

# Sound Propagation in Elongated Bose-Einstein Condensed Clouds

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We consider propagation of sound pulses along the long axis of a Bose-Einstein condensed cloud in a very anisotropic trap. In the linear regime, we demonstrate that the square of the velocity of propagation is given by the square of the local sound velocity,  $c^2 = nU_0/m$ , averaged over the cross section of the cloud. We also carry out calculations in the nonlinear regime, and determine how the speed of the pulse depends on its amplitude.

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In a beautiful recent experiment, Andrews *et al.* have reported measurements of the propagation of sound pulses along the axis of an elongated cloud of Bose-Einstein condensed Na atoms [1]. At the center of the trap a disturbance was created by turning a laser on or off. Due to the energy shift caused by the interaction of atoms with the laser field, this led to a change in the potential experienced by atoms, and thereby to a density disturbance, whose time development was studied by phase-contrast imaging. In a dilute, homogeneous Bose gas at zero temperature, the speed of sound,  $c_0$ , is given by  $c_0^2 = dP/d\rho = n_0U_0/m$ , where  $P$  is the pressure,  $\rho = mn$  is the mass density,  $m$  is the atomic mass,  $n$  is the density of atoms, and  $U_0 = 4\pi\hbar^2a/m$  is the strength of the effective two-body interaction, with  $a$  being the scattering length for atom-atom collisions [2,3]. The experimental data seemed to be in accord with this result, if the density was taken to be that at the center of the cloud. Within the accuracy of the measurements, no evidence for a dependence of propagation speed on amplitude was seen.

The purpose of this Letter is to investigate how propagation of a sound pulse is influenced by the inhomogeneity of the cloud, and by nonlinear effects. The important result of our calculations is that, in the linear regime, the sound speed is given by the expression for the homogeneous gas, but evaluated for the average density over the cross section of the cloud perpendicular to the direction of propagation. A second result is that nonlinear effects are substantial, and can give rise to changes in the velocity of the pulse of order of 10%, for relative density perturbations of order 15%.

Consider a Bose-Einstein condensed cloud whose transverse dimensions ( $\sim R_\perp$ ) are much less than the axial dimension ( $\sim Z$ ). We study the propagation along the axis of the trap, of pulses whose characteristic spatial scale,  $l$ , in the axial direction is large compared with  $R_\perp$ , but small compared with  $Z$ . That this condition is satisfied in the experiments may easily be seen by inspection of the density profiles in Fig. 2 of Ref. [1]. For the moment we make no detailed assumptions about the

confining potential perpendicular to the long direction of the trap, which we take to be the  $z$ -axis, apart from the fact that its variations with respect to  $z$  should be small on a length scale  $l$ . The fact that the transverse dimensions of the cloud are small compared with the scale of the pulse, means that the density profile across the trap may be assumed to have its equilibrium form appropriate to the local number of particles per unit length. This follows from the fact that the timescale for adjustment of the profile ( $\sim R_\perp/c$ ), where  $c$  is a typical sound speed, is short compared with the time for passage of the pulse,  $\sim l/c$ . The problem then becomes a one-dimensional one, and the sound pulse may be specified in terms of a local velocity,  $v(z)$ , and a local density of particles,  $\sigma(z)$ , per unit length. The latter is given in terms of the particle density,  $n(x, y, z)$ , where  $x$  and  $y$  are coordinates perpendicular to the axis of the trap, by

$$\sigma(z) = \int dx dy n(x, y, z). \quad (1)$$

The equations governing the motion of the condensate may be derived from the Gross-Pitaevskii equation, and they are just the equations of perfect fluid hydrodynamics, with the addition of a “quantum pressure” term in the Euler equation [4]. The quantum pressure term is important only on length scales of order the coherence length,  $\xi = (8\pi na)^{-1/2}$ , which is typically of order 0.2–0.4  $\mu\text{m}$  at the center of the cloud [1]. This is considerably less than the lengths of the pulses in the experiment, which are of order 10  $\mu\text{m}$ , and therefore we shall neglect the quantum pressure term in these initial studies. However, this term will be important when shocks or other sharp structures develop. The equation of motion for  $\sigma$  is obtained from the continuity equation by integrating over the cross section of the cloud, and is

$$\frac{\partial \sigma}{\partial t} + \frac{\partial(\sigma v)}{\partial z} = 0. \quad (2)$$

The equation of motion for  $v$  is found by integrating the Euler equation over the cross section of the cloud,

$$m\sigma \frac{dv}{dt} = - \int dxdy \frac{\partial P}{\partial z} - \sigma \frac{\partial V}{\partial z}, \quad (3)$$

where  $V$  is the external potential due to the trap and the interaction with the laser field. In deriving this result, we have assumed that the dependence of  $V$  on  $x$  and  $y$  may be neglected. We shall assume that the number of particles is so large that the Thomas-Fermi approximation is valid [5]. For a spatially-uniform, dilute Bose gas at zero temperature, the energy per unit volume (and also the pressure) are given by  $E = n^2 U_0 / 2$ . The change in the local pressure is thus given by  $dP = nU_0 dn$ , where  $dn$  is the local change in density. The chemical potential,  $\mu$ , neglecting the effects of the trapping potential, is  $nU_0$ , and in the Thomas-Fermi approximation the sum of this and the trap potential  $V$  is a constant for a given value of  $z$ . Thus changes in the chemical potential must likewise be functions only of  $z$ , and, since  $d\mu = U_0 dn$ , it follows that changes in the particle density are independent of  $x$  and  $y$ . The change in density is thus given simply in terms of the change in the number of particles per unit length by  $dn = d\sigma/A$ , where  $A$  is the cross-sectional area of the cloud, and therefore the Euler equation becomes

$$\begin{aligned} m\sigma \frac{dv}{dt} &= -U_0 \int dxdy n \frac{\partial n}{\partial z} - \sigma \frac{\partial V}{\partial z} \\ &= -U_0 \frac{\sigma}{A} \frac{\partial \sigma}{\partial z} - \sigma \frac{\partial V}{\partial z}. \end{aligned} \quad (4)$$

Let us begin by considering the linear regime. If we assume that the variation of the potential in the  $z$ -direction may be neglected, the problem becomes translationally invariant in this direction, and from the linearized versions of the continuity equation, Eq. (2), and the Euler equation, Eq. (4), we find

$$\frac{\partial^2 \sigma}{\partial t^2} - \frac{\bar{n}U_0}{m} \frac{\partial^2 \sigma}{\partial z^2} = 0, \quad (5)$$

where  $\bar{n} = \sigma/A$  is the mean density over the cross section of the cloud. This result demonstrates that the pulse satisfies the wave equation with a velocity,  $c(z)$ , given by

$$c^2(z) = \frac{\bar{n}U_0}{m}. \quad (6)$$

For the case of confinement by a harmonic trap potential in the transverse directions, this result has been derived independently by Zaremba [6] using different methods. In this case  $\bar{n} = n(\rho = 0)/2$ , where  $n(\rho = 0)$  is the density on the axis of the trap. Here  $\rho$  is the radius in cylindrical polar coordinates. In the presence of a confining potential along the axis of the trap, Eq. (5) with the average density replaced by its local value,  $n(\rho = 0, z)$ , will be a good approximation, provided the characteristic length of the disturbance is small compared with the axial dimension of the cloud.

The above derivation underlines the generality of the result for the sound speed, but it is instructive to consider an alternative approach for the specific case of a

harmonic confining potential in the transverse direction. The basic idea is to treat the system as a one-dimensional “elastic” medium. The energy of the system consists of the kinetic energy due to the bulk motion of the condensate, an “elastic” energy that takes into account contributions from particle interactions and the trap potential in the transverse direction, and a contribution due to the  $z$ -component of the trap potential. For simplicity we shall assume that the confining potential in the transverse direction is given by  $U(\rho) = m\omega^2\rho^2/2$ , where  $\omega$  is the frequency of transverse oscillations of a single particle in the trap. The kinetic energy,  $T$ , per unit length is

$$T = \frac{1}{2} \int dz \sigma \left( \frac{\partial \zeta}{\partial t} \right)^2, \quad (7)$$

where  $\zeta(z)$  is the displacement of a particle of the fluid from its equilibrium position. We next evaluate the elastic energy, which may be calculated assuming the cross section of the cloud to be locally uniform. Per unit length, the energy due to the dependence of the trap potential on the transverse coordinates is given by

$$E_{\text{trap}} = \frac{1}{2} m\omega^2 \int 2\pi \rho d\rho n(z, \rho) \rho^2, \quad (8)$$

where the integral is to be taken over the cross section of the cloud. The interaction energy per unit volume is given by

$$E_{\text{int}} = \frac{1}{2} U_0 \int 2\pi \rho d\rho n^2(z, \rho). \quad (9)$$

It is convenient to express quantities in terms of the number of particles per unit length. For the harmonic oscillator trap, the density profile in Thomas-Fermi theory is given by [5]

$$n(\rho, z) = n(0, z) \left( 1 - \frac{\rho^2}{R^2(z)} \right), \quad (10)$$

where  $R(z)$  is the radius of the cross section of the cloud. Thus the number of particles per unit length is given by

$$\sigma(z) = \frac{1}{2} n(0, z) \pi R^2. \quad (11)$$

The density on the axis of the trap is given by

$$n(0, z) U_0 = \frac{1}{2} m\omega^2 R^2, \quad (12)$$

and thus one finds that both  $n(0, z)$  and  $R^2$  vary as  $\sigma^{1/2}$ . On performing the integrations in Eqs. (8) and (9), one arrives at the following expression for the elastic energy per unit length:

$$E_{\text{el}} = \frac{\pi}{3} U_0 n^2(0, z) R^2 = \frac{2}{3} \left( \frac{m\omega^2 U_0}{\pi} \right)^{1/2} \sigma^{3/2}. \quad (13)$$

Note that this increases less rapidly with increasing particle number per unit length than it would in the case of confinement by a rigid pipe with infinitely high walls, in which case the potential energy scales as  $\sigma^2$ . This reflects the fact that for a harmonic confining potential an increase in the number of particles per unit length results in expansion of the cloud in the radial direction, thereby leading to a less rapid increase of the total potential energy than would be the case for confinement by rigid walls.

Let us now use the result of Eq. (13) to calculate the velocity of sound in a Bose-condensed cloud when there is no potential acting along the axis. The force parallel to the axis of the trap is given by  $F = \sigma^2 d(E_{\text{el}}/\sigma)/d\sigma$ , and the sound speed is given by  $mc^2 = dF/d\sigma = \sigma d^2 E_{\text{el}}/d\sigma^2$ , results completely analogous to those for the pressure and the sound speed in a bulk medium. Inserting the expression Eq. (13) into this expression, one finds  $c^2 = n(\rho = 0)U_0/(2m)$ . Since for a trap which is harmonic in the transverse directions the average density is half the density on the axis, this result agrees with Eq. (6).

We now consider how a pulse will propagate along the axis of the trap when allowance is made for the varying density. In the Thomas-Fermi approximation, the local density on the axis of the trap is given by  $n(0, z) = n_{\text{max}}(1 - z^2/Z^2)$ , where  $n_{\text{max}}$  is the density at the center of the cloud, and  $Z$  is the axial dimension of the cloud. Thus the local sound speed varies as  $(1 - z^2/Z^2)^{1/2}$ . In the WKB approximation a disturbance initially at  $z = 0$  will at time  $t$  have propagated a distance  $z$  determined by

$$t(z) = \int_0^z \frac{dz}{c(z)} = \frac{Z}{c(0)} \sin^{-1} \frac{z}{Z}, \quad (14)$$

or  $z = Z \sin[c(0)t/Z]$ , where  $c(0) = [n_{\text{max}}U_0/(2m)]^{1/2}$  is the sound velocity, Eq. (6), at  $z = 0$ .

In the experiments [1], relative density disturbances were as large as 100%, so it is of interest to investigate nonlinear effects. The basic equations are Eqs. (2) and (4), which are a closed set of equations for  $v$  and  $\sigma$ , since from Eqs. (11) and (12) it follows that the area  $A$  is given by  $A = 2[\pi\sigma U_0/(m\omega^2)]^{1/2}$ . We have integrated these equations by employing the method of characteristics [7] for a number of cases that correspond to the experimental conditions. In one set of calculations the cloud was initially in equilibrium in a potential due to the trap and an extra repulsive potential representing the interaction of atoms with the radiation field of the laser. The extra potential was taken to have the form  $V_{\text{rad}} = V_0 \exp(-2z^2/l^2)$ , where  $l = 12 \mu\text{m}$ , as in the experiments, and we took the axial extent of the condensate,  $2Z$ , to be  $450 \mu\text{m}$ . At time  $t = 0$  the repulsive potential was turned off. In other runs we started with a configuration in equilibrium in the presence of the trap alone, and turned the laser field on at  $t = 0$ . The strength of the potential due to the laser could be varied to produce different perturbations in the density.

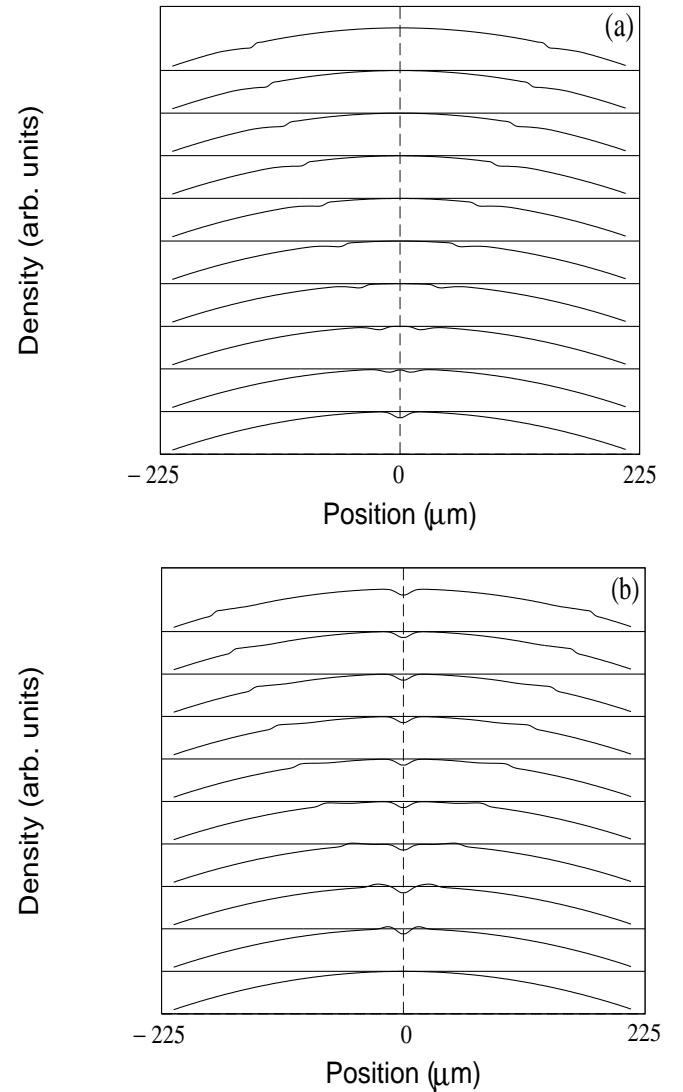


FIG. 1. Density profiles of the gas along the  $z$ -axis for various times for cases when the perturbing potential is turned off, (a), or turned on (b). The profiles are shown for times 0, 3, 6, 12, 18, 24, 30, 36, 42 and 48 ms.

In Fig. 1 we show density profiles along the  $z$ -axis of the cloud for various times for a laser intensity that, in equilibrium, would create a relative density perturbation of 15% at the center of the cloud. We observe that positive density pulses travel faster than negative ones, as one would expect from the fact that the sound speed increases with density. In Fig. 2 we plot the positions of extrema of the pulses as functions of time. For reference we also show corresponding results for low amplitudes, which were obtained by taking the average of the results for switching off a potential having the same spatial dependence, but whose strength was such as to create a relative density perturbation of either +5%, or -5%. The agreement with the result of the linear theory, Eq. (14), which is shown as a solid line, is very good, and the small difference compared with the numerical results is proba-

bly due to the varying cross section of the cloud. Note that pulses become distorted as they propagate. This is a consequence of nonlinear effects, and the distance a pulse can travel before it is distorted significantly may be estimated by observing that the sound velocity varies as  $n^{1/2}$ , where  $n$  is the density, and therefore the sound velocity change associated with a density change  $\delta n$  is approximately  $(c/2)\delta n/n$ . In a time  $t$  a disturbance in the region at density  $n+\delta n$  will move a distance  $\sim (ct/2)\delta n/n$  relative to what it would if the density were  $n$ , and consequently distortion of the pulse will be appreciable if this distance is greater than or of order the length of the pulse,  $l$ . In terms of the distance travelled by the pulse,  $z \approx ct$ , the condition for there to be significant distortion is  $z \gtrsim 2ln/\delta n$ , an estimate which appears to be in accord with the numerical calculations.

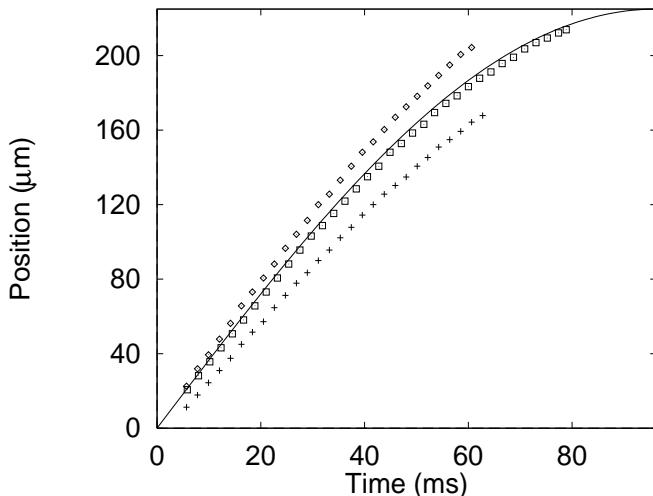


FIG. 2. Distance of the extrema of the density from the center of the cloud as a function of time. Squares give results for low amplitude pulses, as explained in the text, and the solid line is the analytical result, Eq. (14). Diamonds give results for the case shown in Fig. 1a, while crosses correspond to that shown in Fig. 1b. The central density is taken to be  $10^{14} \text{ cm}^{-3}$ , which corresponds to a time  $\pi Z/[2c(0)] = 96.3 \text{ ms}$  for a linear pulse to reach the end of the cloud.

To exhibit stronger nonlinear effects, we show in Fig. 3, results similar to those in Fig. 1 but for a perturbing potential that would create a 30% density perturbation in equilibrium. The steepening of the leading edge of positive density perturbations and of the trailing edge of negative ones is apparent. At times later than those for which results are shown, shocks develop. To understand later stages of the evolution, it is necessary to develop a basic understanding of the physics of shocks in superfluids. In the MIT experiments density perturbations were as large as 100%, and under such conditions nonlinear effects are expected to be extremely strong. An important future experimental challenge is to put in evidence the predicted effects on sound propagation of nonuniformity

of the particle density over the cross section of the trap, and of nonlinearity. Our results indicate that, because of their high compressibilities, low density atomic Bose-Einstein condensates are useful systems for investigation of nonlinear phenomena.

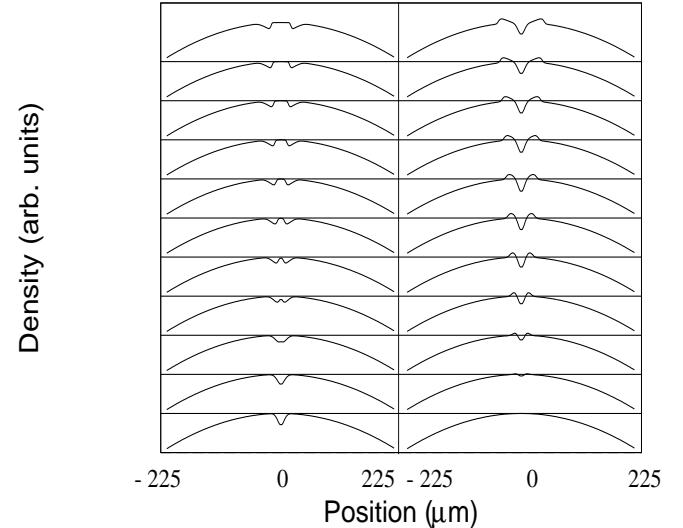


FIG. 3. Density profiles of the gas along the  $z$ -axis for various times for a larger density perturbation. The situations correspond to those in Figs. 1a and 1b, apart from the fact that the strength of the laser field was chosen so that in equilibrium it would create a 30% density perturbation. The profiles are shown for times equal to multiples of 0.85 ms, beginning at  $t = 0$ .

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