V$ is the projection onto V, and Ω is the curvature form associated to the connection ω induced by the metric on $P \times V$.

The structural equation gives the relation $\Omega = d\omega + \frac{1}{2}[\omega \wedge \omega]$, but $\omega = 0$ on horizontal vectors (by definition of horizontal subspace). Since the Cartan map then only evaluates on the horizontal projection of tangent vectors, we can replace Ω by $d\omega$ in the Cartan map.

Manipulation 2: Replace $d\omega$ with $R^{-1}dC^*$

Recall that $C_p : \mathfrak{g} \to V_pP$ defined by $C_p(X) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} (p \cdot \exp(tX))$ is the canonical identification of the Lie algebra with the vectical tangent space at $p \in P$. Define $C_p^{-1} : T_pP \to \mathfrak{g}$ on $T_pP = H_pP \oplus V_pP$ by defining the image of the horizontal subspace to be zero. Therefore if $\omega \in \Omega(P, \mathfrak{g})$ is the connection 1-form, then

$\omega_p(X) = C_p^{-1}(X)$ since they agree on all tangent vectors.

Both $\mathfrak{g}$ and T_P are inner product vector spaces, so C_p has an adjoint $C_p^* : T_pP \to \mathfrak{g}$. For $X, Y \in \mathfrak{g}$, it satisfies

$$\langle C_p(X), C_p(Y)\rangle = \langle C_p^* C_p(X), Y\rangle_{\mathfrak{g}} = \langle R_p X, Y\rangle_{\mathfrak{g}}$$

where $R_p := C_p^* \circ C_p : \mathfrak{g} \to \mathfrak{g}$. It is clear that R_p is self-adjoint and an isomorphism since C_p is an isomorphism. Also note that C_p^* vanishes on horizontal vectors, as $\langle X, C_p^*(v)\rangle = \langle C_p(X), v\rangle$ vanishes on all horizontal $v \in T_p$ due to $C_p(X) \in V_pP$.

Hence, we have the pointwise matrix equation $C^* = R\omega$, or $\omega = R^{-1}C^*$. From this, we compute its differential to be

$$d\omega = R^{-1}dC^* + dR^{-1} \wedge C^*$$

The last term vanishes on a pair of horizontal vectors, so again, we can replace $d\omega$ with $R^{-1}dC^*$, since the Cartan map will only evaluate on the horizontal vectors.

Manipulation 3: Double Fourier transform to avoid inverting R

The next objective is to remove the explicit inverse R^{-1} by using the Fourier inversion formula. Recall that the Fourier transform is an automorphism of the Schwartz space on any vector space W. If $dw \in \bigwedge^n W$ is a volume element with corresponding dual volume element $dy \in \bigwedge^n W^*$, then the Fourier inversion formula states that

$$f(w) = (2\pi)^{-n} \int_{W^*} \int_W e^{i\langle w,y\rangle} e^{-i\langle z,y\rangle} f(z)\,\mathrm{d}z\,\mathrm{d}y$$

Note that for a real vector space the integral is the same if we multiply both exponents by -1. If we identify W with W^* via some inner product, this becomes a double integral over W. For a self-adjoint matrix R with positive determinant, we can compute $f(R^{-1}w)$ by making the change of variables $w \to R^{-1}w$ and $y \to Ry$, in which case $\langle R^{-1}w, Ry\rangle = \langle w, y\rangle$ and $d(Ry) = \det R dy$. The inversion formula becomes

$$f(R^{-1}w) = (2\pi)^{-n} \iint_W e^{i\langle w,y\rangle} e^{-i\langle z,Ry\rangle} f(z) \det R\,\mathrm{d}z\,\mathrm{d}y$$

We now consider the universal Mathai-Quillen element as a $\Omega(V)$-valued function $U : \mathfrak{g} \to \Omega(V)$ on the vector space $\mathfrak{g}$,

$$U(\phi) = (2\pi)^{-m} e^{-|x|^2} \int^B \exp\left(\frac{1}{2}\chi^\intercal \rho_*(\phi)\chi - idx^\intercal \chi\right) \mathrm{d}\chi$$

which we wish to evaluate at $R^{-1}dC^* \in \Omega^2(P, \mathfrak{g})$. By the inversion formula above,

$$U(R^{-1}dC^*) = (2\pi)^{-d-m} e^{-|x|^2/2} \iint_{\mathfrak{g}} \int^B \exp\left(\frac{1}{2}\chi^\intercal \rho_*(\phi)\chi - idx^\intercal \chi \right. \\ \left. + i\langle dC^*, \lambda\rangle - i\langle \phi, R\lambda\rangle\right) \det R \, \mathrm{d}\chi \, \mathrm{d}\phi \, \mathrm{d}\lambda \tag{5.4}$$

where $\lambda, \phi \in \mathfrak{g}$ are Lie algebra variables.

Remark 5.4. Note that the original function $f(X)$ is a polynomial in $X \in \mathfrak{g}$, and therefore not in the Schwartz space. This can be made more precise by inserting a rapidly decaying test function $e^{-\epsilon\langle X,X\rangle}$ and taking the limit as $\epsilon \to 0$.

Manipulation 4: Horizontal projection via integration along G-orbits
The final step is to take the horizontal projection of the invariant element in $\Omega^{2m}(P \times V)$ into $\Omega^{2m}(P \times_\rho V)$ using Proposition 5.3. The intuition is that taking the wedge product with the invariant volume form $W \in \Omega^d(P \times V)$ introduced in the previous section kills all the vertical components. Hence we are only left with terms which did not have a vertical part but now with a factor of W. After integrating out the vertical part of these terms, the result is the horizontal part of the original element in $\Omega(P \times V)$.

In order to construct $W \in \Omega^d(P \times V)$ defined earlier, we use the connection ω on $P \times V$, and assume M is an oriented manifold. Also we extend the metric on $\mathfrak{g}$ to a left-invariant Riemannian metric on G as before. Recall that the invariant volume form W can be written as the Berezin integral

$$W = \int^B \exp(\omega) \, \mathrm{d}\eta_1 \cdots \mathrm{d}\eta_d$$

where $\eta_1, \ldots, \eta_d$ is an orthonormal basis for $\mathfrak{g}$ consistent with the orientation. We wish to rewrite W using the relation $\omega = R^{-1}C^*$, and we claim that this gives

$$W = (\det R)^{-1} \int^B \exp(C^*) \, \mathrm{d}\eta_1 \cdots \mathrm{d}\eta_d$$

Let us write $C^* = a_i \eta_i$, where $a_i \in \Omega^1(P \times V)$. If the top degree term of $\exp(C^*)$ is $A\eta_1 \wedge \cdots \eta_d$, then the top degree term in $\exp(R^{-1}C^*)$ is

$$A(R^{-1}\eta_1) \wedge \cdots \wedge (R^{-1}\eta_d) = (\det R^{-1}) A\eta_1 \wedge \cdots \wedge \eta_d$$

which proves the claim, since $\det R^{-1} = (\det R)^{-1}$. Taking the wedge product of the form $U(R^{-1}dC^*) \in \Omega^{2m}(P \times V)$ in equation (5.4) with W,

$$U(R^{-1}dC^*) \wedge W = (2\pi)^{-d-m} e^{-|x|^2/2} \iint_{\mathfrak{g}} \iint^B \exp\left(\frac{1}{2}\chi^\intercal \rho_*(\phi)\chi - idx^\intercal \chi \right. \\ \left. + i\langle dC^*, \lambda\rangle - i\langle \phi, R\lambda\rangle + C^*\right) \mathrm{d}\eta \, \mathrm{d}\chi \, \mathrm{d}\phi \, \mathrm{d}\lambda \tag{5.5}$$

Note that C^* should be interpreted as 1-form in the η basis, so we will write it as the sum $\langle C^*, \eta\rangle$ to avoid confusion. In this step we have only taken the wedge product with W, so we would be left with a differential form on $P \times V$, which we would need to integrate along the fibers to obtain the Thom form on $P \times_\rho V$.

Manipulation 5: Euler number of associated vector bundle

Recall that the pullback by any section $\sigma : M \to P \times_\rho V$ will give a representative of the Euler class. Then the integral over M (assuming M is compact) is the Euler number of the vector bundle, which is a topological invariant by the Chern-Weil homomorphism.

Recall that as an application of the isomorphism $\Omega^k_\rho(P, V) \simeq \Omega(M, P \times_\rho V)$, sections of the associated bundle are in bijection with ρ-equivariant maps $P \to V$. More concretely, a section $\sigma : M \to P \times_\rho V$ is of the form $\sigma(x) = [s(x), S(s(x))]$ where $s \in \Gamma(M, P)$ and $S : P \to V$ can be extended to a ρ-equivariant map $S(p \cdot g) = \rho(g^{-1})S(p)$. The next result shows that we can eliminate the fiber integral if we instead pull back the V component by S, and integrate over P.

Proposition 5.5. *For any invariant differential form $\eta \in \Omega^k(P \times V)$, and a section $\sigma = [s, S \circ s] : M \to P \times_\rho V$,*

$$\int_M \sigma^* \pi_*(\eta \wedge W) = \int_P S^*(\eta \wedge W)$$

where S^* only acts on the V component.

***Proof*.** Let $\omega \in \Omega^1(P, \mathfrak{g})$ be the connection on P, and $\omega' = p_1^*\omega \in \Omega^1(P \times V, \mathfrak{g})$. We have $W = \omega'_1 \wedge \cdots \wedge \omega'_d \in \Omega^d(P \times V)$, and $S^*W = \omega_1 \wedge \cdots \wedge \omega_d \in \Omega^d(P)$ is the invariant volume form on P, since each ω_i does not have any components in V. Furthermore, $S^*\eta$ is again invariant because $SR_g(v) = S(v \cdot g) = \rho(g^{-1})S(v)$, so $R_g^* S^* \eta_V = S^*(\rho(g)^{-1})^*\alpha = S^*\alpha$ where α is any form with only V components. Taking the wedge product with S^*W only leaves the horizontal terms in $S^*\eta$, so by equation (5.1), the right side of the equation is

$$\int_P S^*(\eta \wedge W) = \int_P \operatorname{Hor}_\omega(S^*\eta) \wedge S^*W = \int_M s^* \operatorname{Hor}_\omega(S^*\eta)$$

where $\operatorname{Hor}_\omega(S^*\eta) \in \Omega(P)$ is the horizontal projection, so taking the pullback by any section of P gives the corresponding form on M.

On the other hand, from Proposition 5.3, $\pi_*(\eta \wedge W) = i^* \operatorname{Hor}_{\omega'}(\eta) \in \Omega^k(P \times_\rho V)$, where $i : P \times_\rho V \to P \times V$ is the section $[p, v] \mapsto (s(x), S(s(x)))$ where $x = \pi(p)$. This section is chosen so that it has the property $i \circ \sigma(x) = (s(x), S(s(x)))$. So the left side of the equation is

$$\int_M \sigma^* \pi_*(\eta \wedge W) = \int_M \sigma^* i^* \operatorname{Hor}_{\omega'}(\eta)$$

It should be stressed that the choice of i and s in the two equations above are arbitrary, since the pullback of a basic form by any section gives the corresponding differential form on the base space. Comparing the two equations, we see that it suffices to prove the following diagram commutes

$$\begin{tikzcd} \Omega^k(P\times V)^G \arrow[r, "\mathrm{Hor}_{\omega'}"] \arrow[d, "S^*"] & \Omega^k(P\times_\rho V) \arrow[d, "\sigma^*"] \\ \Omega^k(P)^G \arrow[r, "\mathrm{Hor}_\omega"] & \Omega^k(M) \end{tikzcd}$$

If we include the pullbacks to the base spaces, we need to show $\sigma^* i^* \operatorname{Hor}_{\omega'} = s^* \operatorname{Hor}_\omega S^*$ on invariant elements $\eta \in \Omega(P\times V)$. The terms in η can be written locally in the form $f(p,v)dx_I \wedge de_I$, where $dx_I \in \Omega(P), de_I \in \Omega(V)$. Since the vertical subspace is contained in the P component, the horizontal terms in $S^*\eta$ come from horizontal terms in η. These are of the form $f(p,S(p))dx_I \wedge dS_I$, where $S_i = e_i \circ S$. Then pullback by s gives $f(s(x),S(s(x)))d(x\circ s)\wedge d((S\circ s)_I)$. It is now apparent that this is exactly the same as pulling back the horizontal terms by the section $x \mapsto (s(x), S(s(x)))$ as required. $\square$

With $\eta = U(R^{-1}dC^*)$, we obtain the following element in $\Omega(P)$ whose integral over P is also the Euler number:

$$\begin{aligned} S^*(U(R^{-1}dC^*)\wedge W) =&(2\pi)^{-d-m} \iint_{\mathfrak{g}} \iint^B \exp\Big(-\frac{1}{2}\left|S\right|^2 + \frac{1}{2}\chi^\intercal \rho_*(\phi)\chi \\ &- idS^\intercal\chi + i\left\langle dC^*, \lambda\right\rangle - i\left\langle \phi, R\lambda\right\rangle + \left\langle C^*, \eta\right\rangle\Big) \mathrm{d}\eta\, \mathrm{d}\chi\, \mathrm{d}\phi\, \mathrm{d}\lambda \end{aligned} \tag{5.6}$$

This result is called the Atiyah-Jeffrey formula for the Euler number.

Manipulation 6: Successive fermionic and bosonic integrals

In order to integrate over P, there is a common notational tool in supersymmetric physics where the integral of a differential form is written as a Berezin (fermionic) integral followed by an ordinary (bosonic) integral. Suppose $\alpha \in \Omega^{2m+d}(P)$ is a top rank form, and dV is a volume form on P. Then we can write $\alpha = f(p)dV$. Let us denote the same form $d\psi = dV$ to indicate Grassman variables for the purpose of the following Berezin integral:

$$f(p) = \int^B \alpha(p,\psi)d\psi$$

where the integral is defined locally by writing ψ and α in terms of any basis $\{dx_i\}$. This naturally isolates $f(p)$, which can then be integrated against the volume form:

$$\int_P \alpha = \int_P \int^B \alpha(p,\psi)d\psi dV_p$$

Therefore, the integral over P of the element in $\Omega(P)$ in equation (5.6) can be written as

$$\begin{aligned}
&\int_M \chi(P \times_\rho V) \\
=&(2\pi)^{-d-m} \int_P \iint_{\mathfrak{g}} \iiint^B \exp\Big(-\frac{1}{2}\,|S(p)|^2 + \frac{1}{2}\chi^\intercal \rho_*(\phi)\chi \\
&- idS_p^\intercal(\psi)\chi + i\langle dC_p^*(\psi), \lambda\rangle - i\,\langle \phi, R_p\lambda\rangle + \langle C_p^*(\psi), \eta\rangle \Big) \,\mathrm{d}\psi\,\mathrm{d}\eta\,\mathrm{d}\chi\,\mathrm{d}\phi\,\mathrm{d}\lambda\,\mathrm{d}V_p
\end{aligned} \tag{5.7}$$

Note that terms like $C_p^*(\psi)$ merely indicates that we write the covectors in $T_p^* P$ in terms of the variables ψ.

5.3 Application to Donaldson-Witten theory

We can now apply the Atiyah-Jeffrey formula for the Euler number to the infinite dimensional setting of Donaldson-Witten theory. As eloquently put by Naber [26], the content of this section is not mathematics, and certainly not physics. The objective is to find an analogue of the Euler number of the infinite-dimensional vector bundle associated with the Donaldson invariant. In the process, the well defined integrals transmute into Feynman integrals over spaces of fields, with their accompanying mathematical difficulties. The objects we consider are

- principal bundle: $\mathcal{A}^* \to \mathcal{A}^*/\mathcal{G}$ where $\mathcal{A}^* \subset \Omega^1(M, \operatorname{ad} P)$ is the space of irreducible connections on a principal SU(2)-bundle $P \to M$ over a compact oriented 4-manifold M, whose elements are called gauge fields.
- structure group: gauge group $\mathcal{G} \simeq \Omega^0(M, \operatorname{Ad} P)$
- vector space: self dual 2-forms $\Omega^{2,+}(M, \operatorname{ad} P)$
- associated vector bundle: $\mathcal{A}^* \times_{\mathcal{G}} \Omega^{2,+}(M, \operatorname{ad} P) \to \mathcal{A}^*/\mathcal{G}$
- section: self-dual part of curvature $\sigma : \mathcal{A}^*/\mathcal{G} \to \mathcal{A}^* \times_{\mathcal{G}} \Omega^{2,+}(M, \operatorname{ad} P)$ defined by $[A] \mapsto [A, -F_A^+]$.
- equivariant section: self-dual part of curvature $S : \mathcal{A}^* \to \Omega^{2,+}(M, \operatorname{ad} P)$ defined by $S(A) = -F_A^+$

We are interested in the moduli space of ASD connections, which is why we have defined the section of the associated vector bundle so that its zero set is identified with $\mathcal{M}^*$. Note that the $\mathcal{G}$-equivariant map $S : \mathcal{A}^* \to \Omega^{2,+}(M, \operatorname{ad} P)$ is the unique map associated to σ.

The main idea of Atiyah and Jeffrey is that we can obtain an interpretation of the Donaldson invariants as the localisation of the integral of certain differential forms to $\sigma^{-1}(0)$.

We need a metric on the principal bundle $\mathcal{A}^*$ which is invariant under the isometries in $\mathcal{G}$. The natural $\operatorname{Ad}\mathrm{SU}(2)$-invariant metric on $\mathfrak{su}(2)$ is $\langle X, Y\rangle = -\operatorname{Tr}(XY)$. There is also a natural L^2 inner product on each of the vector spaces $\Omega^k(M, \operatorname{ad} P)$, which can be written using equation (4.3) as

$$\langle \alpha, \beta\rangle = \int_M \langle \alpha, \beta\rangle_{\operatorname{ad} P}\, dV_g = -\int_M \operatorname{Tr}(\alpha \wedge \star\beta)$$

for $\alpha, \beta \in \Omega^k(M, \operatorname{ad} P)$. Since the metric on $\mathfrak{su}(2)$ is $\operatorname{Ad}\mathrm{SU}(2)$ invariant, the above metric is $\operatorname{Ad}\mathcal{G}$ invariant. And since $\Omega^0(M, \operatorname{Ad} P)$ acts on curvature forms by conjugation, $\mathcal{G}$ acts by isometries on the vector space $\Omega^2(M, \operatorname{ad} P)$. As usual, this metric also defines a connection on $\mathcal{A}^*$ whose horizontal spaces are orthogonal to the gauge orbits.

Next, we need to work out the analogues of the maps C, C^* and R. Recall that $\operatorname{Lie}(\mathcal{G}) \simeq \Omega^0(M, \operatorname{ad} P)$ and $T_A\mathcal{A} \simeq \Omega^1(M, \operatorname{ad} P)$. By Proposition 4.17, given $A \in \mathcal{A}$, the map $C_A : \operatorname{Lie}(\mathcal{G}) \to T_A\mathcal{A}$ is given by d_A. Then relative to the inner products, the formal adjoint of this operator is $C_A^* = d_A^* : T_{\mathcal{A}} \to \operatorname{Lie}(\mathcal{G})$. Therefore the operator R on $\Omega^0(M, \operatorname{ad} P)$ is the Laplacian $\Delta_A = d_A^* d_A$.

Finally, we need to interpret the Berezin and ordinary integrals. The Berezin integral on infinite dimensional vector spaces is well defined, but the issue is that our expression now has an infinite number of Berezin integrals over the bases $\chi \in \Omega^{2,+}(M, \operatorname{ad} P)$ and $\eta \in \Omega^0(M, \operatorname{ad} P)$, and we no longer have a top degree form $\chi_1 \wedge \cdots \wedge \chi_n$ arising from the exponential. In quantum field theory, the infinite dimensional analogue is of this called the fermionic functional integral, where the integral is taken over the space of all fields/sections. Quantities like χ are regarded as a single element, while quantities like $\chi^\intercal dS$ are interpreted as an inner product $\langle \chi, dS\rangle$ with the element. The path integral of analytic functions of the fields is defined to be a formal limit of the finite dimensional case, e.g. a limit of the identities in section A.3. For instance, the limit of Lemma A.4 becomes

$$\int \exp(\overline{\eta}^\intercal K\eta)\, \mathrm{d}\overline{\eta}_i\, \mathrm{d}\eta_i = \det(K) \to \int \exp(\langle \overline{\psi}, K\psi\rangle)\mathcal{D}\overline{\psi}\mathcal{D}\psi = \det(K)$$

A similar interpretation holds for the ordinary integrals over $\phi, \lambda \in \operatorname{Lie}(\mathcal{G})$. In fact, the formal limit of the ordinary integral analogue of Prop A.8 is considered to be central to quantum field theory (see Appendix A or Chapter I.2 in [35]).

We can now interpret each term in equation (5.6). We will evaluate the terms at the point $A \in \mathcal{A}^*$, corresponding to $p \in P$ in the formula.

- $-\frac{1}{2}|S(p)|^2$: The norm on the vector space $\Omega^{2,+}(M, \operatorname{ad} P)$ is the L^2 norm, so this term is
$$-\frac{1}{2}\int_M |F_A^+|^2 dV_g = -\frac{1}{4}\int_M |F_A|^2\, dV_g + 2\pi^2 k$$

where we have used the property in equation (4.5) for the second Chern number k of the associated vector bundle.

- $-i\,\langle\phi, R_p\lambda\rangle$: We know ϕ and λ are Lie algebra variables in $\Omega^0(M, \mathrm{ad}\,P)$. From above, we found that R is the Laplacian Δ_A. So we write this term as
$$-i\,\langle\phi, \Delta_A\lambda\rangle = i\int_M \mathrm{Tr}(\phi\wedge\star(\Delta_A\lambda)) = i\int_M \mathrm{Tr}(\star(\phi\,\Delta_A\lambda))$$
- $\langle C_p^*, \eta\rangle$: We have determined C_A^* to be the formal adjoint d_A^*. This term is meant to be interpreted as a 1-form in $\Omega^1(\mathcal{A}, \mathrm{Lie}(\mathcal{G}))$, written as a sum over the basis η for $\mathrm{Lie}(\mathcal{G})$. By defintion of the adjoint, we can also write this term as
$$\langle d_A^*, \eta\rangle = \langle -, d_A\eta\rangle = -\int_M \mathrm{Tr}(-\wedge\star d_A\eta)$$
- $i\langle dC_p^*, \lambda\rangle$: Recall that dC^* is interpreted as a 2-form $\Omega^2(\mathcal{A}, \mathrm{Lie}(\mathcal{G}))$. Our first goal is to compute dC^* at the point $A\in\mathcal{A}^*$. Fix $a_1, a_2 \in T_A\mathcal{A}^* \subset \Omega^1(M, \mathrm{ad}\,P)$. Since $\mathcal{A}^*$ is contained in an affine space, a_1 and a_2 are associated to constant vector fields, i.e. $a_i(A) = A + a_i$. Then
$$\begin{aligned} dC^*(a_1,a_2) &= a_1(C^*a_2) - a_2(C^*a_1) - C^*([a_1,a_2]) \\ &= a_1(C^*a_2) - a_2(C^*a_1) \end{aligned}$$
since the Lie bracket of constant vector fields is zero. To evaluate these terms, note that C^*a_2 is a map from points $A\in\mathcal{A}^*$ to $C_A^*a_2 = d_A^*a_2$. Hence,
$$a_1(C^*a_2)_A = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}(C^*_{A+ta_1}a_2) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}(d^*_{A+ta_1}a_2)$$
since C_A^* acts as the codifferential on the affine space. We can compute $d^*_{A+ta_1}$ as follows. For any $\lambda\in\Omega^0(M,\mathrm{ad}\,P)$, from Proposition 4.9 we have
$$d_{A+ta_1}(\lambda) = d_A\lambda + t[a_1,\lambda]$$
Recall that the formal adjoint is explicitly described by $d^*_{A+ta_1} = -\star d_{A+ta_1}\star$ for a four-manifold, so in this case,
$$d^*_{A+ta_1}(a_2) = d_A^*a_2 - t\star[a_1,\star a_2]$$
and computing the derivative at $t=0$ gives $a_1(C^*a_2) = -\star[a_1,\star a_2]$. Similarly, $a_2(C^*a_1) = -\star[a_2,\star a_1] = \star[a_1,\star a_2]$, where the Lie bracket introduces a minus sign. Note that these are independent of A. Therefore, $dC^*(a_1,a_2) = -2\star[a_1,\star a_2]$. Finally, our interpretation of the term $i\,\langle dC_A^*,\lambda\rangle$ is
$$-2i\,\langle\star[-,\star-],\lambda\rangle = 2i\int_M \mathrm{Tr}(\lambda\wedge\star^2[-,\star-]) = -2i\int_M \mathrm{Tr}([-,\star-]\lambda)$$

- $-idS_p^\intercal \chi$: The remaining two terms come from the Berezin integral in the Mathai-Quillen formula, involving the odd degree variable χ which is a basis for the vector space $\Omega^{2,+}(M, \operatorname{ad} P)$. So commuting the variables gives $i\chi^\intercal dS_p$. Recall that we interpret the Berezin integral by treating χ as an element $\Omega^{2,+}(M, \operatorname{ad} P)$, and sum $\chi^\intercal dS$ as the inner product $\langle \chi, dS \rangle$.
  From Proposition 4.16, the differential of the curvature operator $A \mapsto F_A$ is the covariant derivative d_A. Therefore, the differential of the section is $dS = -d_A^+$. Since χ is self-dual, our interpretation of this term is
  $$i\langle \chi, d_A^+ \rangle = i\,\langle \chi, d_A \rangle = -i \int_M \operatorname{Tr}(d_A(-) \wedge \chi)$$
- $\frac{1}{2}\chi^\intercal \rho_*(\phi)\chi$: Here $\phi \in \operatorname{Lie}(\mathcal{G})$ is the other Lie algebra variable that we integrate over. The representation ρ corresponds to the invariant action of $\mathcal{G}$ on $\Omega^{2,+}(M, \operatorname{ad} P)$, given by pointwise conjugation. Hence, the induced action on the Lie algebra acts on $\chi \in \Omega^{2,+}(M, \operatorname{ad} P)$ pointwise by $\rho_*(\phi)\chi = [\chi, \phi]$. Then as before, the sum involving $\chi^\intercal$ is interpreted as the inner product on the vector space, and our result is
  $$\frac{1}{2}\,\langle \chi, [\chi, \phi] \rangle = -\frac{1}{2}\int_M \operatorname{Tr}([\chi, \phi] \wedge \chi)$$

Not that all of the terms above are written in terms of the inner product on $\Omega^k(M, \operatorname{ad} P)$, i.e. an integral over M of the trace of a $\operatorname{ad} P$-valued form. We can interpret the terms inside the trace integral as minus the terms in the Lagrangian density. So putting together all of the terms above, we have the Donaldson-Witten Lagrangian density

$$\begin{aligned} \mathcal{L}_{\mathrm{DW}} = &-\frac{1}{4}F_A \wedge \star F_A - \frac{1}{4}F_A \wedge F_A + \frac{1}{2}[\chi, \phi] \wedge \chi + id_A\psi \wedge \chi \\ &+ 2i[\psi, \star\psi]\lambda - \star(i\phi\Delta_A\lambda) + \psi \wedge \star d_A\eta \end{aligned} \tag{5.8}$$

where we have replaced the $(-)$ signs for the argument of differential forms on the principal bundle with $\psi \in \Omega^1(M, \operatorname{ad} P)$, since the Berezin integral over ψ in equation (5.7) becomes a fermionic path integral, in the same way as for the variables η and χ. Thus, our "Euler number" for the vector bundle $\mathcal{A}^* \times_{\mathcal{G}} \Omega^{2,+}(M, \operatorname{ad} P)$ is proportional to

$$\int \exp\left(-\int_M \operatorname{Tr}\mathcal{L}_{\mathrm{DW}}\right) \mathrm{d}\psi\, \mathrm{d}\eta\, \mathrm{d}\chi\, \mathrm{d}\phi\, \mathrm{d}\lambda\, \mathrm{d}V_A \tag{5.9}$$

Note that we have omitted the $(2\pi)^{-d-m}$ factor because both m and d would be infinite in this circumstance.

5.4 Interpretation of Donaldson-Witten theory

When the dimension of the zero set of the section $\sigma : \mathcal{A}^* \to \Omega^{2,+}(M, \mathrm{ad}\, P)$ (i.e. the moduli space) $\dim \mathcal{M}^*$ is zero, the significance of the result is that after Wick rotation, our Lagrangian and Euler number are precisely Witten's Lagrangian and associated partition function for his TQFT for Donaldson theory.[33] In this case, Wick rotation is performed by replacing $\phi \in \mathrm{Lie}(\mathcal{G})$ with $i\phi$, corresponding to analytic continuation. Witten showed that the partition function computes the 0-dimensional Donaldson invariant of a smooth 4-manifold, and hence we can interpret this invariant as the intersection number of the section σ and the zero section of the vector bundle, analogous to the Poincaré-Hopf theorem.

In the more general case $\dim \mathcal{M}^* \neq 0$, we can take the analogy a step further, hoping for a representation of the Donaldson invariants as the Euler class integrated against other differential forms on $\mathcal{A}^*$, analogous to Corollary 3.13. Note that $\mathcal{M}^*$ is the zero set of the section σ, so we can say the "codimension" of the Euler class on $\mathcal{A}^*$ is $\dim \mathcal{M}^*$, as in the case of a transverse section of a finite dimensional vector bundle. Let us briefly describe Witten's formulation of the Donaldson invariants.

Let $\xi = \mathcal{A}^* \times_{\mathcal{G}} P \to \mathcal{B}^* \times M$ be the principal SO(3)-bundle introduced in chapter 4. Recall that the map $\mu : H_2(M, \mathbb{Z}) \to H^2(\mathcal{M}^*, \mathbb{Z})$ is defined by the slant product $-\frac{1}{4}p_1(\xi)/\gamma$, which we can denote as an integral over γ for convenience. There is a connection on this bundle induced by a metric on ξ, which comes from the metric on $\mathcal{A}^* \times P$. The curvature of this connection $\mathcal{F} \in \Omega^2(\xi, \mathfrak{so}(3))$ can be written in terms of the components $(2,0)$, $(1,1)$ and $(0,2)$ in terms of the product $\mathcal{A}^* \times_{\mathcal{G}} P$, describing the multiplicity of cotangent vectors in the two components. Similarly, the Pontryagin class $\mathrm{Tr}(\mathcal{F} \wedge \mathcal{F}) \in \Omega^4(\xi)$ can be written in terms of five components $(4,0), (3,1), (2,2), (1,3)$ and $(0,4)$. The three components of $\mathcal{F}$ can be written in terms of the fields to obtain $\mathcal{F} = F_A + \psi + i\phi$, and thus Witten writes the five components of $\mathrm{Tr}(\mathcal{F} \wedge \mathcal{F})$ as

$$W_0 = \frac{1}{2}\,\mathrm{Tr}\,\phi^2 \qquad W_1 = \mathrm{Tr}(\phi \wedge \psi)$$

$$W_2 = \mathrm{Tr}\left(\frac{1}{2}\psi \wedge \psi + i\phi \wedge F_A\right) \quad W_3 = i\,\mathrm{Tr}(\psi \wedge F_A) \quad W_4 = -\frac{1}{2}\,\mathrm{Tr}(F_A \wedge F_A)$$

See [4] for details on how $\mathcal{F}$ can be written this way. Note that the double Fourier transform turned any expression $f(R^{-1}dC^*)$ into $f(\phi)$ which is why we expect ϕ to appear. The ψ term appears because we evaluate all terms in the path integral at that tangent vector. Let us denote

$$W_\gamma(A, \psi, \phi) = \int_\gamma \mathrm{Tr}(\mathcal{F} \wedge \mathcal{F}) = \int_\gamma W_i$$

Given $\gamma_1, \ldots, \gamma_d \in H^2(M, \mathbb{Z})$, where $d = \frac{1}{2}\dim \mathcal{M}^*$, Witten formulates the associated Donaldson invariant as the expectation value of the operator $W_{\gamma_1} \cdots W_{\gamma_d}$ in terms of the path integral and action [33, eq. 3.40]:

$$\langle W_{\gamma_1} \cdots W_{\gamma_d} \rangle = \int W_{\gamma_1} \cdots W_{\gamma_d} \exp\left(-\int_M \operatorname{Tr} \mathcal{L}_{\mathrm{DW}}\right) \mathrm{d}\psi \, \mathrm{d}\eta \, \mathrm{d}\chi \, \mathrm{d}\phi \, \mathrm{d}\lambda \, \mathrm{d}V_A$$

up to some constant factor. This should be interpreted as the integral over $\mathcal{A}^*$ of the product of the Euler class with classes $\mu(\gamma_i)$. Recall that by manipulations 5 and 6, this is the same as instead integrating over the moduli space, after taking the horizontal projection and pull back by the section σ. Thus if we formally apply Corollary 3.13, the integral localises to the moduli space, and this provides another way to understand Donaldson invariants.

Whether this makes rigorous mathematical sense or not, this path integral representation coincides with Witten's formulation of these invariants. This would be viewed as merely a curiosity if there were not a great deal more to this story. Its importance in topology is that the SUSY TQFT formulation of these invariants have been undeniably effective in producing deep mathematics. In particular, techniques of supersymmetric QFT (Seiberg-Witten duality) has given us a new set of invariants, and a similar interpretation of the theory led to its mathematical description in terms of a moduli space. Tools in this area of physics such as the u-plane integral also gives an effective method for deriving the expression for the Donaldson invariants in the general case.[23]

Moreover, the Atiyah-Jeffrey framework is not limited to the theories we have mentioned here: other examples include the Casson invariant of homology 3-spheres and a two dimensional analogue of Donaldson theory describing intersection theory on a moduli space of flat connections. Hence, the Atiyah-Jeffrey interpretation is a useful geometric setting that can translate between the techniques of TQFT and differential topology, and mathematicians stand to gain intuition from learning it.

Bibliographical notes

- The lecture notes by Cordes, Moore, and Ramgoolam [10] explores this topic from the perspective of physics, using the language of twisted $\mathcal{N} = 2$ SUSY YM theory.
- The conference paper by Naber [26] provides more detailed explanations of Atiyah and Jeffrey's paper [2].

Appendix A

Grassman Calculus

Definition A.1. Let V be a complex or real vector space. A Grassman number is an element $x \in \bigwedge(V)$ in the exterior algebra of V. Given a choice of basis $\{\theta_i\}_{i=1}^{\infty}$ for V, a Grassman variable is an element of the basis set.

The Grassman numbers are typically equipped with a linear involution operator $*$ that extends complex conjugation in the following way: for all $\lambda, \mu \in \mathbb{C}$

$$(\lambda x + \mu y)^* = \lambda^* x^* + \mu^* y^*, \qquad \theta^* = \theta, \qquad (\theta_1 \theta_2 \cdots \theta_k)^* = \theta_k \cdots \theta_2 \theta_1$$

where $\theta, \theta_1, \ldots, \theta_k$ are in the basis set, and x, y are Grassman numbers.

It is standard to omit the wedge symbol $\wedge$ when writing a Grassman number. The Grassman numbers have a $\mathbb{Z}_2$-grading given by $\bigwedge(V) = \bigwedge_+(V) \oplus \bigwedge_-(V)$ which are subspaces generated by products of an even (resp. odd) number of Grassman variables. Elements in $\bigwedge_-(V)$ anti-commute, so are called a-numbers, while elements in $\bigwedge_+(V)$ commute, so are called c-numbers.

A.1 Differentiation

Let $A \in \bigwedge(V)$. For a given Grassman variable θ_i, after some commutations we can uniquely write $A = A_1 + \theta_i A_2$ where A_1, A_2 do not contain θ_i. Then the differential operator with respect to θ_i is defined as

$$\frac{\partial A}{\partial \theta_i} = A_2$$

In particular, we have $\frac{\partial \theta_j}{\partial \theta_i} = \delta_{ij}$.

Note that the action of $\frac{\partial}{\partial \theta_i}$ consists of commuting θ_i to the left in all monomials before suppressing it. So the derivaive of a product does not satisfy the usual Leibnitz rule.

Let P be the algebra automorphism which satisfies $P(\theta_i) = -\theta_i$ for each Grassman variable. Note that this relation uniquely determines P, which has the properties $P(\theta_{i_1} \cdots \theta_{i_p}) = (-1)^p \theta_{i_1} \cdots \theta_{i_p}$ and $A\theta_i = \theta_i P(A)$.

We find that the Leibniz rule is replaced by

$$\frac{\partial}{\partial \theta_i}(AB) = \frac{\partial A}{\partial \theta_i} B + P(A) \frac{\partial B}{\partial \theta_i}$$

because if A contains θ_i then nothing in B is commuted, but if B contains θ_i then it needs to be commuted through all the variables in A. This can be proved by simply expanding $A = A_1 + \theta_i A_2$ and $B = B_1 + \theta_i B_2$.

A.2 Integration

To define integration, we consider what properties it should have in relation to differentiation. Let D denote differentiation and I denote integration with respect to the same Grassman variable. We would like the following relations

(1) $I(A+B) = I(A) + I(B)$. Integration is linear
(2) $ID = 0$. Integral of the derivative vanishes, a property that allows integration by parts
(3) $DI = 0$. After integration over a variable, the result does not depend on this variable any more.
(4) $D(A) = 0$ implies $I(BA) = I(B)A$

where A, B are arbitrary Grassman numbers. If we are integrating with respect to θ_i, and $A = A_1 + \theta_i A_2$, then $I(A) = I(A_1) + I(\theta_i)A_2 = I(\theta_i)A_2$ by linearity and property (4). Note that $I(A_1) = 0$ because $A_1 = \frac{\partial}{\partial \theta_i}(\theta_i A_1)$. Since $I(\theta_i)$ should not depend on θ_i anymore, we are left to choose a constant in $\mathbb{C}$. If we set $I(\theta_i) = 1$ for each Grassman variable, this is exactly the same definition as the Grassman derivative!

Definition A.2. The Berezin integral over the sole Grassman variable θ is an operator $\int d\theta : \bigwedge V \to \bigwedge V$. Given a Grassman number $A = A_1 + \theta A_2$ where A_1, A_2 do not depend on θ, define $\int A d\theta = \frac{\partial A}{\partial \theta} = A_2$.

We can extend the definition to integrate over any Grassman number in $\bigwedge V = T(V)/I$, by choosing a representative and integrating in that order. This is well defined because the integral vanishes on the ideal I generated by $\theta \otimes \theta$ for $\theta \in V$.

For example, if $\dim V = n$ and $\{\theta_i\}_{i=1}^n$ is a basis, the Berezin integral over $\theta_1 \wedge \ldots \wedge \theta_n$ takes the form $\int d\theta : \bigwedge V \to \mathbb{R}$, which kills any $\omega \in \bigwedge^k(V)$ with $k < n$, and maps $\theta_1 \wedge \ldots \wedge \theta_n$ to 1.

A.2.1 Integration on vector-valued differential forms

If $E \xrightarrow{\pi} M$ is an oriented vector bundle, the Berezin integral can be extended to a map on $\bigwedge E$-valued differential forms on a manifold M. We have a non-vanishing section $\nu \in \Gamma(M, \bigwedge^n E)$, and $\int^B : \Omega^*(M, \bigwedge E) \to \Omega^*(M)$ is defined at x by integrating over $\nu(x) \in \bigwedge^n E$.

Another way to formulate this is in terms of a metric on E. The induced metric on $\bigwedge E$ given by

$$\langle v_1 \wedge \ldots \wedge v_k, w_1 \wedge \ldots \wedge w_l \rangle := \begin{cases} \det(\langle v_i, w_j \rangle) & k = l \\ 0 & k \neq l \end{cases}$$

Then the Berezin integral on $\bigwedge E$ valued forms can be defined at $x \in M$ as $\int^B = \langle \nu(x), - \rangle$. Note that this is equivalent to integrating over $\nu(x)$.

If in addition there is a metric compatible connection ∇ on E, this induces a connection on $\bigwedge^n E$, and we choose the section $\nu \in \Gamma(M, \bigwedge^n E)$ such that $\nabla \nu = 0$. To see why this can be done, recall that the induced connection is given by

$$\nabla_X(\sigma_1 \wedge \sigma_2 \wedge \cdots \wedge \sigma_n) = \nabla_X \sigma_1 \wedge \sigma_2 \wedge \cdots \wedge \sigma_n + \sigma_1 \wedge \nabla_X \sigma_2 \wedge \sigma_3 \wedge \cdots \wedge \sigma_n + \cdots \\ + \sigma_1 \wedge \cdots \wedge \sigma_{k-1} \wedge \nabla_X \sigma_n$$

This connection is compatible with the induced metric on $\bigwedge^n E$ because to evaluate $D \langle v_1 \wedge \ldots \wedge v_n, w_1 \wedge \ldots \wedge w_n \rangle$, we can apply the product rule to each term in the determinant, and collect the terms with ∇v_i and ∇w_j which gives the action of the induced connection. Since $\bigwedge^n E$ is trivial, there is a global orthonormal frame $\nu = \sigma_1 \wedge \ldots \wedge \sigma_n \in \Gamma(\bigwedge^n E)$. Moreover, $\langle \nu, \nu \rangle = 1$ implies that $\nabla \nu = 0$ by metric compatibility, which is what we wanted to show.

A.3 Useful identities

A single variable polynomial of degree m of a Grassman number can be written in terms of its Taylor expansion as $p(x) = \sum_{k=0}^{m} \frac{p^{(k)}(a)}{k!}(x-a)^k$. We are free to choose $a \in \mathbb{R}$ for every Grassman number x, because the result still expresses the same polynomial. We would like to extend this for smooth functions, but the choice of $a \in \mathbb{R}$ matters if we truncate the series.

Definition A.3 (Smooth function of Grassman number)**.** Let $f \in C^\infty(\mathbb{R})$ and $z \in \bigwedge(\theta_1, \ldots, \theta_n)$ be a Grassman number. Denote $z_B \in \bigwedge^0 = \mathbb{R}$ to be the projection onto the field. Then define $f(z)$ by

$$f(z) := \sum_{k=0}^{n} \frac{f^{(k)}(z_B)}{k!}(z - z_B)^k \tag{A.1}$$

In other words, a smooth function of a Grassman number z is defined to be its Taylor series at z_B. A particular smooth function we will be interested in is the exponential function e^x. In this case, the point at which the Taylor expansion is taken does not matter because $z_B \in \mathbb{R}$ commutes with all other numbers:

$$e^z = \sum_{k=0}^{n} \frac{e^{z_B}}{k!}(z - z_B)^k = \left(\sum_{k=0}^{\infty} \frac{1}{k!} z_B^k\right)\left(\sum_{k=0}^{n} \frac{1}{k!}(z - z_B)^k\right) = \sum_{k=0}^{\infty} \frac{1}{k!} z^k \tag{A.2}$$

If $\theta = [\theta_1, \ldots, \theta_n]$ are Grassman variables then we can also consider multivariable polynomials $f(\theta) \in \bigwedge(\theta_1, \ldots, \theta_n)$. For example, if there are two arguments the function is of the form $f(\theta_1, \theta_2) = A + B\theta_1 + C\theta_2 + D\theta_1\theta_2$.

Lemma A.4. *If $\theta = [\theta_1, \ldots, \theta_n]$ and $\eta = [\eta_1, \ldots, \eta_n]$ are two lists of Grassman variables such that $\theta \cup \eta$ is linearly independent, and K is a $n \times n$ matrix,*

$$\int \exp(\theta^\intercal K \eta)\, \mathrm{d}\theta_1\, \mathrm{d}\eta_1 \cdots \mathrm{d}\theta_n\, \mathrm{d}\eta_n = \det K$$

Proof. The terms from the series expansion of the exponential that give a non-vanishing contribution to the integral are those that contain the product $\theta_1 \cdots \theta_n \eta_1 \cdots \eta_n$, up to some permutation. This can only come from the term $\frac{1}{n!}(\theta_i K_{ij} \eta_j)^n$. Note that we can commute pairs of Grassman variables $\theta_{i_k}\eta_{j_k}$, so this gives

$$\begin{aligned} &\frac{1}{n!} \sum_{i \in S_n} \sum_{j \in S_n} K_{i_1 j_1} K_{i_2 j_2} \cdots K_{i_n j_n} \theta_{i_1} \eta_{j_1} \cdots \theta_{i_n} \eta_{j_n} \\ &= \sum_{j \in S_n} K_{1 j_1} K_{2 j_2} \cdots K_{n j_n} \theta_1 \eta_{j_1} \cdots \theta_n \eta_{j_n} \end{aligned}$$

After commuting the variables to cast each product into the order $\theta_1 \cdots \theta_n \eta_1 \cdots \eta_n$, this introduces the sign $\operatorname{sgn}(j)$. Then after integrating each of the variables, we are left with the determinant.

$$\sum_{j \in S_n} \operatorname{sgn}(j) K_{1 j_1} K_{2 j_2} \cdots K_{n j_n} = \det(K)$$

□

Lemma A.5. *If $\theta = [\theta_1, \ldots, \theta_n]$ are Grassman variables, and M is a skew-symmetric $n \times n$ matrix,*

$$\int \exp\left(\frac{1}{2}\theta^\intercal M \theta\right) d\theta = \operatorname{Pf}(M)$$

***Proof*.** The only non-vanishing terms in the exponential series are those that contain that product $\theta_1\theta_2\cdots\theta_n$ up to some permutation. But since all the terms contain an even number of Grassman variables, when n is odd the integral vanishes. Accordingly, the Pfaffian is zero for odd n. Now suppose $n = 2m$. Then non-vanishing terms come from $\frac{1}{2^m m!}(\theta_i M_{ij}\theta_j)^m$, where each permutation of $\theta_1\cdots\theta_n$ occurs once, so we are left with

$$\frac{1}{2^m m!}\sum_{j\in S_n} M_{j_1j_2}M_{j_3j_4}\cdots M_{j_{n-1}j_n}\theta_{j_1}\theta_{j_2}\cdots\theta_{j_{n-1}}\theta_{j_n}$$

Commutating the variables to cast each product into a standard order introduces $\mathrm{sgn}(j)$, and integrating gives us the Pfaffian

$$\mathrm{Pf}(M) = \frac{1}{2^m m!}\sum_{j\in S_n}\mathrm{sgn}(j)M_{j_1j_2}M_{j_3j_4}\cdots M_{j_{n-1}j_n}$$

□

Remark A.6. Sometimes the two lemmas above are written with a minus sign in the exponential. This is explained by the order of integration. For the Pfaffian, assuming n even, we can alternatively write it as

$$\begin{aligned}
\mathrm{Pf}(M) &= \int \exp\left(\frac{1}{2}\theta^\intercal M\theta\right) d\theta \\
&= \int \mathrm{d}\theta_n\cdots\mathrm{d}\theta_1 \exp\left(\frac{1}{2}\theta^\intercal M\theta\right) \\
&= (-1)^{\frac{n(n-1)}{2}}\int \mathrm{d}\theta_1\cdots\mathrm{d}\theta_n \exp\left(\frac{1}{2}\theta^\intercal M\theta\right) \\
&= (-1)^{\frac{n}{2}}\int \mathrm{d}\theta_1\cdots\mathrm{d}\theta_n \exp\left(\frac{1}{2}\theta^\intercal M\theta\right) \\
&= \int \mathrm{d}\theta \exp\left(-\frac{1}{2}\theta^\intercal M\theta\right)
\end{aligned}$$

Note that the sign of the permutation $(1,\ldots,n)\mapsto(n,\ldots,1)$ is given by $(-1)^{\frac{n(n-1)}{2}}$, which is $(-1)^{\frac{n}{2}}$ when n is even. Also the discussion still works for odd n, because the integral vanishes regardless.

Proposition A.7 (Invariance under translation [7, Prop A.12])**.** *Let $\theta = [\theta_1,\ldots,\theta_n]$ be Grassman variables, and $\xi = [\xi_1,\ldots,\xi_n]$ be odd Grassman numbers that do not involve θ_i. Then for any smooth function f,*

$$\int f(\theta+\xi)\,\mathrm{d}\theta = \int f(\theta)\,\mathrm{d}\theta$$

Proof. We can write $f(\theta)$ as a finite linear combination of elements of the form θ^J, where $J \subset [n]$. Hence, it suffices to prove this for $f(\theta) = \theta^J$. In this case, $f(\theta+\xi) = (\theta+\xi)^J$. We have

$$\frac{\partial}{\partial\theta_i}(\theta_j + \xi_j) = \frac{\partial}{\partial\theta_i}\theta_j = \delta_{ij}$$

Therefore, in $\frac{\partial}{\partial\theta_n}\cdots\frac{\partial}{\partial\theta_1}(\theta+\xi)^J$, each operator $\frac{\partial}{\partial\theta_j}$ acts successively on $(\theta_j+\xi_j)$ to give an element that is the same as if ξ^j was set to zero. Hence,

$$\frac{\partial}{\partial\theta_n}\cdots\frac{\partial}{\partial\theta_1}(\theta+\xi)^J = \frac{\partial}{\partial\theta_n}\cdots\frac{\partial}{\partial\theta_1}\theta^J$$

$\square$

Proposition A.8. *If $\theta = [\theta_1, \ldots, \theta_n]$ are Grassman variables, $J = [J_1, \ldots, J_n]$ are odd Grassman numbers that do not involve θ_i, and M is an invertible skew-symmetric $n \times n$ matrix,*

$$\int \exp\left(\frac{1}{2}\theta^\intercal M\theta + J^\intercal\theta\right) d\theta = \mathrm{Pf}(M)\exp\left(\frac{1}{2}J^\intercal M^{-1}J\right)$$

Proof. Since the variables in θ and J anticommute, we can complete the square for $\frac{1}{2}\theta^\intercal M\theta + J^\intercal\theta$ in the following way:

$$\begin{aligned}
&\frac{1}{2}(\theta - M^{-1}J)^\intercal M(\theta - M^{-1}J) + \frac{1}{2}J^\intercal M^{-1}J \\
&= \frac{1}{2}(\theta^\intercal + J^\intercal M^{-1})M(\theta - M^{-1}J) + \frac{1}{2}J^\intercal M^{-1}J \\
&= \frac{1}{2}\theta^\intercal M\theta - \frac{1}{2}\theta^\intercal J + \frac{1}{2}J^\intercal\theta - \frac{1}{2}J^\intercal M^{-1}J + \frac{1}{2}J^\intercal M^{-1}J \\
&= \frac{1}{2}\theta^\intercal M\theta + J^\intercal\theta
\end{aligned}$$

where we have used the fact that $(M^{-1})^\intercal = (M^\intercal)^{-1} = -M^{-1}$ and $\theta^\intercal J = -J^\intercal\theta$.

Substituting $\chi = \theta - M^{-1}J$, the integral $\int \exp(\frac{1}{2}\theta^\intercal M\theta + J^\intercal\theta)\,\mathrm{d}\theta$ becomes

$$\begin{aligned}
\int \exp\left(\frac{1}{2}\chi^\intercal M\chi + \frac{1}{2}J^\intercal M^{-1}J\right)\mathrm{d}\theta &= \int \exp\left(\frac{1}{2}\chi^\intercal M\chi\right)\mathrm{d}\theta \exp\left(\frac{1}{2}J^\intercal M^{-1}J\right) \\
&= \int \exp\left(\frac{1}{2}\theta^\intercal M\theta\right)\mathrm{d}\theta \exp\left(\frac{1}{2}J^\intercal M^{-1}J\right) \\
&\qquad\qquad\text{(by Prop A.7)} \\
&= \mathrm{Pf}(M)\exp\left(\frac{1}{2}J^\intercal M^{-1}J\right) \quad \text{(by Lemma A.5)}
\end{aligned}$$

Note that in general for $x, y \in \bigwedge V$, $\exp(x+y) = \exp(x)\exp(y)$ does not hold. But it holds in this case because $J^\intercal MJ$ is an even Grassman number. $\square$

Appendix B

Transversality

Theorem B.1. *Let S be a subset of a smooth manifold M. Then S is a k-dimensional submanifold if and only if S is locally the zero locus of a submersion, i.e. for every $x \in S$ there is a neighbourhood of $U \subset M$ of x and a submersion $G : U \to \mathbb{R}^{n-k}$ such that $S \cap U = G^{-1}(0)$.*

Theorem B.2 ([21, Proposition 5.38])**.** *If $S \subset E$ is a submanifold, and $f : U \to N$ is any local defining submersion map for S, then $T_x S = \ker Df|_x$ for each $x \in S \cap U$.*

Suppose $f : N^n \to M^m$ is a smooth map and $S^k \subset M^m$ is a submanifold.

Definition B.3. A smooth map $f : N \to M$ is <u>transversal</u> to a submanifold $S \subset M$, denoted $f \pitchfork S$, if for every $x \in f^{-1}(S)$

$$Df|_x(T_x N) + T_{f(x)}S = T_{f(x)}M$$

Theorem B.4 ([21, Theorem 6.30])**.** *If $f \pitchfork S$, then $f^{-1}S \hookrightarrow N$ is a submanifold of dimension $n - m + k$.*

Theorem B.5 (Thom transversality theorem)**.** *Let M and E be smooth manifolds, and let $S \subset E$ be a submanifold. The subspace $C^\infty_{\pitchfork S}(M, E) \subset C^\infty(M, E)$, consisting of smooth maps $f : M \to E$ which are transverse to S, is dense.*

Theorem B.6 (Transversality homotopy theorem [21, Theorem 6.36])**.** *Suppose M and E are smooth manifolds and $S \subset E$ is a submanifold. For any smooth map $f : M \to E$, there exists a smooth map $g : M \to E$ homotopic to f such that $g \pitchfork S$.*

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