



$$2.3) b) \theta \in (0, 1)$$

$n$  is fixed

$$\theta \rightarrow 0 \Rightarrow (1-\theta)^n \rightarrow 1 \Rightarrow P_\theta(\hat{\theta}_n = 0) \rightarrow 1 \Rightarrow$$

$$\Rightarrow P_\theta(\sqrt{\hat{\theta}_n \cdot (1-\hat{\theta}_n)} = 0) = P_\theta(\hat{\theta} = 0) \rightarrow 1 \Rightarrow$$

$$\Rightarrow P_\theta(CI_\alpha = [0, 0]) \rightarrow 1 \Rightarrow P_\theta(\theta \in CI_\alpha) \rightarrow 0$$

$$b) \text{cov}_\theta \left( \hat{\mu}_{n1} \frac{1}{n-1} \sum_{i=1}^n (x_i - \hat{\mu}_n)^2 \right) = \frac{n}{n-1} \text{cov}_\theta \left( \hat{\mu}_{n1} x_1^2 - 2 \hat{\mu}_n x_1 + \hat{\mu}_n^2 \right) *$$

$$\text{cov}_\theta(\hat{\mu}_{n1} x_1^2) = \frac{1}{n} \text{cov}_\theta(x_1 x_1^2) = \frac{1}{n} E(x_1^3) - \frac{1}{n} E(x_1) \cdot E(x_1^2) =$$

$$= \frac{1}{n} (E((x_1 - \bar{x})^3)) + 3 \bar{x} E(x_1^2) - 3 \bar{x}^2 E(x_1) + \bar{x}^3 - \bar{x} (\bar{x}^2 + \bar{x}^2) =$$

$$= \frac{1}{n} (0 + 3 \bar{x} (\bar{x}^2 + \bar{x}^2) - 3 \bar{x}^3 + \bar{x}^3 - \bar{x} \bar{x}^2 - \bar{x}^3) = \boxed{\frac{2 \bar{x} \sigma^2 n}{n}}$$

$$\text{cov}_\theta(\hat{\mu}_n, \hat{\mu}_n x_1) = \frac{1}{n^2} \text{cov}_\theta(x_1 + \dots + x_n, x_1^2 + x_1 x_2 + \dots + x_1 x_n) =$$

$$= \frac{1}{n^2} (\text{cov}_\theta(x_1, x_1^2) + (n-1) \text{cov}_\theta(x_1, x_1 x_2) + (n-1) \text{cov}_\theta(x_2, x_1 x_2)) =$$

$$= \frac{1}{n^2} (2 \bar{x} \sigma^2 + (n-2) \cdot \text{cov}_\theta(x_1, x_1 x_2)) = \boxed{\frac{2 \bar{x} \cdot \sigma^2 n}{n}}$$

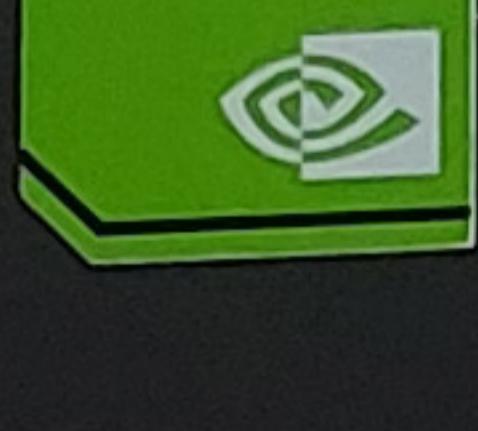
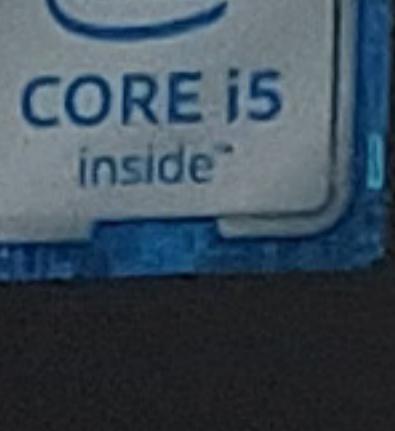
because:

$$\text{cov}_\theta(x_1, x_1 x_2) = E(x_1^2 \cdot x_2) - E(x_1) \cdot E(x_1 x_2) = E(x_1^2) \cdot E(x_1) - E^3(x_1) =$$

$$= (\bar{x}^2 + \bar{x}^2) \bar{x} - \bar{x}^3 = \bar{x}^2 \bar{x}$$

$$\text{cov}_\theta(\hat{\mu}_n, \hat{\mu}_n^2) = \boxed{\frac{2 \bar{x} \sigma^2}{n}}$$

$$* = \frac{n}{n-1} \cdot \left( \frac{2 \bar{x} \sigma^2}{n} - 2 \cdot \frac{2 \bar{x} \sigma^2}{n} + \frac{2 \bar{x} \sigma^2}{n} \right) = 0$$



## EX 2

$$1.2.) \text{ a) } X_1, \dots, X_n \sim \text{i.i.d}$$

$$\mathbb{E}(X_i) = N$$

$$\text{Var}(X_i) = \mathbb{E}(X_i^2) - \mathbb{E}^2(X_i) = \sigma^2 \Rightarrow \mathbb{E}(X_i^2) = \sigma^2 + N^2$$

$$i \neq j \quad \mathbb{E}(X_i \cdot X_j) = \mathbb{E}(X_i) \cdot \mathbb{E}(X_j) = N^2 \Rightarrow \text{i.i.d.}$$

i.i.d.

$$\mathbb{E}_\theta[\hat{\sigma}_n] = \mathbb{E}_\theta \left[ \frac{1}{n-1} \sum_{i=1}^n (X_i - \frac{1}{n} \sum_{i=1}^n X_i)^2 \right] \stackrel{\downarrow}{=} \frac{n}{n-1} \mathbb{E}_\theta \left[ X_1^2 - \frac{2}{n} X_1 \sum_{i=1}^n X_i + \frac{(\sum_{i=1}^n X_i)^2}{n^2} \right] =$$

$$= \frac{n}{n-1} \left( \mathbb{E}_\theta[X_1^2] - \frac{2}{n} \mathbb{E}_\theta[X_1 \sum_{i=1}^n X_i] + \frac{1}{n^2} \mathbb{E}_\theta[(\sum_{i=1}^n X_i)^2] \right) =$$

$$= \frac{n}{n-1} \left( \sigma^2 + n \sigma^2 + \sigma^2 + N^2 - \frac{2}{n} \sigma^2 - 2N^2 + \frac{1}{n} \sigma^2 + \frac{1}{n} N^2 + \frac{n-1}{n} N^2 \right) = \underline{\underline{\sigma^2}}$$

because:

$$\mathbb{E}_\theta \left[ X_1 \cdot \sum_{i=1}^n X_i \right] = \mathbb{E}_\theta \left[ X_1^2 + X_1 \sum_{i=2}^n X_i \right] = \mathbb{E}_\theta[X_1^2] + (n-1) \cdot \mathbb{E}(X_1) \cdot \mathbb{E}(X_1) = \sigma^2 + N^2 + (n-1)N^2 = \sigma^2 + nN^2$$

$$\mathbb{E}_\theta \left[ (\sum_{i=1}^n X_i)^2 \right] = \mathbb{E}_\theta \left[ (X_1 + \dots + X_n)^2 \right] = \mathbb{E}_\theta \left[ X_1^2 + \dots + X_n^2 + 2X_1 X_2 + \dots + 2X_{n-1} X_n \right] =$$

$$= n \cdot \mathbb{E}_\theta[X_1^2] + \binom{n}{2} \cdot 2 \cdot \mathbb{E}[X_1 X_2] = n \cdot \sigma^2 + n \cdot N^2 + n(n-1) \cdot N^2$$

$$\begin{aligned} b) \quad & \text{cov}_\theta(\hat{\mu}_n, \hat{\sigma}_n) = \text{cov}_\theta \left( \frac{X_1 + \dots + X_n}{n}, \frac{1}{n-1} \sum_{i=1}^n (X_i - \frac{X_1 + \dots + X_n}{n})^2 \right) = \\ & = \frac{1}{n(n-1)} \sum_{j=1}^n \text{cov}_\theta \left( X_j, \sum_{i=1}^n (X_i - \frac{X_1 + \dots + X_n}{n})^2 \right) = \frac{1}{n-1} \text{cov}_\theta \left( X_1, \sum_{i=1}^n (X_i - \frac{X_1 + \dots + X_n}{n})^2 \right) = \\ & = \frac{1}{n-1} \cdot \text{cov}_\theta \left( X_1, \left( X_1 - \frac{X_1 + \dots + X_n}{n} \right)^2 \right) + \cancel{\frac{1}{n-1}} \cdot \text{cov}_\theta \left( X_1, \sum_{i=2}^n \left( X_i - \frac{X_1 + \dots + X_n}{n} \right)^2 \right) = \\ & = \frac{1}{n-1} \cdot \text{cov}_\theta \left( X_1, \left( \frac{n-1}{n} X_1 - \frac{X_2 + \dots + X_n}{n} \right)^2 \right) + \cancel{\text{cov}_\theta \left( X_1, \left( \frac{n-1}{n} X_2 - \frac{X_1 + X_3 + \dots + X_n}{n} \right)^2 \right)} = \\ & = \end{aligned}$$

, because  $\rightarrow$



2.2)  $\forall \alpha \in (0,1)$   $\varphi_\alpha$  is a test of  $H_0: \varphi_\alpha(x) \leq \varphi_\alpha(x) \forall x$

$$\forall p \leq \alpha \leq 1$$

$$\text{if } \exists \theta \in \Theta_0: P_\theta(\varphi_\alpha = 1) = \alpha \quad \forall \alpha \in (0,1)$$

$$\Rightarrow$$

$$P_\theta(p \leq u) = u \quad \forall u \in (0,1)$$

1.5 Theorem says that in this case

$$\sup_{\theta \in \Theta_0} P_\theta(p \leq u) = u \quad \forall u \in (0,1)$$

$$\Downarrow$$

$$\forall \theta \in \Theta_0 P_\theta(p \leq u) \leq u \quad \forall u \in (0,1)$$

so for our fixed  $\theta$

$$\textcircled{1} \quad P_\theta(p \leq u) \leq u \quad \forall u \in (0,1)$$

$$P_\theta(p \leq u) = P_\theta(\inf\{\alpha \in (0,1) : \varphi_\alpha(x) = 1\} \leq u) =$$

~~$P_\theta(\exists \alpha \in [0,u] : \varphi_\alpha(x) = 1)$~~

$$= P_\theta(\inf\{z \leq u \mid \inf\{z \in \{\}\}\}) + P_\theta(\inf\{z \leq u \mid \inf\{z \in \{\}\}\}) =$$

$$\geq P_\theta(\inf\{\alpha \in (0,1) : \varphi_\alpha(x) = 1\} \leq u \mid \inf\{z \in \{\}\}) =$$

$$= P_\theta(\exists \alpha \in [0,u] : \varphi_\alpha(x) = 1) = P_\theta(\varphi_u(x) = 1) = u,$$

because if  $u < u_1$ ,  
 $\varphi_u(x) = 1 \Rightarrow \varphi_{u_1}(x) = 1$

so we obtain  $\textcircled{2} \quad P_\theta(p \leq u) \geq u$

$$\textcircled{1} + \textcircled{2} : P_\theta(p \leq u) = u$$

□