Literature Review

Introduction

The problem this project addresses is helping individuals invest their personal wealth to meet the goals they have by using portfolio optimization. Therefore, the problem is two-fold. The first is to identify and formulate the objective function that represents their goal. This can be done with utility theory. The second is to optimize the objective function over multiple periods in the face of uncertainty. For this, we will be using dynamic stochastic programming. This design is based on the paper *Individual asset liability management* [1]. This literature review focuses on these two aspects.

Utility Theory

Daniel Bernoulli initially proposed the expected utility hypothesis in 1738. He theorized that the subjective value of a gamble taken by an individual is the expectation of that individual's valuations of each of the possible outcomes of the gamble.

Von Neumann and Morgenstern, in the landmark book *Theory of Games and Economic Behavior* [2], created the axioms of expected utility theory. There are three axioms known as complete ordering, ordering and combining, and algebra of combining. In order to define these axioms, let $u, v, w \in U$ and $0 < \alpha, \beta < 1$.

Complete ordering:

$$\forall\, u, v \text{, one of the following must hold: } u > v, u < v, u = v.$$

Ordering and combining:

$$u < v \ \rightarrow u < \alpha u + (1-\alpha)v \ \ \forall\, \alpha$$
$$u > v \ \rightarrow u > \alpha u + (1-\alpha)v \ \ \forall\, \alpha$$
$$u < w < v \ \rightarrow u < \exists\, \alpha \ s.t.\ \alpha u + (1-\alpha)v < w$$
$$u > w > v \ \rightarrow u < \exists\, \alpha \ s.t.\ \alpha u + (1-\alpha)v > w$$

Finally, the algebra of combining is simply another name for transitivity.

If these axioms are satisfied for an individual considering a gamble, the individual can be said to be rational and a utility function must exist. A utility function is a function that takes in all possible outcomes of the gamble and returns the individual's valuations of each outcome. By taking the expectation of this function, we can find the value of the gamble to the individual.

They also address the idea of risk aversion. Risk aversion occurs when an individual is willing to pay in order to reduce the riskiness of a gamble. This is the basic premise of insurance and warranty. Because we are dealing with personal wealth management, risk aversion is essential to our definitions of utility. For the same reason that nearly all people prefer to own insurance, nearly all people will be willing to give up wealth in return for lower risk. While it is also possible for an individual to be risk-seeking or risk-neutral, the utility functions considered here will be strictly risk-averse. In order for a function to be risk-averse, it must be concave.

Finally, we also add a condition that more wealth is strictly better than less wealth. That is, we assume that individuals do not prefer to lose money than gain it. Here is a summary of the most important functions that fulfill our criteria.

Quadratic	$U(W) = aW + bW^2$
Logarithmic	$U(W) = \ln W$
Exponential	$U(W) = -e^{-aW}$
Power	$U(W) = W^\gamma; \gamma < 1$
HARA	$U(W) = \frac{1-\gamma}{\gamma}\left(\frac{\beta W}{1-\gamma} + \eta\right)^\gamma ; \beta, \gamma \neq 1, \eta \in \mathbb{R}$

Table 1: The main utility functions [3]

The paper that provides these utility functions also describes a method to estimate the utility function of an individual by finding the midpoint of the utility function at assumed points. For example, let's say that U(0) = 0 and U(100) = 1. This method would ask the user what they would be willing to pay for a fair coin toss where heads resulted in $100 and tails resulted in nothing. If the person said $20, for example, we would now have the midpoint of the utility function. U(20) = 0.5. This method can be done iteratively to find U(x) = 0.25 and 0.75 until a full function exists. In order to estimate our users' utility functions, we will have to make use of a similar methodology.

Finally, Daniel Kahneman, in the book *Thinking Fast and Slow*, which summarizes his work with Amos Tversky, brings forward the idea of prospect theory [4]. The critical difference between prospect theory and utility theory is the notion of loss aversion. Kahneman and Tversky found that in addition to being risk-averse, most people also dislike losing more than they like gaining.

An example given in the book is two individuals who are both offered a raise. They are given a choice of increased paid vacation or an increase in pay, and they are indifferent to the choice (i.e. the utility of each outcome – more pay or more vacation – is the same). Since they are indifferent, the first person is given a pay raise, while the second person is given more vacation. After a year, they are asked if they would be willing to switch to the other alternative. Data suggests that both individuals are likely to reject this offer despite the fact that they were initially indifferent to the two options. They put a higher value on what they already have despite the fact that when they did not have either, both were valued equally.

Accounting for loss aversion can be extremely difficult because the value of the current wealth of an individual must be a parameter in the function rather than just a person's wealth overall. Therefore, we may look into accounting for loss aversion as a secondary objective of our project.

This concludes the utility theory section. Our objective with utility theory is to estimate the utility function of a particular person, accounting for risk aversion. A secondary objective is to also acknowledge the existence of loss aversion.

Portfolio Optimization with Dynamic Stochastic Programming

Dynamic stochastic programming is used for solving time-dependent problems where a decision needs to be made at each time step Δt, and the decision is dependent on all possible outcomes into the future up to a given time T. The general dynamic stochastic problem was defined by Dempster in 1988 [5], and it has since been used in a wide variety of applications.

In particular, the problem we are trying to address is optimizing a portfolio over a particular utility function for an individual for the foreseeable future. This problem can be difficult because of the nature of market data. Statistical analysis shows correlation between assets and the distribution the prices of assets are rarely normal. They often show heavy tails [6].

In order to account for this, we draw on the work of Topaloglou *et al.*, who describe a method to use dynamic stochastic programming for portfolio optimization [7]. The critical difficulty in using dynamic stochastic programming is defining the possible scenarios. Topaloglou *et al.* consider 16 assets, made up of various stocks and bonds from around the world, and start by generating the distribution moments and correlation matrix of the assets. The moments and correlations are inputted into the moment-matching procedure created by Høyland *et al.* [8]. Topaloglou *et al.* match the first four moments as well as the correlation matrix in order to create a finite number of plausible scenarios. This procedure requires a predetermined number of time steps, which is a constraint we will also include in our problem.

Once we have the scenarios defined, we can simply apply them to the general dynamic stochastic programming model just as Topaloglou *et al.* do. They optimize over a VaR and CVaR-based objective function, while we will optimize over a utility function.

Bibliography

[1] E. A. Medova, J. K. Murphy, A. P. Owen, and K. Rehman, "Individual asset liability management," *Quantitative Finance*, vol. 8, no. 6, pp. 547-560, September 2008.

[2] John von Neumann and Morgenstern Oskar, *Theory of Games and Economic Behavior*.: Science Editions, 1944.

[3] Dorin LIXĂNDROIU, "A model for determining the utility function using Fuzzy numbers," *Bulletin of the Transylvania University of Brasov*, vol. 8, 2015.

[4] Daniel Kahneman, *Thinking Fast and Slow*.: Farrar, Strauss and Giroux, 2011.

[5] M. A. H. Dempster, "On stochasic programming: II. Dynamic Problems under Risk," *Stochastics*, vol. 25, no. 1, pp. 15-42, 1988.

[6] A. J. Prakash, C-H. Chang, and T.E. Pactwa, "Selecting a portfolio with skewness: Recent evidence from US, Europe, and Latin American equity markets," *Journal of Banking and Finance*, 2003.

[7] Nikolas Topaloglou, Hercules Vladimirou, and Stavros A. Zenios, "A dynamic stochastic programming model for international portfolio management," *European Journal of Operational Research*, October 2006.

[8] K. Høyland and S. W. Wallace, "Generating scenario trees for multistage decision problems," *Management Science*, 2001.