Chapter 2
Computation of Equilibria in OLG Models with Many Heterogeneous Households

2.1 Introduction

Over the past twenty years infinite horizon general equilibrium models with overlapping generations (OLG) have become an important tool for policy analysis, and have been fruitfully applied in fields such as macroeconomics or public finance (see, e.g., Auerbach and Kotlikoff (1987), and Kotlikoff (2000) for an overview). OLG models naturally involve a large number of variables and equations that describe the equilibrium behavior of economic agents. As a consequence, the development of large-scale OLG models is often limited by the computational capacity of available numerical solution methods. In particular, models that exhibit a rich household side including a variety of household-specific effects, a large number of heterogeneous households, and realistic agent lifetimes typically require "customized solution methods" which may be both costly to implement and difficult to validate.

This chapter[1] develops a decomposition algorithm based on "off the shelf numerical tools" for solving general equilibrium models with many households, of which OLG models are a special case. The presented approach is primarily appropriate for computing equilibria for models in which the number of agents is so large that simultaneous solution methods operating directly on the equilibrium system of equations are infeasible due to the high dimensionality related to income and household-specific effects. The "sequential recalibration" (SR) algorithm presented here is based on the solution of a sequence of nonlinear complementarity problems[2] although in special cases the same procedure may be implemented by solving a

[1] This chapter has been jointly written with Thomas F. Rutherford. GAMS code for the presented applications is available at http://www.mpsge.org.

[2] Rutherford (1995b) and Mathiesen (1985) have shown that a complementary-based approach is convenient, robust, and efficient. A characteristic of many economic models is that they can be cast as a complementary problem, i.e. given a function $F\colon \mathbb{R}^n \longrightarrow \mathbb{R}^n$, find $z \in \mathbb{R}^n$ such that $F(z) \geq 0$, $z \geq 0$, and $z^T F(z) = 0$ The complementarity format embodies weak inequalities and complementary slackness, relevant features for models that contain bounds on specific variables, e.g. activity levels which cannot a priori be assumed to operate at positive intensity. Such features are not easily handled with alternative solution methods.

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sequence of convex nonlinear programming problems. The main idea of the presented decomposition approach is to solve a market economy with many households through the computation of equilibria for a sequence of representative agent economies. Typically, the sequence of prices and quantities converges to the true equilibrium allocation.

The close connection between the allocation of a competitive market economy and the optimal solution to a representative agent's planning problem is well known and widely cited in the economic literature. Negishi (1960) was the first to use an optimization problem to characterize equilibrium allocations in a general equilibrium framework. Negishi's original paper was primarily concerned with optimization as a means of proving existence. Dixon (1975) developed the theory and computational effectiveness of "joint maximization algorithms" for multi-country trade models. Rutherford (1999b) presented the "sequential joint maximization algorithm" (SJM) which provides a simple recursive version of Negishi's method.

There are similarities between the SR and SJM algorithms. Both approaches solve subproblems representing relaxations of the equilibrium conditions. The SJM algorithm ignores consumer budget constraints but retains details of consumer demand systems. The SR algorithm employs a yet looser representation of individual consumer's demand systems by omitting both income constraints and global properties of the individual utility functions. The omission of global characteristics of preferences simplifies the model but can hinder convergence. The appropriateness of the proposed solution method therefore depends on the characteristics of the underlying model.

The decomposition approach can be useful for the computation of equilibria in large-scale general equilibrium models with many households. There are many economic questions for which heterogeneous agent models have to be used to provide answers, and an increasing amount of research employs frameworks that allow for intra-cohort heterogeneity in an OLG setup.[3] We believe that the presented approach can be beneficial for a wide range of economic applications, in particular within the class of OLG models, for the following reasons. First, by significantly reducing the computational overhead of the numerical problem at hand this method facilitates the development of OLG models which feature a complex and rich household side. This strengthens the microfoundation of the models in general and allows to analyze in detail intra- and intergenerational distributive consequences of economic policy. Second, the presented approach enables to solve OLG models that include a "realistic" number of households within each age group since the number of households in the model more closely corresponds to the number of observational units available from household survey data. This approach avoids relying on some ad-

[3] For instance, Conesa and Krueger (1999), Kotlikoff et al. (1999), and Huggett and Ventura (1999) investigate the intra-cohort distributive and welfare consequences of social security reform. Fehr (2000) looks at pension reform during the demographic transition in the case of Germany. Ventura (1999) explores the general equilibrium impact and associated distributional consequences of a revenue neutral tax reform, and Jensen and Rutherford (2002) analyze the intra- and intergenerational welfare effects of fiscal consolidation via debt reduction. This chapter concentrates on applications within the Auerbach-Kotlikoff OLG framework. For a general discussion of economies with heterogeneous agents, see, e.g., Rios-Rull (1995).

hoc aggregation of household groups, and thereby helps to enhance the empirical basis of the model. Third, and more generally, the method can also be effectively applied to OLG models which display a high dimensionality deriving from sources other than the household side. Potential applications may include multi-sectoral and multi-country models, or models which incorporate a detailed government sector.

In addition, the present contribution adds to the recent and growing body of research that deals with the integration of macro and microsimulation models, the "micro-macro" approach to modeling (Bourguignon and Spadaro, 2006). This strand of literature aims to combine the strengths of both the computable general equilibrium (CGE) paradigm and microsimulation models. While CGE models have become standard tools of quantitative policy assessment in the last twenty years, two major criticisms are their reliance on the concept of the "representative agent" and their usage of unclear aggregation procedures. The virtue of the microsimulation approach, on the other hand, is to replace representative agents with "real households" as observed in standard household surveys. This, however, is typically achieved at the cost of ignoring general equilibrium effects that are essential for policy analysis.

The simplest link between economy-wide modeling and the microsimulation approach proceeds top down, i.e. simulated policy changes obtained from an aggregate representation of the economy are passed down to a microsimulation module, as, e.g., in Bourguignon et al. (2005) and Bourguignon and Spadaro (2006). The principal weakness of the top-down approach is the absence of feedback effects from the micro to the macro level. Relatively few studies have attempted to fully integrate both approaches, most of them by means of employing an iterative strategy between the microsimulation and the CGE model (Cockburn and Cororatona (2007), Savard (2003, 2005), Arntz et al. (2006)). Also belonging to this class of models, Rutherford and Tarr (2008) applied the SR algorithm to a large-scale, static general equilibrium model with 25 sectors and 53,000 households to assess the poverty effects of Russia's WTO accession. All of these studies, however, are concerned with applications in a static framework. Clearly, covering complex behavioral responses and potential general equilibrium and macroeconomic effects in a dynamic setup is essential for many policy issues.[4]

The present chapter develops a computational technique that allows to fully integrate a comprehensive system of OLG households, that exhibits a substantial degree of intra-cohort heterogeneity, into a generic Auerbach-Kotlikoff model. We show that the positive experience of the SR algorithm for large-scale static models carries over to dynamic applications and demonstrate its effectiveness for solving OLG models with a large number of heterogeneous households. To find the equilibrium transition path of the OLG economy, the presented algorithm solves a sequence of "related" Ramsey optimal growth problems where the system of overlapping generations is replaced by an infinitely-lived representative agent. An iterative procedure

[4] Available models tend to concentrate on some specific behavior, abstracting from other important components of the demo-economic life cycle. For instance, Townsend (2002), Townsend and Ueda (2003) concentrate on saving/investment behavior under uncertainty and in different financial market environments.

between the macro and micro model is employed based on the successive recalibration of preferences of this artificial representative agent.

In order to characterize limitations of the algorithm, local convergence theory for a simple exchange economy due to Scarf (1960) is developed and a number numerical analyses are carried out to examine global conditions under which the SR algorithm may fail to converge. We show that conditions for local stability of the adjustment process reduce to those of a Walrasian price tâtonnement process, thus SR belongs to a large class of algorithms commonly used in computational economics which are robust and efficient, yet fail to provide global convergence. The presented counterexample illustrates that the SR algorithm may be ill-suited for applications in which there are significant income effects.

After having characterized limitations of the technique, this chapter explores the algorithm's performance when applied to large-scale OLG models. To this end, a prototype Auerbach-Kotlikoff model is considered which includes up to 2000 heterogeneous households within each generation differing with respect to labor productivity over the life cycle and other behavioral parameters. The performance of the SR algorithm is compared to a simultaneous solution method as suggested by Rasmussen and Rutherford (2004). We find that the proposed algorithm can provide improvements in both efficiency and robustness. It is furthermore demonstrated that the decomposition algorithm can routinely solve high-dimensional OLG models which are infeasible for conventional solution methods.

Lastly, it is important to emphasize that the decomposition method is inadequate for approximating equilibria in OLG economies that are generically *Pareto-inferior*, i.e. models in which the economy's growth rate exceeds the real interest rate (see, e.g., Diamond (1965) and Phelps (1961)). For the given model, this corresponds to a situation where population growth dominates discounting. In such circumstances there is no social planner's problem which corresponds to the OLG demand system. Whether this significantly limits the relevance and scope of the presented approach ultimately is an empirical question. Empirical evidence suggests that the incapacity of the method to deal with dynamically inefficient equilibria is of minor practical significance.[5]

The rest of the chapter is organized as follows. Section 2.3 introduces the SR algorithm for the case of a static economy and illustrates its basic logic by means of graphical analysis. Section 2.4 investigates a model where convergence of the SR algorithm fails if income effects are relatively strong. Section 2.5 demonstrates that the algorithm can be effectively applied to solve Auerbach-Kotlikoff OLG models. Furthermore, the performance of the algorithm is compared to computational experience from an integrated simultaneous solution method. Section 2.6 concludes.

[5] Abel et al. (1989) find that for the US economy the condition for dynamic efficiency seems to be satisfied in practice. Similarly, under the weak assumption that rates of return are ergodic, Barbie, Hagedorn and Kaul (2004) reach the conclusion that the US economy does not overaccumulate capital. By means of numerical analysis, Larch (1993) suggests that in the Auerbach-Kotlikoff framework rather implausible values of the pure rate of time preference, the intertemporal elasticity of substitution or the population growth rate are required to obtain non Pareto-optimal market solutions.

2.2 Calibration in CGE models

In order to facilitate the subsequent description of the algorithm and to clarify the algebraic forms used in the associated computer programs, this section reviews some fundamental aspects of calibration which underly most computable general equilibrium (CGE) models.

2.2.1 CES Preferences

Calibration refers to the process of selecting values of model parameters which ensure that the model's reference equilibrium is consistent with given data. Such data are typically obtained in the form of a social accounting matrix for a given base year. CGE models are based on parametric forms which describe technology and preferences. The most common functional form used in empirical applications is the constant-elasticity-of-substitution (CES) function. A CES utility function can be written as:

$$U(\mathbf{C}) = \left(\sum_{i=1}^{n} \alpha_i C_i^{\rho} \right)^{1/\rho} \tag{2.1}$$

where $\mathbf{C}$ denotes the vector of consumption goods C_i, $i = 1, \ldots, n$. There are $n+1$ parameters in this function, with n share parameters $\alpha_i > 0$ and a curvature parameter ρ. The latter is related to the Allen-Uzawa elasticity of substitution σ as: $\rho = 1 - 1/\sigma$ with $\rho < 1$ and $\sigma > 0$.

Consumers in CGE models are typically modeled as budget constrained utility-maximizers, so a model would incorporate the following behavioral subproblem:

$$\begin{aligned} &\max\nolimits_{c_1,\ldots,c_n} \; U(\mathbf{C}) \\ \text{s.t.} \;\; &\textstyle\sum_{i=1}^{n} p_i C_i = M \end{aligned} \tag{2.2}$$

where p_i is the price of consumption good i, and M is consumer income. The consumer problem in closed-form can be solved, obtaining demand functions:

$$C_i(\mathbf{p}, M) = \frac{\alpha_i^{\sigma} M p_i^{-\sigma}}{\sum_{i'} \alpha_{i'}^{\sigma} p_{i'}^{1-\sigma}} \tag{2.3}$$

where $\mathbf{p}$ denotes the price vector, and $i \neq i'$. The calibration of preferences involves inverting this demand function to express the function parameters in terms of an observed set of prices and demands. If a consumer chooses to consume quantities $\overline{C}_i$ when commodity prices are $\overline{p}_i$, it may be concluded that the share parameters have to be given by:

$$\alpha_i = \lambda \, \overline{p}_i \overline{C}_i^{1-\rho} \tag{2.4}$$

where $\lambda > 0$ is an arbitrary scale factor[6], and the elasticity parameter, ρ, is exogenously specified. It is helpful to think of the share and scale parameters as calibrated values, determined by an agent's observed choices in a reference equilibrium, whereas the elasticity parameters are "free parameters" which are typically drawn from econometric estimates of the responsiveness of demand or supply to changes in relative prices. In traditional applied general equilibrium models, the reference quantities $\overline{C}_i$ and prices $\overline{p}_i$ are based on a benchmark equilibrium data set.

2.2.2 The Calibrated Share Form

In applied work it may be convenient to work with a different yet equivalent form of the CES utility function (Rutherford, 1995a). The *calibrated share form* is based on the observed quantities, prices and budget shares. In computational applications the calibrated form is preferable because it provides a simple parameter and functional check that is independent from second-order curvature. Normalizing the benchmark utility index to unity, the utility function can be written as:

$$U(\mathbf{C}) = \left[\sum_i \theta_i \left(\frac{C_i}{\overline{C}_i}\right)^{\rho}\right]^{1/\rho} \tag{2.5}$$

in which:

$$\theta_i = \frac{\overline{p}_i \overline{C}_i}{\sum_{i'} \overline{p}_{i'} \overline{C}_{i'}} \tag{2.6}$$

is defined as the benchmark value share of good i. Similarly, one can express the unit expenditure function as:

$$e(\mathbf{p}) = \left[\sum_i \theta_i \left(\frac{p_i}{\overline{p}_i}\right)^{1-\sigma}\right]^{1/1-\sigma}, \tag{2.7}$$

the indirect utility function as:

$$V(\mathbf{p}, M) = \frac{M}{e(\mathbf{p})\overline{M}}, \tag{2.8}$$

and by Roy's identity the demand function as:

$$C_i(\mathbf{p}, M) = \overline{C}_i \frac{M}{e(\mathbf{p})\overline{M}} \left(\frac{e(\mathbf{p})\overline{p}_i}{p_i}\right)^{\sigma} \tag{2.9}$$

where $\overline{M} = \sum_i \overline{p}_i \overline{C}_i$.

[6] The consumer maximization problem is invariant with respect to positive scaling of U, hence the share parameters may only be determined up to a scale factor.

One can think of the demand function given here as a second-order Taylor approximation to the "true" demand function based on an observation of the true function at the reference point. At that point, $\overline{C}_i$ corresponds to a "zero*th* order approximation" to the utility function, $\overline{p}_i$ corresponds to the "first order approximation", and the "free parameter" σ controls the second (and higher) order properties of preferences. The benchmark prices correspond to the marginal rate of substitution—the slope of the indifference curve—at $\overline{C}$. As long as one remains in the neighborhood of $\overline{\mathbf{p}}$, the elasticity parameter σ only plays a minor role, and calibrated demand is determined largely by $\overline{\mathbf{C}}$ and $\overline{\mathbf{p}}$.

2.3 CGE with Many Households: A Decomposition Approach

This section presents the decomposition method by which a market economy with many heterogeneous households may be solved through the computation for equilibria for a sequence of representative agent economies. While the primary interest is in dynamic models, it is advantageous to introduce the algorithm for the case of a two-sector static economy in which one can provide a graphical description that serves to illustrate its basic logic.

2.3.1 A Static Economy

Consider the following static economy which is populated by a large number of heterogeneous households $h = 1, \ldots, H$ each of whom is endowed with K^h and L^h units of capital and labor, respectively. Households earn income $M^h = rK^h + wL^h$ from supplying their factor endowments inelastically at respective market prices r and w. Household $h = 1, \ldots, H$ solve:

$$\max_{c_1^h, c_2^h} U^h(\mathbf{c}^h) = \left[\sum_{i=1}^{2} \theta_i^h \left(\frac{c_i^h}{\overline{c}_i^h} \right)^{\rho^h} \right]^{1/\rho^h}$$

$$s.t. \qquad \sum_{i=1}^{2} p_i c_i^h = M^h. \tag{2.10}$$

where the utility function is written in calibrated share form. θ_i^h and $\overline{c}_i^h$ denote the benchmark value share and the benchmark consumption of good i for household h. Households are heterogeneous with respect to θ_i^h, ρ^h, K^h, and L^h.

Furthermore, there is a single representative firm which uses capital and labor services to produce two consumption goods X_i, $i, j = 1, 2$, according to a constant returns to scale production function $X_i = f_i(K, L)$, where $K = \sum_h K^h$ and $L = \sum_h L^h$. All goods and factor markets are perfectly competitive.

2.3.2 A Decomposition Algorithm

The main challenge for computing equilibria in a setup where H is very large is dimensionality. Typically, conventional simultaneous solution methods that operate directly on the equilibrium system of equations are infeasible. The proposed algorithm decomposes the corresponding numerical problem into two parts and thereby effectively manages to reduce its dimensionality. The general equilibrium of the underlying economic model is approximated by computing equilibria for a sequence of representative agent economies. In each iteration, first a general equilibrium representation of the underlying economic model is solved in which the household side is replaced by a single representative agent (RA). The second subproblem then consists of solving a partial equilibrium relaxation of the original model that retains the full structure of the household demand system. Given equilibrium prices from the previous solution of the RA economy, it is possible to compute optimal choices for each of the "real" households. In a next step, and to create a basis for successive iterations, the preferences of the "artificial" RA are recalibrated such that given candidate equilibrium prices the RA choices replicate aggregate household choices.

The key departure from the routine use of calibration in the decomposition algorithm is the idea that the calibration of preferences occurs more than once. The first iteration of the algorithm is based on observable benchmark data, but in subsequent iterations the preferences of the RA are sequentially recalibrated to values determined in the iterative process. For the case of the static economy as described above, the SR algorithm involves the following steps:

Step 0: Initialize the Representative Agent Economy

In the computation of equilibria we will portray the choices of H households using a single representative agent. To construct an RA economy of the underlying economic model, replace (2.10) by:

$$\max_{C_1,C_2} U^k(\mathbf{C}) = \left[\sum_{i=1}^{2} \Theta_i^k \left(\frac{C_i}{\overline{C}_i^k}\right)^{\rho}\right]^{1/\rho}$$

$$s.t. \qquad \sum_{i=1}^{2} p_i C_i = w\sum_{h=1}^{H} L^h + r\sum_{h=1}^{H} K^h \tag{2.11}$$

where k is an iteration index. Factor endowments of the RA equal the sum of respective factor endowments across all households. To initialize the RA economy at a consistent data point, the data set has be constructed such that the RA model and the household model share the same optimal consumption quantities in the initial benchmark. This is achieved by setting:

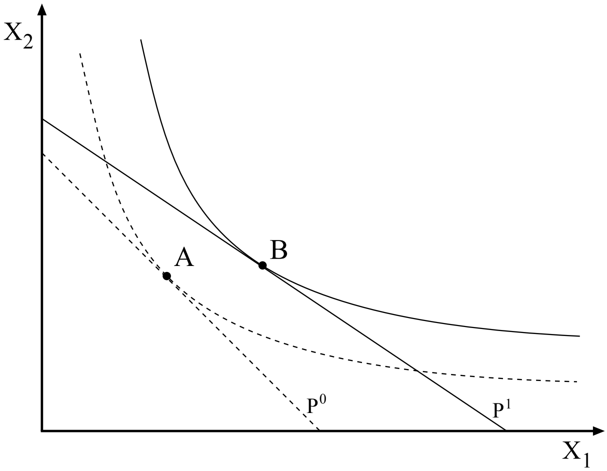

Fig. 2.1 Solution to the initial representative agent model (Step 0 and Step 1)

$$\overline{C}_i^0 = \sum_{h=1}^{H} \overline{c}_i^h \tag{2.12}$$

$$\Theta_i^0 = \frac{\overline{p}_i \overline{C}_i^0}{\sum_{i'} \overline{p}_{i'} \overline{C}_{i'}^0}, \quad i \neq i' \tag{2.13}$$

where $\overline{C}_i^0$ and Θ_i^0 denote initial consumption by the RA and the aggregate value share for good i in iteration $k = 0$, respectively.

This initial consumption point of the RA in the benchmark equilibrium is represented by point A in Figure 2.1 where initial goods prices are denoted by P^0. Benchmark prices and an arbitrary elasticity are used to extrapolate preferences in the neighborhood of the benchmark point to the global preferences of the RA, as indicated by the indifference curve which is tangent to the benchmark budget constraint at point A. The key limitation of the RA model on the demand side is that the "community indifference curve" represented by this indifference curve does not truthfully portray the response of household demand to a comprehensive change in both goods and factor prices.

Step 1: Solve for a General Equilibrium of the RA Economy

The solution to the RA model in the first iteration of the algorithm is illustrated in Figure 2.1. As depicted here, the assumed exogenous policy shock has led to an increase in factor earnings, and a reduction in the relative price of X_1 as compared with

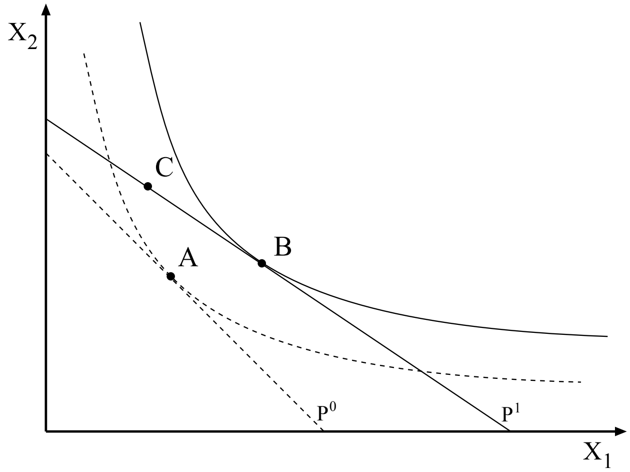

Fig. 2.2 Evaluating household demands at new prices (Step 2)

X_2. This new price situation is denoted by P^1. The RA model, based on the assumed community indifference curve and the associated change in factor and commodity prices returns point B as the optimal consumption point.

Step 2: Evaluate Household Demand Functions

In the solution program equilibrium prices are read from the RA model and evaluate the household demand vector. This produces a different point on the same budget constraint (see Figure 2.2). The household demand model is based on compensated demand functions so the aggregate budget constraint for the household demand system is equivalent to the budget constraint which applies to the RA. Point C corresponds to the aggregate demand which results from solving the individual household optimization problems. The extent to which C differs from B depends on both the difference in implicit substitution elasticities and differences in income effects.

Step 3: Recalibrate Preferences of the Representative Agent

The next step in the algorithm consists of specifying a new set of preferences for the RA model. The algorithm is termed "sequential recalibration" on the basis of this idea. After having solved one RA model, a new RA model is constructed based on a set of preferences which are locally calibrated to the aggregate consumption quantities at point C and the associated relative prices. This ensures that given prices P^1 the optimal consumption point of the new RA, point C in Figure 2.3, is consistent with the aggregated choices by households. Preferences of the RA in iteration k,

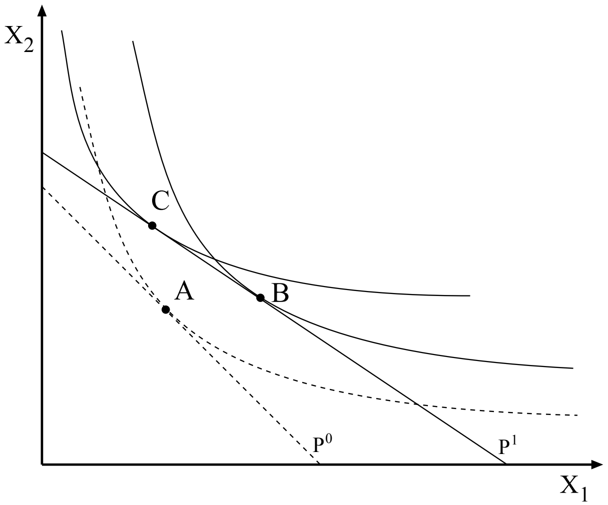

Fig. 2.3 Recalibration of preferences (Step 3)

$U^k(\mathbf{C})$ in (2.11), are based on household demands at the prices returned in iteration $k-1$:

$$\overline{C}_i^k = \sum_h c_i^h(\mathbf{p}^{k-1}, M_h^{k-1}) \tag{2.14}$$

in which $c_i^h(\mathbf{p}^{k-1})$ is the demand for good i by household h evaluated at the candidate price vector from the previous iteration $k-1$, and where factor income of household h in iteration k, $M_h^k(\mathbf{p}^k)$, is a function of prices in iteration k. Likewise, value shares in $U^k(\mathbf{C})$ are updated to:

$$\Theta_i^k = \frac{p_i^{k-1} \sum_h c_i^h(\mathbf{p}^{k-1}, M_h^{k-1})}{\sum_j p_j^{k-1} \sum_h c_j^h(\mathbf{p}^{k-1}, M_h^{k-1})} \tag{2.15}$$

where θ_i^k is the aggregate value share of good i at iteration $k-1$. The indifference curves tangent at A and B are based on identical preferences, but the indifference curve tangent at point C is based on a new set of community indifference curves, hence it may intersect the indifference curves based on RA utility in the previous iteration of the algorithm. Note that the preferences of the "real" households remain unchanged throughout the entire iteration process.

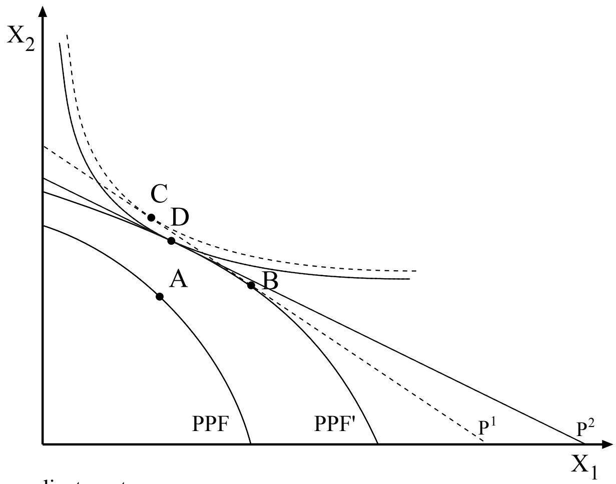

Fig. 2.4 Iterative adjustment

Iterative Adjustment

When the RA model is recalibrated at point C, both the representative agent and all households are in equilibrium at C with prices P^1, but at these prices firms will only supply quantities given by point B. Hence, due to inconsistency with the supply side of the model there is a general disequilibrium. To illustrate this idea, it is convenient to portray the supply side of the economy by a production possibility frontier (*PPF*). Assume that the policy shock produces an expansion in the *PPF* (to *PPF*′) and a substantial change in relative prices from point A to B. The next step in the solution program is to resolve for a general equilibrium of the new RA model with recalibrated preferences at point C. Point C in Figure 2.4 becomes therefore interpreted as point A in the next iteration. The solution of this RA model is then characterized by a new optimal consumption point, here depicted by point D, and prices P^2.

Subsequent iterations involve carrying out Steps 1 to Steps 3 (Step 0 initializes the solution procedure). The process is stopped if some convergence metric, e.g., the 1-norm of the difference between the price vectors from one iteration to the next, is satisfied. Note that subsequent iterations of the algorithm only involve refinements of the demand system and result in much smaller changes in relative prices, as indicated here by the change from C to D as compared with A to B.

2.4 Convergence Theory

This section evaluates the performance of the algorithm for an economy in which the exact equilibrium is known and where the computed allocation can be compared to the true equilibrium allocation. Local convergence theory for the proposed algorithm is developed and conditions under which the adjustment process may fail to converge are identified.

The example is due to Scarf (1960) who considers a pure exchange economy with an equal number of n consumers and goods. Consumer h is endowed with one unit of good h and demands only goods h and $h+1$. Let $d_{i,h}$ denote demand for good i by consumer h. Preferences are represented by CES utility functions with the following structure:

$$U_h(d) = \left(\theta\, d_{h,h}^{\frac{\sigma}{\sigma-1}} + (1-\theta)\, d_{h+1,h}^{\frac{\sigma}{\sigma-1}}\right)^{\frac{\sigma-1}{\sigma}}. \tag{2.16}$$

Scarf (1960) demonstrates that this economy has a unique equilibrium in which all prices are equal to unity.[7]

2.4.1 Local Convergence

As explained in Section 2.3.2, the sequential recalibration algorithm iteratively adjusts the baseline level parameter $\overline{C}_i$ (and $\Theta_i(\overline{C}_i)$) in the utility function of the representative agent. These may be normalized so that $\sum_i \overline{C}_i = n$. Market clearing commodity prices for the RA economy are determined given the baseline level parameters. Let $p_i(\overline{C})$ denote the price of good i consistent with $\overline{C} = (\overline{C}_1, \ldots, \overline{C}_n)$. In this exchange economy, let $\zeta_i\left(p(\overline{C})\right)$ denote the market excess demand function for good i that is obtained from evaluating household demand functions. Given the special structure of preferences and endowments, this function has the form:

$$\zeta_i\left(p(\overline{C})\right) = d_{i,i} + d_{i,i-1} - 1\,. \tag{2.17}$$

Furthermore, let $\xi_i\left(\overline{C}\right)$ denote the value of market excess demand for good i at prices $p_i(\overline{C})$:

$$\xi_i\left(\overline{C}\right) = p_i(\overline{C})\, \zeta_i\left(p(\overline{C})\right). \tag{2.18}$$

Of course, in equilibrium it must be true that $\xi_i(\overline{C}^*) = 0$, $\forall i$. Let the initial estimate $\overline{C}^0$ be selected on the n-simplex, i.e. $\sum_i \overline{C}_i^0 = n$. Walras' law ensures that the adjustment process $\frac{d\overline{C}_i}{dt} = \xi_i\left(\overline{C}\right)$ remains on the n-simplex: $\frac{d\sum_i \overline{C}_i(t)}{dt} = \sum_i p_i(\overline{C}(t))\, \zeta_i\left(p(\overline{C}(t))\right) = 0$.

[7] See Lemma 1 and Lemma 2 (Scarf, 1960, p.164). The parameters of this utility function correspond to Scarf's parameters a and b (Scarf, 1960, p.168) as: $\sigma = \frac{1}{1+a}$ and $\theta = \frac{b}{1+b}$.

Local convergence concerns properties of the Jacobian matrix evaluated at the equilibrium point, $\nabla\xi(\overline{C}^*) = [\xi_{ij}]$. This Jacobian has entries defined as follows:

$$\xi_{ij} \equiv \frac{\partial \xi_i}{\partial \overline{C}_j} = \begin{cases} \frac{\partial \zeta_i(p(\overline{C}))}{\partial \overline{C}_i} & i = j \\ \frac{\partial \zeta_i(p(\overline{C}))}{\partial \overline{C}_j} & i \neq j \end{cases} . \tag{2.19}$$

If all principal minors of $\nabla\xi(\overline{C}^*) = [\xi_{ij}]$ are negative, the adjustment process is locally convergent. If, however, $\frac{\partial \zeta_i(p(\overline{C}))}{\partial \overline{C}_i} > 0$, the process is locally unstable. When an equilibrium is unique and the process is uninterrupted, then local stability implies global instability.

For this model, the tâtonnement price adjustment process is unstable (in the case $n = 3$) when $\frac{\theta}{1-\theta} > \frac{1}{1-2\sigma}$ (Scarf, 1960). In the following, it is shown that the same condition implies instability for the $\overline{C}$-adjustment process of the SR algorithm. Furthermore, it is shown (numerically) that while the tâtonnement and SR price adjustment processes are locally identical, they may be quite different at points in the price space that are sufficiently far away from the equilibrium.

Given the special structure of $\zeta_i\left(p(\overline{C})\right)$, one can write:

$$\frac{\partial \zeta_i}{\partial p_i} = \frac{\partial d_{i,i}}{\partial p_i} + \frac{\partial d_{i,i-1}}{\partial p_i}, \quad \frac{\partial \zeta_i}{\partial p_{i-1}} = \frac{\partial d_{i,i-1}}{\partial p_{i-1}}, \quad \frac{\partial \zeta_i}{\partial p_{i+1}} = \frac{\partial d_{i,i}}{\partial p_{i+1}} . \tag{2.20}$$

Defining the "unit-utility" expenditure function for consumer i as:

$$e_i(p) = \left(\theta\, p_i^{1-\sigma} + (1-\theta)\, p_{i+1}^{1-\sigma}\right)^{\frac{1}{1-\sigma}} \tag{2.21}$$

demand functions are given by:

$$d_{i,i} = \frac{\theta\, p_i}{e_i^{1-\sigma} p_i^{\sigma}}, \quad d_{i+1,i} = \frac{(1-\theta)\, p_i}{e_i^{1-\sigma} p_{i+1}^{\sigma}} . \tag{2.22}$$

Evaluating $\frac{\partial \zeta_i(p(\overline{C}))}{\partial \overline{C}_i}$ at $p^* = 1$, yields

$$\begin{aligned} \frac{\partial \zeta_i\left(p(\overline{C})\right)}{\partial \overline{C}_i} = \frac{\partial p_i}{\partial \overline{C}_i} &\left((2\sigma-2)\theta^2 + (3-2\sigma)\theta - 1\right) \\ &+ \frac{\partial p_{i+1}}{\partial \overline{C}_i}\left(-\theta(1-\theta)(1-\sigma)\right) \\ &+ \frac{\partial p_{i-1}}{\partial \overline{C}_i}\left(-\theta(1-\theta)(1-\sigma) + 1 - \theta\right) . \end{aligned} \tag{2.23}$$

The function $p(\overline{C})$ is defined implicitly by the equation:

$$\widetilde{\zeta}\left(p, \overline{C}\right) = 0 \tag{2.24}$$

where $\widetilde{\zeta}\left(p,\overline{C}\right)$ denotes the vector of market excess demand functions from the representative agent economy. Its *i-th* element is given by:

$$\widetilde{\zeta}_i\left(p,\overline{C}\right)=\frac{\overline{C}_i}{\left(\sum_{i'}\alpha_{i'}\,p_{i'}^{1-\widetilde{\sigma}}\right)p_i^{\widetilde{\sigma}}}-1 \tag{2.25}$$

with $\alpha_{i'}=\frac{\overline{C}_i}{\sum_{i'}\overline{C}_{i'}}$ and where $\widetilde{\sigma}$ denotes the elasticity of substitution for the representative agent. In order to evaluate $\nabla_p\widetilde{\zeta}$ at $\overline{C}^*$, we make a first-order Taylor series expansion:

$$\nabla_p\widetilde{\zeta}\left(p,\overline{C}\right)dp+\nabla_{\overline{C}}\widetilde{\zeta}\left(p,\overline{C}\right)d\overline{C}=0 \tag{2.26}$$

which gives:

$$\frac{dp}{d\overline{C}}=-\nabla_p^{-1}\widetilde{\zeta}\,\nabla_{\overline{C}}\widetilde{\zeta}\,. \tag{2.27}$$

Evaluating gradients at $p^*=\overline{C}^*=1$ yields:

$$\nabla_p\widetilde{\zeta}=\begin{pmatrix}\frac{-(2\widetilde{\sigma}+1)}{3} & \frac{-(1-\widetilde{\sigma})}{3} & \frac{-(1-\widetilde{\sigma})}{3}\\ \frac{-(1-\widetilde{\sigma})}{3} & \frac{-(2\widetilde{\sigma}+1)}{3} & \frac{-(1-\widetilde{\sigma})}{3}\\ \frac{-(1-\widetilde{\sigma})}{3} & \frac{-(1-\widetilde{\sigma})}{3} & \frac{-(2\widetilde{\sigma}+1)}{3}\end{pmatrix} \tag{2.28}$$

$$-\nabla_p^{-1}\widetilde{\zeta}=\begin{pmatrix}\frac{\widetilde{\sigma}+2}{3\widetilde{\sigma}} & \frac{\widetilde{\sigma}-1}{3\widetilde{\sigma}} & \frac{\widetilde{\sigma}-1}{3\widetilde{\sigma}}\\ \frac{\widetilde{\sigma}-1}{3\widetilde{\sigma}} & \frac{\widetilde{\sigma}+2}{3\widetilde{\sigma}} & \frac{\widetilde{\sigma}-1}{3\widetilde{\sigma}}\\ \frac{\widetilde{\sigma}-1}{3\widetilde{\sigma}} & \frac{\widetilde{\sigma}-1}{3\widetilde{\sigma}} & \frac{\widetilde{\sigma}+2}{3\widetilde{\sigma}}\end{pmatrix} \tag{2.29}$$

$$\nabla_{\overline{C}}\widetilde{\zeta}=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}. \tag{2.30}$$

Hence, in the neighborhood of the equilibrium:

$$\frac{\partial p_i(\overline{C})}{\partial\overline{C}_i}=\frac{\widetilde{\sigma}+2}{3\widetilde{\sigma}}>0,\quad \frac{\partial p_i(\overline{C})}{\partial\overline{C}_j}=0\,. \tag{2.31}$$

From (2.23) it therefore follows that the adjustment process in $\overline{C}$ is locally unstable if:

$$(2\sigma-2)\theta^2+(3-2\sigma)\theta-1>0 \tag{2.32}$$

which is equivalent to the condition for instability of a simple price tâtonnement adjustment process as demonstrated by Scarf (1960, p.169).

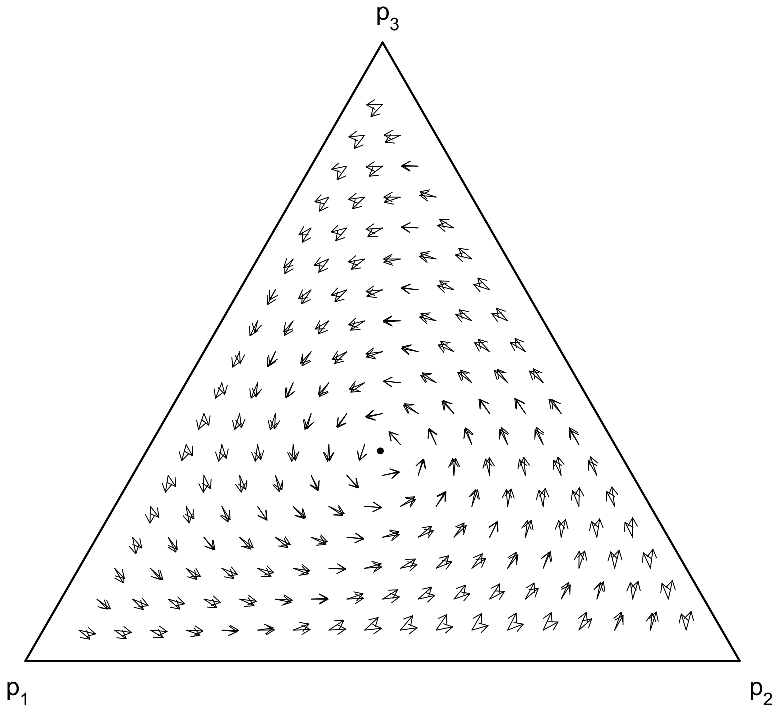

Fig. 2.5 Comparison of SR and tâtonnement fields

2.4.2 Global Convergence

Although the local behavior of the price tâtonnement and the SR algorithm adjustment processes are identical, they produce different search directions away from a neighborhood of the equilibrium. This is apparent in Figure 2.5 where the two vector fields are superimposed. Only local to the equilibrium where price effects dominate income effects, do the two fields coincide exactly, as indicated by (2.32). As one moves further away from the center of the simplex, the vector fields become more divergent. It is found that there are cases in which the SR algorithm does not converge even though the price tâtonnement is globally stable. This convergence failure is a manifestation of the simplifying nature of the adjustment process. By solving a sequence of representative agent economies the SR algorithm omits both income constraints and global properties of the individual utility functions. While the omission of global characteristics of preferences reduces the dimensionality of the model significantly, this may at the same time hinder convergence.

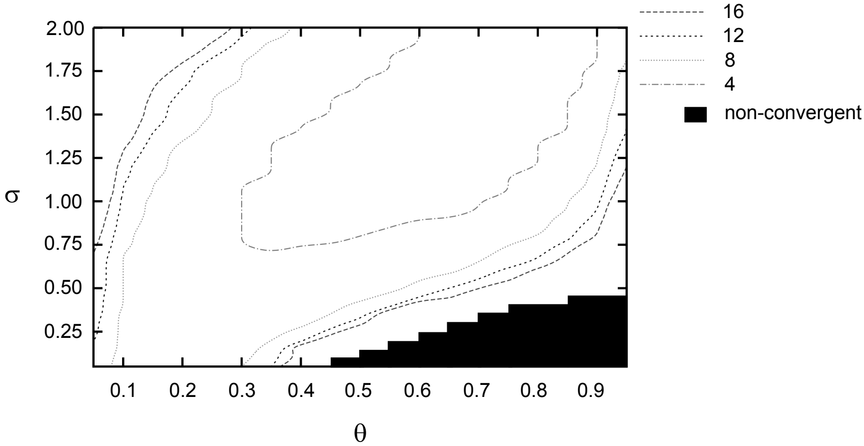

Fig. 2.6 Global convergence behavior for different configurations of θ and σ

To assess the global convergence properties of the SR algorithm, a grid search over the behavioral parameters σ and θ is performed. We let the algorithm start from a disequilibrium point $p = (0.2, 0.2, 0.6)$ where local equilibrium dynamics are absent. Figure 2.6 reveals that convergence of the SR algorithm fails for combinations of small values for σ and high values for θ.[8] For these parameter configurations, income effects are relatively strong vis-à-vis substitution effects. In cases where convergence is achieved, the presence of significant income effects means that more iterations are required to find the true equilibrium. If, however, income effects are relatively weak, the SR algorithm only requires a modest number of iterations. The appropriateness of the presented solution method therefore depends on the characteristics of the underlying model.

One last remark is in order. To guarantee convergence of the SR algorithm, it is necessary to select a sufficiently large value for $\widetilde{\sigma}$, the elasticity of substitution of the representative agent. If $\widetilde{\sigma}$ is too low, convergence may fail even if income effects are relatively weak. Non-convergent behavior, however, that occurs in the bottom right corner of Figure 2.6 is robust with respect to $\widetilde{\sigma}$. The choice of $\widetilde{\sigma}$ is entirely innocuous since this parameter bears no economic significance for the behavior of "real" households in the underlying economic model. Computational experience suggests to use values of order $\widetilde{\sigma} \geq 1$.

[8] For both parameters, a grid resolution of 0.05 is chosen, $\widetilde{\sigma}$ is set to one, and a maximum of 1000 iterations is allowed for. The adjustment process is said to converge if the 1-norm of differences between a computed price vector and the equilibrium point drops below some metric δ, i.e. $\|p_i - p_i^*\|_1 < \delta$, where p_i^* denotes the analytical equilibrium solution. We set $\delta = 0.01$.

2.5 OLG Models with Many Households

This section presents a decomposition algorithm for solving overlapping generations models with many heterogeneous households. The proposed algorithm is an application of the SR approach with a few elaborations specific to the OLG context. As in the static setting, an equilibrium allocation is approximated by computing equilibria for a sequence of representative agent economies. In the case of OLG, the representative agent economies are Ramsey optimal growth problems where the system of overlapping generations is replaced by a single infinitely-lived agent.

The algorithm is demonstrated for a simple prototype Auerbach-Kotlikoff OLG economy with production activities, intra-cohort heterogeneity, a labor-leisure choice, and a government sector.[9] We solve for the effects of a tax reform that is introduced unexpectedly in year zero, and then evaluate the performance of the algorithm against numerical solutions that are available from simultaneous solution methods.

2.5.1 A Prototype Auerbach-Kotlikoff OLG Model

2.5.1.1 Households

Time is discrete and extends from $t = 0,\ldots,\infty$. There is no aggregate or household-specific uncertainty. The economy is populated by overlapping generations of heterogeneous agents. A household of generation g and type $h = 1,\ldots,H$ is born at the beginning of year $t = g$, lives for $N+1$ years, and is endowed with $\omega_{g,h,t} = \omega\,(1+\gamma)^g$ units of time in each period $g \leq t \leq g+N$, and $\pi_{g,h,t}$ is an index of labor productivity over the life cycle.[10] γ denotes the exogenous steady-state growth rate of the economy. Leisure time, $\ell_{g,h,t}$, enters in a CES function with consumption, $c_{g,h,t}$, to create full consumption, $z_{g,h,t}$. Expressed with present value prices, the optimization problem is:

[9] The example is an adapted version of the production model presented in Rasmussen and Rutherford (2004). A closed economy version of their model with intra-cohort heterogeneity is considered. While a single-sector model is investigated here, the logic can be readily extended to a multisectoral framework.

[10] ω is a constant income scaling factor which is determined in the initial calibration procedure to reconcile household behavior with the aggregate benchmark data. For more details see Rasmussen and Rutherford (2004).

$$\begin{aligned}
\max_{c_{g,h,t},\,\ell_{g,h,t}} \; u_{g,h}\left(z_{g,h,t}\right) &= \sum_{t=g}^{g+N} \left(\frac{1}{1+\rho}\right)^{t-g} \frac{z_{g,h,t}^{1-1/\sigma_h}}{1-1/\sigma_h} \\
s.t. \qquad\qquad z_{g,h,t} &= \left(\alpha c_{g,h,t}^{\nu} + (1-\alpha)\,\ell_{g,h,t}^{\nu}\right)^{\frac{1}{\nu}} \\
p_{c,t}\, c_{g,h,t} + p_{y,t}\, i_{g,h,t} &\leq p_{l,t}\, \pi_{g,h,t}\left(\omega_{g,h,t} - \ell_{g,h,t}\right) + p_{r,t}\, k_{g,h,t} + p_{y,t}\, \zeta_{g,h,t} \\
k_{g,h,t+1} &\leq (1-\delta)\, k_{g,h,t} + i_{g,h,t} \\
\ell_{g,h,t} &\leq \omega_{g,h,t} \\
c_{g,h,t}, \ell_{g,h,t} &\geq 0 \\
k_{g,h,g} &\leq \bar{k}_{g,h,g}\,, \quad i_{g,h,t+N} + (1-\delta)\, k_{g,h,t+N} \geq 0\,,
\end{aligned} \tag{2.33}$$

where ρ is the utility discount factor, σ_h is the intertemporal elasticity of substitution for household h, $\sigma_\nu = 1/(1-\nu)$ is the uniform elasticity of substitution between consumption and leisure, α is a share parameter, and $p_{x,t}$, $x = \{y, c, l, r\}$, denotes the price for the single output good, the price for consumption, the wage rate, and the capital rental rate, respectively. All prices refer to after-tax prices. Heterogeneity relates to intra-cohort differences in labor productivity and the intertemporal elasticity of substitution. Households have access to a storage technology: they can use one unit of the output good to obtain one unit of the capital good next period. We denote the investment into this technology by $i_{g,h,t}$. We do not restrict $i_{g,h,t}$, because we want to permit households to borrow against future labor income. Private capital $k_{g,h,t}$ depreciates at an annual rate of δ. $\bar{k}_{g,h,g}$ denotes the capital holdings of generation g at the beginning of life $t = g$. Initial old generations, i.e. generations born prior to period zero, are endowed with a non-zero amount of assets: $\bar{k}_{g,h,0} \neq 0$, $\forall g = -N, \ldots, -1, \forall h$. The initial asset distribution for these generations is selected such that the economy is on a balanced growth path. We assume that newborn households enter with zero assets: $\bar{k}_{g,h,g} = 0$, $\forall g \geq 0, \forall h$. We furthermore rule out that households die in debt. In each period of the life cycle households receive $\zeta_{g,h,t}$ units of the output good as a lump-sum transfer from the government. For simplicity, we assume that these transfers are allocated to each generation and type according to its share in the total population. We moreover assume that the government has no outstanding debt in the initial steady state and hence households' total assets in period zero are equal to the value of the capital stock, i.e. $\sum_{g=-N}^{0} \sum_{h=1}^{H} \bar{k}_{g,h,t} = (1+\bar{r})\, K_0$, where $\bar{r}$ is the steady state interest rate and K_0 is the initial aggregate capital stock.[11]

2.5.1.2 Firms

There is a single representative firm which in each period t uses capital and labor services to produce a single output good Y_t according to a linearly homogeneous

[11] We therefore implicitly assume that the government has no outstanding debt at period zero. A situation with non-zero initial government debt slightly complicates the calibration procedure but is conceptually straightforward (see Rasmussen and Rutherford (2004)).

production function $Y_t = F(K_t, L_t)$. All goods and factor markets are perfectly competitive.

2.5.1.3 Government

The government agent collects revenue from levying taxes on consumption, and on capital and labor income. Tax revenue is spent on government expenditure (G_t) and on total transfers to households ($T_t = \sum_{g=t-N}^{t} \sum_{h=1}^{H} \zeta_{g,h,t}$). We assume that the consumption tax rate (τ_t^c) adjusts such that the government budget is balanced on a period-by-period basis:

$$\tau_t^r \, p_{r,t} \, R_t + \tau_t^l \, p_{l,t} \, L_t + \tau_t^c \, p_{y,t} \, C_t = p_{y,t} \, G_t + p_{y,t} \, T_t \tag{2.34}$$

where τ_t^r and τ_r^l are the net tax rates on capital and labor income, respectively.

2.5.1.4 Aggregate Economy Restrictions

Given inter- and intragenerational heterogeneity, the following feasibility conditions must be satisfied:

$$L_t = \sum_{g=t-N}^{t} \sum_{h=1}^{H} \omega_{g,h,t} - \ell_{g,h,t} \tag{2.35}$$

$$I_t = \sum_{g=t-N}^{t} \sum_{h=1}^{H} i_{g,h,t} \tag{2.36}$$

$$K_t = \sum_{g=t-N}^{t} \sum_{h=1}^{H} k_{g,h,t} \tag{2.37}$$

$$C_t = \sum_{g=t-N}^{t} \sum_{h=1}^{H} c_{g,h,t} \, . \tag{2.38}$$

The law of motion for the aggregate capital stock is given by:

$$K_{t+1} = (1-\delta)K_t + I_t \, . \tag{2.39}$$

Finally, the single output good may be used for household consumption, investment, or government consumption implying the following condition for balance between aggregate supply and demand:

$$F(K_t, L_t) = C_t + I_t + G_t \, . \tag{2.40}$$

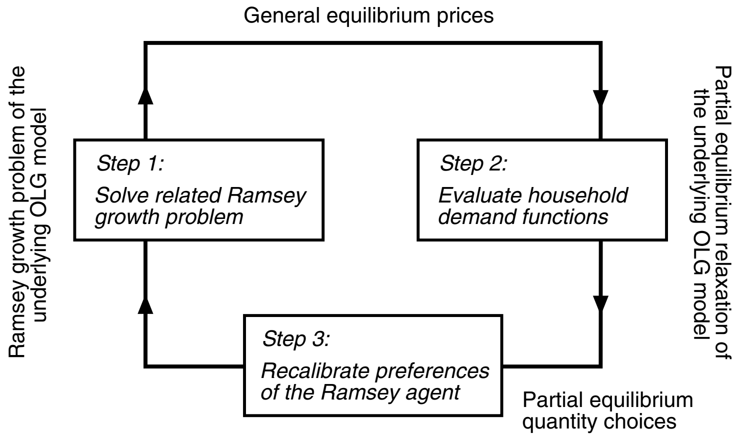

Fig. 2.7 Solving OLG by Ramsey: steps in the decomposition algorithm

2.5.2 A Decomposition Algorithm for OLG Models

The unknown equilibrium allocation of the OLG economy described by (2.33)-(2.40) is approximated by computing equilibria for a sequence of "related" Ramsey (optimal) growth problems. Figure 2.7 provides a schematic exposition of the steps involved in the decomposition procedure. Each iteration comprises the following three steps. In the first step, we solve the general equilibrium of the "related" Ramsey growth problem (Section 2.5.2.1) which retains the full structure of the production side of the model but replaces the system of OLG households by a representative infinitely-lived consumer agent. The second step computes optimal household behavior given the equilibrium prices from the Ramsey economy (Section 2.5.2.3). This step can be viewed as solving a partial equilibrium relaxation of the underlying economy that ignores general equilibrium interactions with the production side of the model but retains the full structure of the OLG demand system. In the third step, we construct a new Ramsey optimal growth problem by recalibrating the preferences of the "artificial" Ramsey agent based on households' choices from Step 2 (Section 2.5.2.4). Subsequent iterations proceed with analogous steps. Typically, the sequence of prices and quantities computed by the algorithm converges to the true equilibrium allocation. In what follows, we provide more details on each of the involved steps.

2.5.2.1 The "Related" Ramsey Growth Problem

As the "related" Ramsey optimal growth problem, we define a model of the underlying OLG economy in which the system of overlapping generations is replaced by a single infinitely-lived representative agent, henceforth called the Ramsey agent. Apart from this modification, the entire economic structure of the OLG model including the behavior of other agents, market structure, the number of sectors etc. is unchanged. The Ramsey agent solves the following optimization problem:

$$
\begin{aligned}
\max_{C_t,\mathscr{L}_t} U(Z_t) &= \left[\sum_{t=0}^{T} \Theta_t^k \left(\frac{Z_t}{\overline{Z}_t^k}\right)^{1-1/\widehat{\sigma}}\right]^{\frac{\widehat{\sigma}}{\widehat{\sigma}-1}} \\
s.t. \qquad Z_t &= \left(\Delta_t^k \left(\frac{C_t}{\overline{C}_t^k}\right)^{\nu} + \left(1-\Delta_t^k\right)\left(\frac{\mathscr{L}_t}{\overline{\mathscr{L}}_t^k}\right)^{\nu}\right)^{\frac{1}{\nu}} \\
C_t + I_t + G_t &= F(K_t, \Omega_t^k - \mathscr{L}_t) \\
K_{t+1} &\leq (1-\delta) K_t + I_t \\
\mathscr{L}_t &\leq \Omega_t^k \\
C_t, \mathscr{L}_t &\geq 0 \\
K_0 &\leq \Psi \\
K_{T+1} &= \hat{K}_{T+1}
\end{aligned}
\tag{2.41}
$$

where C_t, $\mathscr{L}_t$, Z_t, K_t, I_t, and Ω_t^k now denote consumption, leisure time, full consumption, the capital stock, investment, and the time endowment by the Ramsey agent, respectively, and where $\widehat{\sigma}$ is the intertemporal elasticity of substitution. k denotes an iteration index.[12]

The initial capital stock in the Ramsey economy is given by the aggregate capital stock of the OLG economy in year zero:

$$\Psi = \sum_{g=-N}^{0} \sum_{h=1}^{H} \overline{k}_{g,h,0} \,. \tag{2.42}$$

To approximate the infinite-horizon Ramsey economy by a finite-dimensional complementarity problem, we use the *state-variable targetting* method suggested by Lau, Pahlke and Rutherford (2002) in which the target post-terminal capital stock ($\hat{K}_{T+1}$) is chosen at a level such that investments grow at the steady-state rate in the last period:

$$\frac{I_T}{I_{T-1}} = 1 + \gamma. \tag{2.43}$$

[12] Note that the nested lifetime utility function $U(Z_t)$ is written in calibrated share form. We monotonically transform preferences in (2.33) to obtain a linear homogenous CES representation. This does not alter the underlying preference orderings and hence optimization yields the same demand functions.

2.5.2.2 Initialization

In order to initialize the "related" Ramsey growth problem, it is first necessary to characterize a baseline reference path of the OLG economy. For simplicity, we assume that the economy is initially on a balanced growth path and employ a steady-state calibration procedure proposed by Rasmussen and Rutherford (2004) which proceeds in two steps. In the first step, the optimal profile of decision variables for a reference generation of type h is computed subject to given aggregate benchmark data in year zero. The second step involves extrapolating the results from the household calibration model, together with remaining elements in the aggregate dataset, to set up a baseline reference path.

Given this reference path, we choose an initial set of preferences for the representative agent such that the Ramsey growth problem endogenously reproduces the baseline reference path of the underlying OLG economy. This is accomplished by selecting appropriate reference levels and value share parameters for $U(Z_t)$ in (2.41):

$$\overline{C}_t^0 = \sum_{g=t-N}^{t} \sum_{h=1}^{H} \overline{c}_{g,h,t} \tag{2.44}$$

$$\overline{\mathscr{L}}_t^0 = \sum_{g=t-N}^{t} \sum_{h=1}^{H} \overline{\ell}_{g,h,t} \tag{2.45}$$

$$\overline{Z}_t^0 = \overline{C}_t^0 + \overline{\mathscr{L}}_t^0 \tag{2.46}$$

$$\Delta_t^0 = \frac{\overline{p}_t \overline{C}_t^0}{\overline{p}_t \overline{C}_t^0 + \overline{p}_t \overline{\mathscr{L}}_t^0} \tag{2.47}$$

$$\Theta_t^0 = \frac{\overline{p}_t \overline{Z}_t^0}{\sum_{t'} \overline{p}_{t'} \overline{Z}_{t'}^0} \tag{2.48}$$

where $\overline{p}_t, t = 0, \ldots, T$ and $\overline{x}_{g,h,t}, x = \{c, \ell\}$, denote benchmark prices and household quantity choices, respectively. The superscript "0" indicates starting values.

The Ramsey agent is endowed with units of "productive" time, Ω_t, that reflect households' labor productivity in a given period:

$$\Omega_t^0 = \sum_{g=t-N}^{t} \sum_{h=1}^{H} \pi_{g,h,t} \left(\omega_{g,h,t} - \overline{\ell}_{g,h,t}\right) + \overline{\ell}_{g,h,t} \tag{2.49}$$

where $\overline{\ell}_{g,h,t}$ is benchmark leisure time by households.

The algorithm is started off by solving the initial Ramsey growth problem as defined by (2.34), and (2.41)-(2.49). If no policy change is implemented, the given specification of the Ramsey economy ensures that it can reproduce the initial steady state path of the underlying OLG economy as an equilibrium solution.

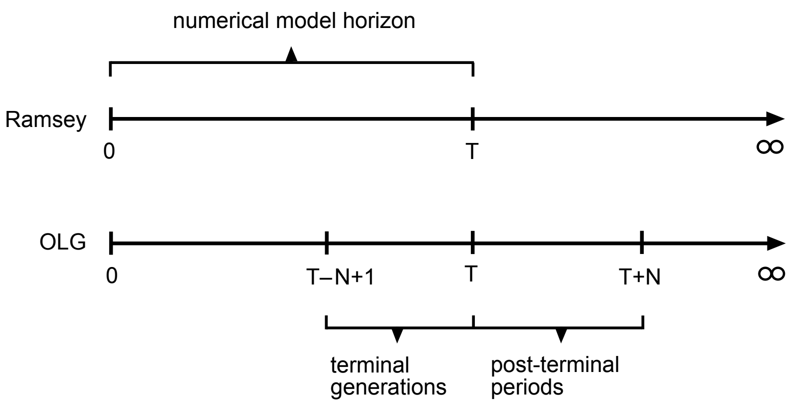

Fig. 2.8 Approximating OLG by Ramsey: the issue of terminal generations

2.5.2.3 The Partial Equilibrium Relaxation

The second step of the algorithm solves a partial equilibrium relaxation of the underlying OLG economy which retains full details of the household demand system but ignores general equilibrium effects. Hence, any interactions via commodity and factor markets and with the production side of the economy are suppressed. Given equilibrium prices from the previous solution of the "related" Ramsey growth problem, we evaluate demand functions for each generation and type that originate from the set of household problems in (2.33).

In order to obtain a good approximation of the underlying OLG economy, it is necessary to compute optimal household demand for all households and types in each period of the numerical model that runs from $t = 0, \ldots, T$. This information then forms the basis for the recalibration of preferences of the Ramsey agent in the subsequent step of the algorithm. A complication arises for periods $T - N + 1 \leq t \leq T$ in which generations are born that live beyond T (henceforth called terminal generations). Figure2.8 illustrates this issue. To compute the optimal decision profiles of these agents, it is essential to account for their behavior over the full life cycle. With the last cohort of households being born in period T, this means that there are N post-terminal periods that have to be included in the analysis, which we denote by $\hat{t} = T, \ldots, T + N$. We resolve this issue by employing a steady-state closure rule which postulates that the economy has reached a steady state by period $T - N + 1$. This additional restriction is not binding if T is chosen sufficiently large.[13] Exploiting this fact, prices for post-terminal periods can be inferred from the following steady-state projection:

[13] The specific choice of T depends on the nature of the policy shock that is considered. In the numerical examples below we set $T = 150$.

$$x_{\hat{t}}^k = \frac{x_T^k}{(1+r_\infty)^{\hat{t}-T}} \tag{2.50}$$

where $x_{\hat{t}}^k = \{pt_{g,h,\hat{t}}^k, pt_{y,\hat{t}}^k, pt_{l,\hat{t}}^k\}$ denote the post-terminal price for full consumption, the output good, and the market wage all obtained in iteration k, respectively. Correspondingly, x_T^k refer to respective prices in the terminal period. The price for full consumption can be obtained as:

$$p_{g,h,t}^k = e_{g,h,t}\left(p_{c,t}^k\,,\,p_{l,t}^k\right) \tag{2.51}$$

where $e_{g,h,t}(\cdot)$ denotes the unit expenditure function for $z_{g,h,t}$. For future reference, let $\overline{p}_{g,h,t}$ denote respective benchmark prices. In (2.50), $r_\infty = p_{T-1}^c/p_T^c - 1$ defines the endogenous (steady-state) interest rate in the terminal period.

Finally, the lifetime income of generation g and type h, evaluated at candidate prices from iteration k, is given by:

$$M_{g,h}^k = \sum_{t=g}^{g+N} \pi_{g,h,t}\, p_{l,t}^k\, \omega_{g,h,t} + p_{y,t}^k\, \zeta_{g,h,t} + p_{r,0}^k\, \overline{k}_{g,h,g}\,. \tag{2.52}$$

A similar formula applies to the lifetime income of terminal generations where for post-terminal periods projected prices according to (2.50) are used. For future reference, let $\overline{M}_{g,h}$ denote the lifetime income at benchmark prices.

We are now in a position to compute household demand. In order to reduce computational complexity, we solve the dual problem making use of formulas (2.7)-(2.9) (see the Appendix). Let $e_{g,h}(\mathbf{p}_{g,h}^k)$ denote the expenditure function for a unit of $u_{g,h}$ which—given the specific structure of preferences[14]—can be constructed using the vector of prices for the full consumption good, $\mathbf{p}_{g,h}^k$ (including projected prices). Indirect utility can then be written as:

$$V_{g,h}(\mathbf{p}_{g,h}^k\,,\,M_{g,h}^k) = \frac{M_{g,h}^k}{e_{g,h}(\mathbf{p}_{g,h}^k)\,\overline{M}_{g,h}} \quad \forall g\,,\forall h\,. \tag{2.53}$$

Applying Roy's identity, optimal household demand (in the context of the partial equilibrium relaxation) for full consumption, goods consumption and leisure, respectively, evaluated at the *candidate price vector* $\mathbf{p}_{g,h}^k$, are updated in each iteration according to:

[14] Note that in the given case of homothetic preferences, the unit expenditure function conveys all information concerning the underlying preferences.

$$z_{g,h,t}(\mathbf{p}^k_{g,h}, M^k_{g,h}) = \bar{z}_{g,h,t}\, V_{g,h}(\mathbf{p}^k_{g,h}, M^k_{g,h}) \left(\frac{e_{g,h}(\mathbf{p}^k_{g,h})\, \overline{p}_{g,h,t}}{p^k_{g,h,t}} \right)^{\sigma_h} \tag{2.54}$$

$$c_{g,h,t}(\mathbf{p}^k_{g,h}, M^k_{g,h}) = \bar{c}_{g,h,t}\, z_{g,h,t}(\mathbf{p}^k_{g,h}, M^k_{g,h}) \left(\frac{p^k_{g,h,t}\, \overline{p}_t}{p^k_{c,t}\, \overline{p}_{g,h,t}} \right)^{\sigma_v} \tag{2.55}$$

$$\ell_{g,h,t}(\mathbf{p}^k_{g,h}, M^k_{g,h}) = \bar{\ell}_{g,h,t}\, z_{g,h,t}(\mathbf{p}^k_{g,h}, M^k_{g,h}) \left(\frac{p^k_{g,h,t}\, \overline{p}_t}{p^k_{l,t}\, \pi_{g,h,t}\, \overline{p}_{g,h,t}} \right)^{\sigma_v}. \tag{2.56}$$

2.5.2.4 Recalibration of the Ramsey Agent's Preferences

The last step in each iteration is to construct a new Ramsey optimal growth problem by recalibrating the Ramsey agent's preferences based on optimal household choices from the previous step. This is accomplished by updating level parameters in (2.41) according to:

$$\overline{C}^{k+1}_t = \sum_{g=t-N}^{t} \sum_{h=1}^{H} c_{g,h,t}(\mathbf{p}^k_{g,h}, M^k_{g,h}) \tag{2.57}$$

$$\overline{\mathscr{L}}^{k+1}_t = \sum_{g=t-N}^{t} \sum_{h=1}^{H} \ell_{g,h,t}(\mathbf{p}^k_{g,h}, M^k_{g,h}) \tag{2.58}$$

$$\overline{Z}^k_t = \overline{C}^k_t + \overline{\mathscr{L}}^k_t \tag{2.59}$$

and value share parameters in (2.41) according to:

$$\Delta^{k+1}_t = \frac{p^k_{c,t}\, \overline{C}^k_t}{p^k_{c,t}\, \overline{C}^k_t + p^k_{l,t}\, \overline{\mathscr{L}}^k_t} \tag{2.60}$$

$$\Theta^{k+1}_t = \frac{p^k_{c,t}\, \overline{C}^k_t + p^k_{l,t}\, \overline{\mathscr{L}}^k_t}{\sum_{t'} \left(p^k_{c,t'}\, \overline{C}^k_{t'} + p^k_{l,t'}\, \overline{\mathscr{L}}^k_{t'} \right)}. \tag{2.61}$$

With varying prices, households adjust their labor supply, and hence the composition of the labor force with respect to age and household type is altered. But since labor productivity depends on these two socio-economic characteristics, aggregate labor productivity in the underlying OLG economy also changes. We therefore adjust the time endowment of the Ramsey agent in each iteration according to:

$$\Omega^{k+1}_t = \sum_{g=t-N}^{t} \sum_{h=1}^{H} \pi_{g,h,t} \left(\omega_{g,h,t} - \ell_{g,h,t}(\mathbf{p}^k_{g,h}, M^k_{g,h}) \right) + \ell_{g,h,t}(\mathbf{p}^k_{g,h}, M^k_{g,h}). \tag{2.62}$$

Thus, the newly constructed Ramsey optimal growth problem in iteration $k+1$ consists of solving (2.41) (subject to (2.42) and (2.43) with updated preference parameters and time endowment as defined by (2.57)–(2.62). This completes the description of the algorithm.

2.5.3 Algorithmic Performance

The OLG economy presented above has no analytical solution. In order to evaluate the algorithm, we therefore compare its performance to those of conventional simultaneous/direct solution methods. As a benchmark, we take a complementarity-based approach as suggested by Rasmussen and Rutherford (2004).

The base case parametrization of the economy is as follows. Households live for 51 years or $N = 50$. We set $\bar{r} = 0.05$, $\gamma = 0.01$, $\delta = 0.07$, $\nu_h = 0.8$, $\beta = 0.32$, and $\alpha = 0.8$. In the numerical analysis, we test the performance of the algorithm for a different number of household types H and also allow for various degrees of intra-cohort heterogeneity. For simplicity, we assume that σ_h, $h = 1,\ldots,H$, are generated by random draws from a uniform distribution defined over $[\underline{\sigma}, \overline{\sigma}]$. Likewise, differences in labor productivity are generated by randomly drawing $a_{g,h}$ from a uniform distribution with support $\underline{a} \leq a_{g,h} \leq \bar{a}$, where the parameter $a_{g,h}$ enters the labor productivity profile over the life cycle as: $\pi_{g,h,t} = \exp\left[4.47 + a_{g,h}(t-g) - 0.00067(t-g)^2\right]$. Furthermore, it is assumed that each type has equal size in the total population. The values for the aggregate data including tax payments in the initial benchmark are based on Input-Output tables for the U.S. economy in 1996 and are presented at the top of the corresponding computer programs. We solve the model for $T = 150$ years.

2.5.3.1 Solving For a Policy Shock: A Fundamental Tax Reform

We now present an illustrative application of the decomposition algorithm by solving for the effects of a policy change that in year zero unexpectedly and permanently reduces the capital income tax and introduces a consumption tax to endogenously balance the government budget. The capital income tax is reduced from a benchmark value of 28.4% to 22.9%.

We start out by considering a case where $H = 1$, $\sigma_h = 1.2$, and $a_{g,h} = 0.04$. Figure 2.9 shows the sequence of time paths for investment that emerges from the iterative procedure of the algorithm. The true transition path to a new steady state of the OLG economy as computed by the benchmark simultaneous solution method is labeled "OLG". "Iteration 1" plots the impact of the tax reform scenario after the first iteration of the solution method. This is equivalent to what would be obtained from solving a Ramsey optimal growth model. Each subsequent iteration of the algorithm produces a new time path for investment that eventually converges to the true solution. We terminate the search process if $\|p^k_{c,t} - p^*_{c,t}\|_1 < 10^{-6}$. For the current model, this is achieved after 34 iterations. Figure 3.7 shows a similar picture for the welfare

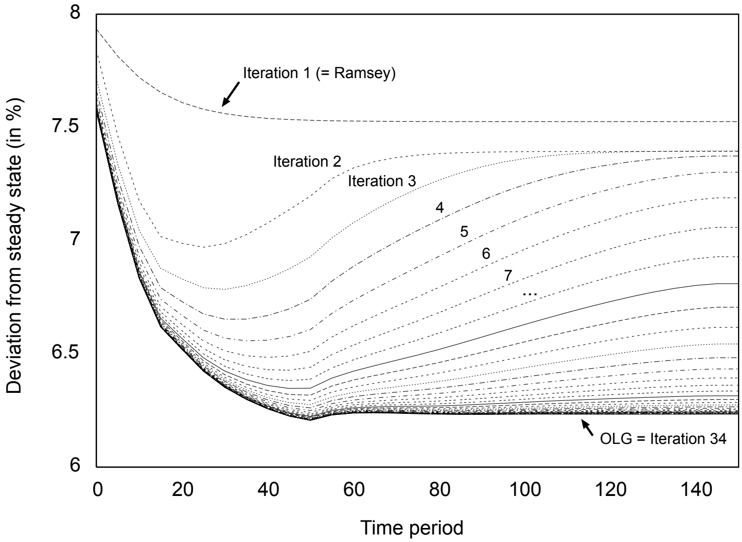

Fig. 2.9 Solving OLG by Ramsey: sequence of investment time paths

change experienced by each generation.[15] Note that in terms of welfare changes, stopping after the first iteration corresponds to a situation which would results from a pure top-down approach that fails to take into account general equilibrium feedback effects from the micro to the macro level.

To assess the quality of the approximation, we use the following two measures. First, the approximation error e^k reports the 1-norm of differences between computed consumption prices and true equilibrium prices as calculated by the benchmark method: $e^k = \|p^k_{c,t} - p^*_{c,t}\|_1$. As e^k constitutes a summary statistic which is defined over the entire model horizon, it says little about whether price deviations of the computed from the true price path lie within a tolerable bandwidth. As a second measure, we therefore report the maximum distance error τ^k which is defined as: $\tau^k = \max\{|p^k_{c,t} - p^*_{c,t}|\}$.

Figure 2.11 plots e^k as a function of the number of iterations. The approximation error quickly decreases and then converges to zero. After the first few iterations the decomposition technique only involves refinements of the demand system, and consequently, subsequent changes in relative prices are small.

[15] Kehoe and Levine (1985) have shown that the OLG framework may permit multiple equilibria for certain parameter values. In such cases, indeterminacy would manifest itself as sensitivity to the truncation date. None of the models presented here are sensitive to T, provided that it is sufficiently large. This and the general robustness of the models provide evidence that the equilibria are unique. Kotlikoff (2000) reaches the same conclusion regarding the uniqueness of equilibria in the OLG models he has been working with.

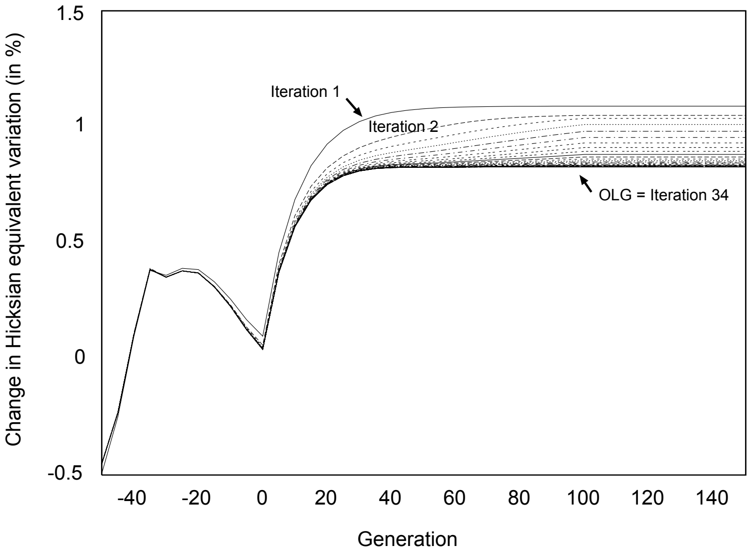

Fig. 2.10 Solving OLG by Ramsey: sequence of welfare changes by generation

2.5.3.2 Robustness and Accuracy

In order to explore the capacity of the algorithm to solve large-scale OLG models, we examine its performance for a different number of households and various degrees of intra-cohort heterogeneity. We look again at the effects of the tax reform scenario as described above. Given the simple specification for the source of intra-cohort differences, we vary the extent of heterogeneity, denoted by Γ, by changing the support for the distributions from which σ_h and $a_{g,h}$ are drawn.[16]

Table 2.1 reports results from a series of runs where the number of households within each generation is increased while holding fixed the degree of intra-cohort heterogeneity. The quality of approximation is excellent (τ_k is around 10^{-4}). As the number of household types increases, the proposed decomposition procedure become advantageous.[17] Most importantly, it is shown that the algorithm can provide improvements in robustness as compared to the benchmark simultaneous solution method which quickly becomes infeasible for models in which $H \geq 100$.

To examine the performance of the algorithm in the presence of a substantial degree of heterogeneity among households, we report results for different configura-

[16] We consider the following sets of choices for $\{(\underline{\sigma},\overline{\sigma}),(\underline{a},\overline{a})\}$ ordered by their implied degree of heterogeneity: $\Gamma_1 = \{(1.00,1.50),(0.2,0.3)\}$, $\Gamma_2 = \{(1.00,1.50),(0.2,0.4)\}$, $\Gamma_3 = \{(0.25,0.75),(0.2,0.3)\}$, $\Gamma_4 = \{(0.25,0.75),(0.2,0.4)\}$, $\Gamma_5 = \{(0.25,1.25),(0.2,0.3)\}$, $\Gamma_6 = \{(0.25,1.25),(0.2,0.4)\}$, $\Gamma_7 = \{(0.25,2.00),(0.2,0.3)\}$, $\Gamma_8 = \{(0.25,2.00),(0.2,0.4)\}$.

[17] All reported running times refer to an implementation on a Dual Core 2 GHz processor machine.

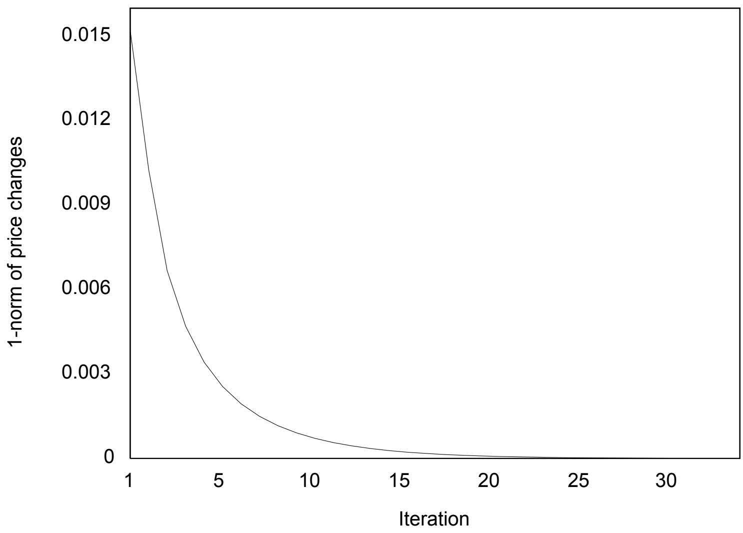

Fig. 2.11 Solving OLG by Ramsey: approximation error e^k

tions of Γ. We set $H = 50$ so that the benchmark solution method is feasible and the calculation of approximation errors is available. Not surprisingly, the approximation quality of the method is decreasing with the degree of heterogeneity. Overall, the quality of approximation is still very good: computed prices fall within a reasonably small interval around the true equilibrium price path (τ^k is around $10^{-5} - 10^{-3}$).

Motivated by the discussion of the potential convergence failure of the SR algorithm in the presence of significant income effects (Section 2.4), we conduct a number of sensitivity analyses for behavioral parameters governing intra-period and intertemporal substitution and income effects (see Table 2.3). Looking first at the intra-period dimension, we find that combinations of too small ν and α can pose serious problems for the decomposition approach. Although the search process is terminated within a modest number of iterations, both approximation measures indicate a rather poor quality of approximation for $\alpha \leq 0.5$. If α is too small, the equilibrium behavior over the life cycle of a household displays periods with zero labor supply in old ages, i.e. there is endogenous retirement. This happens because the shadow price of time exceeds the market wage rate. In the presence of such corner solutions, it is harder to portray the choices of OLG households by using a representative agent which in turn explains why the approximation error increases.

As for the role of intertemporal income effects, we do not experience problems of convergence or a poor quality of approximation (results not shown). However, the speed of convergence (in terms of the number of iterations required for convergence) is the slower, the larger is σ_h. This finding indicates that income effects

Table 2.1 Convergence performance and approximation error

H ($\Gamma = \Gamma_1$)	Number of iterations	Approx. error e^k (last iteration)	Max. distance τ^k (last iteration)	CPU computing time
1	37	10^{-6}	10^{-7}	0 min 13 s (3.78)
10	36	0.002	10^{-4}	0 min 21 s (1.17)
50	35	0.005	10^{-4}	1 min 10 s (0.18)
100	35	—	—	2 min 06 s (×)
500	34	—	—	6 min 14 s (×)
1000	34	—	—	10 min 48 s (×)
2000	29	—	—	30 min 31 s (×)

Note: Figures in parentheses denote running time of the decomposition algorithm expressed as a fraction of the running time as required by the benchmark simultaneous solution method. A "×" indicates infeasibility of the simultaneous solution method.

Table 2.2 Approximation errors for different Γ

Γ ($H = 50$)	Number of iterations	Approx. error e^k (last iteration)	Max. distance τ^k (last iteration)	CPU computing time
Γ_1	35	0.005	10^{-4}	1 min 10 s (0.18)
Γ_2	35	0.005	10^{-4}	0 min 56 s (1.14)
Γ_3	79	0.001	10^{-5}	2 min 14 s (0.39)
Γ_4	74	0.004	10^{-4}	2 min 23 s (0.42)
Γ_5	58	0.005	10^{-4}	6 min 14 s (0.29)
Γ_6	55	0.011	10^{-3}	1 min 44 s (0.26)
Γ_7	41	0.011	10^{-3}	1 min 10 s (0.18)
Γ_8	37	0.021	10^{-3}	1 min 15 s (0.22)

Note: Figures in parentheses denote running time of the decomposition algorithm expressed as a fraction of the running time as required by the benchmark simultaneous solution method.

stemming from an intertemporal reallocation of resources do not cause problems of convergence.

2.6 Concluding Remarks

This chapter develops a decomposition approach which can be applied to solve high-dimensional static and dynamic general equilibrium models with many households. We demonstrate its effectiveness for computing equilibria in large-scale OLG models which are infeasible for conventional simultaneous/direct methods. We find that the proposed algorithm provides an efficient and robust way to approximate general equilibrium in models with a large number of heterogeneous agents if income effects remain sufficiently weak. The appropriateness of the solution method there-

Table 2.3 Convergence behavior in the presence of strong income effects

ν , α ($H = 50$)	Number of iterations	Approx. error e^k (last iteration)	Max. distance τ^k (last iteration)	CPU computing time
0.50 , 0.80	33	0.015	10^{-3}	1 min 06 s (0.17)
0.25 , 0.80	30	0.035	10^{-3}	1 min 05 s (0.18)
0.80 , 0.50	42	0.014	0.013	1 min 27 s (0.19)
0.80 , 0.25	53	0.042	0.037	1 min 33 s (0.25)
0.50 , 0.50	42	0.117	0.011	1 min 15 s (0.23)
0.25 , 0.25	47	0.171	0.015	1 min 42 s (0.25)

Note: Figures in parentheses denote running time of the decomposition algorithm expressed as a fraction of the running time as required by the benchmark simultaneous solution method.

fore depends on the characteristics of the underlying model and the nature of the implemented policy shock.

We believe that the approach can be beneficial for a wide range of economic applications. In particular, it is advantageous for modeling tasks which necessitate to economize on the dimensionality of the corresponding numerical problem. Potential applications may include multi-country and multi-sectoral OLG models, and analyses of relevant policy issues—such as, e.g., population aging, trade policy, and poverty—which require detailed account of the distributional effects on a household level while at the same time taking into account general equilibrium effects. Moreover, the decomposition approach may prove useful for the further development of fully-integrated static and dynamic microsimulation models that incorporate the essential macroeconomic linkages required for a comprehensive policy analysis.

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