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# Entanglement criteria for Dicke states

Marcel Bergmann<sup>1,2</sup> and Otfried Gühne<sup>2</sup>

<sup>1</sup> Institut für Physik, Johannes-Gutenberg-Universität, Staudingerweg 7, D-55128 Mainz, Germany

<sup>2</sup> Naturwissenschaftlich-Technische Fakultät, Universität Siegen, Walter-Flex-Str. 3, D-57068 Siegen, Germany

E-mail: [marcel.bergmann@uni-mainz.de](mailto:marcel.bergmann@uni-mainz.de)

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## Abstract

Dicke states are a family of multi-qubit quantum states with interesting entanglement properties and they have been observed in many experiments. We construct entanglement witnesses for detecting genuine multiparticle entanglement in the vicinity of these states. We use the approach of positive partial transpose mixtures to derive the conditions analytically. For nearly all cases, our criteria are stronger than all conditions previously known.

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(Some figures may appear in colour only in the online journal)

## 1. Introduction

Multiparticle entanglement plays a crucial role for different aspects of quantum information processing. Therefore, many researchers work on the theoretical characterization of entanglement or the experimental observation of entangled states using ions, photons, or solid-state systems [1, 2]. Among the different types of multiparticle quantum states, Dicke states have attracted a lot of attention. These states were first investigated in 1954 by R Dicke for describing light emission from a cloud of atoms [3], and recently several other features have been studied. Dicke states are relatively robust to decoherence [4], their permutational symmetry allows to simplify the task of state tomography [5, 6] and entanglement characterization [7–10]. In addition, they are the symmetric states which are in some sense far away from the separable states [11]. Finally, they are relatively easy to generate in photon experiments, and Dicke states with up to six photons have been observed experimentally [12–14].

In this paper we develop an analytical approach to characterize genuine multiparticle entanglement in Dicke states. Our methods use the idea of positive partial transpose (PPT) mixtures [15]. This means that the set of all biseparable states is approximated by the set of all states that can be written as a mixture of states which have a PPT with respect to some partition. This approach has turned out to be fruitful for characterizing genuine multiparticle

entanglement. For many examples, it delivers the strongest entanglement criteria so far [15], and it can be shown that it solves the problem of entanglement characterization for many families of states, such as permutationally invariant three-qubit states [10] or graph-diagonal four-qubit states [16]. For the family of permutationally invariant states (to which the Dicke states belong) it was shown in [10] that one can decide the question whether a given state is a PPT mixture or not numerically with an effort scaling polynomially with the number of particles. In practice, however, the numerical approach is limited to systems with up to ten qubits, and therefore analytical results for arbitrary qubit numbers are highly desirable.

This paper is structured as follows. In section 2 we explain the main definitions and concepts which we need for our presentation. This includes the concept of genuine multiparticle entanglement and the notions of entanglement witnesses, PPT mixtures and the definition of Dicke states. In section 3 we determine the projective entanglement witnesses for Dicke states with an arbitrary number of excitations. This completes existing work where these witnesses were derived for two special cases [7, 12]; moreover, these witnesses will be a starting point for the further improvement with the PPT mixture approach. In section 4 we first improve the projective entanglement witness for Dicke states with  $N/2$  excitations by a more advanced ansatz and then generalize this to the case of Dicke states with arbitrary excitations. This section contains the main results of this paper. We also compare the resulting noise tolerances with the best known results from the literature. Finally, section 5 is concerned with entanglement witnesses for W states, these are the Dicke states with only one excitation. Besides the characterization via PPT mixtures, we present a different construction which turns out to be stronger for this special case. Finally, we conclude and discuss possible generalizations of our results.

## 2. Basic concepts

### 2.1. Multipartite entanglement and entanglement witnesses

We begin by explaining the structure of the set of entangled states in multiparticle systems. For reasons of simplification, all definitions are given for three-particle states, but it should be stressed that they can be extended to an arbitrary number of particles in a straightforward manner.

For three or more particles, different notions of entanglement exist<sup>3</sup>, but the most investigated class of entanglement is genuine multiparticle entanglement [1, 2]. First, a three-particle state is called biseparable if it can be written as a convex combination of states which are separable for some bipartition,

$$\varrho^{\text{bs}} = p_1 \varrho_{A|BC}^{\text{sep}} + p_2 \varrho_{C|AB}^{\text{sep}} + p_3 \varrho_{B|AC}^{\text{sep}}. \quad (1)$$

That means that the set of biseparable states is given by the convex hull of the states that are separable for a fixed bipartition. Any state that is not a biseparable state is called genuine multiparticle entangled. For more than three particles, this definition can directly be extended, only the number of possible bipartitions increases. To prove that a given state is entangled, one has to check whether it can be written in the above form or not. Unfortunately, it is not efficiently feasible to find all possible decompositions of the desired form.

An efficient tool for the detection of genuine multipartite entanglement which is frequently used in experiments is entanglement witnesses [2]. A witness operator is defined as an observable which has a positive expectation value on all biseparable states, but a negative expectation value on at least one genuine multiparticle entangled state. Experimentally,

<sup>3</sup> A detailed discussion of the various definitions can be found in [2].

entanglement can then be proven by measuring a negative expectation value. Compared to other detection tools, e.g. Bell inequalities, the use of entanglement witnesses is advantageous because for every entangled state there exists an entanglement witness that detects it.

The fact that every entangled state is detected by at least one entanglement witness leads to the question of how to find an suitable witness for a given quantum state. A typical construction goes as follows: for every pure state  $|\psi\rangle$ , the operator

$$\mathcal{W} = \alpha \mathbb{1} - |\psi\rangle\langle\psi| \quad (2)$$

is called the projective entanglement witness [2]. Here  $\alpha$  is the squared maximal overlap between  $|\psi\rangle$  and the biseparable states,  $\alpha = \sup_{|\phi\rangle \in \text{bisep}} |\langle\psi|\phi\rangle|^2$ . The maximal overlap  $\alpha$  can be computed explicitly by taking the square of the maximal Schmidt coefficient, in addition maximized over all possible bipartitions.

## 2.2. Dicke states

The focus of this article is on entanglement witnesses for Dicke states. An  $N$ -qubit Dicke state with  $k$  excitations is defined as [3, 7]

$$|D_k^N\rangle = \frac{1}{\sqrt{\binom{N}{k}}} \sum_j P_j \{ |1\rangle^{\otimes k} \otimes |0\rangle^{\otimes (N-k)} \}, \quad (3)$$

where  $\sum_j P_j \{\cdot\}$  denotes the sum over all possible permutations. For example,  $|D_2^3\rangle = \frac{1}{\sqrt{3}}(|110\rangle + |101\rangle + |011\rangle)$ . The case of  $k = N/2$  excitations for even  $N$  is the most discussed example of a Dicke state. In this case, an analytical expression of the projective entanglement witness was derived [7]

$$\mathcal{W} = \frac{1}{2} \frac{N}{N-1} \mathbb{1} - |D_{\frac{N}{2}}^N\rangle\langle D_{\frac{N}{2}}^N|. \quad (4)$$

Another important case is the Dicke state with  $k = 1$  excitations, these states are also called W states [17],

$$|W_N\rangle = \frac{1}{\sqrt{N}} (|100\dots 0\rangle + |010\dots 0\rangle + \dots + |000\dots 1\rangle). \quad (5)$$

In this case, the projective witness is known to be [12]

$$\mathcal{W} = \frac{N-1}{N} \mathbb{1} - |W_N\rangle\langle W_N|. \quad (6)$$

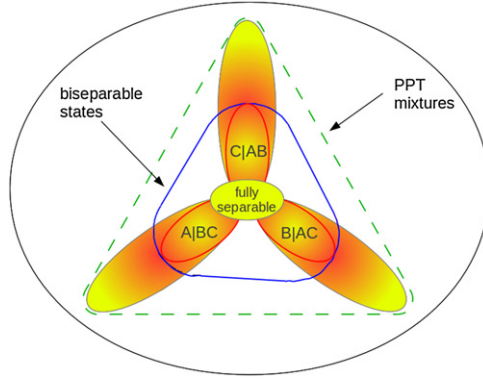
We stress that other types of witnesses and entanglement criteria for Dicke states are known [9], later we will also compare them with our results.

## 2.3. The method of PPT mixtures

In order to improve the existing entanglement criteria for Dicke states, we consider a more general type of witness than the projective witnesses, the so-called fully decomposable entanglement witnesses. They are derived from the approach of PPT mixtures [15], and how this method is described in the following.

In this approach, instead of considering biseparable states, one considers a superset of the set of biseparable states, the PPT mixtures. This set is given by those states that can be written as convex combination of states that have a positive partial transposition (PPT) with respect to a fixed bipartition [18]. For three particles, these states are of the form

$$\varrho^{\text{pmix}} = p_1 \varrho_{A|BC}^{\text{ppt}} + p_2 \varrho_{C|AB}^{\text{ppt}} + p_3 \varrho_{B|AC}^{\text{ppt}}. \quad (7)$$



**Figure 1.** Schematic view of the PPT mixtures (green, dashed lines) as an outer approximation of the set of biseparable states (blue, solid lines). See text for further details.

Since every separable state is necessarily a PPT state, the PPT mixtures are indeed a superset of the biseparable states (see figure 1). In order to characterize PPT mixtures in a better way, it was then shown in [15] that all states which are not PPT mixtures can be detected by entanglement witnesses with a simple form.

**Observation 1.** *Every state  $\rho$  that is not a PPT mixture can be detected by a witness that can be written as*

$$\mathcal{W} = P_M + Q_M^{T_M} \text{ with } P_M \geq 0, \quad Q_M \geq 0, \quad (8)$$

*for all possible bipartitions  $M$ . Witnesses of this type are also positive on all PPT mixtures. These witnesses are called fully decomposable witnesses.*

Therefore, the characterization of PPT mixtures boils down to a characterization of fully decomposable witnesses. For a given multiparticle quantum state  $\rho$ , the question whether it is detected by a fully decomposable witness can be formulated as the optimization problem [15]

$$\begin{aligned} &\text{minimize: } \text{Tr}(\mathcal{W}\rho), \\ &\text{subject to: } \text{Tr}(\mathcal{W}) = 1 \text{ and for all partitions } M: \\ &\quad \mathcal{W} = P_M + Q_M^{T_M}, \quad Q_M, P_M \geq 0. \end{aligned} \quad (9)$$

This problem is a convex optimization problem that can efficiently be solved by semidefinite programming and a ready-to-use implementation which can in practice be applied to systems up to six qubits is freely available [19].

The approach of PPT mixtures for permutationally invariant states was recently investigated by Novo *et al* [10]. With the help of the symmetry of such states, a simplified formulation was found, which leads to a numerical optimization problem, where the number of parameters scales only polynomially as  $\mathcal{O}[N^7]$ . This allows to investigate system sizes of about ten qubits for which the original implementation fails. Though this is an advance, numerical solutions are always restricted to a limited system size and analytical solutions are needed. This is exactly the aim of the present paper.

In the following, we mainly deal with fully PPT witnesses, which is a special kind of fully decomposable witnesses. Fully PPT witnesses have the additional property that the operators  $P_M$  equal zero for any bipartition. Thus, they are of the structure

$$\mathcal{W} = Q_M^{T_M} \text{ with } Q_M \geq 0. \quad (10)$$

A fully PPT witness is easier to characterize than a fully decomposable witness because it is sufficient to show that any partial transposition with respect to fixed bipartition is positive. Clearly, the class of fully PPT witnesses does not detect as many states as the class of fully decomposable witnesses. But, as we will see below, the resulting entanglement criteria are often already better than all other constructions known so far.

### 3. Projective entanglement witnesses for arbitrary Dicke states

As mentioned above, projective entanglement witnesses for Dicke states are known analytically only for two special cases. In this section we will fill this gap, but first we derive a useful result on the structure of general projective entanglement witnesses<sup>4</sup>.

**Proposition 2.** *Every projective entanglement witness is a fully PPT witness, therefore it detects only states which are NPT (that is, they are not PPT) with respect to all bipartitions.*

**Proof.** We consider the Schmidt decomposition of the state  $|\psi\rangle$  with respect to a fixed bipartition  $|\psi\rangle = \sum_i a_i |ii\rangle$ , where the Schmidt coefficients are defined in decreasing order, i.e.,  $a_0 \geq a_1 \geq \dots \geq a_N$ . Then the corresponding projective entanglement witness is given by  $\mathcal{W} = \alpha \mathbb{1} - |\psi\rangle\langle\psi|$ , and the coefficient  $\alpha$  fulfils  $\alpha \geq a_0^2$ . Calculating the partial transposition of  $\mathcal{W}$  for the given bipartition results in a matrix that can be brought, via permuting columns and rows, into block structure consisting of  $2 \times 2$ -blocks of the form

$$X = \begin{pmatrix} \alpha & -a_i a_j \\ -a_i a_j & \alpha \end{pmatrix}.$$

These blocks have no negative eigenvalues since  $\alpha \geq a_i a_j$  for all  $i, j$  and therefore this partial transposition, as well as the transpositions for the other bipartitions, are non-negative.  $\square$

Based on the Schmidt decomposition, we are able to calculate the projective entanglement witness for Dicke states with arbitrary  $k$ . We formulate it for any  $k < N/2$ , the expression in the case of  $k = N/2$  was already given in [7], and the case  $k > N/2$  corresponds to the case  $k' = N - k < N/2$ .

**Proposition 3.** *For an arbitrary Dicke state  $|D_k^N\rangle$  with  $1 < k < \frac{N}{2}$ , the corresponding projective entanglement witness  $\mathcal{W}_{\text{proj}}$  is given by*

$$\mathcal{W}_{\text{proj}} = \frac{N-k}{N} \mathbb{1} - |D_k^N\rangle\langle D_k^N|. \quad (11)$$

**Proof.** We determine the Schmidt decomposition of  $|D_k^N\rangle$  for the bipartition  $\hat{A}|\hat{B}$ , where  $\hat{A}$  denotes the first  $A$  qubits, and  $\hat{B}$  denotes the last  $B = N - A$  qubits. We can rewrite the Dicke state as

$$\begin{aligned} |D_k^N\rangle &= \frac{1}{\sqrt{\binom{N}{k}}} \sum_j P_j \{ |1\rangle^{\otimes k} \otimes |0\rangle^{\otimes (N-k)} \} \\ &= \frac{1}{\sqrt{\binom{N}{k}}} \sum_{\alpha=0}^k \sqrt{\binom{A}{\alpha}} \sqrt{\binom{B}{k-\alpha}} |\alpha\rangle_{\hat{A}} \otimes |k-\alpha\rangle_{\hat{B}}. \end{aligned} \quad (12)$$

Here,  $|\alpha\rangle_{\hat{A}}$  (resp.,  $|k-\alpha\rangle_{\hat{B}}$ ) denotes a normalized and symmetrized superposition of  $\alpha$  (resp.,  $k-\alpha$ ) excitations on the system  $\hat{A}$  (resp.,  $\hat{B}$ ); in other words,  $|\alpha\rangle_{\hat{A}}$  is the Dicke state  $|D_\alpha^A\rangle$  on the

<sup>4</sup> We stress that this statement has occurred in the literature, the implicit proof from [20] is, however, incomplete.

first  $A$  qubits. For the maximal overlap  $\alpha$  with the biseparable states, we now have to compute the maximally possible squared Schmidt coefficient,

$$\alpha = \max_{1 \leq A \leq N-1} \max_{0 \leq \beta \leq k} \frac{\binom{A}{\beta} \binom{N-A}{k-\beta}}{\binom{N}{k}} = \frac{N-k}{N}. \quad (13)$$

Details of this maximization are described in appendix A.1.  $\square$

We note that the projective witness can be improved since the full identity operator  $\mathbb{1}$  is not needed in order to guarantee positivity on all biseparable states. Indeed, as already known for the W state [12], one can substitute the  $\mathbb{1}$  by  $\mathbb{1}_{2k}$  which is the identity on the space with maximally  $2k$  excitations. Using the same argument as in the proof of proposition 2 one finds that also with this substitution the witness is a fully PPT witness. So the refined projective witness is given by

$$\mathcal{W}'_{\text{proj}} = \frac{N-k}{N} \mathbb{1}_{2k} - |D_k^N\rangle\langle D_k^N|. \quad (14)$$

Entanglement witnesses are a frequently used tool in experiments, and in order to compare the strength of different witnesses, the resistance to white noise is often used. For that, one considers noisy states of the form

$$\rho_{\text{noise}} = p \frac{\mathbb{1}}{2^N} + (1-p) |D_k^N\rangle\langle D_k^N|, \quad (15)$$

and asks for the critical value  $p_{\text{crit}}$  when the state is not detected anymore. For the witness  $\mathcal{W}'_{\text{proj}}$  the noise tolerance is

$$p_{\text{crit}} = \frac{1}{1 + \frac{1}{2^N} \left[ \frac{N-k}{k} \sum_{i=0}^{2k} \binom{N}{i} - \frac{N}{k} \right]}. \quad (16)$$

For any  $k < N/2$  the value of  $p_{\text{crit}}$  tends to one if particle number  $N$  increases, meaning that a large fraction of white noise can be added and still the entanglement is preserved. Therefore, the genuine multiparticle entanglement in Dicke states with  $k < N/2$  is very robust for large particle numbers.

#### 4. Advanced entanglement witnesses for general Dicke states

In this section, we will present our main idea to construct strong entanglement criteria for Dicke states. To explain our idea, we start with a discussion of the Dicke state with  $k = N/2$  excitations and derive a simple improvement of the projector based witness in equation (4). Then, we introduce a general construction for arbitrary Dicke states.

To start, we make the following ansatz for a witness for Dicke states with  $k = N/2$  excitations:

$$\mathcal{W}_{\frac{N}{2}} = \sum_{i=0}^N \omega_i \Pi_i - |D_{\frac{N}{2}}^N\rangle\langle D_{\frac{N}{2}}^N|. \quad (17)$$

Here,  $\Pi_i$  is the projector onto the space with exactly  $i$  excitations. The task is now to determine the parameters  $\omega_i$  such that the witness is positive on all biseparable states. One possible choice is to take  $\omega_i = N/(2N-2)$ , since this leads to the witness in equation (4). For some values of  $i$  one can choose smaller  $\omega_i$ , however, and improve the witness in equation (4) in this way.

To do so, the parameters  $\omega_i$  are determined by the requirement that all partial transpositions of  $\mathcal{W}_{\frac{N}{2}}$  should be positive semidefinite (see equation (10)). Since the calculation of the partial transposition of the witness turns out to be demanding, we first use an approximation known as Gershgorin circles [21].

**Lemma 4.** Let  $A = (a_{ij})$  a complex  $n \times n$ -matrix,  $R_i = \sum_{j \neq i} |a_{ij}|$  the sum of the off-diagonal elements in the  $i$ th row, and  $D_i = D(a_{ii}, R_i)$  the closed disc around  $a_{ii}$  with radius  $R_i$ . Then, all eigenvalues of  $A$  lie in the union of the discs  $D_i$ .

Since the eigenvalues of a Hermitian matrix  $A$  are real, it is sufficient that all Gershgorin circles lie in the half space with positive real part in the complex plane in order to guarantee that  $A$  is positive semidefinite. This will be used in the following.

**Proposition 5.** The operator  $\mathcal{W}_{\frac{N}{2}}$  is a fully PPT entanglement witness if

$$\omega_i = \min \left\{ \frac{\binom{N}{i}}{\binom{N}{N/2}}, \frac{1}{2} \frac{N}{N-1} \right\}. \quad (18)$$

holds for all  $i = 0, 1, \dots, N$ .

**Proof.** The condition follows by calculating the partial transposition of  $\mathcal{W}_{\frac{N}{2}}$  with respect to the first  $\delta$  qubits, where  $\delta$  is fixed but arbitrary in the interval  $1 \leq \delta \leq \frac{N}{2}$ . This transposition has a block structure where the blocks are of the form

$$X_\delta = \begin{pmatrix} \omega_{N/2+\delta} \mathbb{1} & A \\ A & \omega_{N/2-\delta} \mathbb{1} \end{pmatrix}. \quad (19)$$

Here, all the four sub-blocks of the matrix  $X_\delta$  are  $\binom{N}{i} \times \binom{N}{i}$  matrices (with  $i = N/2 + \delta$  or  $i = N/2 - \delta$ ) and the entries in the block  $A$  are zero or  $1/\binom{N}{N/2}$ , coming from the normalization of the state  $|D_{\frac{N}{2}}^N\rangle$ . Then, lemma 4 implies that if  $\omega_i \binom{N}{N/2} \geq \binom{N}{i}$  is fulfilled, then the matrix  $X_\delta$  is positive semidefinite. The choice in equation (18) is then justified because we know from proposition 1 that the projective witness in equation (4) is also a fully PPT witness. Therefore, also for the choice  $\omega_i = N/(2N-2)$  the block matrices  $X_\delta$  are positive semidefinite. Clearly, taking the minimum of both expressions results in the strongest witness.  $\square$

To generalize the ansatz of equation (17) to arbitrary Dicke states, we make the ansatz

$$\mathcal{W}_{N,k}^{\text{opti}} = \sum_{i=0}^{2k} \omega_i \Pi_i - |D_k^N\rangle \langle D_k^N| \quad (20)$$

for a fully PPT entanglement witness for Dicke states with  $k \neq \frac{N}{2}$ . To obtain a fully PPT witness, the parameters  $\omega_i$  have to fulfil certain conditions which can be found by a detailed singular value decomposition of matrices similar to matrix  $A$  in equation (19). One finds:

**Theorem 6.** The operator  $\mathcal{W}_{N,k}^{\text{opti}}$  is a fully PPT witness, if the following two conditions are fulfilled.

(i) For  $i \neq k$  the parameters  $\omega_i$  obey

$$\omega_{k-\delta} \omega_{k+\delta} \geq (\lambda_{\max})^2 := \max_{x_1, x_2} \{\lambda^2(x_1, x_2)\}, \quad (21)$$

where the coefficients  $\lambda(x_1, x_2)$  are given by

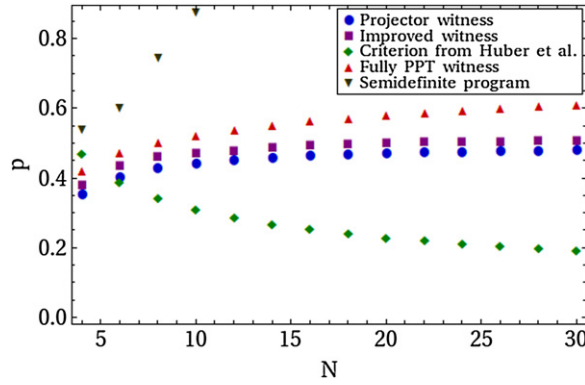
$$\lambda^2(x_1, x_2) = \frac{\binom{N-x_1-x_2}{k-x_2} \binom{x_1+x_2}{x_2} \binom{N-x_1-x_2}{k-\delta-x_2} \binom{x_1+x_2}{x_2+\delta}}{\binom{N}{k}^2}. \quad (22)$$

For the maximization of  $\lambda^2(x_1, x_2)$ , it suffices to consider the case that  $x_1 \geq \delta$  and  $x_1 + x_2 \leq N/2$ .

(ii) For  $i = k$  the parameter  $\omega_i$  is given by

$$\omega_i = \mu(N, k) := \begin{cases} \frac{N-k}{N} & \text{for } k < N/2 \\ \frac{N}{2(N-1)} & \text{for } k = N/2 \end{cases}. \quad (23)$$





**Figure 2.** Comparison of the noise robustness of the different entanglement detection methods for Dicke states with  $k = N/2$ . See text for further details.

**Proof.** The proof for this statement can be found in section A.2 of the [appendix](#). It should be noted that the alternative estimate  $\omega_{k\pm\delta} \geq \mu(N, k)$ , as it was used in equation (18), is not useful here, and the estimate from equation (21) is always stronger. This follows from the previous result in proposition 3, see also equations (A.5) and (A.6) in section A.1 of the [appendix](#). Applied to the definition of  $\lambda^2(x_1, x_2)$  these estimates show that  $\lambda^2(x_1, x_2) \leq \mu^2(N, k)$ .  $\square$

It remains to find the optimal choice of the coefficients  $\omega_i$  in order to maximize the noise tolerance. If we consider white noise, the optimal noise tolerance is achieved, if the trace of  $\mathcal{W}_{N,k}^{\text{opti}}$  is minimal. This is the case if for any  $\delta \geq 1$  the value  $\binom{N}{k-\delta}\omega_{k-\delta} + \binom{N}{k+\delta}\omega_{k+\delta}$  is minimal. Using Lagrange multipliers one can directly derive the following result.

**Corollary 7.** Consider the witness  $\mathcal{W}_{N,k}^{\text{opti}}$  from theorem 6. The optimal noise robustness with respect to white noise is achieved if the  $\omega_{k\pm\delta}$  are given by

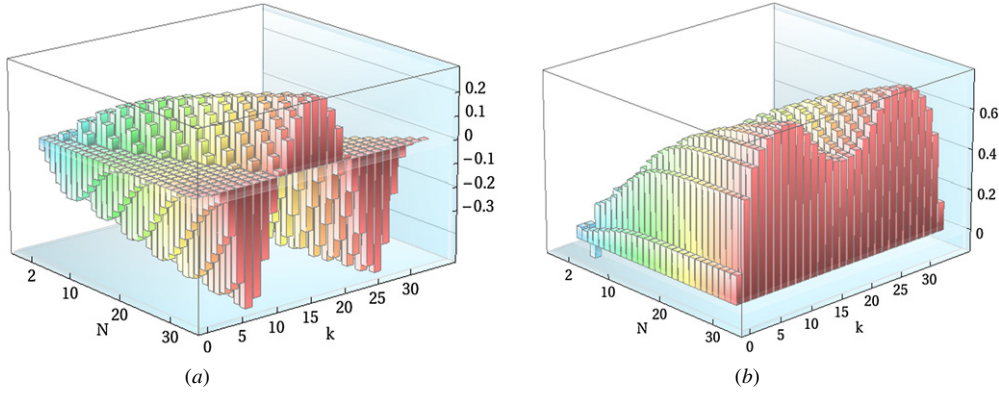
$$\omega_{k-\delta} = \sqrt{\frac{\binom{N}{k+\delta}}{\binom{N}{k-\delta}}} \lambda_{\max} \quad \text{and} \quad \omega_{k+\delta} = \sqrt{\frac{\binom{N}{k-\delta}}{\binom{N}{k+\delta}}} \lambda_{\max}. \quad (24)$$

Now it is time to compare our new criteria with existing criteria from the literature. Here, the work of Huber *et al* [9] is important, where interesting entanglement criteria for Dicke states have been derived, which are for many cases the best known results thus far. The noise tolerance of these criteria is given by

$$p_{\text{crit}} = \frac{1}{1 + \frac{1}{2^N} [(2N - 2k - 1) \binom{N}{k}]}. \quad (25)$$

In addition to these criteria we consider for  $N \leq 10$  also the direct solution of the semidefinite program in equation (9), obtained with the methods of [10].

In figure 2 we compare the criteria for the case  $k = N/2$ . It can be seen that the improved witness from proposition 5 is better than the projective witness by some amount, but the fully PPT witness from theorem 6 results in a further significant improvement. The full solution of the semidefinite program is, for the cases that it can be computed, clearly the best criterion. This shows that there is still space for improving the analytical criteria of this paper.



**Figure 3.** (a) Comparison between the projective entanglement witness (proposition 3 and equations (14), (16)) and the criterion of [9]. Here, the difference of the noise robustnesses  $p_{\text{crit}}$  is shown. For the cases with  $k \approx N/2$ , equation (14) results in a better criterion, if  $k \ll N/2$  or  $k \gg N/2$  the results from [9] are stronger. (b) Comparison between the PPT entanglement witness (theorem 6) and the criterion of [9]. Here, for all cases apart from  $(N, k) = (4, 2)$  the criteria from theorem 6 are stronger. The figure displays  $(p_{\text{crit}}^{\text{Thm.6}} - p_{\text{crit}}^{[9]}) / (1 - p_{\text{crit}}^{[9]})$ . This quantifies the fraction of the states not detected by [9], which are in addition detected by theorem 6.

In figure 3(a) we compare the projective witness from proposition 3 and equations (14), (16) with the results of [9]. For the case  $k \approx N/2$ , the criterion from equation (14) is better, but if  $k \ll N/2$  or  $k \gg N/2$  the results from [9] are superior. In figure 3(b) we compare the PPT entanglement witness (theorem 6) and the criterion of [9]. Here, for all cases [apart from  $(N, k) = (4, 2)$ ] the criteria from theorem 6 are stronger. Finally, a detailed comparison for the Dicke states with  $k = 1$  can be found in the following section.

## 5. Results for W states

In this section, we discuss in some detail entanglement witnesses for Dicke states with only one excitation, the so-called W states. Besides the construction from theorem 6 we will introduce a different type of witness, which does not rely on the approach of PPT mixtures.

The set of W states is given by  $|W_N\rangle = |D_1^N\rangle = (|100\dots 0\rangle + |010\dots 0\rangle + \dots + |000\dots 1\rangle) / \sqrt{N}$ . For these states, we use the same ansatz for a fully PPT entanglement witness as in the section before (see equation (20)),

$$\mathcal{W}_N^{\text{opti}} = \sum_{i=0}^2 \omega_i \Pi_i - |W_N\rangle \langle W_N|. \quad (26)$$

Note that it was shown in [22] that for the experimental determination of  $\mathcal{W}_N^{\text{opti}}$  only  $2N - 1$  local measurements are necessary. First, we can now explicitly write down the witnesses following the method in theorem 6. We have:

**Corollary 8.** Consider the  $N$ -qubit W state. If  $N$  is even, the witness with the highest noise tolerance according to corollary 7 is given by

$$\omega_0 = \frac{\sqrt{N(N-1)}}{\sqrt{8}}, \quad \omega_1 = \frac{N-1}{N}, \quad \omega_2 = \frac{1}{\sqrt{2N(N-1)}}. \quad (27)$$

For the case that  $N$  is odd, the optimal coefficients are

$$\omega_0 = \frac{\sqrt{(N-1)^2(N+1)}}{\sqrt{8N}}, \quad \omega_1 = \frac{N-1}{N}, \quad \omega_2 = \frac{\sqrt{N+1}}{\sqrt{2N^3}}. \quad (28)$$

For both cases, the noise tolerance is given by

$$p_{\text{crit}} = \frac{1}{1 + \frac{1}{2^N} [2N\omega_0 + N(N-2)]}. \quad (29)$$

**Proof.** Starting from theorem 6, we have to compute  $\lambda_{\text{max}}^2$ . In the definition of  $\lambda^2(x_1, x_2)$ , we have  $k = \delta = 1$  and therefore  $x_2 = 0$ . Then,  $\lambda^2(x_1, 0) = x_1(N - x_1)/N^2$ . If  $N$  is even, this is maximal for  $x_1 = N/2$  and therefore  $\lambda_{\text{max}}^2 = 1/4$ . If  $N$  is odd, the optimum is attained at  $x_1 = (N-1)/2$ , resulting in  $\lambda_{\text{max}}^2 = 1/4(1 - 1/N^2)$ . Then the claim follows from a simple calculation.  $\square$

We can improve this result by going back to the original definition of an entanglement witness as an observable being positive on all biseparable states. Using the ansatz in equation (26), the parameters  $\omega_i$  can be determined by requiring that the minimal overlap of  $\mathcal{W}_N^{\text{opti}}$  with the pure biseparable states is non-negative. This defines a new witness  $\tilde{\mathcal{W}}_N$ . In the following, we show that this optimization problem can be reduced to a simple numerical optimization, in principle, also other methods might be feasible [23]. Note that the following proposition is a generalization of the method used in [12] to construct witnesses for W states.

**Proposition 9.** For given parameters  $\omega_0, \omega_1$  and  $\omega_2$ , the operator  $\tilde{\mathcal{W}}_N$  defines an entanglement witness iff the result of the following optimization,

$$\Upsilon := \max_{K,L} \max_{\alpha,\beta} \max_{\delta,\gamma} \left\{ \frac{(K\beta\gamma + L\delta\alpha)^2}{N} - \omega_0\alpha^2\gamma^2 - \omega_1(\alpha^2 + \gamma^2 - 2\alpha^2\gamma^2) - \omega_2(1 - \alpha^2 - \gamma^2 + \alpha^2\gamma^2) \right\} \quad (30)$$

is not positive,  $\Upsilon \leq 0$ . Here, the optimization must be carried out under the constraints

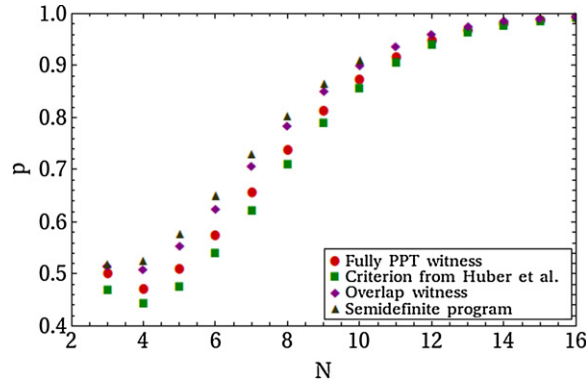
$$K + L = N, \quad \alpha^2 + K\beta^2 = 1 \quad \text{and} \quad \gamma^2 + L\delta^2 = 1. \quad (31)$$

**Proof.** The proof is given in section A.3 of the appendix.  $\square$

Before comparing this result with the previous witnesses, let us discuss its implications. First, note that the maximizations in the definition of  $\Upsilon$  are lengthy, but elementary. They may be rewritten as a linear optimization with polynomial constraints, then they belong to a class of problems, where the *global* maximum can be obtained in an efficient manner [24]. Also, the complexity of the optimization procedure does not increase with the number of qubits.

Second, note that shifting all the  $\omega_i$  by a constant amount, i.e.  $\omega_i \rightarrow \omega_i + \varepsilon$  results in a decrease of  $\Upsilon$  by the same constant  $\varepsilon$ . Hence, proposition 9 constitutes a constructive way to determine the  $\omega_i$  for an good witness. One starts with arbitrary  $\omega_i$ , performs the maximization resulting in a constant  $\tilde{\Upsilon}$ , then shifts the  $\omega_i$  by  $\tilde{\Upsilon}$  to achieve that finally  $\Upsilon = 0$ .

In figure 4 we compare the witness from corollary 8, the criterion from [9], the overlap witness from proposition 9 and the solution of the semidefinite program [10]. One can clearly see that proposition 9 results in a stronger criterion than corollary 8. This is not surprising. The entanglement criterion from proposition is more general than the fully PPT entanglement witness, since it does not rely on any structural approximation. It should be noted, however, that for large  $N$  all three criteria detect nearly all W states affected by white noise.



**Figure 4.** Comparison of the different criteria (corollary 8, proposition 9 and [9, 10]) for  $N$ -qubit  $W$  states. See text for further details.

## 6. Conclusion

Using the approach of PPT mixtures we have derived entanglement criteria which are suited for Dicke states. In most cases, these criteria are stronger than entanglement conditions previously known. In this way, we have demonstrated how the method of PPT mixtures, being at first sight a numerical tool, can be used to gain analytical insight into the structure of entanglement.

There are several ways to extend our results. First, one can try to derive analytical conditions for an operator as in equation (20) to be a fully decomposable witness. This, however, seems very demanding to us. A more direct research problem is the application of the method of PPT mixtures to other families of states with symmetries, e.g. states with  $U \otimes U \otimes \dots \otimes U$ -symmetry [25]. This may finally help to answer the question for which types of states the criterion of PPT mixtures constitutes a necessary and sufficient criterion of genuine multiparticle entanglement.

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## Appendix

### A.1. A lemma required for the proof of proposition 3

**Lemma A.1.** Consider a pair of integer numbers  $A, B \geq 1$  with  $A + B = N$  and a second pair  $\alpha, \beta \geq 0$  with  $\alpha + \beta = k$ , where  $k < N/2$ . Then the inequality

$$\binom{A}{\alpha} \binom{B}{\beta} \leq \binom{N-1}{k} \quad (\text{A.1})$$

holds, and equality is assumed for  $A = 1, B = N - 1$  and  $\alpha = 0$  and  $\beta = k$ .

**Proof.**<sup>5</sup> First, consider the case that  $A = 1$  (or, equivalently,  $B = 1$ ). Then, for  $\alpha$  only the possibilities  $\alpha = 0$  or  $\alpha = 1$  have to be considered, but for both cases the statement in equation (A.1) is clear. Therefore, we can assume in the following that  $A \geq 2$  and  $B \geq 2$ . Then, using the fact that

$$\binom{N-1}{k} = \binom{N}{k} - \binom{N-1}{k-1} = \binom{N}{k} - \left[ \binom{N-2}{k-2} + \binom{N-2}{k-1} \right] \quad (\text{A.2})$$

the statement of the lemma is equivalent to

$$\underbrace{\binom{A}{\alpha} \binom{B}{\beta}}_{T_1} + \underbrace{\binom{N-2}{k-1}}_{T_2} + \underbrace{\binom{N-2}{k-2}}_{T_3} \leq \binom{N}{k}. \quad (\text{A.3})$$

Since  $k < N/2$  we have that

$$\frac{\binom{N-2}{k-2}}{\binom{N-2}{k-1}} = \frac{k-1}{N-k} < 1, \quad (\text{A.4})$$

which implies that  $T_2 > T_3$  and consequently  $T_3 \leq T_2 - 1$ .

In the following, we will prove equation (A.3) using a combinatorial argument. The right-hand side of equation (A.3) is the number of possibilities to choose  $k$  elements out of a set  $\hat{N}$  with  $N$  elements, and we show that each of the three terms appearing on the left-hand side of equation (A.3) can be interpreted as the number of *different* ways to choose  $k$  objects out of  $N$ . To start, we consider a partition of the set  $\hat{N}$ , i.e. two disjoint subsets  $\hat{A}$  and  $\hat{B}$  satisfying  $\hat{A} \cup \hat{B} = \hat{N}$  with the cardinalities  $|\hat{A}| = A$  and  $|\hat{B}| = B$ . We proceed in three steps.

- Step 1. The first term  $T_1$  equals the number of ways to choose  $\alpha$  elements out of  $\hat{A}$  and  $\beta$  elements out of  $\hat{B}$ .
- Step 2. To catch the second term  $T_2$ , we fix two elements,  $a_0 \in \hat{A}$  and  $b_0 \in \hat{B}$  and choose  $k-1$  objects out of the remaining  $N-2$  objects. This results in  $a$  elements out of  $\hat{A}$  and  $b$  elements out of  $\hat{B}$  with  $a+b = k-1$ . Due to the fact that  $a+1 = \alpha$  and  $b+1 = \beta$  cannot hold at the same time, we can choose to add either  $a_0$  or  $b_0$  (leading to a selection of  $k$  elements out of  $N-1$ ), which is different from the selections in step 1, since now a different number of terms from  $\hat{A}$  and  $\hat{B}$  is chosen. The number of different selections obtained in that way is equal to the second term  $T_2$ .
- Step 3. In order to justify the third term, a more detailed analysis is required. Let us start with a given choice of a subset  $\hat{a}$  with  $a$  elements out of  $\hat{A} \setminus \{a_0\}$  and a subset  $\hat{b}$  with  $b$  elements out of  $\hat{B} \setminus \{b_0\}$ , which we used in step 2. Note that since  $A \geq 2$  and  $B \geq 2$  the sets  $\hat{A} \setminus \{a_0\}$  and  $\hat{B} \setminus \{b_0\}$  are not empty. We can distinguish two cases.
  - (i) If  $a \geq 1$  and  $b \geq 1$ , then the sets  $\hat{a}$  and  $\hat{b}$  are not empty, and we can choose elements  $a_+ \in \hat{a}$  and  $b_+ \in \hat{b}$ . Then, we can interchange both  $a_+$  with  $a_0$  and  $b_+$  with  $b_0$  and afterwards we can decide (as in step 2) to add again  $a_+$  or  $b_+$  to arrive at a selection of  $k$  elements out of  $N$ . This choice will be different from the one in step 2, since now both  $a_0$  and  $b_0$  are in. It will also be different from the choice in step 1, since we can choose  $a_+$  or  $b_+$ , similar to in step 2.
  - (ii) If  $a \neq A-1$  and  $b \neq B-1$ , then the sets  $\hat{a}$  and  $\hat{b}$  are not equal to the respective supersets  $\hat{A} \setminus \{a_0\}$  and  $\hat{B} \setminus \{b_0\}$ . Then we can choose elements  $a_- \in \hat{A} \setminus (\{a_0\} \cup \hat{a})$  and  $b_- \in \hat{B} \setminus (\{b_0\} \cup \hat{b})$  and interchange both  $a_-$  with  $a_0$  and  $b_-$  with  $b_0$ . Afterwards, we can

<sup>5</sup> From an anonymous referee, we learned that our proof of lemma A.1 can be made significantly shorter. First, one can assume without loss of generality that  $\alpha < A/2$  and consider the case  $\alpha \geq 1, \beta \geq 1$ . Then, one has the estimate  $\binom{A}{\alpha} \binom{B}{\beta} = \binom{A}{\alpha} \binom{B-1}{\beta} + \binom{A}{\alpha} \binom{B-1}{\beta-1} \leq \binom{A}{\alpha} \binom{B-1}{\beta} + \binom{A}{\alpha+1} \binom{B-1}{\beta-1} < \binom{N-1}{k}$ , where also Vandermonde's identity [26] has been used. This already proves the claim.

decide (as in step 2) to add  $a_-$  or  $b_-$  to arrive at a selection of  $k$  elements out of  $N$ . Again, this choice will be different from the one in step 2, since now neither  $a_0$  nor  $b_0$  are in. It is also different from the choice in step 1.

For a given choice of  $\hat{a}$  and  $\hat{b}$  these cases are not exclusive, in fact, often both cases are true. The only situation where *none* of the cases applies is if  $a = A - 1$  and simultaneously  $b = 0$  (or vice versa), corresponding to the case that when choosing  $k - 1$  elements out of  $\hat{N} \setminus \{a_0, b_0\}$  in step 2, the complete set  $A \setminus \{a_0\}$  was chosen. This can only happen if  $A = k$ . Note, however, that then only for a single example of the different choices in step 2 none of the conditions in step 3 applies, for all other choices at least one of the cases is true.

This leads to our final estimate: If  $A \neq k$  and  $B \neq k$  we can apply all three steps outlined above, leading to  $T_1 + T_2 + T_3 \leq \binom{N}{k}$ , which implies equation (A.3). If  $A = k$  or  $B = k$  the same procedure results in  $T_1 + T_2 + (T_3 - 1) \leq \binom{N}{k}$ , also proving equation (A.3).  $\square$

As a final remark, note that from our discussion it follows that in the situation of lemma A.1 for  $k < N/2$  the estimate

$$\binom{A}{\alpha} \binom{B}{\beta} / \binom{N}{k} \leq \frac{N - k}{N} \quad (\text{A.5})$$

follows, while for  $k = N/2$  from [7] the bound

$$\binom{A}{\alpha} \binom{B}{\beta} / \binom{N}{k} \leq \frac{N}{2(N - 1)} \quad (\text{A.6})$$

is known. These properties are useful for the construction of general PPT witnesses for Dicke states, see also the discussion in the proof of theorem 6.

## A.2. Proof of theorem 6

**Proof.** As in the case of proposition 5, the partial transposition of the witness  $\mathcal{W}_{N,k}^{\text{opti}}$  can, after appropriate reordering, be written in a block diagonal form, and the blocks are of the form

$$X_{\delta}^{(k)} = \begin{pmatrix} \omega_{k-\delta} \mathbb{1} & A \\ A^T & \omega_{k+\delta} \mathbb{1} \end{pmatrix}. \quad (\text{A.7})$$

We have to prove that this matrix is positive semidefinite and for that we first characterize the off-diagonal block  $A$ . Note that here the matrix  $A$  is rectangular and not quadratic as in the proof of proposition 5. In the end, we want to determine the singular value decomposition of  $A$ , but this requires several steps.

Step 1. Let us start with the first row of  $A$ . The entries in this first column are of the type

$$|\chi\rangle\langle\eta| = |0 \cdots 0 \underbrace{0 \cdots 0}_{x_1} \underbrace{1 \cdots 1}_{x_2} \underbrace{1 \cdots 1}_{k-\delta-x_2}\rangle\langle\eta|. \quad (\text{A.8})$$

This notation should be understood as follows: the vector  $|\chi\rangle$  contains  $k - \delta$  entries ‘1’, which are, since we are considering the first row of  $A$ , aligned on the left. The partial transposition affects the qubits denoted by  $x$ . In the first  $x_1$  of these qubits, the vector  $|\chi\rangle$  has the entries ‘0’, while for the last  $x_2$  qubits of  $x$  the vector  $|\chi\rangle$  has the entries ‘1’. In the following, it will be useful to denote a set of qubits by  $\hat{x}$ , and the corresponding number of qubits by  $x = |\hat{x}|$ . Clearly, we are considering a special type of the transposition, since the set  $\hat{x}$  consists of neighboring qubits, but, as we will see, this is no restriction, and in the end only the numbers  $x_1$  and  $x_2$  matter, not their position.

Now we consider the possible values for  $\langle \eta |$ . Clearly,  $\langle \eta |$  has  $k + \delta$  entries with the value ‘1’. These entries must be distributed in such a way that, if the transposition on the qubits  $\hat{x}$  is applied a second time on  $|\chi\rangle\langle\eta|$  (that is, one considers  $|\chi'\rangle\langle\eta'| = (|\chi\rangle\langle\eta|)^{T_i}$ ), then  $|\chi'\rangle$  and  $\langle\eta'|$  have both  $k$  entries with the value ‘1’. Let  $\alpha$  be the number of ‘1’ in the set  $\hat{x}_1$  for  $\langle\eta|$  and let  $\beta'$  be the number of ‘0’ in the set  $\hat{x}_2$  for  $\langle\eta|$ . The condition on  $|\chi'\rangle\langle\eta'|$  is fulfilled, iff  $\alpha - \beta' = \delta$ . There are  $\binom{x_1}{\alpha}\binom{x_2}{\beta'}$  possibilities for that, and  $\binom{N-x_1-x_2}{k+\delta-\alpha-(x_2-\beta')}$  possibilities for distributing the remaining  $[k + \delta - \alpha - (x_2 - \beta')]$  entries ‘1’ on the remaining  $N - x_1 - x_2$  qubits. Since  $\alpha$  is not fixed, the total number of possible  $\langle\eta|$  is given by

$$\begin{aligned} n(x_1, x_2) &= \sum_{\alpha} \binom{N-x_1-x_2}{k+\delta-\alpha-(x_2-\beta')} \binom{x_1}{\alpha} \binom{x_2}{\beta'} \\ &= \binom{N-x_1-x_2}{k-x_2} \sum_{\alpha} \binom{x_1}{\alpha} \binom{x_2}{x_2-\beta'} \\ &= \binom{N-x_1-x_2}{k-x_2} \sum_{\alpha} \binom{x_1}{\alpha} \binom{x_2}{x_2+\delta-\alpha} \\ &= \binom{N-x_1-x_2}{k-x_2} \binom{x_1+x_2}{x_2+\delta}, \end{aligned} \quad (\text{A.9})$$

where we have used Vandermonde’s identity [26] in the last step. The number  $n(x_1, x_2)$  is the number of entries in the first row of  $A$ , for the given partition.

Summarizing the first step, we can write the first row of  $A$  as  $|\phi\rangle\langle\psi|$ , where  $\langle\psi|$  is the sum over all possible  $\langle\eta|$ . The norm of  $\langle\psi|$  is  $\|\langle\psi|\| = \sqrt{n(x_1, x_2)/\binom{N}{k}}$ , since  $1/\binom{N}{k}$  is the global factor in the entries of  $A$ , originating from the normalization of the Dicke state.

Step 2. In the previous step we characterized the possible  $\langle\eta|$  (or their sum  $\langle\psi|$ ) for the first row. However, many other rows lead to the same  $\langle\psi|$ . Indeed, let us consider a fixed set  $\hat{x}$ , to which the partial transposition is applied. A given row of  $A$  is labeled by a vector  $|\chi\rangle$  with  $k - \delta$  entries equal to ‘1’. We can define the set  $\hat{x}_1$  as the subset of  $\hat{x}$ , where the entries of  $|\chi\rangle$  are ‘0’, and  $\hat{x}_2$  as the subset of  $\hat{x}$ , where the entries of  $|\chi\rangle$  are ‘1’. All rows which lead to the same sets  $\hat{x}_1$  and  $\hat{x}_2$  lead to the same  $\langle\psi|$  in step 1. For given  $\hat{x}_1$  and  $\hat{x}_2$  there are  $\binom{N-x_1-x_2}{k-\delta-x_2}$  rows compatible with that. Therefore, we can sum over these rows, and can write  $A$  as

$$A = \sum_{\hat{x}_1 \cup \hat{x}_2 = \hat{x}} \mu(\hat{x}_1, \hat{x}_2) |\phi(\hat{x}_1, \hat{x}_2)\rangle \langle\psi(\hat{x}_1, \hat{x}_2)|. \quad (\text{A.10})$$

Here, the vectors  $|\phi(\hat{x}_1, \hat{x}_2)\rangle$  and  $\langle\psi(\hat{x}_1, \hat{x}_2)|$  are normalized, and the coefficients are given by  $\mu(\hat{x}_1, \hat{x}_2) = \sqrt{n(x_1, x_2) \binom{N-x_1-x_2}{k-\delta-x_2} / \binom{N}{k}}$ . The vectors  $|\phi(\hat{x}_1, \hat{x}_2)\rangle$  are clearly orthogonal, the vectors  $\langle\psi(\hat{x}_1, \hat{x}_2)|$ , however, are not yet orthogonal.

Step 3. Now we describe the orthogonality relations of the vectors  $\langle\psi(\hat{x}_1, \hat{x}_2)|$  in some more detail. First, we consider two vectors  $\langle\psi(\hat{x}_1, \hat{x}_2)|$  and  $\langle\psi(\hat{x}'_1, \hat{x}'_2)|$  and prove that they are orthogonal, if  $x_1 \neq x'_1$  (and consequently  $x_2 \neq x'_2$ ). Consider the terms  $\langle\eta|$  as discussed in step 1 for  $\langle\psi(\hat{x}_1, \hat{x}_2)|$  and  $\langle\psi(\hat{x}'_1, \hat{x}'_2)|$  and assume that two of them ( $\langle\eta|$  from  $\langle\psi(\hat{x}_1, \hat{x}_2)|$  and  $\langle\eta|$  from  $\langle\psi(\hat{x}'_1, \hat{x}'_2)|$ ) are *not* orthogonal. Let  $\alpha$  the number of ‘1’ in  $\hat{x}_1$  in  $\langle\eta|$ ,  $\alpha'$  the number of ‘0’ in  $\hat{x}_1$  in  $\langle\eta|$ ,  $\beta$  the number of ‘1’ in  $\hat{x}_2$  in  $\langle\eta|$ ,  $\beta'$  the number of ‘0’ in  $\hat{x}_2$  in  $\langle\eta|$ , and let  $\underline{\alpha}$ ,  $\underline{\alpha}'$ ,  $\underline{\beta}$ ,  $\underline{\beta}'$  be the analogous quantities for  $\langle\eta|$ . If  $\langle\eta|\eta\rangle \neq 0$  we must have  $\alpha + \beta = \underline{\alpha} + \underline{\beta}$  and  $\alpha' + \beta' = \underline{\alpha}' + \underline{\beta}'$ , together with  $\alpha - \beta' = \delta = \underline{\alpha} - \underline{\beta}'$  this implies that  $\beta - \alpha' = \underline{\beta} - \underline{\alpha}'$ . On the other hand,



let us assume that  $\underline{x}_1 = \underline{\alpha} + \underline{\alpha}' > x_1 = \alpha + \alpha'$ . Adding  $\beta - \alpha' = \underline{\beta} - \underline{\alpha}'$  leads to  $\alpha + \beta < \underline{\alpha} + \underline{\beta}$  and so to a contradiction.

Second, let us consider two vectors  $|\psi(\hat{x}_1, \hat{x}_2)\rangle$  and  $|\psi(\hat{x}_1, \hat{x}_2)\rangle$  with  $x_1 = \underline{x}_1$  and  $x_2 = \underline{x}_2$ . In this case, they are identical. Given a term  $\langle \eta |$  from  $|\psi(\hat{x}_1, \hat{x}_2)\rangle$  one can see that this is also a valid term  $\langle \eta |$  for  $|\psi(\hat{x}_1, \hat{x}_2)\rangle$  as follows. First, let us assume that  $\hat{x}_1$  contains only one particle in addition to  $\hat{x}_1$  (and, consequently, one particle in  $\hat{x}_1$  is missing in  $\hat{x}_1$ ). Then, one can transform  $\hat{x}_1$  to  $\hat{x}_1$  by exchanging the two particles, and it is clear that  $\alpha - \beta' = \delta = \underline{\alpha} - \underline{\beta}'$  remains valid. By iterating this procedure, one finds that this can be done for arbitrary  $\hat{x}_1$  and  $\hat{x}_1$ , as long as  $x_1 = \underline{x}_1$ .

Therefore, we can sum the different vectors  $|\phi(\hat{x}_1, \hat{x}_2)\rangle$  when  $x_1 = \underline{x}_1$ , and this sum contains  $\binom{x_1+x_2}{x_1} = \binom{x_1+x_2}{x_2}$  terms. So, the singular value decomposition of  $A$  is given by

$$A = \sum_{x_1+x_2=x} \lambda(x_1, x_2) |\phi(x_1, x_2)\rangle \langle \psi(x_1, x_2)|, \quad (\text{A.11})$$

and the singular values are

$$\lambda(x_1, x_2) = \sqrt{\binom{N-x_1-x_2}{k-x_2} \binom{x_1+x_2}{x_2+\delta} \binom{N-x_1-x_2}{k-\delta-x_2} \binom{x_1+x_2}{x_2}} / \binom{N}{k}. \quad (\text{A.12})$$

Step 4. Finally, we can derive the precise conditions for the matrix  $X_\delta^{(k)}$  to be positive semidefinite. In our situation, this block matrix is positive, iff its Schur complement

$$Y_\delta^{(k)} = \omega_{k-\delta} \mathbb{1} - A \frac{1}{\omega_{k+\delta} \mathbb{1}} A^T \quad (\text{A.13})$$

is positive semidefinite [27]. From this it follows that  $X_\delta^{(k)}$  is positive semidefinite iff

$$\omega_{k-\delta} \omega_{k+\delta} \geq \max_{x_1, x_2, x_1+x_2=x} \{\lambda^2(x_1, x_2)\}. \quad (\text{A.14})$$

Together with the result on the projective entanglement witnesses (proposition 3), this proves theorem 6.  $\square$

### A.3. Proof of proposition 9

**Proof.** In order to show that  $\tilde{\mathcal{W}}_N$  is an entanglement witness, we have to show that

$$\max_{\psi} \langle \psi | \tilde{\mathcal{W}}_N | \psi \rangle \geq 0, \quad (\text{A.15})$$

where the minimization is taken over all biseparable states. Let  $|\psi\rangle = |a\rangle \otimes |b\rangle$  a pure biseparable state with  $|a\rangle$  being a state on  $K$  qubits and  $|b\rangle$  a state on  $L$  qubits. It is sufficient to restrict ourselves to states with one excitation at most, since otherwise the overlap with the  $W$  state vanishes, so that  $\langle \psi | \tilde{\mathcal{W}}_N | \psi \rangle$  cannot become negative. We can write

$$\begin{aligned} |a\rangle &= \alpha |0000 \dots 0\rangle + \beta_1 |100 \dots\rangle + \beta_2 |010 \dots\rangle + \dots + \beta_K |00 \dots 1\rangle, \\ |b\rangle &= \gamma |0000 \dots 0\rangle + \delta_1 |100 \dots\rangle + \delta_2 |010 \dots\rangle + \dots + \delta_L |00 \dots 1\rangle, \end{aligned} \quad (\text{A.16})$$

with complex coefficients  $\alpha, \gamma, \beta_i, \delta_i$ . In the following, we show that the coefficients can be chosen to be real and that furthermore  $\beta_i = \beta_j = \beta$  and  $\delta_i = \delta_j = \delta$  can be assumed. To prove this statement, we use the following notation. We denote by  $|\beta\rangle$  the column vector of the  $\beta_i$  with  $K$  entries and  $|\delta\rangle$  is the  $L$  component vector of the  $\delta_i$ . In addition to that,  $E_{M \times N}$  denotes



a matrix with  $M$  rows and  $N$  columns where all entries are equal to 1. With that notation, one obtains after a short calculation:

$$\begin{aligned} \langle \psi | \tilde{W} | \psi \rangle &= \omega_0 |\alpha|^2 |\gamma|^2 + \omega_1 (|\alpha|^2 \langle \delta | \delta \rangle + |\gamma|^2 \langle \beta | \beta \rangle) + \omega_2 \langle \delta | \delta \rangle \langle \beta | \beta \rangle \\ &\quad - \frac{1}{N} (|\alpha|^2 \langle \delta | E_{L \times L} | \delta \rangle + |\gamma|^2 \langle \beta | E_{K \times K} | \beta \rangle + \alpha^* \gamma \langle \delta | E_{L \times K} | \beta \rangle + \alpha \gamma^* \langle \beta | E_{K \times L} | \delta \rangle). \end{aligned} \quad (\text{A.17})$$

Let us fix  $|\alpha|$  and  $|\gamma|$  which implies that the norm of  $|\delta\rangle$  and  $|\beta\rangle$  is also fixed, because  $|a\rangle$  and  $|b\rangle$  are normalized vectors. The term  $\langle \delta | E_{L \times L} | \delta \rangle$  is maximal, if  $|\delta\rangle$  is the vector corresponding to the maximal singular value of  $E_{L \times L}$ . This is the case iff  $\delta_i = \delta_j = \delta \in \mathbb{R}$ . For the same reason,  $\langle \beta | E_{K \times K} | \beta \rangle$  is maximal iff  $\beta_i = \beta_j = \beta \in \mathbb{R}$ . At the same time, we maximize  $\alpha^* \gamma \langle \delta | E_{L \times K} | \beta \rangle + \alpha \gamma^* \langle \beta | E_{K \times L} | \delta \rangle = 2 \operatorname{Re}(\alpha^* \gamma \langle \delta | E_{L \times K} | \beta \rangle)$  with this choice of  $|\beta\rangle$  and  $|\delta\rangle$  since the singular vectors of  $E_{L \times K}$  are of the same type. In addition to that, it is optimal to take  $\alpha$  and  $\gamma$  real because then  $\alpha^* \gamma \langle \delta | E_{L \times K} | \beta \rangle$  is real.

So in the end we have only to perform a minimization over real coefficients  $\alpha$ ,  $\beta$ ,  $\gamma$  and  $\delta$  with the normalization constraints  $\alpha^2 + K\beta^2 = \gamma^2 + L\delta^2 = 1$ . This leads after a straightforward calculation to the given formula in proposition 9.  $\square$

## References

- [1] Horodecki R, Horodecki P, Horodecki M and Horodecki K 2009 *Rev. Mod. Phys.* **81** 865
- [2] Gühne O and Tóth G 2009 *Phys. Rep.* **474** 1
- [3] Dicke R H 1954 *Phys. Rev.* **93** 99
- [4] Gühne O, Bodoky F and Blaauuboer M 2008 *Phys. Rev. A* **78** 060301
- [5] Tóth G *et al* 2010 *Phys. Rev. Lett.* **105** 250403
- [6] Moroder T *et al* 2012 *New J. Phys.* **14** 105001
- [7] Tóth G 2007 *J. Opt. Soc. Am. B* **24** 275
- [8] Tóth G *et al* 2009 *New J. Phys.* **11** 083002
- [9] Huber M *et al* 2011 *Phys. Rev. A* **83** 040301
- Huber M *et al* 2011 *Phys. Rev. A* **84** 039906 (erratum)
- [10] Novo L, Moroder T and Gühne O 2013 *Phys. Rev. A* **88** 012305
- [11] Christandl M, König R, Mitchison G and Renner R 2007 *Commun. Math. Phys.* **273** 473
- [12] Häffner H *et al* 2005 *Nature* **438** 643
- [13] Kiesel N *et al* 2007 *Phys. Rev. Lett.* **98** 063604
- [14] Wieczorek W *et al* 2009 *Phys. Rev. Lett.* **103** 020504
- [15] Jungnitsch B, Moroder T and Gühne O 2011 *Phys. Rev. Lett.* **106** 190502
- [16] Gühne O, Jungnitsch B, Moroder T and Weinstein Y S 2011 *Phys. Rev. A* **84** 052319
- [17] Dür W, Vidal G and Cirac J I 2000 *Phys. Rev. A* **62** 062314
- [18] Peres A 1996 *Phys. Rev. Lett.* **77** 1413
- [19] See the program PPTmixer [www.mathworks.com/matlabcentral/fileexchange/30968](http://www.mathworks.com/matlabcentral/fileexchange/30968)
- [20] Gühne O, Tóth G, Hyllus P and Briegel H J 2005 *Phys. Rev. Lett.* **95** 120405
- [21] Horn R and Johnson C 1999 *Matrix Analysis* (Cambridge: Cambridge University Press)
- [22] Gühne O, Lu C, Gao W and Pan J 2007 *Phys. Rev. A* **76** 030305
- [23] Sperling J and Vogel W 2013 arXiv:1303.6403
- [24] Lasserre J B 2001 *SIAM J. Optim.* **11** 796
- [25] Eggeling T and Werner R F 2001 *Phys. Rev. A* **63** 042111
- [26] Shijie Z 1303 *Siyuan Yujian*
- [27] Bhatia R 2007 *Positive Definite Matrices (Princeton Series in Applied Mathematics)* (Princeton, NJ: Princeton University Press)