P.8

$$A = \begin{pmatrix} 5 & 1 & 6 \\ 1 & 2 & 4 \\ 6 & 4 & 12 \end{pmatrix}$$

a) find A's eigenvalues = the solutions of the equation

$$|A - \lambda I| = 0$$

$$A - \lambda I = \begin{pmatrix} 5-\lambda & 1 & 6 \\ 1 & 2-\lambda & 4 \\ 6 & 4 & 12-\lambda \end{pmatrix}$$

$$|A - \lambda I| = (5-\lambda)(2-\lambda)(12-\lambda) + 24 + 24 - 36(2-\lambda) - 16(5-\lambda) - (12-\lambda) =$$

$$= 120 - 10\lambda - 60\lambda + 5\lambda^2 - 24\lambda + 2\lambda^2 + 12\lambda^2 - \lambda^3 + 48 - 72 + 36\lambda - 80 + 16\lambda - 12 + \lambda =$$

$$= -\lambda^3 + 19\lambda^2 - 41\lambda + 4$$

$\hookrightarrow$ the solutions of this equation are

$\lambda_1 \approx 16{,}535$

$\lambda_2 \approx 2{,}363$ $\Rightarrow$ all eigenvalues are positive

$\lambda_3 \approx 0{,}102$

$\Downarrow$

A is positive (semi-)definite

b) let's compute v_1 $\quad A v_1 = \lambda_1 v_1 \quad v_1 = (x, y, z)^T$

$$\begin{pmatrix} 5x + y + 6z \\ x + 2y + 4z \\ 6x + 4y + 12z \end{pmatrix} = \begin{pmatrix} 16{,}535x \\ 16{,}535y \\ 16{,}535z \end{pmatrix} \Rightarrow z = \frac{6x + 4y}{4{,}535} \Rightarrow$$

$$\Rightarrow x + 2y + \frac{24x + 16y}{4{,}535} = 16{,}535y$$

$$\frac{28{,}535x}{4{,}535} = \left(14{,}535 - \frac{16}{4{,}535}\right) y$$

$$y = \frac{28{,}535x}{4{,}535 \cdot 14{,}535 - 16} \Rightarrow \begin{pmatrix} 1 \\ 0{,}5717 \\ 1{,}8273 \end{pmatrix}$$

is an eigenvector

P.9

$$M = \begin{pmatrix} 0 & 9 & 0 \\ 16 & 0 & 0 \end{pmatrix}$$

a) SVD: $$M^T M = \begin{pmatrix} 0 & 16 \\ 9 & 0 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 9 & 0 \\ 16 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 256 & 0 & 0 \\ 0 & 81 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

the eigenvalues of this matrix are trivially

$$\lambda_1 = 256, \quad \lambda_2 = 81 \left(, \quad \lambda_3 = 0 \right)$$

for v_1 we need $M^T M v_1 = \lambda v_1$

$$\begin{pmatrix} 256x \\ 81y \\ 0 \end{pmatrix} = \begin{pmatrix} 256x \\ 256y \\ 256z \end{pmatrix} \Rightarrow v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$v_2: \quad \begin{pmatrix} 256x \\ 81y \\ 0 \end{pmatrix} = \begin{pmatrix} 81x \\ 81y \\ 81z \end{pmatrix} \Rightarrow v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\left(v_3: \quad \begin{pmatrix} 256x \\ 81y \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right)$$

$$V = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\sigma_1 = 16, \quad \sigma_2 = 9$$

$$\Downarrow$$

$$\Sigma = \begin{pmatrix} 16 & 0 \\ 0 & 9 \end{pmatrix}$$

$$u_1 = \frac{1}{\sigma_1} M v_1 = \frac{1}{16} \begin{pmatrix} 0 & 9 & 0 \\ 16 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$u_2 = \frac{1}{\sigma_2} M v_2 = \frac{1}{9} \begin{pmatrix} 0 & 9 & 0 \\ 16 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\Downarrow$$

$$U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

So $$M = U \Sigma V^T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 16 & 0 \\ 0 & 9 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\mathcal{L} = D^{-\frac{1}{2}} L D^{-\frac{1}{2}}$$

$$D^{-\frac{1}{2}} = \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}^{-\frac{1}{2}} \Rightarrow \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}$$

$$\Downarrow$$

$$\mathcal{L} = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix} \cdot \begin{pmatrix} 4 & -3 & -1 & 0 \\ -3 & 4 & 0 & -1 \\ -1 & 0 & 4 & -3 \\ 0 & -1 & -3 & 4 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 & -\frac{3}{4} & -\frac{1}{4} & 0 \\ -\frac{3}{4} & 1 & 0 & -\frac{1}{4} \\ -\frac{1}{4} & 0 & 1 & -\frac{3}{4} \\ 0 & -\frac{1}{4} & -\frac{3}{4} & 1 \end{pmatrix}$$

the eigenvalues of this matrix $\mathcal{L}$: $0, \frac{1}{2}, 2, \frac{3}{2}$

the second smallest is $\frac{1}{2}$, let's find v_2:

$$\mathcal{L} v_2 = \frac{1}{2} v_2$$

$$\begin{cases} x - \frac{3}{4}y - \frac{1}{4}z = \frac{1}{2}x \\ -\frac{3}{4}x + y - \frac{1}{4}w = \frac{1}{2}y \\ -\frac{1}{4}x + z - \frac{3}{4}w = \frac{1}{2}z \\ -\frac{1}{4}y - \frac{3}{4}z + w = \frac{1}{2}w \end{cases} \Rightarrow w = \frac{1}{2}y + \frac{3}{2}z$$

$$\Rightarrow -\frac{1}{4}x + z - \frac{3}{8}y - \frac{9}{8}z = \frac{1}{2}z$$

$$-\frac{2}{5}x - \frac{3}{5}y = z$$

$$\Rightarrow x - \frac{3}{4}y + \frac{1}{10}x + \frac{3}{20}y = \frac{1}{2}x$$

$$x = y$$

So if $x = 1 \Rightarrow y = 1 \Rightarrow z = -1 \Rightarrow w = -1$

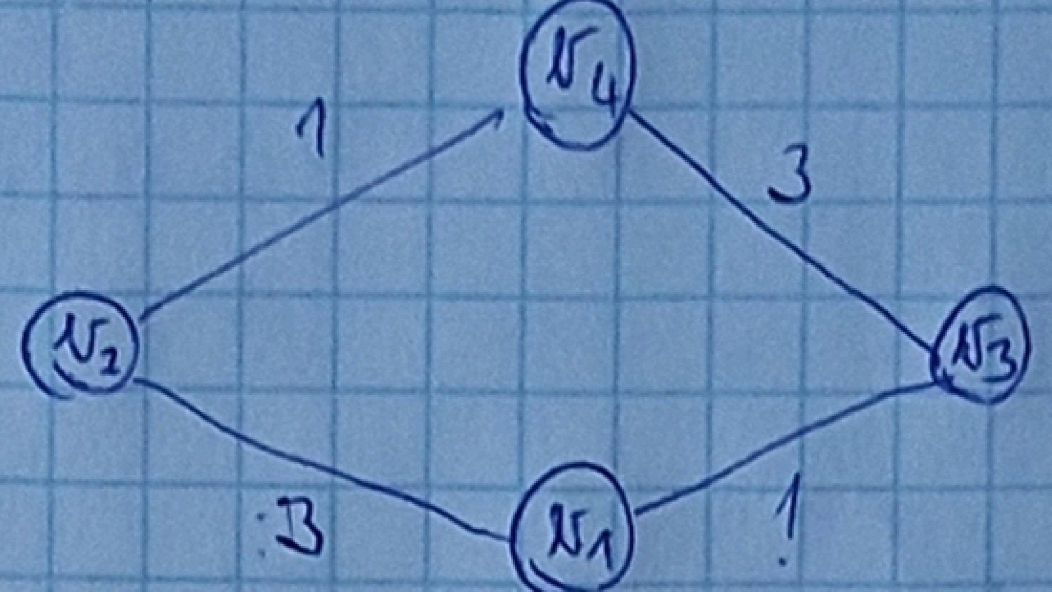
$$v_2 = \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix} \Rightarrow \text{our graph cut:}$$

$$A = \{v_1, v_2\}$$

$$B = \{v_3, v_4\}$$

$\text{Ncut}(A,B) = \frac{1}{2}$ same as before

P. 13 G:



$$W = \begin{pmatrix} 0 & 3 & 1 & 0 \\ 3 & 0 & 0 & 1 \\ 1 & 0 & 0 & 3 \\ 0 & 1 & 3 & 0 \end{pmatrix} \qquad D = \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}$$

$$L = D - W = \begin{pmatrix} 4 & -3 & -1 & 0 \\ -3 & 4 & 0 & -1 \\ -1 & 0 & 4 & -3 \\ 0 & -1 & -3 & 4 \end{pmatrix}$$

the eigenvalues of this L: $0, 2, 6, 8$

the second smallest is 2, let's find v_2:

$$L v_2 = 2 v_2$$

$$\begin{cases} 4x - 3y - z = 2x \\ -3x + 4y - w = 2y \\ -x + 4z - 3w = 2z \\ -y - 3z + 4w = 2w \end{cases} \Rightarrow w = \frac{y}{2} + \frac{3}{2} z$$

$$\Rightarrow -x + 4z - \frac{3}{2} y - \frac{3}{2} z = 2z$$

$$-\frac{2}{3} x - y = z$$

$$\Rightarrow 4x - 3y + \frac{2}{3} x + y = 2x$$

$$\frac{8}{3} x = 2y$$

So if $x = 1 \Rightarrow y = \frac{4}{3} \Rightarrow z = -2 \Rightarrow w = -\frac{7}{3}$

$$v_2 = \begin{pmatrix} 1 \\ \frac{4}{3} \\ -2 \\ -\frac{7}{3} \end{pmatrix} \Rightarrow$$ our graph cut:

$A = \{v_1, v_2\}$ $\quad cut(A,B) = 2$

$vol(A) = 4 + 4 = 8$

$B = \{v_3, v_4\}$ $\quad vol(B) = 4 + 4 = 8$

$$Ncut(A,B) = 2 \cdot \left(\frac{1}{8} + \frac{1}{8}\right) = \frac{1}{2}$$ seems reasonable

b) pseudo-inverse:

$$M^{\#} = V \Sigma^{-1} U^T$$

$$\Sigma = \begin{pmatrix} 16 & 0 \\ 0 & 9 \end{pmatrix} \Longrightarrow \Sigma^{-1} = \begin{pmatrix} \frac{1}{16} & 0 \\ 0 & \frac{1}{9} \end{pmatrix}$$

$$M^{\#} = V \cdot \Sigma^{-1} \cdot U^T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{16} & 0 \\ 0 & \frac{1}{9} \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{16} & 0 \\ 0 & \frac{1}{9} \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{16} \\ \frac{1}{9} & 0 \\ 0 & 0 \end{pmatrix}$$

P.11 $k: \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$

$$k(x,y) = (\langle x,y \rangle + 1)^2 = (x_1 \cdot y_1 + x_2 \cdot y_2 + 1)^2 = x_1^2 y_1^2 + x_2^2 y_2^2 + 1 + 2x_1y_1 + 2x_2y_2 + 2x_1y_1x_2y_2$$

$\Phi: \mathbb{R}^2 \to \mathbb{R}^6$

$\Phi(x_1, x_2) = (x_1^2, x_2^2, 1, \sqrt{2}x_1, \sqrt{2}x_2, \sqrt{2}x_1x_2)$ will lead to $k(x,y) = \langle \Phi(x), \Phi(y) \rangle$

$$\langle \Phi(x_1,x_2), \Phi(y_1,y_2) \rangle = (x_1^2, x_2^2, 1, \sqrt{2}x_1, \sqrt{2}x_2, \sqrt{2}x_1x_2) \cdot (y_1^2, y_2^2, 1, \sqrt{2}y_1, \sqrt{2}y_2, \sqrt{2}y_1y_2)^T =$$
$$= x_1^2 y_1^2 + x_2^2 y_2^2 + 1 + 2x_1y_1 + 2x_2y_2 + 2x_1y_1x_2y_2 = k(x,y)$$

P.12 $\mathcal{X} \neq \emptyset$

$k_i: \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ for $i = 1,2$ positive-definite kernels

as $\mathbb{K} = \mathbb{R}$, this is equivalent to

$\forall (x_j)_{j=1}^n \subseteq \mathcal{X}$ and $\forall (c_j)_{j=1}^n \subseteq \mathbb{R}$

$$\sum_{i=1}^n \sum_{j=1}^n c_i c_j k_i(x_i, x_j) \geq 0$$

but it is equivalent to finding $\Phi: \mathcal{X} \to \mathbb{R}^n : k(x,y) = \langle \Phi(x), \Phi(y) \rangle$

for k_1 we have Φ_1, for k_2 we have Φ_2

let $\Phi: \mathcal{X} \to \mathbb{R}^{n^2}$ such that Φ is denoted from 11 to nn and $\Phi_{ij}(x) = \Phi_1(x)_i \Phi_2(x)_j$

$$k(x,y) = k_1(x,y) \cdot k_2(x,y) = \Phi_1(x)^T \cdot \Phi(y) \cdot \Phi_2(x)^T \cdot \Phi_2(y) = \left(\sum_{i=1}^n \Phi_1(x)_i \cdot \Phi_1(y)_i\right) \left(\sum_{j=1}^n \Phi_2(x)_j \Phi_2(y)_j\right)$$
$$= \sum_{i=1}^n \sum_{j=1}^n (\Phi_1(x)_i \cdot \Phi_2(x)_j) \cdot (\Phi_1(y)_i \cdot \Phi_2(y)_j) = \sum_{i=1}^n \sum_{j=1}^n \Phi(x)_{ij} \Phi(y)_{ij} = \Phi(x)^T \cdot \Phi(y)$$

$\square$