University of Florida
Dept. of Computer & Information Science & Engineering

# COT 3100

# Applications of Discrete Structures

Dr. Michael P. Frank

Slides for a Course Based on the Text
*Discrete Mathematics & Its Applications*
($5^{th}$ Edition)
by Kenneth H. Rosen

# Module #18: **Relations**

Rosen 5$^{th}$ ed., ch. 7

~32 slides (in progress), ~2 lectures

# Binary Relations

- Let $A$, $B$ be any two sets.
- A *binary relation* $R$ from $A$ to $B$, written (with signature) $R{:}A\leftrightarrow B$, is a subset of $A\times B$.
  - *E.g.*, let $\mathbf{<} : \mathbf{N}\leftrightarrow\mathbf{N} :\equiv \{(n,m) \mid n < m\}$
- The notation $a\ R\ b$ or $aRb$ means $(a,b)\in R$.
  - *E.g.*, $a \mathbf{<} b$ means $(a,b)\in \mathbf{<}$
- If $aRb$ we may say "$a$ is related to $b$ (by relation $R$)", or "$a$ relates to $b$ (under relation $R$)".
- A binary relation $R$ corresponds to a predicate function $P_R{:}A\times B\rightarrow\{\mathbf{T},\mathbf{F}\}$ defined over the 2 sets $A,B$; *e.g.*, "eats" $:\equiv \{(a,b)|$ organism $a$ eats food $b\}$

# Complementary Relations

- Let $R:A\leftrightarrow B$ be any binary relation.
- Then, $\not R:A\leftrightarrow B$, the *complement* of $R$, is the binary relation defined by
  $$\not R :\equiv \{(a,b) \mid (a,b)\notin R\} = (A\times B) - R$$
  Note this is just $\overline{R}$ if the universe of discourse is $U = A\times B$; thus the name *complement*.
- Note the complement of $\not R$ is $R$.

Example: $\not< \; = \{(a,b) \mid (a,b)\notin <\} = \{(a,b) \mid \neg a<b\} = \;\geq$

# Inverse Relations

- Any binary relation $R{:}A\leftrightarrow B$ has an *inverse* relation $R^{-1}{:}B\leftrightarrow A$, defined by

  $$R^{-1} :\equiv \{(b,a) \mid (a,b)\in R\}.$$

  *E.g.*, $<^{-1} = \{(b,a) \mid a<b\} = \{(b,a) \mid b>a\} = >$.
- *E.g.*, if $R$:People→Foods is defined by $aRb \Leftrightarrow a \text{ eats } b$, then:
  $b\, R^{-1}\, a \Leftrightarrow b \text{ is eaten by } a$. (Passive voice.)

# Relations on a Set

- A (binary) relation from a set $A$ to itself is called a relation *on* the set $A$.
- *E.g.*, the "**<**" relation from earlier was defined as a relation *on* the set **N** of natural numbers.
- The *identity relation* $\mathbf{I}_A$ on a set $A$ is the set $\{(a,a)|a\in A\}$.

# Reflexivity

- A relation $R$ on $A$ is *reflexive* if $\forall a \in A$, $aRa$.
  - *E.g.*, the relation $\geq :\equiv \{(a,b) \mid a \geq b\}$ is reflexive.
- A relation is *irreflexive* iff its complementary relation is reflexive.
  - Note "*irreflexive*" ≠ "*not reflexive*"!
  - Example: $<$ is irreflexive.
  - Note: "likes" between people is not reflexive, but not irreflexive either. (Not everyone likes themselves, but not everyone dislikes themselves either.)

# Symmetry & Antisymmetry

- A binary relation $R$ on $A$ is symmetric iff $R = R^{-1}$, that is, if $(a,b)\in R \leftrightarrow (b,a)\in R$.
  - *E.g.*, = (equality) is symmetric. < is not.
  - "is married to" is symmetric, "likes" is not.
- A binary relation $R$ is *antisymmetric* if $(a,b)\in R \rightarrow (b,a)\notin R$.
  - < is antisymmetric, "likes" is not.

# Transitivity

- A relation $R$ is *transitive* iff (for all $a$,$b$,$c$)
  $(a,b)\in R \wedge (b,c)\in R \rightarrow (a,c)\in R$.
- A relation is *intransitive* if it is not transitive.
- Examples: "is an ancestor of" is transitive.
- "likes" is intransitive.
- "is within 1 mile of" is… ?

# Totality

- A relation $R:A\leftrightarrow B$ is *total* if for every $a\in A$, there is at least one $b\in B$ such that $(a,b)\in R$.
- If $R$ is not total, then it is called *strictly partial*.
- A *partial relation* is a relation that *might* be strictly partial. Or, it might be total. (In other words, all relations are considered "partial.")

# Functionality

- A relation $R{:}A\leftrightarrow B$ is *functional* (that is, it is also a partial function $R{:}A\rightarrow B$) if, for any $a\in A$, there is *at most 1* $b\in B$ such that $(a,b)\in R$.
- $R$ is *antifunctional* if its inverse relation $R^{-1}$ is functional.
  - Note: A functional relation (partial function) that is also antifunctional is an invertible partial function.
- $R$ is a *total function* $R{:}A\rightarrow B$ if it is both functional and total, that is, for any $a\in A$, there is *exactly* 1 $b$ such that $(a,b)\in R$. If $R$ is functional but not total, then it is a *strictly partial function*.

# Composite Relations

- Let $R:A\leftrightarrow B$, and $S:B\leftrightarrow C$. Then the *composite* $S\circ R$ of $R$ and $S$ is defined as:

$$S\circ R = \{(a,c) \mid aRb \wedge bSc\}$$

- Note function composition $f\circ g$ is an example.
- The $n^{\text{th}}$ power $R^n$ of a relation $R$ on a set $A$ can be defined recursively by:

$$R^0 :\equiv \mathbf{I}_A ; \qquad R^{n+1} :\equiv R^n\circ R \quad \text{for all } n\geq 0.$$

  – Negative powers of $R$ can also be defined if desired, by $R^{-n} :\equiv (R^{-1})^n$.

# §7.2: *n*-ary Relations

- An *n*-ary relation $R$ on sets $A_1,\ldots,A_n$, written $R{:}A_1,\ldots,A_n$, is a subset $R \subseteq A_1\times \ldots \times A_n$.
- The sets $A_i$ are called the *domains* of $R$.
- The *degree* of $R$ is $n$.
- $R$ is *functional in domain* $A_i$ if it contains at most one $n$-tuple $(\ldots, a_i ,\ldots)$ for any value $a_i$ within domain $A_i$.

# Relational Databases

- A *relational database* is essentially an $n$-ary relation $R$.
- A domain $A_i$ is a *primary key* for the database if the relation $R$ is functional in $A_i$.
- A *composite key* for the database is a set of domains $\{A_i, A_j, \ldots\}$ such that $R$ contains at most 1 $n$-tuple $(\ldots,a_i,\ldots,a_j,\ldots)$ for each composite value $(a_i, a_j,\ldots)\in A_i \times A_j \times \ldots$

# Selection Operators

- Let $A$ be any $n$-ary domain $A=A_1\times\ldots\times A_n$, and let $C:A\rightarrow\{\mathbf{T},\mathbf{F}\}$ be any condition (predicate) on elements ($n$-tuples) of $A$.
- Then, the *selection operator* $s_C$ is the operator that maps any ($n$-ary) relation $R$ on $A$ to the $n$-ary relation of all $n$-tuples from $R$ that satisfy $C$.
  - *I.e.*, $\forall R\subseteq A$, $s_C(R) = R\cap\{a\in A \mid s_C(a) = \mathbf{T}\}$

# Selection Operator Example

- Suppose we have a domain
  $A$ = StudentName × Standing × SocSecNos
- Suppose we define a certain condition on $A$,
  $UpperLevel(name,standing,ssn) :\equiv$
  $[(standing = \text{junior}) \vee (standing = \text{senior})]$
- Then, $s_{UpperLevel}$ is the selection operator that takes any relation $R$ on $A$ (database of students) and produces a relation consisting of *just* the upper-level classes (juniors and seniors).

# Projection Operators

- Let $A = A_1 \times \ldots \times A_n$ be any *n*-ary domain, and let $\{i_k\}=(i_1,\ldots,i_m)$ be a sequence of indices all falling in the range 1 to $n$,
  - That is, where $1 \le i_k \le n$ for all $1 \le k \le m$.
- Then the *projection operator* on *n*-tuples

$$P_{\{i_k\}} : A \to A_{i_1} \times \ldots \times A_{i_m}$$

is defined by:

$$P_{\{i_k\}}(a_1, \ldots, a_n) = (a_{i_1}, \ldots, a_{i_m})$$

# Projection Example

- Suppose we have a ternary (3-ary) domain *Cars*=*Model*×*Year*×*Color*. (note $n$=3).
- Consider the index sequence $\{i_k\}$= 1,3. ($m$=2)
- Then the projection $P_{\{i_k\}}$ simply maps each tuple $(a_1,a_2,a_3) = (model,year,color)$ to its image:

$$(a_{i_1}, a_{i_2}) = (a_1, a_3) = (model, color)$$

- This operator can be usefully applied to a whole relation $R \subseteq Cars$ (database of cars) to obtain a list of model/color combinations available.

# Join Operator

- Puts two relations together to form a sort of combined relation.
- If the tuple $(A,B)$ appears in $R_1$, and the tuple $(B,C)$ appears in $R_2$, then the tuple $(A,B,C)$ appears in the join $J(R_1,R_2)$.
  - $A$, $B$, $C$ can also be sequences of elements rather than single elements.

# Join Example

- Suppose $R_1$ is a teaching assignment table, relating *Professors* to *Courses*.
- Suppose $R_2$ is a room assignment table relating *Courses* to *Rooms*,*Times*.
- Then $J(R_1,R_2)$ is like your class schedule, listing (*professor*,*course*,*room*,*time*).

# §7.3: Representing Relations

- Some ways to represent *n*-ary relations:
  - With an explicit list or table of its tuples.
  - With a function from the domain to {**T**,**F**}.
    - Or with an algorithm for computing this function.
- Some special ways to represent binary relations:
  - With a zero-one matrix.
  - With a directed graph.

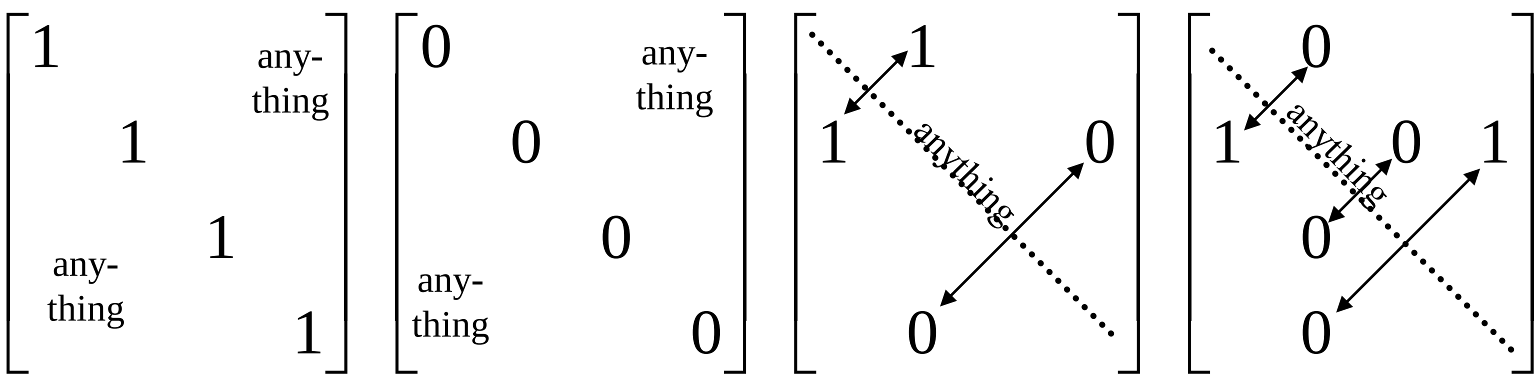
# Using Zero-One Matrices

- To represent a relation $R$ by a matrix $\mathbf{M}_R = [m_{ij}]$, let $m_{ij} = 1$ if $(a_i,b_j)\in R$, else 0.
- *E.g.*, Joe likes Susan and Mary, Fred likes Mary, and Mark likes Sally.
- The 0-1 matrix representation of that "Likes" relation:

$$\begin{array}{r c} & \begin{array}{ccc} \text{Susan} & \text{Mary} & \text{Sally} \end{array} \\ \begin{array}{r} \text{Joe} \\ \text{Fred} \\ \text{Mark} \end{array} & \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

# Zero-One Reflexive, Symmetric

- Terms: *Reflexive*, *non-Reflexive*, *irreflexive*, *symmetric*, *asymmetric*, *and antisymmetric*.
  - These relation characteristics are very easy to recognize by inspection of the zero-one matrix.

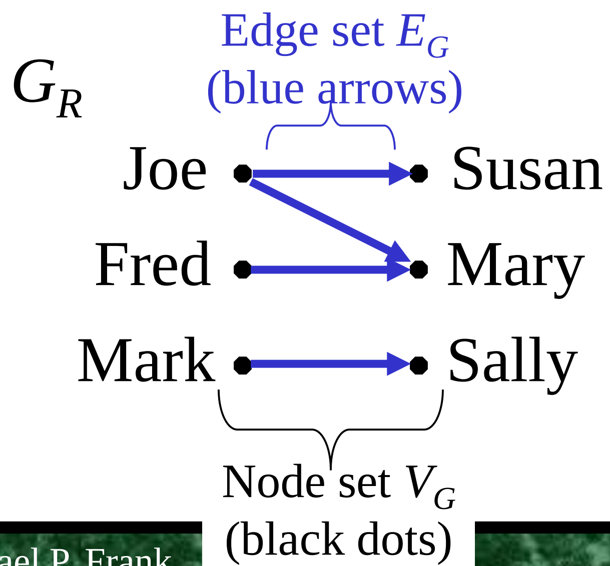



| | | | |
|---|---|---|---|
| *Reflexive*: all 1's on diagonal | *Irreflexive*: all 0's on diagonal | *Symmetric*: all identical across diagonal | *Antisymmetric*: all 1's are across from 0's |

# Using Directed Graphs

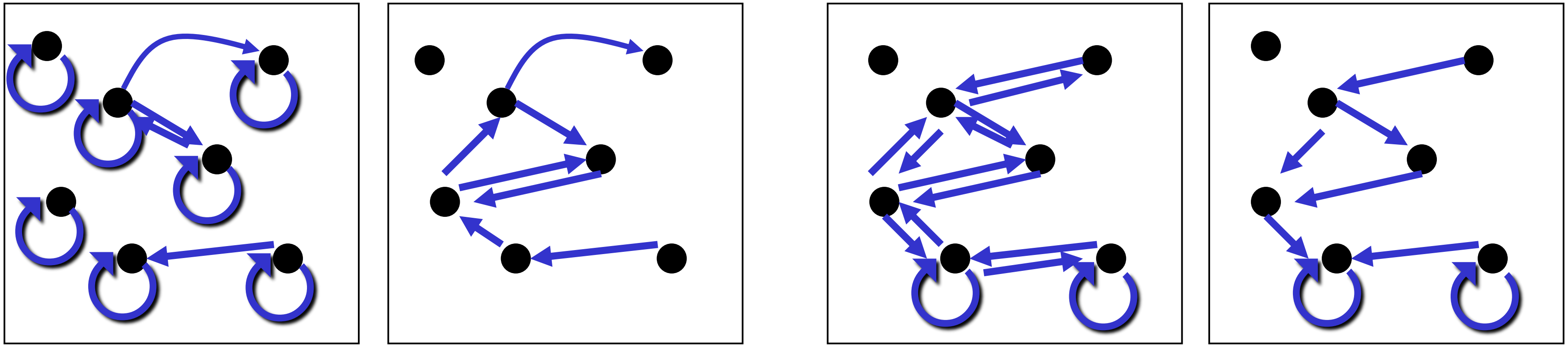
- A *directed graph* or *digraph* $G=(V_G,E_G)$ is a set $V_G$ of *vertices (nodes)* with a set $E_G \subseteq V_G \times V_G$ of *edges (arcs,links)*. Visually represented using dots for nodes, and arrows for edges. Notice that a relation $R{:}A \leftrightarrow B$ can be represented as a graph $G_R=(V_G=A \cup B,\ E_G=R)$.

$$\mathbf{M}_R \quad \begin{array}{c} \\ \text{Joe} \\ \text{Fred} \\ \text{Mark} \end{array} \begin{array}{c} \begin{array}{ccc} \text{Susan} & \text{Mary} & \text{Sally} \end{array} \\ \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$



# Digraph Reflexive, Symmetric

It is extremely easy to recognize the reflexive/irreflexive/ symmetric/antisymmetric properties by graph inspection.



Reflexive: Every node has a self-loop

Irreflexive: No node links to itself

Symmetric: Every link is bidirectional

Antisymmetric: No link is bidirectional

Asymmetric, non-antisymmetric

Non-reflexive, non-irreflexive

# §7.4: Closures of Relations

- For any property *X*, the "*X* closure" of a set *A* is defined as the "smallest" superset of *A* that has the given property.
- The *reflexive closure* of a relation *R* on *A* is obtained by adding $(a,a)$ to *R* for each $a \in A$. *I.e.*, it is $\boxed{R \cup I_A}$
- The *symmetric closure* of *R* is obtained by adding $(b,a)$ to *R* for each $(a,b)$ in *R*. *I.e.*, it is $\boxed{R \cup R^{-1}}$
- The *transitive closure* or *connectivity relation* of *R* is obtained by repeatedly adding $(a,c)$ to *R* for each $(a,b),(b,c)$ in *R*.
  - *I.e.*, it is

$$\boxed{R^* = \bigcup_{n \in \mathbf{Z}^+} R^n}$$

# Paths in Digraphs/Binary Relations

- A *path* of length $n$ from node $a$ to $b$ in the directed graph $G$ (or the binary relation $R$) is a sequence $(a,x_1), (x_1,x_2), \ldots, (x_{n-1},b)$ of $n$ ordered pairs in $E_G$ (or $R$).
  - An empty sequence of edges is considered a path of length 0 from $a$ to $a$.
  - If any path from $a$ to $b$ exists, then we say that $a$ is *connected to b*. ("You can get there from here.")
- A path of length $n \geq 1$ from $a$ to $a$ is called a *circuit* or a cycle.
- Note that there exists a path of length $n$ from $a$ to $b$ in $R$ if and only if $(a,b) \in R^n$.

# Simple Transitive Closure Alg.

A procedure to compute $R^*$ with 0-1 matrices.

**procedure** *transClosure*($\mathbf{M}_R$:rank-$n$ 0-1 mat.)

$\mathbf{A} := \mathbf{B} := \mathbf{M}_R$;

**for** $i := 2$ to $n$ **begin**

$\mathbf{A} := \mathbf{A} \odot \mathbf{M}_R$; $\mathbf{B} := \mathbf{B} \vee \mathbf{A}$ {join}

**end** {note **A** represents $R^i$}

**return B** {Alg. takes $\Theta(n^4)$ time}

# Roy-Warshall Algorithm

- Uses only $\Theta(n^3)$ operations!

**Procedure** *Warshall*($\mathbf{M}_R$ : rank-$n$ 0-1 matrix)

$\mathbf{W} := \mathbf{M}_R$

**for** $k := 1$ **to** $n$

**for** $i := 1$ **to** $n$

**for** $j := 1$ **to** $n$

$w_{ij} := w_{ij} \vee (w_{ik} \wedge w_{kj})$

**return W** {this represents $R^*$}

$w_{ij} = 1$ means there is a path from $i$ to $j$ going only through nodes $\leq k$

# §7.5: Equivalence Relations

- An *equivalence relation* (e.r.) on a set $A$ is simply any binary relation on $A$ that is reflexive, symmetric, and transitive.
  - *E.g.*, = itself is an equivalence relation.
  - For any function $f:A\rightarrow B$, the relation "have the same $f$ value", or $=_f :\equiv \{(a_1,a_2) \mid f(a_1)=f(a_2)\}$ is an equivalence relation, *e.g.*, let $m$= "mother of" then $=_m$ = "have the same mother" is an e.r.

# Equivalence Relation Examples

- “Strings *a* and *b* are the same length.”
- “Integers *a* and *b* have the same absolute value.”
- “Real numbers *a* and *b* have the same fractional part (*i.e.*, $a - b \in \mathbf{Z}$).”
- “Integers *a* and *b* have the same residue modulo *m*.” (for a given $m>1$)

# Equivalence Classes

- Let $R$ be any equiv. rel. on a set $A$.
- The *equivalence class* of $a$,
  $[a]_R :\equiv \{ b \mid aRb \}$ (optional subscript $R$)
  - It is the set of all elements of $A$ that are "equivalent" to $a$ according to the eq.rel. $R$.
  - Each such $b$ (including $a$ itself) is called a *representative* of $[a]_R$.
- Since $f(a)=[a]_R$ is a function of $a$, any equivalence relation $R$ be defined using $aRb :\equiv$ "$a$ and $b$ have the same $f$ value", given that $f$.

# Equivalence Class Examples

- "Strings $a$ and $b$ are the same length."
  - $[a]$ = the set of all strings of the same length as $a$.
- "Integers $a$ and $b$ have the same absolute value."
  - $[a]$ = the set $\{a, -a\}$
- "Real numbers $a$ and $b$ have the same fractional part (*i.e.*, $a - b \in \mathbf{Z}$)."
  - $[a]$ = the set $\{\ldots, a-2, a-1, a, a+1, a+2, \ldots\}$
- "Integers $a$ and $b$ have the same residue modulo $m$." (for a given $m>1$)
  - $[a]$ = the set $\{\ldots, a-2m, a-m, a, a+m, a+2m, \ldots\}$

# Partitions

- A *partition* of a set $A$ is the set of all the equivalence classes $\{A_1, A_2, \ldots\}$ for some e.r. on $A$.
- The $A_i$'s are all disjoint and their union = $A$.
- They "partition" the set into pieces. Within each piece, all members of the set are equivalent to each other.

# §7.6: Partial Orderings

- Def 1. A relation R on a set S is called a *partial ordering* or *partial order* if it is reflexive, anti-symmetric, and transitive. A set S together with a partial ordering R is called a *partially ordered set*, or *poset*.
- Ex.
  - (>=) on Z.
  - Divisibility ( | ) on Z.
  - Inclusion ( $\subseteq$ ) on the power set of a set.
- The notation $\preccurlyeq$ denotes $(a,b) \in R$ , $\prec$ denotes a $\preccurlyeq$ b but a $\neq$ b.
- Def 2. The elements a,b are *comparable* if a $\preccurlyeq$ b or b $\preccurlyeq$ a, otherwise a,b are *incomparable*.

# Partial ordering

- Def 3. If (S, $\preccurlyeq$) is a poset and every 2 elements of S are comparable, S is called a *totally ordered set* or *linearly ordered set*, $\preccurlyeq$ is called a *total order* or *linear order*. A totally ordered set is also called a *chain*.
- Ex. (Z, $\leq$) ; (Z+, |)
- Def 4. (S, $\preccurlyeq$) is a well-ordered set if it is a poset such that $\preccurlyeq$ is a total ordering and every nonempty subset of S has a least element.
- **Lexicographic ordering** is a common arrangement for ordering strings.

# Partial ordering

- Hasse Diagrams: simplified representation of a poset
  - 1. remove all loops.
  - 2. remove all edges because of the transitivity.
  - 3. Arrange each edge so that initial vertex is below its terminal vertex.
  - 4. remove arrows.

# Maximal/Minimal Elements

- Def 5. An element *a* is maximal if there's no *b* in S such that $a \prec b$. It is minimal if no *b* such that $b \prec a$.
- Ex. ({2,4,5,10,12,20,25}, |), what are maximals and minimals.
- Def 6. An element a in A is the greatest element (maximum) if ∀ b, $b \preccurlyeq a$. a is the least element (minimum) if ∀ b, $a \preccurlyeq b$.
- Upper bound *a*: ∀ b, $b \preccurlyeq a$, but *a* not necessarily in *A*. Similarly for lower bound.

# Lattices

- Least upper bound: the minimum of all upper bounds.
- Greatest lower bound: The maximum of all lower bounds.
- Def 6. A poset in which every pair of elements has both a least upper bound and a greatest lower bound is called a ***lattice***.
- Ex. (Z+, |) a lattice? Yes. (P(S), $\subseteq$)?
- **Topological sorting**: to assign a ***compatible*** total ordering for a given poset.