NYU CS Bridge to Tandon Course: Homework #11

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Problem 5

Use mathematical induction to prove that for any positive integer n, 3 divide $n^3 + 2n$ (leaving no remainder)

(a) 3 Divide $n^3 + 2n$

Proof. By induction on n

Base case: n = 1
we will prove true for n = 1, 3 evenly divide $n^3 + 2n$

$$\begin{aligned} n^3 + 2n &= 1^3 + 2(1) \\ &= 1 + 2 \\ &= 3 \end{aligned}$$

3 evenly divide 3 Therefore P(1) is True.

Inductive Step
Assume n = k, 3 evenly divide P(k)
we will prove for any positive integer $k \geq 1$ if 3 evenly divide $k^3 + 2k$ then 3 also evenly divide $(k+1)^3 + 2(k+1)$

$$\begin{aligned} (k+1)^3 + 2(k+1) &= k^3 + 3k^2(1) + 3k(1^2) + 1^3 + 2k + 2 \\ &= k^3 + 3k^2 + 3k + 2k + 3 \\ reorder &= (k^3 + 2k) + 3k^2 + 3k + 3 \end{aligned}$$

By inductive hypothesis we prove 3 divides $k^3 + 2k$ therefore 3 evenly divide $3k^2 + 3k + 3$ Thus $(k+1)^3 + 2(k+1)$ evenly divide 3 ■

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Use strong induction to prove that any positive integer $n (n \geq 2)$ can be written as a product of primes.

(b) Prove any positive integer $n \geq 2$ can be written as product of prime

Proof. By strong induction on n

Base case: n = 2
since 2 is a prime number its already product of 2 only 1 * 2 = 2
Therefore P(2) is True

Inductive Step
Assume that for $k \geq 2$ for any integer j in the range from 2 through k can be expressed as a product of prime number. Then we will show k +1 can be expressed as a product of prime number.

if $k + 1$ is prime, then it is product of itself and 1. if $k + 1$ is not a prime the its a product of two integer a, b. as $a \geq 2$ and $b \geq 2$ which are prime numbers $(k + 1)^3 + 2(k + 1)$

$$\begin{aligned} k + 1 &= a * b \\ a &= \frac{k+1}{b} \\ b &= \frac{k+1}{a} \end{aligned}$$

since $a \geq 2$ and $b \geq 2$

$$\begin{aligned} k + 1 &= a * b \\ a &= \frac{k+1}{b} < k + 1 \\ b &= \frac{k+1}{a} < k + 1 \end{aligned}$$

Therefore k +1 can be expressed as product of primes

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Problem 6

7.4.1: Components of an inductive proof.

$$\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$$

(A) Verify that P(3) is true.

$$\begin{aligned}\sum_{j=1}^{n} j^2 = 1^2 + 2^2 + 3^2 = 14 \\ &= \frac{3(3+1)+2(3)+1}{6} \\ &= \frac{12 * 7}{6} \\ &= 14\end{aligned}$$

84 / 6 = 14 Therefore P(3) is True.

(B) Express P(k).

$$\sum_{j=1}^{k} j^2 = \frac{k(k+1)+(2k+1)}{6}$$

(C) Express P(k +1).

$$\sum_{j=1}^{k+1} j^2 = \frac{(k+1)(k+2)+(2k+3)}{6}$$

(D) In an inductive proof that for every positive integer n, what must be proven in the base case?
Since n is an positive integer n = 1

$$\sum_{j=1}^{1} j^2 = 1^2 = 1$$

$$\frac{1(1+1)+(2(1)+1)}{6} = \frac{6}{6} = 1$$

Therefore P(1) is True.

(E) In an inductive proof that for every positive integer n,what must be proven in the inductive step?

The inductive step shows for that any positive integer K if P(K) is True, then P(K+1) is true. that is for any $K \geq 1$

$$\sum_{j=1}^{k} j^2 = \frac{k(k+1)+(2k+1)}{6}$$

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then

$$\sum_{j=1}^{k+1} j^2 = \frac{(k+1)(k+2)+(2k+3)}{6}$$

(F) What would be the inductive hypothesis in the inductive step from your previous answer?

$$P(K+1) = \sum_{j=1}^{k+1} j^2 = \sum_{j=1}^{k} j^2 + (K+1)^2$$

(G) Prove by induction that for any positive integer n,

(G) *Proof.* By induction on n n

Base case: n = 1

$$\sum_{j=1}^{1} j^2 = 1^2 = 1$$

$$\frac{1(1+1)+(2(1)+1)}{6} = \frac{6}{6} = 1$$

Therefore P(1) is True.

Inductive Step

We assume $k \geq 1$ if p(K) is true, then p(k+1) is also true.

$$\begin{aligned}
\sum_{j=1}^{k+1} j^2 &= \sum_{j=1}^{k} j^2 + (K+1)^2 \\
&= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\
&= \frac{k(k+1)(2k+1)+6(k+1)^2}{6} \\
&= \frac{(k+1)(2k^2+7k+6}{6} \\
&= \frac{(k+1)(k+2)+(2k+3)}{6}
\end{aligned}$$

Therefore P(k +1) is true

■

Exercise 7.4.3: Proving inequalities by induction .

(b) Prove that for $n \geq 1$

$$\sum_{j=1}^{n} \frac{1}{j^2} \leq 2 - \frac{1}{n}$$

Proof. By induction on n

Base case: n = 1

$$\sum_{j=1}^{n} \frac{1}{j^2} = \leq 2 - \frac{1}{n}$$

$$\sum_{j=1}^{1} \frac{1}{1^2} = \leq 2 - \frac{1}{1}$$

$1 \leq 1$ Therefore P(1) is True.

Inductive Step

we will show for any positive integer $k \geq 1$ if

$$\sum_{j=1}^{k} \frac{1}{j^2} \leq 2 - \frac{1}{k}$$

is True then

$$\sum_{j=1}^{k+1} \frac{1}{j^2} \leq 2 - \frac{1}{k+1}$$

is also true. If p(k) is true then p(k+1) is also true.

$$\begin{aligned}
\sum_{j=1}^{k+1} \frac{1}{j^2} &= \sum_{j=1}^{k} \frac{1}{j^2} + \frac{1}{(k+1)^2} \\
&= \frac{1}{(k+1)^2} + (2 - \frac{1}{k}) \\
&= \frac{1}{k(k+1)} + (2 - \frac{1}{k}) \\
&= 2 - \frac{1}{k}(1 - \frac{1}{k+1}) \\
&= 2 - \frac{1}{k}(\frac{k}{k+1}) = 2 - \frac{1}{k+1}
\end{aligned}$$

Therefore $P(k \geq 1)$ is True. ■

7.5.1: Proving divisibility results by induction. .

(a) Prove that for any positive integer n, 4 evenly divides $3^{2n} - 1$.

Proof. By induction on n

Base case: n = 1

$$\begin{aligned} 3^{2(1)} - 1 &= 9 - 1 = 8 \\ &= 8/4 \\ &= 2 \end{aligned}$$

since 4 evenly divide 8 Theorem hold true.

Inductive Step
we assume true for k: we will show for any positive integer $k \geq 1$, 4 divides $3^{2k} - 1$ then 4 also divides $3^{2k-1} - 1$

$$\begin{aligned} 3^2k + 1 - 1 & \\ &= 3^{2k+2} - 1 \\ &= 3^{2k} * 9 - 1 \\ &= 3^{2k} * 8 + (3^{2k} - 1) \\ reorder &= 8 * 3^{2k} + (3^{2k} - 1) \end{aligned}$$

by inductive hypothesis 4 divides $3^{2k} - 1$ thus 4 also divide $8 * 3^{2k}$ therefore 4 will divide$8 * 3^{2k} + (3^{2k} - 1)$ ∎