CONTENTS

A. Kepler Problem 1
1. Category 1

B. Mechanics, symmetries, currents 2
1. Hydrogen and SO(4) symmetry 2
2. Action and time evolution 2
3. Orbit stability 2

C. Non-inertial system 2
1. Tidal Force 2
2. Coriolis Force 3

D. Thermaldynamics and Hydrodynamics 3
1. Fluid Roche limit 3
2. Waterball without gravity 4

E. Scenarios 4
1. Wander around space station 4
2. Effective gravity induced by rotation of facility 5
3. Lagrangian points 5
4. Epicycle model 6
5. Space channel in gravitational fields 6

References 6

Appendix A: Kepler Problem

1. Category

- Effective potential
- Stability
- $O(4)$ Symmetry

- Bertland Theorem

Appendix B: Mechanics, symmetries, currents

1. Hydrogen and SO(4) symmetry

Laplace-Runge-Lenz vector does not necessarily commute with angular momentum L in the $\mathcal{L}$ notion. It is conserved given that $\mathcal{H}$ is time-independent. In other words, it is invariant under a different (more strict) variational condition.

2. Action and time evolution

Time evolution operators classically (failed)

3. Orbit stability

Bertland Theorem

Appendix C: Non-inertial system

1. Tidal Force

$$(1+x)^{-\frac{1}{2}} = 1 + (-\frac{1}{2})x + (\frac{3}{8})x^2 + o(x^3) \tag{C1}$$

$$(1+x)^{-\frac{3}{2}} = 1 + (-\frac{3}{2})x + (\frac{15}{8})x^2 + o(x^3) \tag{C2}$$

$$
\begin{aligned}
\boldsymbol{a}_g &= -\frac{k}{|\boldsymbol{r}|^3}\boldsymbol{r} \\
&= -k(\boldsymbol{r}^2)^{-\frac{3}{2}}\boldsymbol{r} \\
&= -\frac{k}{r_0^3}(1+\frac{(\boldsymbol{r_0}+\boldsymbol{\Delta r})^2-\boldsymbol{r_0}^2}{\boldsymbol{r_0}^2})^{-\frac{3}{2}}(\boldsymbol{r_0}+\boldsymbol{\Delta r}) \\
&= -\frac{k}{r_0^3}(1+\frac{2\boldsymbol{r_0}\cdot\boldsymbol{\Delta r}+(\boldsymbol{\Delta r})^2}{\boldsymbol{r_0}^2})^{-\frac{3}{2}}(\boldsymbol{r_0}+\boldsymbol{\Delta r}) \\
&= -\frac{k}{r_0^3}(1-\frac{3\boldsymbol{r_0}\cdot\boldsymbol{\Delta r}}{\boldsymbol{r_0}^2}+\frac{15(\boldsymbol{r_0}\cdot\boldsymbol{\Delta r})^2}{2\boldsymbol{r_0}^4}-\frac{3(\boldsymbol{\Delta r})^2}{2(\boldsymbol{r_0})^2})(\boldsymbol{r_0}+\boldsymbol{\Delta r})
\end{aligned} \tag{C3}
$$

$$
\begin{aligned}
\boldsymbol{a}_e &= -\frac{k}{r_0^3}\boldsymbol{r_0} \\
\boldsymbol{a}_{Tide} &= \boldsymbol{a}_g - \boldsymbol{a}_e \\
&= o((\boldsymbol{\Delta r})^3) - \frac{k}{r_0^3}(\boldsymbol{\Delta r}-\frac{3\boldsymbol{r_0}\cdot\boldsymbol{\Delta r}}{\boldsymbol{r_0}^2}\boldsymbol{r_0}) \\
&\quad - \frac{k}{r_0^3}(\frac{15(\boldsymbol{r_0}\cdot\boldsymbol{\Delta r})^2}{2\boldsymbol{r_0}^4}-\frac{3(\boldsymbol{\Delta r})^2}{2(\boldsymbol{r_0})^2})\boldsymbol{r_0} - \frac{k}{r_0^3}(-\frac{3\boldsymbol{r_0}\cdot\boldsymbol{\Delta r}}{\boldsymbol{r_0}^2})\boldsymbol{\Delta r} \\
&= \boldsymbol{a}^1_{Tide} + \boldsymbol{a}^2_{Tide} + o((\boldsymbol{\Delta r})^3)
\end{aligned} \tag{C4}
$$

$$
\boldsymbol{a}^1_{Tide} = -\frac{k}{r_0^3}(\boldsymbol{\Delta r}_\perp - 2\boldsymbol{\Delta r}_\parallel) \tag{C5}
$$

$$
\begin{aligned}
V_g &= -\frac{k}{|\boldsymbol{r}|} \\
&= -k(\boldsymbol{r}^2)^{-\frac{1}{2}} \\
&= -\frac{k}{r_0}(1+\frac{(\boldsymbol{r_0}+\boldsymbol{\Delta r})^2-\boldsymbol{r_0}^2}{\boldsymbol{r_0}^2})^{-\frac{1}{2}} \\
&= -\frac{k}{r_0}(1+\frac{2\boldsymbol{r_0}\cdot\boldsymbol{\Delta r}+(\boldsymbol{\Delta r})^2}{\boldsymbol{r_0}^2})^{-\frac{1}{2}} \\
&= -\frac{k}{r_0}(1-\frac{\boldsymbol{r_0}\cdot\boldsymbol{\Delta r}}{\boldsymbol{r_0}^2}+\frac{3(\boldsymbol{r_0}\cdot\boldsymbol{\Delta r})^2}{2\boldsymbol{r_0}^4}-\frac{(\boldsymbol{\Delta r})^2}{2(\boldsymbol{r_0})^2}) + o((\boldsymbol{\Delta r})^3) \\
&= V_0 + V_1 + V_2 + o((\boldsymbol{\Delta r})^3)
\end{aligned} \tag{C6}
$$

2. Coriolis Force

In system with rotation $(\boldsymbol{\omega}, \boldsymbol{\beta})$ around point O, Non-inertial forces are

$$
\begin{aligned}
\boldsymbol{a}_c &= -2(\boldsymbol{\omega} \times \boldsymbol{\delta v}) \\
\boldsymbol{a}_\beta &= -\beta \times \boldsymbol{\delta r}
\end{aligned}
\tag{C7}
$$

Appendix D: Thermaldynamics and Hydrodynamics

1. Fluid Roche limit

To be continued.

2. Waterball without gravity

Ossilation?

Appendix E: Scenarios

1. Wander around space station

Space station O is orbiting $\boldsymbol{r} : (r, \theta)$ around the earth and an astronaut is wandering around O with displacement $\boldsymbol{\Delta r}$ in non-rotating system and $\boldsymbol{\delta r}$ in rorating system.

Dynamic in the rotating system is described in Eqn. E1

$$
\begin{aligned}
\boldsymbol{a}_\delta &= \boldsymbol{a}_{Tide} + \boldsymbol{a}_c + \boldsymbol{a}_\beta \\
\boldsymbol{a}_{Tide} &= -\frac{k}{r_0^3}(-2\delta r_r \hat{r} + \delta r_\theta \hat{\theta}) \\
\boldsymbol{a}_c &= 2\dot{\theta}(\dot{\delta r_\theta} \hat{r} - \dot{\delta r_r} \hat{\theta}) \\
\boldsymbol{a}_\beta &= \ddot{\theta}(\delta r_\theta \hat{r} - \delta r_r \hat{\theta})
\end{aligned}
\tag{E1}
$$

Assume O is orbiting on a circle $(\dot{\theta}, \ddot{\theta}) = (\omega, 0)$, we have $\omega = \frac{k}{r_0^3}$, then Eqn. E1 is simplified as Eqn. E2

$$
\begin{aligned}
\frac{d^2}{dt^2}\boldsymbol{\delta r} &= \boldsymbol{a}_{Tide} + \boldsymbol{a}_c \\
&= -\omega^2(-2\delta r_r \hat{r} + \delta r_\theta \hat{\theta}) + 2\omega(\dot{\delta r_\theta} \hat{r} - \dot{\delta r_r} \hat{\theta})
\end{aligned}
\tag{E2}
$$

Qualitative analysis of the dynamics: Assume the astronaut orbits around the space station with $\delta\dot{\theta} < 0$ and a period same as the space station orbit T_0

- Averagely, orbit of $\boldsymbol{\delta r}$ operates with $-\boldsymbol{\omega}$;
- At vertex along the $\hat{r}$ direction, $\boldsymbol{a}_{Tide}$ points outwards, so velocity should be large to generate massive $\boldsymbol{a}_c$;
- At vertex along the $\hat{\theta}$ direction, $\boldsymbol{a}_{Tide}$ points inwards, enough to keep the orbit bound, so velocity should be small;
- Quantitave description of the orbit dynamic remains mystery;

2. Effective gravity induced by rotation of facility

Only second-order effect is present at a perturbative level.

3. Lagrangian points

We consider the effective dynamics in a Non-inertial system of two-body gravity system.

Consider a two-body system consists of two celestial bodies M_1 and M_2 with distance of D. The two-body effective mass and orbiting angular velocity are

$$
\begin{aligned}
M &= \frac{M_1 M_2}{M_1 + M_2} \\
\frac{G M_1 M_2}{D^2} &= \frac{M_1 M_2}{\omega}^2 D \\
\omega^2 &= \frac{G(M_1 + M_2)}{D^3}
\end{aligned}
\tag{E3}
$$

The two celestial bodies are orbiting around the centroid O. Now we consider the Lagrangian point L_4 locating at the vertex of an equilateral triangle connecting M_1 and M_2, thus we have $\boldsymbol{r} = \frac{M_1 \boldsymbol{r_1} + M_2 \boldsymbol{r_2}}{M_1 + M_2}$

In the rotating celestial system, the effective potential around L_4 is

$$
\begin{aligned}
V_{eff} &= V_0(\boldsymbol{r_1}) + V_1(\boldsymbol{r_1}) + V_2(\boldsymbol{r_1}) + V_0(\boldsymbol{r_2}) + V_1(\boldsymbol{r_2}) + V_2(\boldsymbol{r_2}) + o((\boldsymbol{\Delta r})^3) + V_\omega \\
&= V_{eff}^0 + V_{eff}^1 + V_{eff}^2 + o((\boldsymbol{\Delta r})^3) \\
V_\omega &= -\frac{1}{2}\omega^2(\boldsymbol{r} + \boldsymbol{\Delta r})^2 \\
V_{eff}^1 &= -\frac{GM_1}{D}(-\frac{\boldsymbol{r_1}\cdot\boldsymbol{\Delta r}}{|\boldsymbol{r_1}|^2}) - \frac{GM_2}{D}(-\frac{\boldsymbol{r_2}\cdot\boldsymbol{\Delta r}}{|\boldsymbol{r_2}|^2}) - \omega^2\boldsymbol{r}\cdot\boldsymbol{\Delta r} \\
&= \frac{G}{D^3}(M_1\boldsymbol{r_1} + M_2\boldsymbol{r_2})\cdot\boldsymbol{\Delta r} - \frac{G}{D^3}(M_1\boldsymbol{r_1} + M_2\boldsymbol{r_2})\cdot\boldsymbol{\Delta r} \\
&= 0 \\
V_{eff} &= V_{eff}^0 + V_{eff}^2 + o((\boldsymbol{\Delta r})^3)
\end{aligned}
\tag{E4}
$$

In fact V_{eff}^2 is convex around L_4, thus L_4 is a smooth maximum in the rotating system. Celestial bodies around L_4 are bounded by Coriolis force under certain condition (the mass ratio boundary $\frac{M_1}{M_2}$ actually)

4. Epicycle model

Fitting Kepler orbit of binary star system to epicycle system?

5. Space channel in gravitational fields

More fictional rather than a realistic one. Few materials are found.