vwio ESKF Theory

1 Kalman Filter Formula Review

1.1 Construct Predict Model And Mesure Model Of System

1) Predict Model Equations

$$x_t = F_t x_{t-1} + B_t u_{t-1} + w_k$$

$$w_k \sim \mathcal{N}(0, Q)$$

2) Mesure Model Equations

$$z_t = H x_t + v_k$$

$$v_k \sim \mathcal{N}(0, R)$$

1.2 Predict Step

1) Priori Estimate

$$\hat{x}_t^- = F_t \hat{x}_{t-1} + B_t u_{t-1}$$

2) Priori Status Estimate Covariance

$$P_t^- = F_t P_{t-1} F_t^T + Q$$

1.3 Status Update Step

1) Kalman Gein

$$K_t = \frac{P_t^- H^T}{H P_t^- H^T + R}$$

2) Posterior Estimate

$$\hat{x}_t = \hat{x}_t^- + K_t (z_t - H \hat{x}_t^-)$$

3) Posterior Estimate Covariance

$$P_t = (I - K_t H) P_t^-$$

2 Analysis Of Algorithms ESKF For WO & IMU & VO

2.1 ESKF Init Step

1. Set **intial position value** from WO(Wheel Odometry) according to the formula (1).

$$\mathbf{p}_t = [\mathbf{p}_{xt}^{wo}, \mathbf{p}_{yt}^{wo}, \mathbf{p}_{zt}^{wo}]^{\mathrm{T}} \tag{1}$$

2. Initialize the **State Estimate Covariance** matrix and set the noise for the position/velocity/posture estimation of the robot's initial state according to the formula (2).

$$\mathbf{P}_t^{init} = \begin{pmatrix} \sigma_{\mathrm{p}}^2 \mathbf{I}_{3\times3} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \sigma_{\mathrm{v}}^2 \mathbf{I}_{3\times3} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \sigma_{\theta}^2 \mathbf{I}_{3\times3} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \sigma_{\mathrm{a}}^2 \mathbf{I}_{3\times3} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \sigma_{\omega}^2 \mathbf{I}_{3\times3} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \sigma_{g}^2 \mathbf{I}_{3\times3} \end{pmatrix} \in \mathbb{R}^{18\times18} \tag{2}$$

Where $\sigma_{\mathrm{p}}^2 = 1.2$, $\sigma_{\mathrm{v}}^2 = 10.0$ and $\sigma_{\theta}^2 = 1.0$ while $\sigma_a^2 = \sigma_\omega^2 = \sigma_g^2 = 0$.

2.2 ESKF Predict Step

2.2.1 Construct Predict Model Of System

1. Set the **ESKF truth state** to the formula (3)

$$\mathbf{x}_t = [\mathbf{p}_t, \mathbf{v}_t, \mathbf{R}_t, \mathbf{b}_{at}, \mathbf{b}_{\omega t}, \mathbf{g}_t]^{\mathrm{T}} \tag{3}$$

2. Set the **ESKF error state** to the formula (4)

$$\delta\mathbf{x} = \begin{bmatrix} \delta\mathbf{p} \\ \delta\mathbf{v} \\ \delta\boldsymbol{\theta} \\ \delta\mathbf{b}_a \\ \delta\mathbf{b}_\omega \\ \delta\mathbf{g} \end{bmatrix} \in \mathbb{R}^{18\times 1} \tag{4}$$

3. State transition equation in discrete time to formula set (5).

$$\begin{cases} \mathbf{p}(t+\Delta t) = \mathbf{p}(t) + \mathbf{v}\Delta t + \frac{1}{2}\left(\mathbf{R}(\tilde{\mathbf{a}} - \mathbf{b}_a)\right)\Delta t^2 + \frac{1}{2}\mathbf{g}\Delta t^2 \\ \mathbf{v}(t+\Delta t) = \mathbf{v}(t) + \mathbf{R}(\tilde{\mathbf{a}} - \mathbf{b}_a)\Delta t + \mathbf{g}\Delta t \\ \mathbf{R}(t+\Delta t) = \mathbf{R}(t)\mathrm{Exp}\left((\tilde{\boldsymbol{\omega}} - \mathbf{b}_\omega)\Delta t\right) \\ \mathbf{b}_a(t+\Delta t) = \mathbf{b}_a(t) \\ \mathbf{b}_\omega(t+\Delta t) = \mathbf{b}_\omega(t) \\ \mathbf{g}(t+\Delta t) = \mathbf{g}(t) \end{cases} \tag{5}$$

where $\mathbf{u}_m = [\tilde{\mathbf{a}}, \mathbf{b}_a, \tilde{\boldsymbol{\omega}}, \mathbf{b}_\omega]^{\mathrm{T}}$ **from the information of the IMU's linear acceleration and angular acceleration in the WIO odometer and their deviation values**.

4. Error state equation in discrete time to formula set (6).

$$\begin{cases} \delta\mathbf{p}(t+\Delta t) = \delta\mathbf{p} + \delta\mathbf{v}\Delta t \\ \delta\mathbf{v}(t+\Delta t) = \delta\mathbf{v} + \left(-\mathbf{R}(\tilde{\mathbf{a}} - \mathbf{b}_a)^\wedge\delta\boldsymbol{\theta} - \mathbf{R}\delta\mathbf{b}_a + \delta\mathbf{g}\right)\Delta t + \boldsymbol{\eta}_v \\ \delta\boldsymbol{\theta}(t+\Delta t) = \mathrm{Exp}\left(-(\tilde{\boldsymbol{\omega}} - \mathbf{b}_\omega)\Delta t\right)\delta\boldsymbol{\theta} - \delta\mathbf{b}_\omega\Delta t - \boldsymbol{\eta}_\theta \\ \delta\mathbf{b}_a(t+\Delta t) = \delta\mathbf{b}_a + \boldsymbol{\eta}_a \\ \delta\mathbf{b}_\omega(t+\Delta t) = \delta\mathbf{b}_\omega + \boldsymbol{\eta}_\omega \\ \delta\mathbf{g}(t+\Delta t) = \delta\mathbf{g} \end{cases} \tag{6}$$

5. $f(.)$ is the nonlinear state function of the system and $\mathbf{Q}$ is the **Variance of Process Noise matrix**. The motion equations of Error status $\delta\mathbf{x}$ in Discrete Time to formula (7).

$$\delta\mathbf{x}_{(t+\Delta t)} = f(\delta\mathbf{x}) + \mathbf{w}, \mathbf{w} \sim \mathcal{N}(0, \mathbf{Q}) \tag{7}$$

$$\mathbf{Q} = \begin{pmatrix} \Delta t^2\sigma_a^2\mathbf{I}_{3\times 3} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \Delta t^2\sigma_\omega^2\mathbf{I}_{3\times 3} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \Delta t\sigma_{ba}^2\mathbf{I}_{3\times 3} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \Delta t\sigma_{b\omega}^2\mathbf{I}_{3\times 3} \end{pmatrix} \in \mathbb{R}^{12\times 12} \tag{8}$$

Where $\sigma_a^2 = \sigma_\omega^2 = \sigma_{ba}^2 = \sigma_{b\omega}^2 = 5 \times 10^{-4}$.

6. Linearization of Equations of Motion in Discrete Time to formula (9)

$$\delta\mathbf{x} = \mathbf{F_x}\delta\mathbf{x} + \mathbf{F_i}\mathbf{w} \tag{9}$$

where F_x and F_i equal to formula (10), formula(11) respectively.

$$\mathbf{F_x} = \left.\frac{\partial f}{\partial \delta\mathbf{x}}\right|_{\mathbf{x},\mathbf{u}_m} = \begin{bmatrix} \mathbf{I} & \mathbf{I}\Delta t & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & -\mathbf{R}(\tilde{\mathbf{a}} - \mathbf{b}_a)^\wedge\Delta t & -\mathbf{R}\Delta t & \mathbf{0} & \mathbf{I}\Delta t \\ \mathbf{0} & \mathbf{0} & \mathrm{Exp}\left(-(\tilde{\boldsymbol{\omega}} - \mathbf{b}_\omega)\Delta t\right) & \mathbf{0} & -\mathbf{I}\Delta t & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix} \in \mathbb{R}^{18\times 18} \tag{10}$$

$$\mathbf{F_i} = \frac{\partial f}{\partial \mathbf{i}}\bigg|_{\mathbf{x},\mathbf{u}_m} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \mathbf{I} & 0 & 0 & 0 \\ 0 & \mathbf{I} & 0 & 0 \\ 0 & 0 & \mathbf{I} & 0 \\ 0 & 0 & 0 & \mathbf{I} \\ 0 & 0 & 0 & 0 \end{bmatrix} \in \mathbb{R}^{18\times 12} \tag{11}$$

2.2.2 Priori Estimate

$$\delta\hat{\mathbf{x}}_t^- = \mathbf{F_x}\delta\mathbf{x}_t \tag{12}$$

2.2.3 Priori State Estimate Covariance

$$\mathbf{P}_t^- = \mathbf{F_x}\mathbf{P}_t{\mathbf{F_x}}^{\mathrm{T}} + \mathbf{F_i}\mathbf{Q}{\mathbf{F_i}}^{\mathrm{T}} \tag{13}$$

2.3 ESKF Correct Step

2.3.1 Construct The Observe Model Of The System

1. Assume that the **visual sensor (Visual Odometry)** can observe state variables, and its nonlinear observation function is $h(.)$, can be written as the formula (14), where $\mathbf{z} = [\mathbf{p}_{xt}^{vo}, \mathbf{p}_{yt}^{vo}, \mathbf{p}_{zt}^{vo}]^{\mathrm{T}}$ is the **observation data** from the visual odometry, v is the observation noise, V is the **Variance of Collection Noise matrix**.

$$\mathbf{z} = h(\mathbf{x}) + \mathbf{v}, \mathbf{v} \sim \mathcal{N}(0, \mathbf{V}) \tag{14}$$

$$\mathbf{V} = \begin{pmatrix} \sigma^2_{\text{observe}} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \sigma^2_{\text{observe}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \sigma^2_{\text{observe}} \end{pmatrix} \in \mathbb{R}^{3\times 3},\ (\sigma^2_{observe} = 1.2) \tag{15}$$

In the traditional **Extended Kalman Filter (EKF)**, the observation equation is linearized, **the Jacobian matrix of the observation equation with respect to the state variable is found**, and then the Kalman filter is updated.

In the **Error State Kalman Filter (ESKF)**, an estimate of the nominal state $\mathbf{x}$ and an estimate of the error state $\delta\mathbf{x}$ are available. To update the error state, **the Jacobian matrix H of the observation equation with respect to the error state is calculated**.

$$\mathbf{H} = \frac{\partial h}{\partial \delta\mathbf{x}} = \frac{\partial h}{\partial \mathbf{x}} \cdot \frac{\partial \mathbf{x}}{\partial \delta\mathbf{x}} \tag{16}$$

2.3.2 Calcurate Klaman Gein

$$\mathbf{K} = \frac{\mathbf{P}_t^- \cdot \mathbf{H}^T}{\mathbf{H}\cdot\mathbf{P}_t^- \cdot \mathbf{H}^T + \mathbf{V}} \tag{17}$$

2.3.3 Posterior Estimate

$$\delta\hat{\mathbf{x}}_t = \mathbf{K}(\mathbf{z} - \mathbf{H}\cdot\delta\hat{\mathbf{x}}_t^-) \tag{18}$$

2.3.4 Posterior State Estimate Covariance

$$\mathbf{P}_t = (\mathbf{I} - \mathbf{K}\mathbf{H})\mathbf{P}_t^- \tag{19}$$

2.4 ESKF State Update

The true state $\mathbf{x}_t$ equal to the **Combination Operations** of nomal state $\mathbf{x}$ and error state $\delta\hat{x}_t$.

$$\mathbf{x}_t = \mathbf{x} \oplus \delta\hat{\mathbf{x}}_t \tag{20}$$

The formula (15) equal to the formula set (16) below.

$$\begin{cases} \mathbf{p}_t = \mathbf{p} + \delta\mathbf{p}_t \\ \mathbf{v}_t = \mathbf{v} + \delta\mathbf{v}_t \\ \mathbf{R}_t = \mathbf{R}\mathrm{Exp}(\delta\boldsymbol{\theta}_t) \\ \mathbf{b}_{a,t} = \mathbf{b}_a + \delta\mathbf{b}_{a,t} \\ \mathbf{b}_{\omega,t} = \mathbf{b}_\omega + \delta\mathbf{b}_{\omega,t} \\ \mathbf{g}_t = \mathbf{g} + \delta\mathbf{g}_t \end{cases} \tag{21}$$

2.5 ESKF Error State Reset

The error state δx will be **RESET to ZERO** after ESKF State Update according to the formula (22) below.

$$\delta \mathbf{x}_t = \begin{bmatrix} \delta \mathbf{p}_t \\ \delta \mathbf{v}_t \\ \delta \boldsymbol{\theta}_t \\ \delta \mathbf{b}_{a,t} \\ \delta \mathbf{b}_{\omega,t} \\ \delta \mathbf{g}_t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{22}$$