

Random Variables and Expectation

Random Variable

A Random Variable is a variable that probabilistically takes on different values. You can also think of it similarly to how you think about variables in programming languages. They have type and domains over which they are applicable. Importantly we can ask the probability of the event that a random variable takes on a particular value.

As an example, let's say we flip three fair coins. Let Y be a random variable that is equal to the number of "heads" on the three coins. We can ask about the probability of Y taking on different values like so:

- $P(Y = 0) = 1/8$ (T, T, T)
- $P(Y = 1) = 3/8$ (H, T, T), (T, H, T), (T, T, H)
- $P(Y = 2) = 3/8$ (H, H, T), (H, T, H), (T, H, H)
- $P(Y = 3) = 1/8$ (H, H, H)
- $P(Y \geq 4) = 0$
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Using random variables is a convenient notation technique that assists in decomposing problems. There are many different types of random variables (indicator, binary, choice, bernoulli, etc). The two main families of random variable types are discrete and continuous.

Probability Mass Function

Probability mass functions (PMF) allow us to ask what is the relative probability of a random variable taking on a particular value. For discrete random variables the PMF is simply a mapping from values to their probabilities. We can plot PMF graphs:

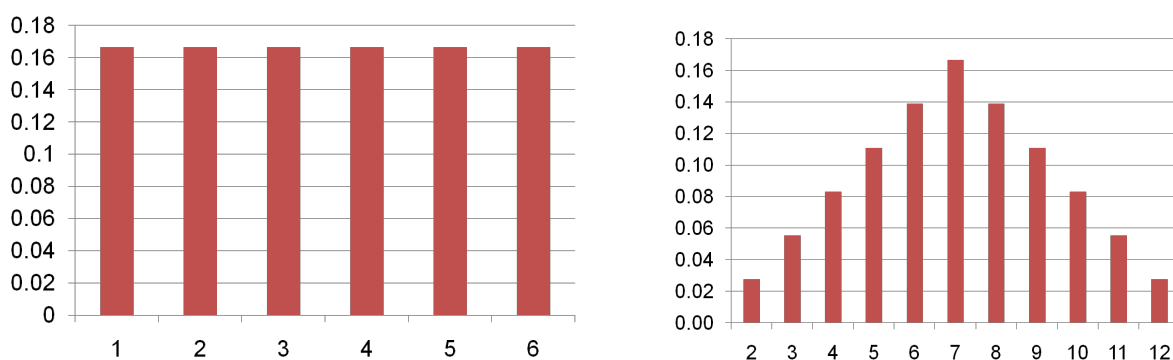


Figure: On the left, the PMF of a single 6 sided dice roll. On the right, the PMF of the sum of two dice rolls.

Cumulative Distribution Function

For a random variable X , the Cumulative Distribution Function (CDF) is defined as:
 $F(a) = P(X \leq a)$ where $-\infty < a < \infty$

Expected Value

The Expected Values for a discrete random variable X is defined as:

$$E[X] = \sum_{x:P(x)>0} xP(x)$$

It goes by many other names: Mean, Expectation, Weighted Average, Center of Mass, 1st Moment.

Example 1

Lets say you roll a 6-Sided Die and that a random variable X represents the outcome of the roll. What is the $E[X]$? This is the same as asking what is the average value.

$$E[X] = 1 (1/6) + 2 (1/6) + 3(1/6) + 4 (1/6) + 5 (1/6) + 6(1/6) = 7/2$$

Example 2

As an example, lets say a school has 3 classes with 5, 10, and 150 students. If we randomly choose a class with equal probability and let X = size of the chosen class:

$$\begin{aligned} E[X] &= 5 (1/3) + 10(1/3) + 150(1/3) \\ &= 165/3 = 55 \end{aligned}$$

If instead we randomly choose a student with equal probability and let Y = size of the class the student is in

$$\begin{aligned} E[Y] &= 5 (5/165) + 10 (10/165) + 150 (150/165) \\ &= 22635/165 = 137 \end{aligned}$$

Example 3

Consider a game played with a fair coin which comes up heads with $p = 0.5$. Let n = the number of coin flips before the first “tails”. In this game you win $\$2^n$. How many dollars do you expect to win? Let X be a random variable which represents your winnings.

$$E[X] = \left(\frac{1}{2}\right)^1 2^0 + \left(\frac{1}{2}\right)^2 2^1 + \left(\frac{1}{2}\right)^3 2^2 + \left(\frac{1}{2}\right)^4 2^3 + \dots = \sum_{i=0}^{\infty} \left(\frac{1}{2}\right)^{i+1} 2^i$$

$$E[X] = \sum_{i=0}^{\infty} \frac{1}{2} = \infty$$

Linearity

Expectations preserve linearity which means that $E[aX + b] = aE[X] + b$

Indicator Random Variable

A variable I is called an indicator variable for an event A if $I = 1$ if A occurs and $I = 0$ if A does not occur. $P(I = 1) = P(A)$ and $E[I] = P(A)$. Indicator variables are useful for cleaning up notation.

Disclaimer: This handout was made fresh just for you. Did you notice any mistakes? Let Chris know and he will fix them.