LINEAR ALGEBRA FOR DATA SCIENCE IN R

Eigenvalues and Eigenvectors

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Motivation - Face Recognition

Matrix-Vector Multiplication

- Rotations
- Reflections
- Dilations
- Contractions
- Projections
- Every Imaginable Combination of These

Scalar Multiplication

The scalar multiplication of c times the vector $\vec{x}$ is denoted by:

$$c\vec{x}$$

which simply multiplies each element of $\vec{x}$ by c.

Scalar Multiplication

```
> x
[1] 3 2 3
```

```
> c <- 4
```

```
> c*x
[1] 12  8 12
```

Scalar Multiplication Achieved Through Matrix Multiplication

Scalar multiplication can be replicated by a special matrix multiplication:

$$cI\vec{x} = c\vec{x}$$

However, there are many other matrices that, when applied to the correct vector or collection of vectors, act exactly like scalar multiplication.

These scalars and vectors are called eigenvalues and eigenvectors!

Scalar Multiplication Achieved Through Matrix Multiplication

```
> A
> print(A)
     [,1] [,2] [,3]
[1,]    3    0    0
[2,]    0    3    0
[3,]    0    0    3
>
> x <- c(3, 2, 3)
```

```
> A%*%x
     [,1]
[1,]    9
[2,]    6
[3,]    9
```

```
> 3*x
[1] 9 6 9
```

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Let's practice!

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Eigenvalue/Eigenvector Definition

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Definition

For a matrix A, the the scalar λ is an *eigenvalue* of A, with *associated eigenvector* $\vec{v} \neq \vec{0}$ if the following equation is true:

$$A\vec{v} = \lambda\vec{v}.$$

In other words:

The matrix multiplication $A\vec{v}$, a matrix-vector operation, produces the same vector as $\lambda\vec{v}$ a scalar multiplication acting on a vector.

This matrix *does not* have to be like the matrices in the last lecture.

Example

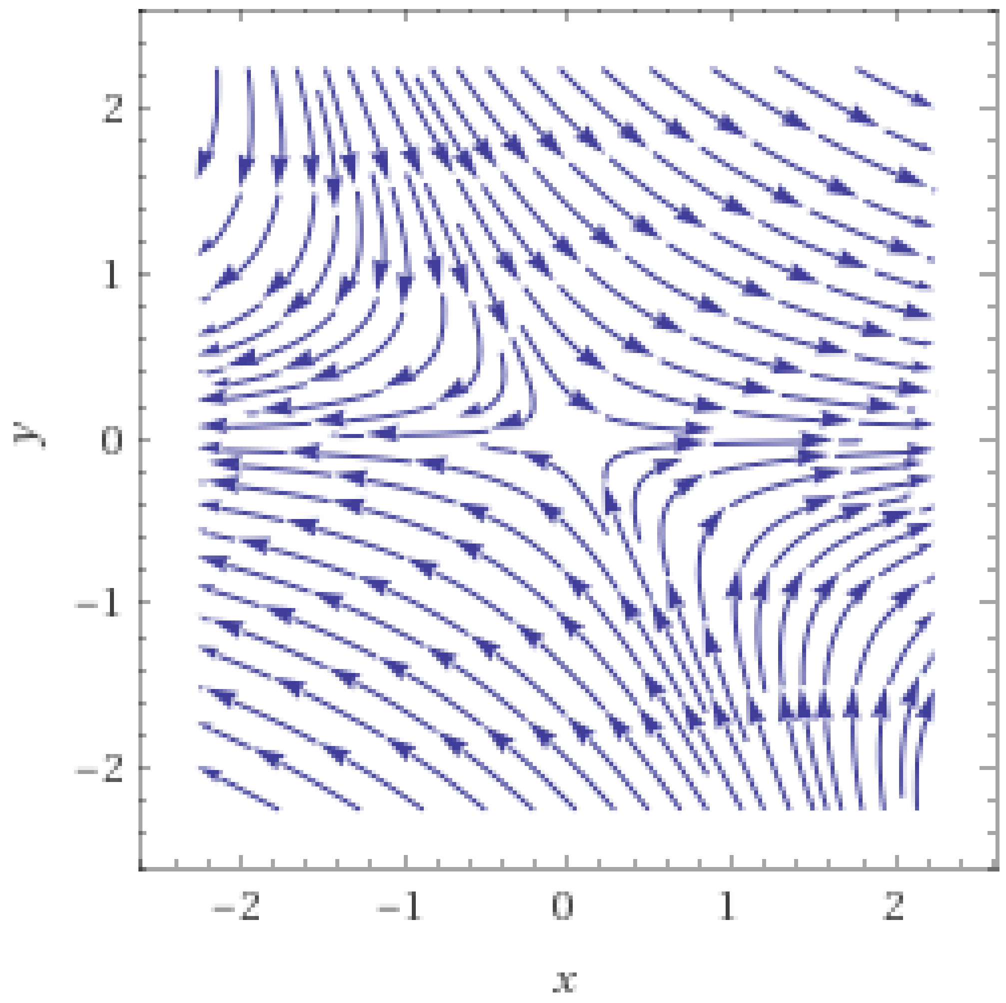
```
> A
     [,1] [,2]
[1,]    2    3
[2,]    0    1
```

Notice that $\lambda = 2$ is an eigenvalue of A with eigenvector $\vec{v} = (1,0)^T$:

```
> A%*%c(1,0)
     [,1]
[1,]    2
[2,]    0
```

```
> 2*c(1, 0)
[1] 2 0
```

Geometric Motivation



Example, cont'd

Notice that $\lambda = 2$ is an eigenvalue of A with eigenvector $\vec{v} = (1, 0)^T$ *and* $\vec{v} = (4, 0)^T$:

```
> A%*%c(1,0)
     [,1]
[1,]    2
[2,]    0
>
> 2*c(1, 0)
[1] 2 0
```

```
> A%*%c(4,0)
     [,1]
[1,]    8
[2,]    0
>
> 2*c(4, 0)
[1] 8 0
```

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Solving Eigenvalue/Eigenvector Problems

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Properties of Solutions to Eigenvalue/Eigenvector Problems

- An n by n matrix A has, up to multiplicity, n eigenvalues.
- Even if A is a matrix consisting entirely of real numbers, some (or all) of its eigenvalues could be complex numbers.
- All complex eigenvalues must come in conjugate pairs, though, like $1 + 2i$ and $1 - 2i$.

Solving Eigenvalue/Eigenvector Problems in R

```
> A
     [,1] [,2] [,3]
[1,]   -1    2    4
[2,]    0    7   12
[3,]    0    0   -4
```

```
> eigen(A)
eigen() decomposition
$`values`
[1]  7 -4 -1

$vectors
          [,1]       [,2] [,3]
[1,] 0.2425356 -0.3789810    1
[2,] 0.9701425 -0.6821657    0
[3,] 0.0000000  0.6253186    0
```

Solving Eigenvalue/Eigenvector Problems in R

```
> A
     [,1] [,2] [,3]
[1,]   -1    2    4
[2,]    0    7   12
[3,]    0    0   -4
```

Extracting eigenvalue and eigenvector information:

```
> E <- eigen(A)
> E$values[1]
[1] 7
> E$vectors[, 1]
[1] 0.2425356 0.9701425 0.0000000
```

Example with Complex Eigenvalues

```
> A
     [,1] [,2]
[1,]    1    2
[2,]   -2   -1
```

```
> eigen(A)
eigen() decomposition
$`values`
[1] 0+1.732051i 0-1.732051i

$vectors
                      [,1]                   [,2]
[1,]  0.3535534+0.6123724i  0.3535534-0.6123724i
[2,] -0.7071068+0.0000000i -0.7071068+0.0000000i
```

```
> eigen(A)$values[1]*eigen(A)$values[2]
[1] 3+0i
```

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Some More on Eigenvalues and Eigenvectors

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What's Going On?

- If the eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$ of A are *distinct* and $\vec{v}_1, \vec{v}_2, ..., \vec{v}_n$ is a set of associated eigenvectors, then this set of vectors forms a *basis* for the set of n-dimensional vectors.

- In other words, if we assume that the matrix A has a basis of eigenvectors $\vec{v}_1, \vec{v}_2, ...\vec{v}_n$, with associated, distinct eigenvalues $\lambda_1, \lambda_2, ...\lambda_n$, then *every* n-dimensional vector can be expressed as a linear combination of these vectors, i.e.

$$\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2 + ... + c_n\vec{v}_n.$$

What's Going On?

Applying the matrix A to $\vec{x}$, and using the fact that $A\vec{v}_j = \lambda_j \vec{v}_j$, notice the following simple decomposition

$$A\vec{x} = c_1\lambda_1\vec{v}_1 + c_2\lambda_2\vec{v}_2 + ... + c_n\lambda_n\vec{v}_n.$$

Hence, eigenpairs turn matrix multiplication into a linear combination of scalar multiplications!

Iterating the Matrix

If we iteratively multiply by the matrix A:

$$AA\vec{x} =$$
$$= A(c_1\lambda_1\vec{v}_1 + c_2\lambda_2\vec{v}_2 + ... + c_n\lambda_n\vec{v}_n)$$
$$= c_1\lambda_1^2\vec{v}_1 + c_2\lambda_2^2\vec{v}_2 + ... + c_n\lambda_n^2\vec{v}_n,$$

or, in general:

$$A^t\vec{x} = c_1\lambda_1^t\vec{v}_1 + c_2\lambda_2^t\vec{v}_2 + ... + c_n\lambda_n^t\vec{v}_n.$$

Thus, successive matrix multiplication is not successive scalar multiplication (exponentiation)!

Also, if one of the eigenvalues is larger than all of the others, these differences will be exacerbated as t grows.

Example with Allele Frequencies

$$
\begin{array}{c c}
 & \begin{array}{cccc} \mathrm{A} & \mathrm{T} & \mathrm{C} & \mathrm{G} \end{array} \\
\begin{array}{c} \mathrm{A} \\ \mathrm{T} \\ \mathrm{C} \\ \mathrm{G} \end{array} &
\begin{pmatrix} 0.980 & 0.005 & 0.005 & 0.010 \\ 0.005 & 0.980 & 0.010 & 0.005 \\ 0.005 & 0.010 & 0.980 & 0.005 \\ 0.010 & 0.005 & 0.005 & 0.980 \end{pmatrix}
\end{array}
\times \begin{pmatrix} p_A \\ p_T \\ p_C \\ p_G \end{pmatrix}_t = \begin{pmatrix} p_A \\ p_T \\ p_C \\ p_G \end{pmatrix}_{t+1}
$$

Example with Allele Frequencies

```
> M
      [,1]  [,2]  [,3]  [,4]
[1,] 0.980 0.005 0.005 0.010
[2,] 0.005 0.980 0.010 0.005
[3,] 0.005 0.010 0.980 0.005
[4,] 0.010 0.005 0.005 0.980
```

```
> eigen(M)
eigen() decomposition
$`values`
[1] 1.00 0.98 0.97 0.97

$vectors
     [,1] [,2]          [,3]          [,4]
[1,] -0.5  0.5  0.000000e+00  7.071068e-01
[2,] -0.5 -0.5 -7.071068e-01  1.132427e-14
[3,] -0.5 -0.5  7.071068e-01 -2.442491e-15
[4,] -0.5  0.5 -1.382228e-14 -7.071068e-01
```

Example with Allele Frequencies

```
> M
      [,1]  [,2]  [,3]  [,4]
[1,] 0.980 0.005 0.005 0.010
[2,] 0.005 0.980 0.010 0.005
[3,] 0.005 0.010 0.980 0.005
[4,] 0.010 0.005 0.005 0.980
```

```
> Lambda <- eigen(M)
> v1 <- Lambda$vectors[, 1]/sum(Lambda$vectors[, 1])
> v1
[1] 0.25 0.25 0.25 0.25
```

$$\begin{pmatrix} p_A \\ p_T \\ p_C \\ p_G \end{pmatrix}_{\infty} = \begin{pmatrix} 0.25 \\ 0.25 \\ 0.25 \\ 0.25 \end{pmatrix}$$

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