**CONTENTS**



## Appendix A: Kepler Problem

### 1. Category

- Effective potential
- Stability
- $O(4)$ Symmetry
- Bertland Theorem

## Appendix B: Mechanics, symmetries, currents

### 1. Hydrogen and SO(4) symmetry

Laplace-Runge-Lenz vector does not necessarily commute with angular momentum $L$ in the $\mathcal{L}$ notion. It is conserved given that $\mathcal{H}$ is time-independent. In other words, it is invariant under a different (more strict) variational condition.

### 2. Action and time evolution

Time evolution operators classically (failed)

### 3. Orbit stability

Bertland Theorem

## Appendix C: Non-inertial system

### 1. Tidal Force

$$(1+x)^{-\frac{1}{2}} = 1 + (-\frac{1}{2})x + (\frac{3}{8})x^2 + o(x^3) \tag{C1}$$

$$(1+x)^{-\frac{3}{2}} = 1 + (-\frac{3}{2})x + (\frac{15}{8})x^2 + o(x^3) \tag{C2}$$

$$\begin{aligned}
\boldsymbol{a}_g &= -\frac{k}{|\boldsymbol{r}|^3}\boldsymbol{r} \\
&= -k(\boldsymbol{r}^2)^{-\frac{3}{2}}\boldsymbol{r} \\
&= -\frac{k}{r_0^3}(1 + \frac{(\boldsymbol{r_0} + \boldsymbol{\Delta r})^2 - \boldsymbol{r_0}^2}{\boldsymbol{r_0}^2})^{-\frac{3}{2}}(\boldsymbol{r_0} + \boldsymbol{\Delta r}) \\
&= -\frac{k}{r_0^3}(1 + \frac{2\boldsymbol{r_0}\cdot\boldsymbol{\Delta r} + (\boldsymbol{\Delta r})^2}{\boldsymbol{r_0}^2})^{-\frac{3}{2}}(\boldsymbol{r_0} + \boldsymbol{\Delta r}) \\
&= -\frac{k}{r_0^3}(1 - \frac{3\boldsymbol{r_0}\cdot\boldsymbol{\Delta r}}{\boldsymbol{r_0}^2} + \frac{15(\boldsymbol{r_0}\cdot\boldsymbol{\Delta r})^2}{2\boldsymbol{r_0}^4} - \frac{3(\boldsymbol{\Delta r})^2}{2(\boldsymbol{r_0})^2})(\boldsymbol{r_0} + \boldsymbol{\Delta r})
\end{aligned} \tag{C3}$$

$$
\begin{aligned}
\boldsymbol{a}_e &= -\frac{k}{r_0^3}\boldsymbol{r_0} \\
\boldsymbol{a}_{Tide} &= \boldsymbol{a}_g - \boldsymbol{a}_e \\
&= o((\boldsymbol{\Delta r})^3) - \frac{k}{r_0^3}(\boldsymbol{\Delta r} - \frac{3\boldsymbol{r_0}\cdot\boldsymbol{\Delta r}}{\boldsymbol{r_0^2}}\boldsymbol{r_0}) \\
&- \frac{k}{r_0^3}(\frac{15(\boldsymbol{r_0}\cdot\boldsymbol{\Delta r})^2}{2\boldsymbol{r_0^4}} - \frac{3(\boldsymbol{\Delta r})^2}{2(\boldsymbol{r_0})^2})\boldsymbol{r_0} - \frac{k}{r_0^3}(-\frac{3\boldsymbol{r_0}\cdot\boldsymbol{\Delta r}}{\boldsymbol{r_0^2}})\boldsymbol{\Delta r} \\
&= \boldsymbol{a}_{Tide}^1 + \boldsymbol{a}_{Tide}^2 + o((\boldsymbol{\Delta r})^3) \qquad (C4) \\
\boldsymbol{a}_{Tide}^1 &= -\frac{k}{r_0^3}(\boldsymbol{\Delta r}_\perp - 2\boldsymbol{\Delta r}_\parallel) \qquad (C5)
\end{aligned}
$$

$$
\begin{aligned}
V_g &= -\frac{k}{|\boldsymbol{r}|} \\
&= -k(\boldsymbol{r}^2)^{-\frac{1}{2}} \\
&= -\frac{k}{r_0}(1 + \frac{(\boldsymbol{r_0}+\boldsymbol{\Delta r})^2 - \boldsymbol{r_0^2}}{\boldsymbol{r_0^2}})^{-\frac{1}{2}} \\
&= -\frac{k}{r_0}(1 + \frac{2\boldsymbol{r_0}\cdot\boldsymbol{\Delta r} + (\boldsymbol{\Delta r})^2}{\boldsymbol{r_0^2}})^{-\frac{1}{2}} \\
&= -\frac{k}{r_0}(1 - \frac{\boldsymbol{r_0}\cdot\boldsymbol{\Delta r}}{\boldsymbol{r_0^2}} + \frac{3(\boldsymbol{r_0}\cdot\boldsymbol{\Delta r})^2}{2\boldsymbol{r_0^4}} - \frac{(\boldsymbol{\Delta r})^2}{2(\boldsymbol{r_0})^2}) + o((\boldsymbol{\Delta r})^3) \\
&= V_0 + V_1 + V_2 + o((\boldsymbol{\Delta r})^3) \qquad (C6)
\end{aligned}
$$

### 2. Coriolis Force

In system with rotation $(\boldsymbol{\omega}, \boldsymbol{\beta})$ around point $O$, Non-inertial forces are

$$
\begin{aligned}
\boldsymbol{a}_c &= -2(\boldsymbol{\omega}\times\boldsymbol{\delta v}) \\
\boldsymbol{a}_\beta &= -\beta\times\boldsymbol{\delta r} \qquad (C7)
\end{aligned}
$$

## Appendix D: Thermaldynamics and Hydrodynamics

### 1. Fluid Roche limit

To be continued.

### 2. Waterball without gravity

Ossilation?

## Appendix E: Scenarios

### 1. Wander around space station

Space station $O$ is orbiting $\boldsymbol{r} : (r, \theta)$ around the earth and an astronaut is wandering around $O$ with displacement $\boldsymbol{\Delta r}$ in non-rotating system and $\boldsymbol{\delta r}$ in rorating system.

Dynamic in the rotating system is described in Eqn. E1

$$
\begin{aligned}
\boldsymbol{a}_\delta &= \boldsymbol{a}_{Tide} + \boldsymbol{a}_c + \boldsymbol{a}_\beta \\
\boldsymbol{a}_{Tide} &= -\frac{k}{r_0^3}(-2\delta r_r \hat{r} + \delta r_\theta \hat{\theta}) \\
\boldsymbol{a}_c &= 2\dot{\theta}(\delta \dot{r}_\theta \hat{r} - \delta \dot{r}_r \hat{\theta}) \\
\boldsymbol{a}_\beta &= \ddot{\theta}(\delta r_\theta \hat{r} - \delta r_r \hat{\theta})
\end{aligned} \tag{E1}
$$

Assume $O$ is orbiting on a circle $(\dot{\theta}, \ddot{\theta}) = (\omega, 0)$, we have $\omega = \frac{k}{r_0^3}$, then Eqn. E1 is simplified as Eqn. E2

$$
\begin{aligned}
\frac{d^2}{dt^2}\boldsymbol{\delta r} &= \boldsymbol{a}_{Tide} + \boldsymbol{a}_c \\
&= -\omega^2(-2\delta r_r \hat{r} + \delta r_\theta \hat{\theta}) + 2\omega(\delta \dot{r}_\theta \hat{r} - \delta \dot{r}_r \hat{\theta})
\end{aligned} \tag{E2}
$$

Qualitative analysis of the dynamics: Assume the astronaut orbits around the space station with $\dot{\delta\theta} < 0$ and a period same as the space station orbit $T_0$

- Averagely, orbit of $\boldsymbol{\delta r}$ operates with $-\boldsymbol{\omega}$;
- At vertex along the $\hat{r}$ direction, $\boldsymbol{a}_{Tide}$ points outwards, so velocity should be large to generate massive $\boldsymbol{a}_c$;
- At vertex along the $\hat{\theta}$ direction, $\boldsymbol{a}_{Tide}$ points inwards, enough to keep the orbit bound, so velocity should be small;
- Quantitave description of the orbit dynamic remains mystery;

### 2. Effective gravity induced by rotation of facility

Only second-order effect is present at a perturbative level.

### 3. Lagrangian points

We consider the effective dynamics in a Non-inertial system of two-body gravity system.

Consider a two-body system consists of two celestial bodies $M_1$ and $M_2$ with distance of $D$. The two-body effective mass and orbiting angular velocity are

$$\begin{aligned} M &= \frac{M_1 M_2}{M_1 + M_2} \\ \frac{G M_1 M_2}{D^2} &= \frac{M_1 M_2}{\omega}^2 D \\ \omega^2 &= \frac{G(M_1 + M_2)}{D^3} \end{aligned} \tag{E3}$$

The two celestial bodies are orbiting around the centroid $O$. Now we consider the Lagrangian point $L_4$ locating at the vertex of an equilateral triangle connecting $M_1$ and $M_2$, thus we have $\boldsymbol{r} = \frac{M_1 \boldsymbol{r_1} + M_2 \boldsymbol{r_2}}{M_1 + M_2}$

In the rotating celestial system, the effective potential around $L_4$ is

$$\begin{aligned} V_{eff} &= V_0(\boldsymbol{r_1}) + V_1(\boldsymbol{r_1}) + V_2(\boldsymbol{r_1}) + V_0(\boldsymbol{r_2}) + V_1(\boldsymbol{r_2}) + V_2(\boldsymbol{r_2}) + o((\boldsymbol{\Delta r})^3) + V_\omega \\ &= V_{eff}^0 + V_{eff}^1 + V_{eff}^2 + o((\boldsymbol{\Delta r})^3) \\ V_\omega &= -\frac{1}{2}\omega^2(\boldsymbol{r} + \boldsymbol{\Delta r})^2 \\ V_{eff}^1 &= -\frac{G M_1}{D}\left(-\frac{\boldsymbol{r_1} \cdot \boldsymbol{\Delta r}}{|\boldsymbol{r_1}|^2}\right) - \frac{G M_2}{D}\left(-\frac{\boldsymbol{r_2} \cdot \boldsymbol{\Delta r}}{|\boldsymbol{r_2}|^2}\right) - \omega^2 \boldsymbol{r} \cdot \boldsymbol{\Delta r} \\ &= \frac{G}{D^3}(M_1 \boldsymbol{r_1} + M_2 \boldsymbol{r_2}) \cdot \boldsymbol{\Delta r} - \frac{G}{D^3}(M_1 \boldsymbol{r_1} + M_2 \boldsymbol{r_2}) \cdot \boldsymbol{\Delta r} \\ &= 0 \\ V_{eff} &= V_{eff}^0 + V_{eff}^2 + o((\boldsymbol{\Delta r})^3) \end{aligned} \tag{E4}$$

In fact $V_{eff}^2$ is convex around $L_4$, thus $L_4$ is a smooth maximum in the rotating system. Celestial bodies around $L_4$ are bounded by Coriolis force under certain condition (the mass ratio boundary $\frac{M_1}{M_2}$ actually)

### 4. Epicycle model

Fitting Kepler orbit of binary star system to epicycle system?

### 5. Space channel in gravitational fields

More fictional rather than a realistic one. Few materials are found.