Trigonometry

by Ethan and Friends !

Introduction

What is Trigonometry

- The study of the relationship between the sides and angles of a triangle.

- Supposed to learn it from Grade 11

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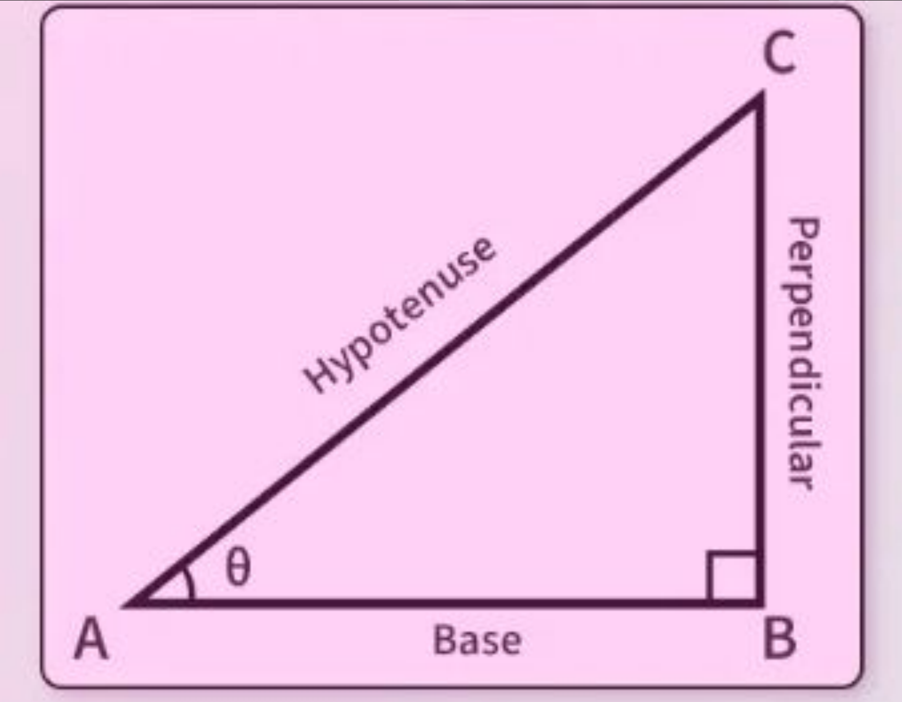


Solving Trigonometric Equations Checklist

- ❑ Check for Restrictions (A value that results in dividing by 0, ex 1/x and x = 0)
- ❑ Simplify the Equation (Combine like terms and use trig identities)
- ❑ Isolate the Trigonometric Function (Rearrange equation so one trig function (eg. sinx, cosx, tanx) is by itself on one side of equation = 0)
- ❑ Solve for the Angle (Remember quadrant rules, CAST and keep restrictions in mind)
- ❑ Find Additional Solution(s) Within the Domain (Use the period of the function to write a general equation or find all solutions within the range by adding and subtracting periods.)

Additional:

- ❑ Verify Solution(s) (Plug your solutions back into the equation to make sure they are valid and do not violate any restrictions)

Squaring both sides

Make sure both sides are the same SIGN while squaring.

Both sides need to be the same sign because in the original equation a positive can't equal a negative or vice versa.

By squaring you may encounter solutions that result in what is stated above. We need to reject those solutions because we are limited by the original equation (extraneous solutions).

Practice

1. $3\sqrt{(2x + 1)} - \sqrt{(1 - x)} = 0$

2. $\cos x + 2\sin x = 1$

Let's now go through some examples!

Example 1

$$8\sin^3 x * \cos^2 x - 4\sin^3 x = 2\sin x * \cos^2 x - \sin x, \text{ for } 0 \leq x \leq 2\pi$$

Solution

Solve

$8\sin^3 x\cdot\cos^2 x-4\sin^3 x=2\sin x\cdot\cos^2 x-\sin x$, for $0\le x\le 2\pi$

$\sin x(8\sin^2 x\cdot\cos^2 x-4\sin^2 x-2\cos^2 x+1)=0$

$\sin x((4\sin^2 x)(2\cos^2 x-1)-(2\cos^2 x-1))=0$

$\sin x(4\sin^2 x-1)(2\cos^2 x-1)=0$

$\sin x(2\sin x+1)(2\sin x-1)(\sqrt{2}\cos x+1)(\sqrt{2}\cos x-1)=0$ This step is not necessary

$\sin x=0$

$\therefore x=0,\pi,2\pi$

$\sin x=\pm\frac{1}{2}$

$\alpha=\sin^{-1}\left(\left|\pm\frac{1}{2}\right|\right)$

$=\frac{\pi}{6}$

$\therefore x=\frac{\pi}{6},\frac{5\pi}{6},\frac{7\pi}{6},\frac{11\pi}{6}$

$\cos x=\pm\frac{\sqrt{2}}{2}$

$\alpha=\cos^{-1}\left(\left|\pm\frac{\sqrt{2}}{2}\right|\right)$

$=\frac{\pi}{4}$

$\therefore x=\frac{\pi}{4},\frac{3\pi}{4},\frac{5\pi}{4},\frac{7\pi}{4}$

Example 2

$$\tan x = \sin^3 x * \sec x,\ x \in [0, 2\pi]$$

hint : restrictions

Solution

$\tan x = \sin^3 x * \sec x$

Make sure to state RESTRICTIONS, $\cos x \neq 0$, $x \neq \pi/2, 3\pi/2$

<u>Step 1:</u> $\sin x/\cos x = \sin^3 x/\cos x$

<u>Step 2:</u> $(\sin x)(\sin^2 x - 1)/\cos x = 0$

<u>Step 3:</u> $\sin x = 0$ and $\sin x = \pm 1$

<u>Step 5:</u> $x = 0, \pi, 2\pi$ and $x = \pi/2, 3\pi/2$

*remember unit circle, y is 0 at 0, π, 2π and y is 1 at π/2, etc

<u>Step 6:</u> reject $x = \pi/2, 3\pi/2$ since restr

$\therefore x = 0, \pi, 2\pi$

Example 3

Solve: $\cos x + 8\sin x - 7 = 0, x \in \mathbb{R}$

Hint: isolate cosx and sinx to separate sides

Solution

Step 1: $8\sin x - 7 = \cos x$

Make sure both sides are the same SIGN while squaring. In this case, right side is always positive and left side is positive in Quadrants 1 and 4.

Step 2: $(8\sin x - 7)^2 = \cos^2 x$

Step 3: $64\sin^2 x - 112\sin x + 49 = 1 - \sin^2 x$

Step 4: $65\sin^2 x - 112\sin x + 48 = 0$

Step 5: $(13\sin x - 12)(5\sin x - 4) = 0$

Step 6: $\sin x = 12/13$ and $\sin x = 4/5$

Step 7: $\alpha = \sin^{-1}(12/13)$ and $\alpha = \sin^{-1}(4/5)$

Since sine is positive in Quadrant 1 and 2, take solutions in Quadrant 1.

Step 8: Q1: $x = \alpha$ so $x = \sin^{-1}(12/13)$

and $x = \sin^{-1}(4/5)$

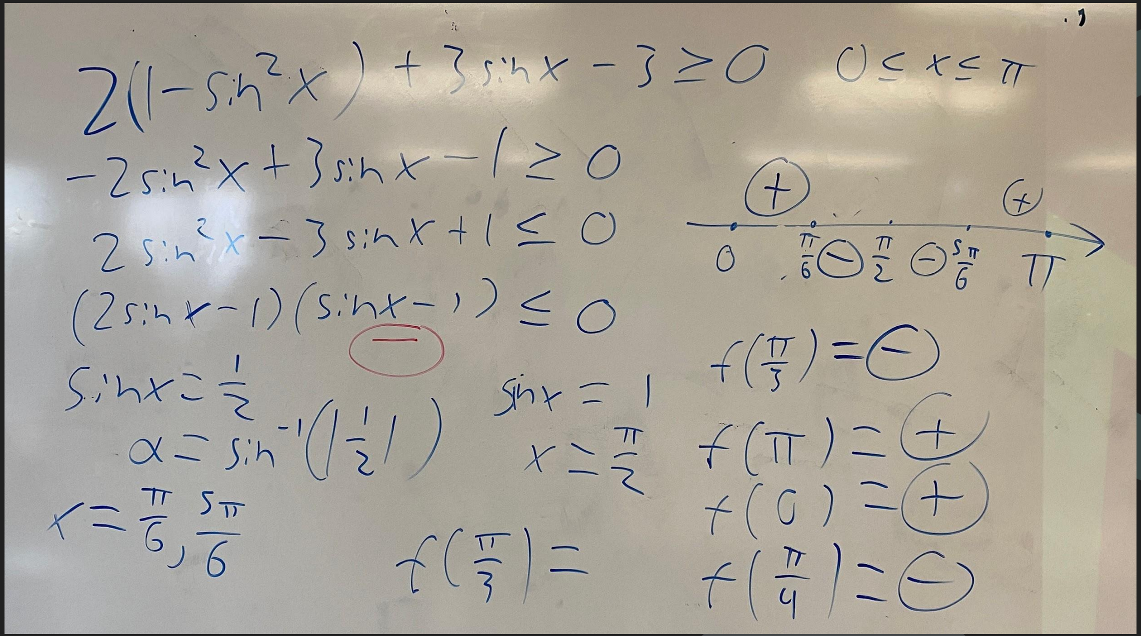
Step 9: $x = \sin^{-1}(12/13) + 2k\pi$

$x = \sin^{-1}(4/5) + 2k\pi$

for $k \in \mathbb{Z}$

Example 4

Solve: $2\cos^2 x + 3\sin x \geq 3$, for $0 \leq x \leq \pi$



$\therefore \pi/6 \leq x \leq 5\pi/6$

Example Question

2. Prove that: $\cos\left(\sin^{-1}(x)\right) = \sqrt{1-x^2}$.

3. Prove that: $\sin\left(\tan^{-1}(x)\right) = \dfrac{x}{\sqrt{1+x^2}}$.

Solution

Prove: $\cos(\sin^{-1}(x)) = \sqrt{1-x^2}$

let a be $\sin^{-1}(x)$ $\therefore \sin a = x$

$LS = \cos a$

$RS = \sqrt{1-x^2} = \sqrt{1-\sin^2 a} = \sqrt{\cos^2 a} = |\cos a|$

'a' will always be an acute angle since $\sin^{-1}(x)$ will always return the relative acute angle. Therefore $-\pi/2 <= a <= \pi/2$ AND since $\cos(-a) = \cos(a)$, $\cos(a)$ is always positive.

so $RS = \cos(a)$ not $\pm\cos(a)$ ∴ **LS = RS**

Example Question

Case Discussion: $3\cot x - a = 0, x \in R$

The Slide

Solving Trigonometric Equations

1. Solve for θ within the specified domain. Keep answers exact whenever possible.
   a. $\sqrt{2}\sin(\theta) + 2 = 1, 0 \le \theta \le 2\pi$
   b. $3\csc(2\theta) - 4 = 0, 0^\circ \le \theta \le 360^\circ$
   c. $5\cos\left(\theta - \frac{\pi}{3}\right) - 4 = 0, 0 \le \theta \le 2\pi$
   d. $4\sin(\theta) + 3 = 2(\sin(\theta) + 1), 0 \le \theta \le 2\pi$
   e. $2\tan(\theta) - 4 = 3(\tan(\theta) - 2), -2\pi \le \theta \le 2\pi$
   f. $4\cot\left(\frac{\theta}{2}\right) - 3\sqrt{3} = \cot\left(\frac{\theta}{2}\right), -2\pi \le \theta \le 2\pi$

2. Solve for x within the specified domain. Keep answers exact whenever possible.
   a. $\tan(x)\cos(x) = 0, 0 \le x \le 2\pi$
   b. $\left(2\cos(x) + \sqrt{3}\right)\left(\csc(x) - \sqrt{2}\right) = 0, 0 \le x \le 2\pi$
   c. $2\cos^2(x) + \sqrt{3}\cos(x) = 0, -\pi \le x \le \pi$
   d. $2\sin(x)\sec(x) = 6\sin(x), 0^\circ \le x \le 360^\circ$

3. Determine the roots of each equation within the specified domain. Keep answers exact whenever possible.
   a. $2\cos^2(\theta) - 1 = 0, 0 \le \theta \le 2\pi$
   b. $2\sin^2(x) + 3\sin(x) - 2 = 0, -2\pi \le x \le 2\pi$
   c. $2\cos^2(x) = 5\cos(x) + 4, 0^\circ \le x \le 360^\circ$
   d. $4\csc^2(\theta) - 4\csc(\theta) = 11, 0 \le \theta \le 2\pi$
   e. $3\tan^4(\theta) - 4\tan^2(\theta) + 1 = 0, -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

4. Solve for x in the interval $[0, 2\pi]$. Make any necessary substitutions using identities. Keep answers exact whenever possible.
   a. $8\sin(x) + 7 = 4\cos^2(x)$
   b. $5\cos(2x) + 3\cos(x) + 4 = 0$
   c. $2\tan^2(x) = \sec(x) + 4$
   d. $\tan(x)\cos^2(x) = \sin(x)$
   e. $\sin(2x) - 1 = \cos(2x)$
   f. $4\cos(2x) = 2(\sin(x) + 1)$

5. Given $f(x) = 2\cos(x)$ and $g(x) = \sqrt{2}\sin(2x)$
   a. Find the points of intersection of the two functions for $0 \le x \le 2\pi$. Illustrate the situation graphically.
   b. Solve $2\cos(x) > \sqrt{2}\sin(2x)$ for $0 \le x \le 2\pi$.

6. Determine the exact values of a and b such that the quadratic trigonometric equation $a\cos^2(x) + b\cos(x) - 3 = 0$ has the solutions $\frac{\pi}{4}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{7\pi}{4}$ in the interval $0 \le x \le 2\pi$.

7. Determine roots of each equation within the domain specified. Keep answers exact whenever possible.
   a. $\tan^2(x) - 2\tan(x)\sin(x) = 0, 0 \le \theta \le 2\pi$
   b. $\sin(3x) - \sin(6x) = 0, 0 \le \theta \le \pi$
   c. $\cos(3\theta) + \cos(2\theta) + \cos(\theta) = 0, 0 \le \theta \le 2\pi$

8. Determine roots of $\sin(x) + \cos(x) = \sqrt{\frac{3}{2}}$ for $0 \le x \le 2\pi$. Verify your answer using graphing technology.

9. Determine all possible values of x such that $\sin\left(x + \frac{\pi}{6}\right) = \cos(x)$.

10. For what values of θ, $0 \le \theta \le 2\pi$, does the equation $2x^2 + (4\sin(\theta))x + \cos(2\theta) = 0$ have real roots?

Useful Trigonometric Identities

Half-Angle Formulas

- $\sin\left(\frac{A}{2}\right) = \pm\sqrt{\frac{1-\cos A}{2}}$
- $\cos\left(\frac{A}{2}\right) = \pm\sqrt{\frac{1+\cos A}{2}}$
- $\tan\left(\frac{A}{2}\right) = \pm\sqrt{\frac{1-\cos A}{1+\cos A}} = \frac{\sin A}{1+\cos A} = \frac{1-\cos A}{\sin A}$

Double-Angle Formulas

(i) $\sin 2A = 2\sin A\cos A = \frac{2\tan A}{1+\tan^2 A}$

(ii) $\cos 2A = \cos^2 A - \sin^2 A$

$= 2\cos^2 A - 1$

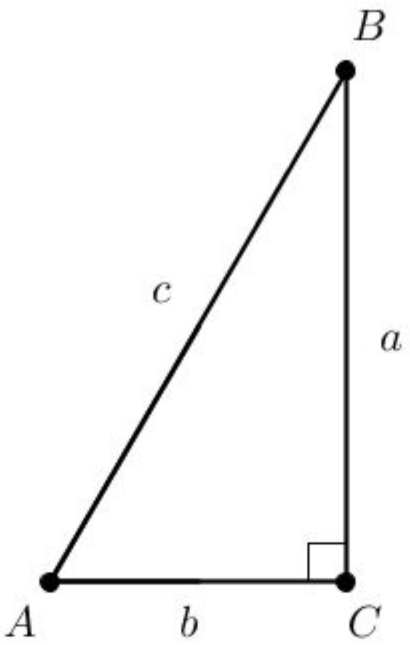
$= 1 - 2\sin^2 A$

$= \frac{1-\tan^2 A}{1+\tan^2 A}$

(iii) $\tan 2A = \frac{2\tan A}{1-\tan^2 A}$

Fun identity

9. For a right triangle ABC (shown to the right) with $\angle C = \frac{\pi}{2}$, show that the area of the triangle is given by $\frac{1}{4}c^2 \sin(2A)$ or $\frac{1}{4}c^2 \sin(2B)$.



$5 if you can solve this identity

$$LS = \frac{\left(\sec(x)\sin(x)-\sin(x)+\sec(x)-1\right)\left(\sec(x)+1\right)}{\left(\csc^2(x)+\tan^2(x)-\cot^2(x)\right)\left(\sin(x)+1\right)} + \frac{\tan^2(x)+\tan(x)-\sec^2(x)}{\tan(x)^3-\tan^2(x)+\tan(x)-1}$$

$$RS = \frac{\sin^2(x)\left(1+2\csc^2 x+\csc(x)^4\right)(\cot x-1)}{\left(\cot(x)^3-\cot^2 x+\cot x-1\right)\left(\sin^2 x-\cos^2(x)+2\right)} - \frac{\tan(x)^4\left(\cot^2 x+\csc x+1\right)^2\left(2-2\sin x-\cos^2 x\right)}{\left(\cot^2 x+1\right)\left(1+\csc^2 x\right)+\cot^2 x+1}$$

End