Objective 2 - Vertical Asymptotes

Link to section in online textbook.

Introduction video describing holes/vertical asymptotes *without limits*.

When we learned about the domain of rational functions, we set the denominator equal to 0 and solved. This gave us all values that the function is not defined for. As we saw in Objective 1, some of these x-values are holes of the function. The rest are **vertical asymptotes** of the function, or vertical lines where the function approaches positive or negative infinity.

$$\text{Graph of } f(x) = 1/((x-1)(x-3))$$

Unlike holes, vertical asymptotes affect the function around where they are defined. With left- and right-sided limits, we can determine how the function is behaving near these vertical lines.

Theorem 1. *Vertical Asymptotes of a Rational Function*

A rational function has a vertical asymptote $x = a$ (vertical line) if

$$\lim_{x \to a^-} f(x) = \pm\infty \ \ or \ \ \lim_{x \to a^+} f(x) = \pm\infty$$

Thus, to determine if a rational function has any vertical asymptotes, we need to factor the denominator and evaluate the limit. If a one-sided limit is positive or negative infinity, it is a vertical asymptote.

Practice with the questions below.

Question 1 *Find all vertical asymptotes of the rational function below. If they do not exist, answer "NA".*

$$f(x) = \frac{x-4}{2x-8}$$

Vertical Asymptote: the vertical line $x = \boxed{NA}$.

Question 2 *Find all vertical asymptotes of the rational function below. If they do not exist, answer "NA".*

$$f(x) = \frac{4}{x-4}$$

Vertical Asymptote: the vertical line $x = \boxed{4}$.

Question 3 *Find all vertical asymptotes of the rational function below. If they do not exist, answer "NA".*

$$f(x) = \frac{-3}{-6x^2 - 5x + 25}$$

Vertical Asymptote: the vertical line $x = \boxed{-\dfrac{5}{2}}$ *and* $x = \boxed{\dfrac{5}{3}}$.

Question 4 *Find all vertical asymptotes of the rational function below. If they do not exist, answer "NA".*

$$f(x) = \frac{6x^2 - 17x + 12}{6x^3 - 25x^2 + 3x + 4}$$

Vertical Asymptote: the vertical line $x = \boxed{-\dfrac{1}{3}}$, $x = \boxed{\dfrac{1}{2}}$, *and* $x = \boxed{4}$.

Question 5 *Find all vertical asymptotes of the rational function below. If they do not exist, answer "NA".*

$$f(x) = \frac{-3x^2 - 10x - 8}{9x^3 - 18x^2 - 115x - 100}$$

Vertical Asymptote: the vertical line $x = \boxed{-\dfrac{5}{3}}$ *and* $x = \boxed{5}$.

Question 6 *Find all vertical asymptotes of the rational function below. If they do not exist, answer "NA".*

$$f(x) = \frac{x^2 - 25}{x^3 + 5x^2 + 13x + 65}$$

Vertical Asymptote: the vertical line $x = \boxed{NA}$.

Main takeaway: Values not in the domain of the function can be one of two things:

- Vertical Asymptotes: Values the function will come close to, but will not touch at. These are vertical lines $x = a$, where a makes the denominator zero. The limit of the function at these points are $\pm\infty$.

- Holes: Values that only affect the function exactly at that point (rather than nearby by vertical asymptotes). The limit of the function at these points are $\frac{0}{0}$.