LINEAR ALGEBRA FOR DATA SCIENCE IN R

Solving Matrix-Vector Equations

Eric Eager
Data Scientist at Pro Football Focus

Motivation - Can These Vectors Make That Vector?

$$A\vec{x} = \begin{pmatrix} 1 & -1 \\ 2 & 1 \\ 4 & -2 \end{pmatrix} \times \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Motivation - Can These Vectors Make That Vector?

$$= \begin{pmatrix} 1 \times 1 & + & (-1) \times 2 \\ 2 \times 1 & + & 1 \times 2 \\ 4 \times 1 & + & (-2) \times 2 \end{pmatrix}$$

Motivation - Can These Vectors Make That Vector?

$$= 1 \times \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} + 2 \times \begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix}$$

Motivation - Can These Vectors Make That Vector?

```
> A%*%x
     [,1]
[1,]   -1
[2,]    4
[3,]    0
```

```
> A[, 1]*x[1] + A[,2]*x[2]
[1] -1  4  0
```

Motivation - Can These Vectors Make That Vector?

$$A\vec{x} = \vec{b}$$

$$\begin{pmatrix} 4 & -2 \\ -3 & 2 \end{pmatrix} \times \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$x_1 \times \begin{pmatrix} 4 \\ -3 \end{pmatrix} + x_2 \times \begin{pmatrix} -2 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

Example of a Matrix-Vector Equation

Teams	Johns Hopkins	F & M	Gettysburg	Dickinson	McDaniel
Johns Hopkins	-	Loss, 12 - 14	Win 49-35	Win 49-0	Win 49-7
F & M	Win, 14 - 12	-	Loss, 31-38	Win 36-28	Win 35-10
Gettsyburg	Loss 35-49	Win, 38-31	-	Loss 13-23	Win 35-3
Dickinson	Loss 0-49	Loss 28-36	Win 23-13	-	Win 38-31
McDaniel	Loss 7-49	Loss 10-35	Loss 3-35	Loss 31-38	-

Example of a Matrix-Vector Equation

$$
\begin{array}{c}
\\
\text{JH} \\
\text{F\&M} \\
\text{G} \\
\text{D} \\
\text{McD}
\end{array}
\begin{array}{c}
\begin{array}{ccccc}
\text{JH} & \text{F\&M} & \text{G} & \text{D} & \text{McD}
\end{array} \\
\begin{pmatrix}
4 & -1 & -1 & -1 & -1 \\
-1 & 4 & -1 & -1 & -1 \\
-1 & -1 & 4 & -1 & -1 \\
-1 & -1 & -1 & 4 & -1 \\
-1 & -1 & -1 & -1 & 4
\end{pmatrix}
\end{array}
$$

Example of a Matrix-Vector Equation

$$
\begin{array}{c|ccccc}
 & \text{JH} & \text{F\&M} & \text{G} & \text{D} & \text{McD} \\
\end{array}
$$

$$
\begin{array}{c}
\text{JH} \\ \text{F\&M} \\ \text{G} \\ \text{D} \\ \text{McD}
\end{array}
\begin{pmatrix}
4 & -1 & -1 & -1 & -1 \\
-1 & 4 & -1 & -1 & -1 \\
-1 & -1 & 4 & -1 & -1 \\
-1 & -1 & -1 & 4 & -1 \\
-1 & -1 & -1 & -1 & 4
\end{pmatrix}
\times
\begin{pmatrix}
r_{\text{JH}} \\ r_{\text{F\&M}} \\ r_{\text{G}} \\ r_{\text{D}} \\ r_{\text{McD}}
\end{pmatrix}
=
\begin{pmatrix}
103 \\ 28 \\ 15 \\ -40 \\ -106
\end{pmatrix}
$$

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Matrix-Vector Equations - Some Theory

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A Matrix-Vector Equation Without a Solution

- Inconsistent

$$\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \times \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

$$\begin{aligned} x_1 - x_2 &= -1 \\ 0 &= 2 \end{aligned}$$

A Matrix-Vector Equation with Infinitely-Many Solutions

- Consistent (but infinitely-many solutions)

$$\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \times \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

$$\begin{aligned} x_1 - x_2 &= -1 \\ 0 &= 0 \end{aligned}$$

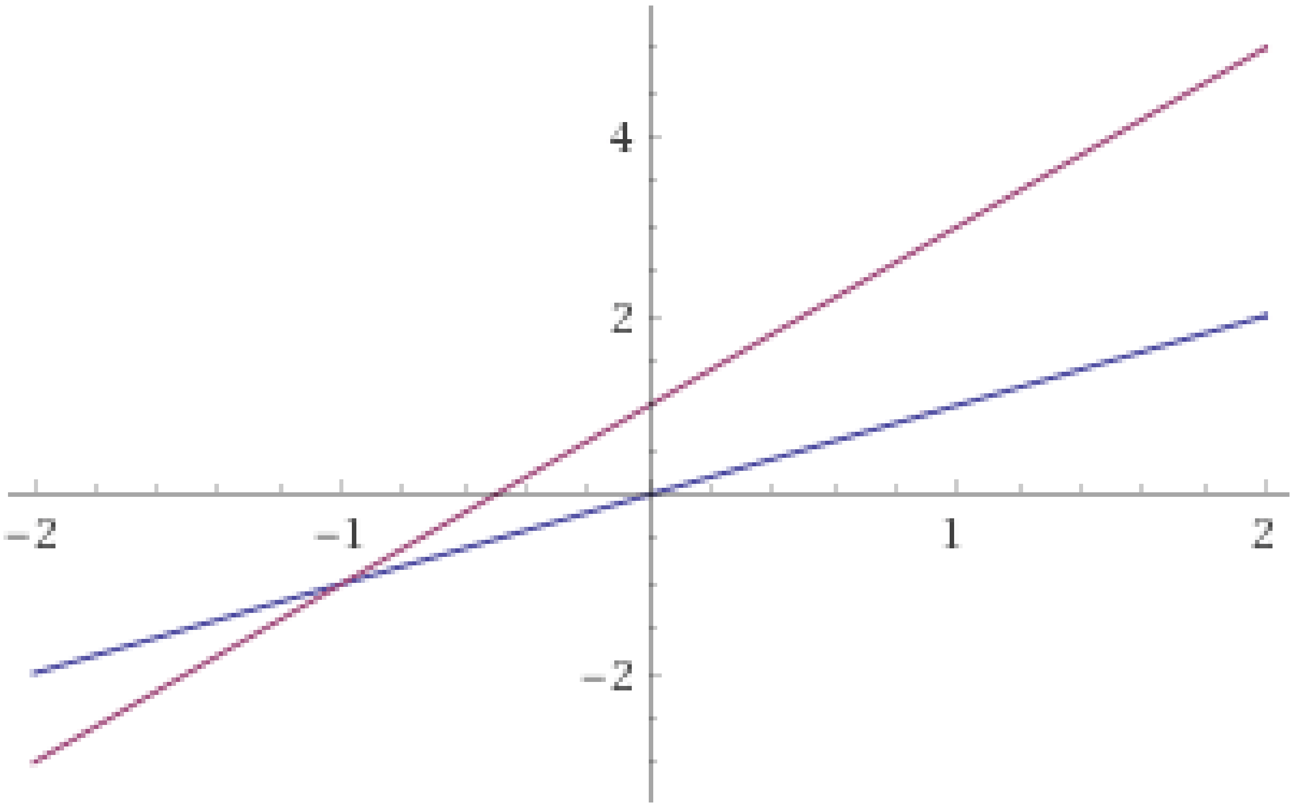
A Matrix-Vector Equation with a Unique Solution

- Consistent (unique solution)

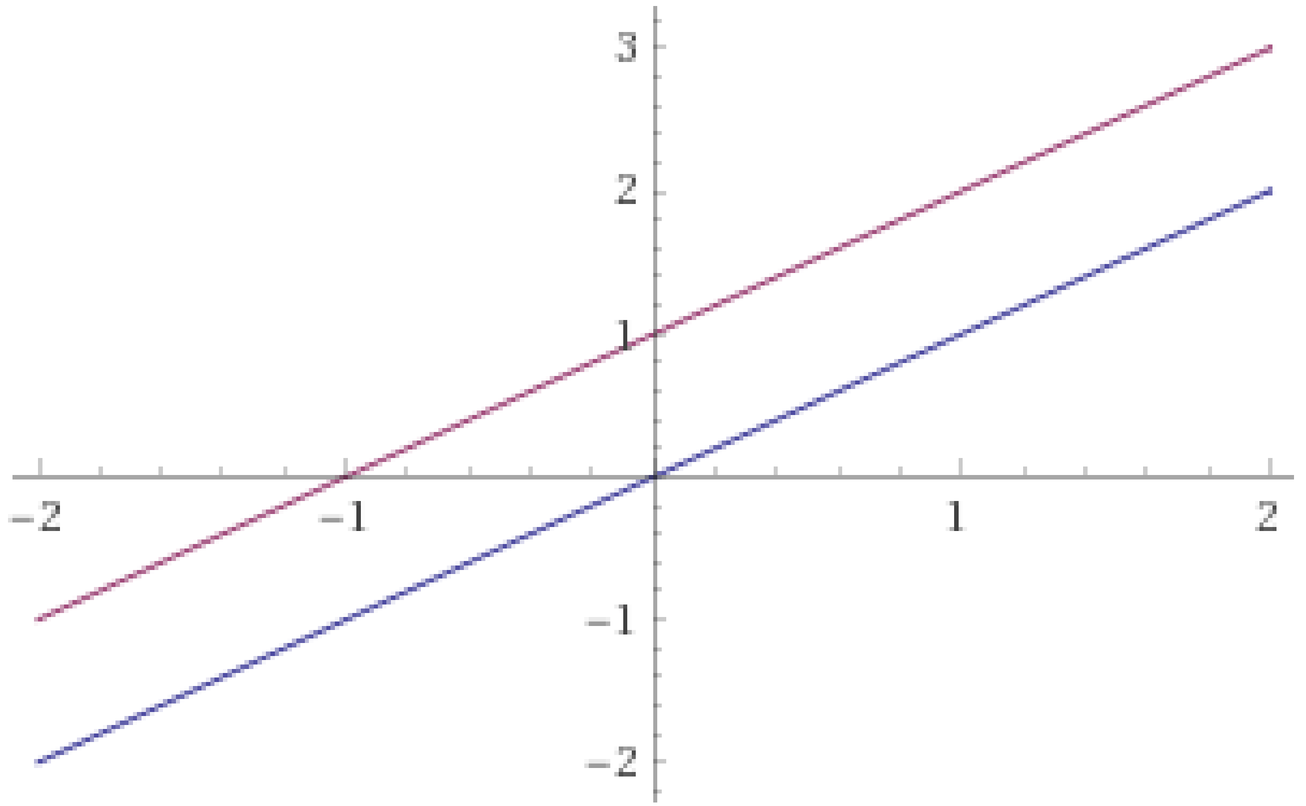
$$\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \times \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

$$\begin{aligned} x_1 - x_2 &= -1 \\ x_1 &= 0 \end{aligned}$$

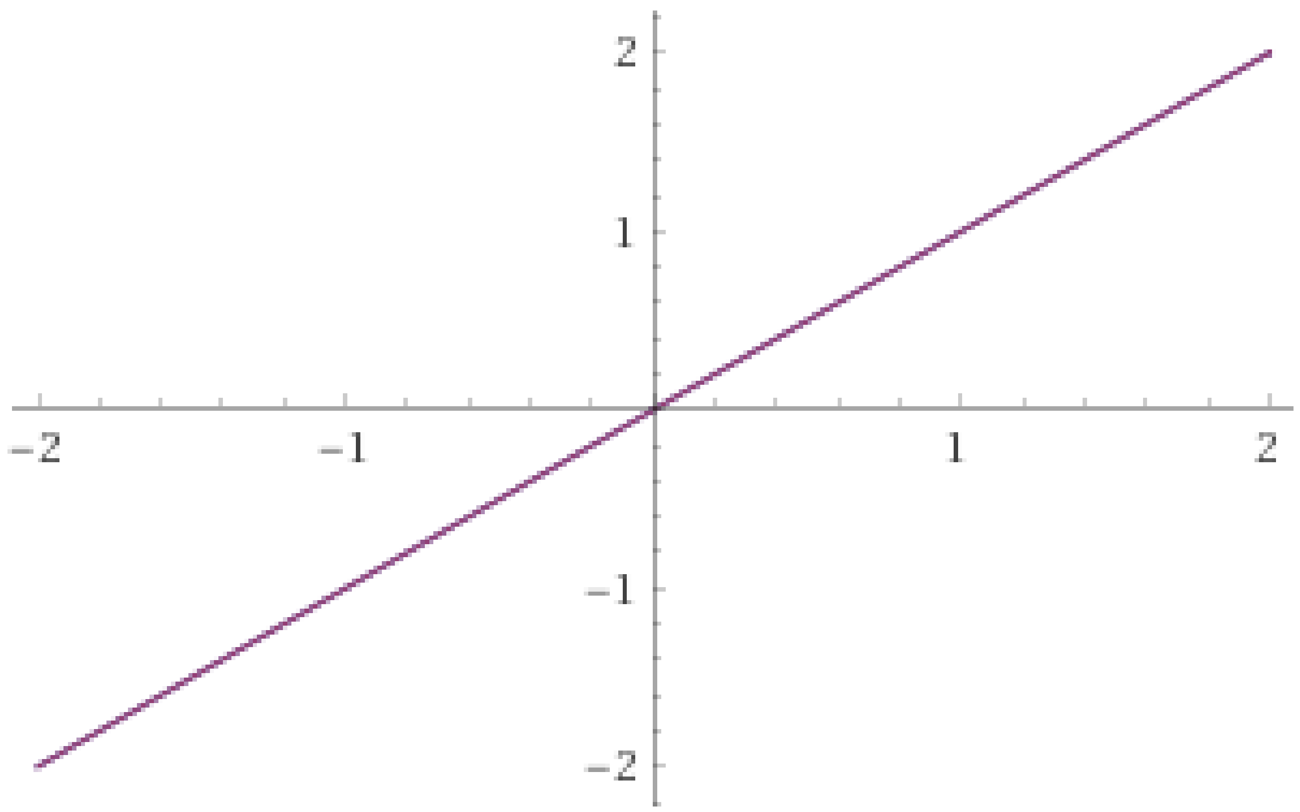
Properties of Solutions to Matrix-Vector Equations - Exactly One Solution



Properties of Solutions to Matrix-Vector Equations - No Solutions



Properties of Solutions to Matrix-Vector Equations - Infinitely-Many Solutions



Properties to Ensure A Unique Solution to $A\vec{x} = \vec{b}$

If A is an n by n square matrix, then the following conditions are equivalent and imply a unique solution to

$$A\vec{x} = \vec{b}:$$

- The matrix A has an inverse (is *invertible*)
- The *determinant* of A is nonzero
- The rows and columns of A form a *basis* for the set of all vectors with n elements

Properties to Ensure A Unique Solution to $A\vec{x} = \vec{b}$

```
A
     [,1] [,2]
[1,]    1   -2
[2,]    0    4
```

- Computing the Inverse of A (if it Exists)

```
solve(A)
   [,1] [,2]
[1,]    1 0.50
[2,]    0 0.25
```

- Computing the Determinant of A

```
det(A)
[1] 4
```

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Solving Matrix-Vector Equations

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Solving Matrix-Vector Equations

$$
\begin{aligned}
5x &= 7 \\
\frac{1}{5}5x &= \frac{1}{5}7 \\
1x &= \frac{7}{5} \\
x &= \frac{7}{5}
\end{aligned}
$$

Solving Matrix-Vector Equations

$$
\begin{aligned}
A\vec{x} &= \vec{b} \\
A^{-1}A\vec{x} &= A^{-1}\vec{b} \\
I\vec{x} &= A^{-1}\vec{b} \\
\vec{x} &= A^{-1}\vec{b}
\end{aligned}
$$

Solving Matrix-Vector Equations

```
A
     [,1] [,2]
[1,]    1   -2
[2,]    0    4

b
[1]  1 -2
```

Solving $A\vec{x} = \vec{b}$ using $\vec{x} = A^{-1}\vec{b}$:

```
x <- solve(A)%*%b

print(x)
     [,1]
[1,]  0.0
[2,] -0.5
```

Solving Matrix-Vector Equations

```
x <- solve(A)%*%b

print(x)
      [,1]
[1,]   0.0
[2,]  -0.5
```

Checking your solution by plugging in the solution $\vec{x}$:

```
A%*%x
     [,1]
[1,]    1
[2,]   -2
```

Which is equal to the given $\vec{b}$:

```
print(b)
[1]  1 -2
```

Additional Conditions for Unique Solutions

$$\begin{aligned} A\vec{x} &= \vec{0} \\ A^{-1}A\vec{x} &= A^{-1}\vec{0} \\ I\vec{x} &= A^{-1}\vec{0} \\ \vec{x} &= A^{-1}\vec{0} \\ \vec{x} &= \vec{0} \end{aligned}$$

Thus, the only solution to the *homogeneous* equation $A\vec{x} = \vec{0}$ is the *trivial* solution $\vec{x} = \vec{0}$.

Additional Conditions for Unique Solutions

```
A
      [,1] [,2]
[1,]     1   -2
[2,]     0    4

b <- rep(0, 2)
print(b)
[1]   0  0
```

```
> solve(A)%*%b
      [,1]
[1,]     0
[2,]     0
```

Conditions for a Unique Solution to Matrix-Vector Equations

If A is an n by n square matrix, then the following conditions are equivalent and imply a unique solution to

$$A\vec{x} = \vec{b}:$$

- The matrix A has an inverse (is *invertible*)
- The *determinant* of A is nonzero
- The rows and columns of A form a *basis* for the set of all vectors with n elements
- The homogeneous equation $A\vec{x} = \vec{0}$ has just the trivial ($\vec{x} = 0$) solution

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Other Considerations for Matrix-Vector Equations

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More Equations than Unknowns

$$2x_1 + 3x_2 = 7$$
$$-x1 + 4x_2 = 1$$
$$x_1 + 7x_2 = 0$$

$$\begin{pmatrix} 2 & 3 \\ -1 & 4 \\ 1 & 7 \end{pmatrix} \times \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} =$$

$$x_1 \times \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} + x_2 \times \begin{pmatrix} 3 \\ 4 \\ 7 \end{pmatrix} = \begin{pmatrix} 7 \\ 1 \\ 0 \end{pmatrix}$$

More Equations than Unknowns

$$2x_1 + 3_x2 = 7$$
$$-x1 + 4x_2 = 1$$
$$x_1 + 7x_2 = 0$$

$$\begin{pmatrix} 2 & 3 \\ -1 & 4 \\ 1 & 7 \end{pmatrix} \times \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} =$$

$$x_1 \times \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} + x_2 \times \begin{pmatrix} 3 \\ 4 \\ 7 \end{pmatrix} = \begin{pmatrix} 7 \\ 1 \\ 0 \end{pmatrix}$$

Fewer Equations than Unknowns

$$x_1 - x2 + 4x_3 = 0$$
$$2x_1 + 5x_2 + 7x_3 = 1$$

$$\begin{pmatrix} 1 & -1 & 4 \\ 2 & 5 & 7 \end{pmatrix} \times \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} =$$

$$x_1 \times \begin{pmatrix} 1 \\ 2 \end{pmatrix} + x_2 \times \begin{pmatrix} -1 \\ 5 \end{pmatrix} + x_3 \times \begin{pmatrix} 4 \\ 7 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Some Options for Non-Square Matrices

- Row Reduction (By Hand, Difficult for Big Problems)
- Least Squares (If More Rows Than Columns - Used in Linear Regression)
- Singular Value Decomposition (If More Columns Than Rows - Used in Principal Component Analysis)
- Generalized or Pseudo-Inverse

Moore-Penrose Generalized Inverse

```
library(MASS)
A
     [,1] [,2]
[1,]    2    3
[2,]   -1    4
[3,]    1    7
```

```
ginv(A)
          [,1]        [,2]       [,3]
[1,] 0.3333333 -0.30303030 0.03030303
[2,] 0.0000000  0.09090909 0.09090909
```

```
ginv(A)%*%A
     [,1]          [,2]
[1,]    1 -1.110223e-16
[2,]    0  1.000000e+00

A%*%ginv(A)
           [,1]       [,2]      [,3]
[1,]  0.6666667 -0.3333333 0.3333333
[2,] -0.3333333  0.6666667 0.3333333
[3,]  0.3333333  0.3333333 0.6666667
```

Moore-Penrose Generalized Inverse

```
A
     [,1] [,2]
[1,]    2    3
[2,]   -1    4
[3,]    1    7

b
[1] 1 7 8
```

`{r} <- ginv(A)%*%b A%*%x [,1] [1,] 1 [2,] 7 [3,] 8` {{2}}

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Let's Practice