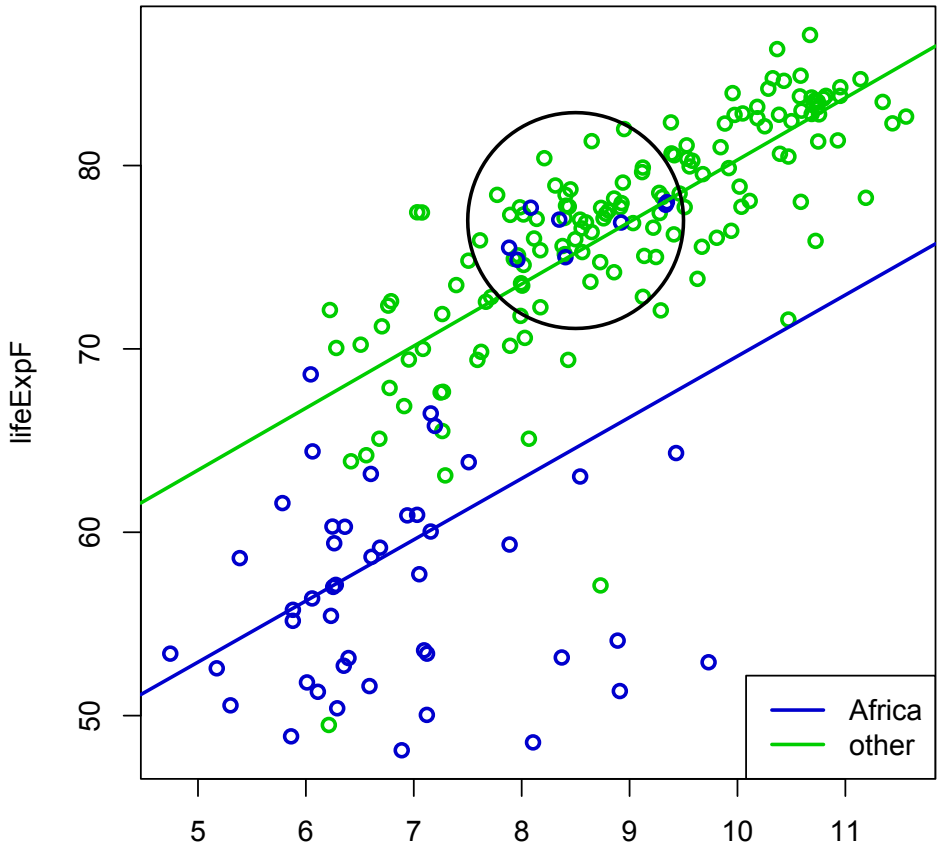
UN DATA: INTERACTION EFFECT

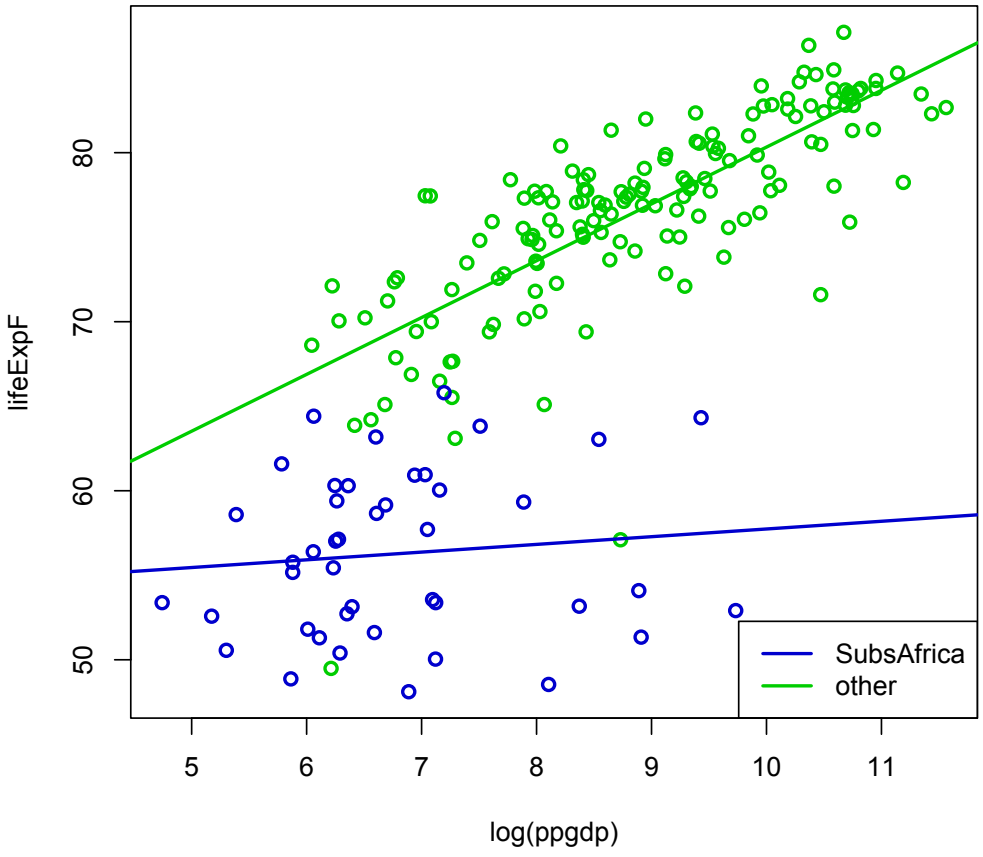


Algeria, Cape Verde, Egypt, Libya, Mauritius, Morocco, Seychelles, Tunisia

North African countries or islands!

	Estimate	Std. Error	t value	Pr(>|t|)
(Intercept)	36.22882	4.18145	8.664	1.73e-15
log(ppgdp)	3.33752	0.58428	5.712	4.12e-08
groupother	10.24281	5.08235	2.015	0.0452
log(ppgdp):groupother	0.04578	0.66539	0.069	0.9452

UN DATA: INTERACTION EFFECT



sub-Saharan Africa:

$$\mathbb{E}[Y_i] = \beta_1 + X_{i2}\beta_2$$

other:

$$\mathbb{E}[Y_j] = \beta_1 + X_{j2}\beta_2 + \beta_3 + X_{j2}\beta_4 = \beta_1 + \beta_3 + X_{j2}(\beta_2 + \beta_4)$$

Test: Are the slopes different?

$H_0: \beta_4 = 0$

Does GDP actually influence female life expectancy in sub-Saharan Africa?

		Estimate	Std. Error	t value	Pr(>|t|)
(Intercept)	β_1	53.4042	3.9035	13.681	< 2e-16
log(ppgdp)	β_2	0.4330	0.5656	0.766	0.4448
groupSSother	β_3	-7.9811	4.4871	-1.779	0.0769
log(ppgdp):groupSSother	β_4	3.0627	0.6167	4.967	1.48e-06

$\alpha = 0.05$ $\alpha = 0.01$

Looking at the data and then changing the model or the question to ask is actually not allowed!!

More on that later...

Exploratory data analysis vs. statistical inference!

Philosophical question: What actually is the population here?

SIMPSON'S PARADOX

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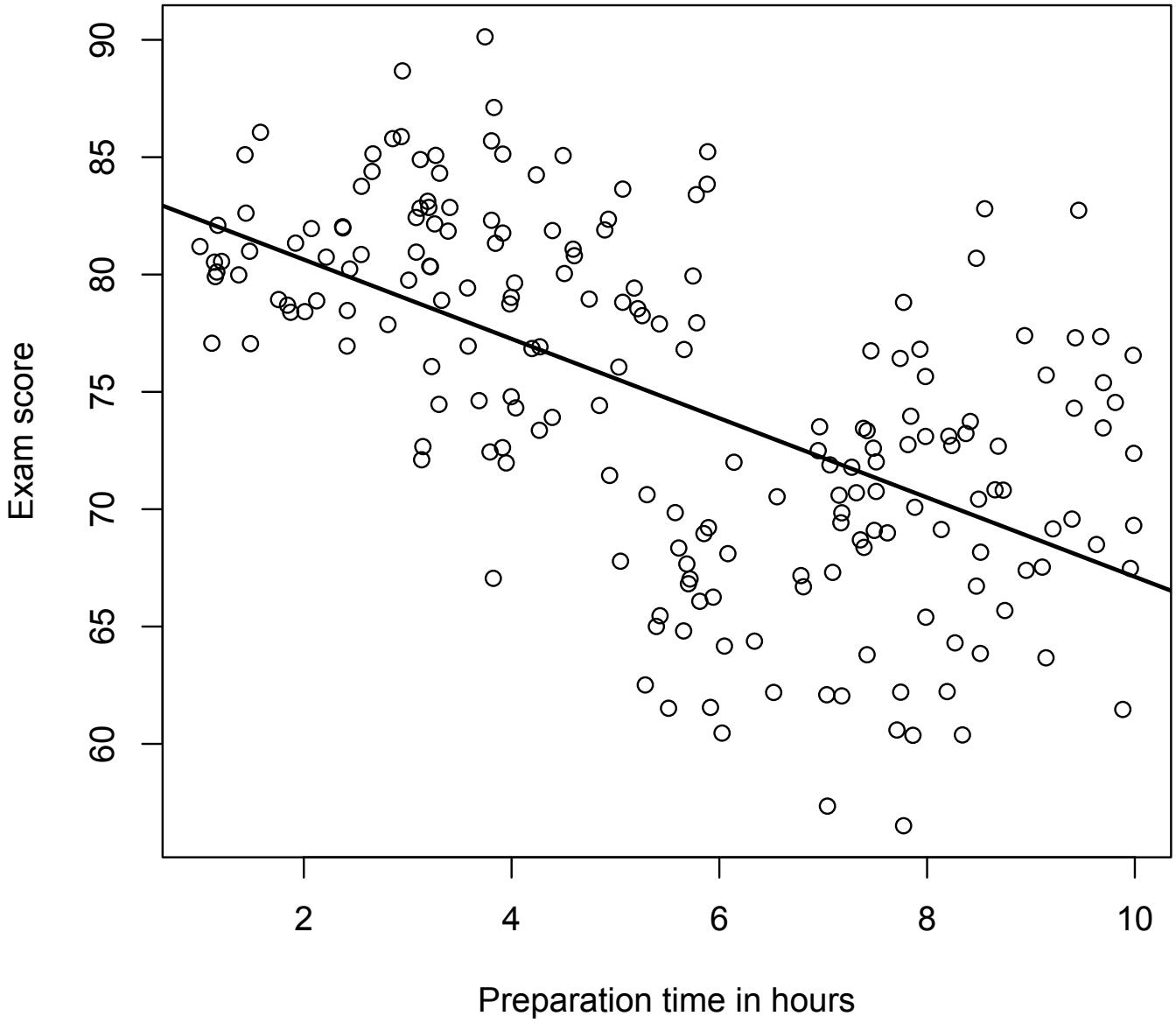

Simpson's paradox

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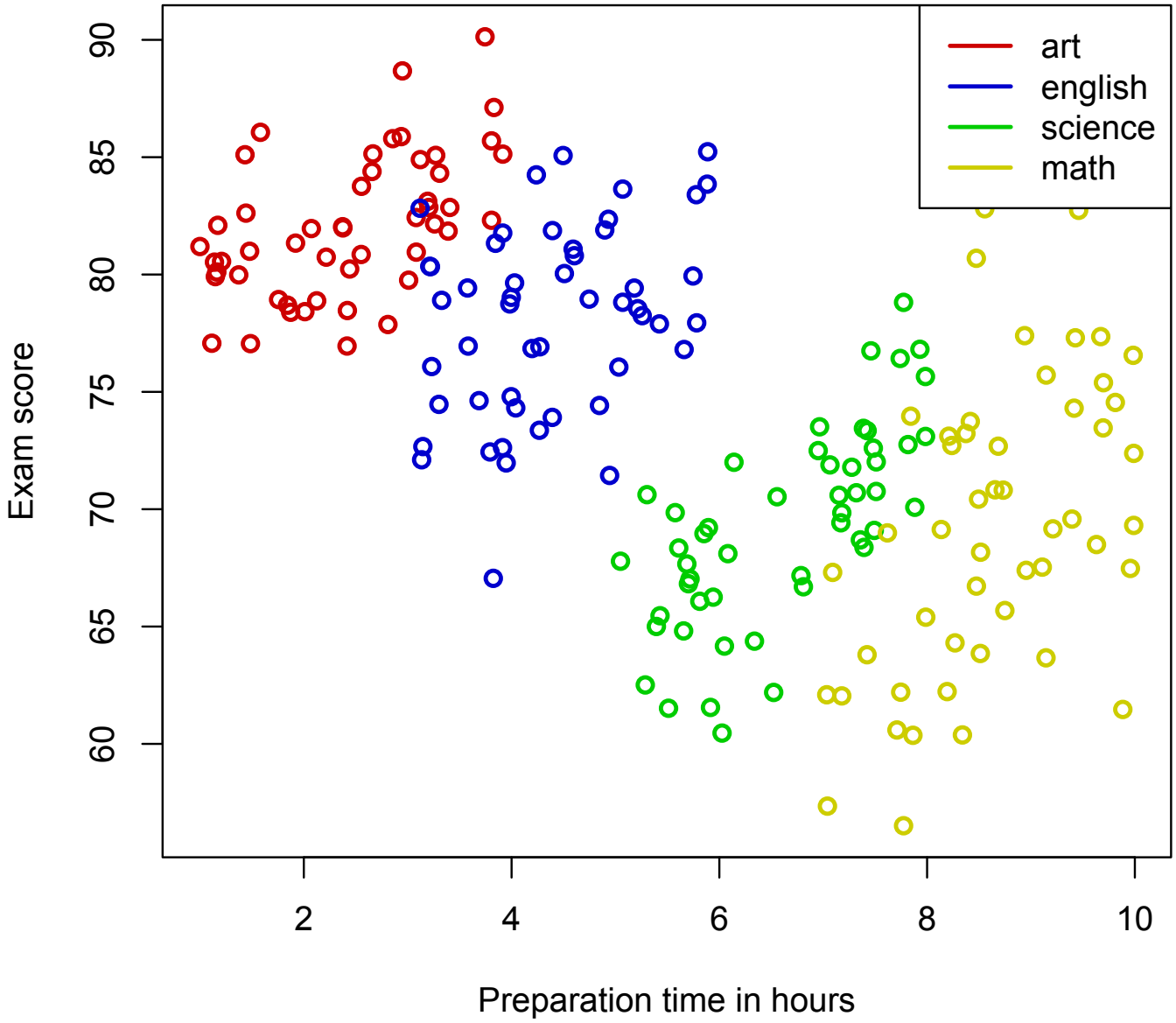

Simpson's paradox

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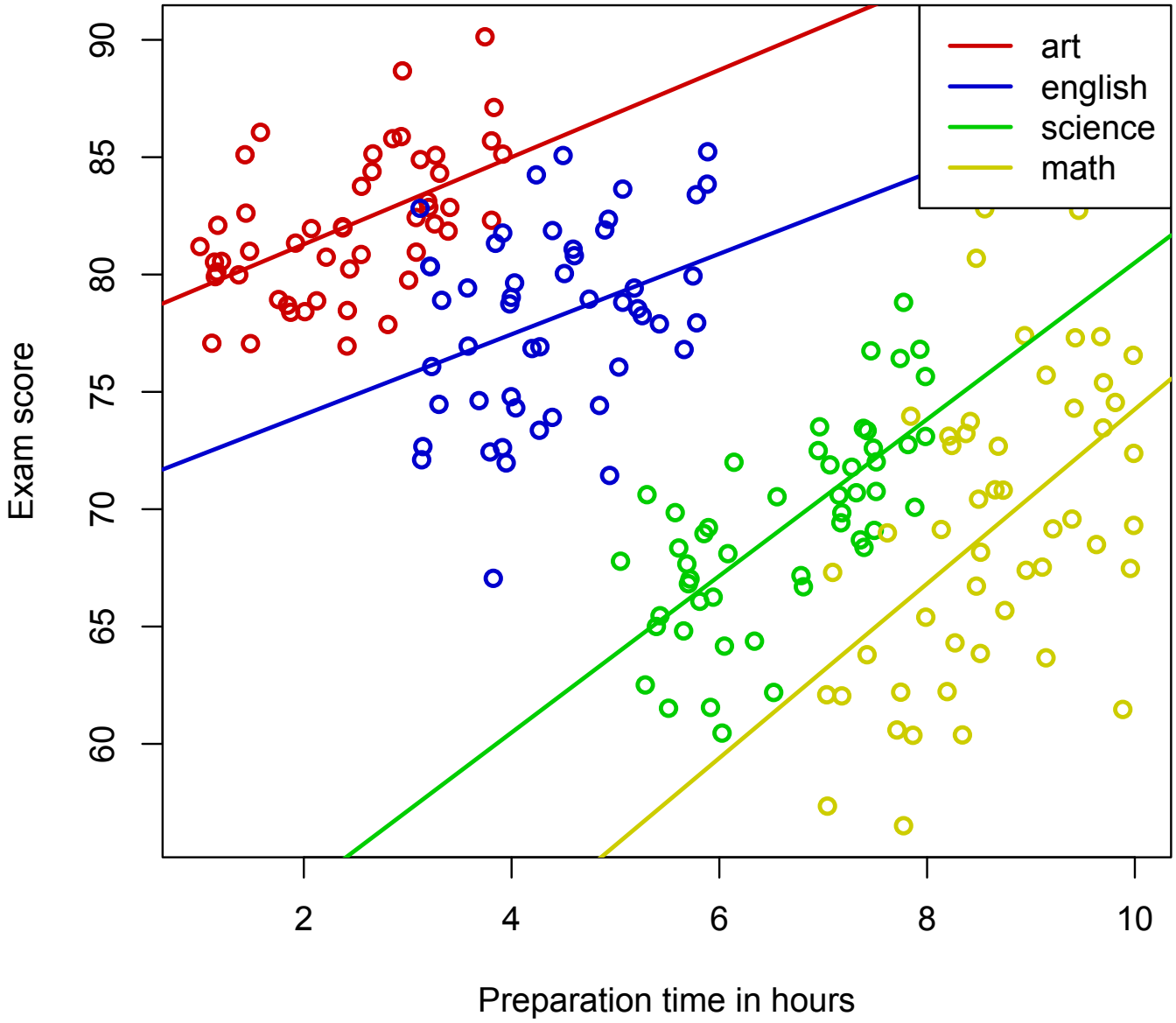
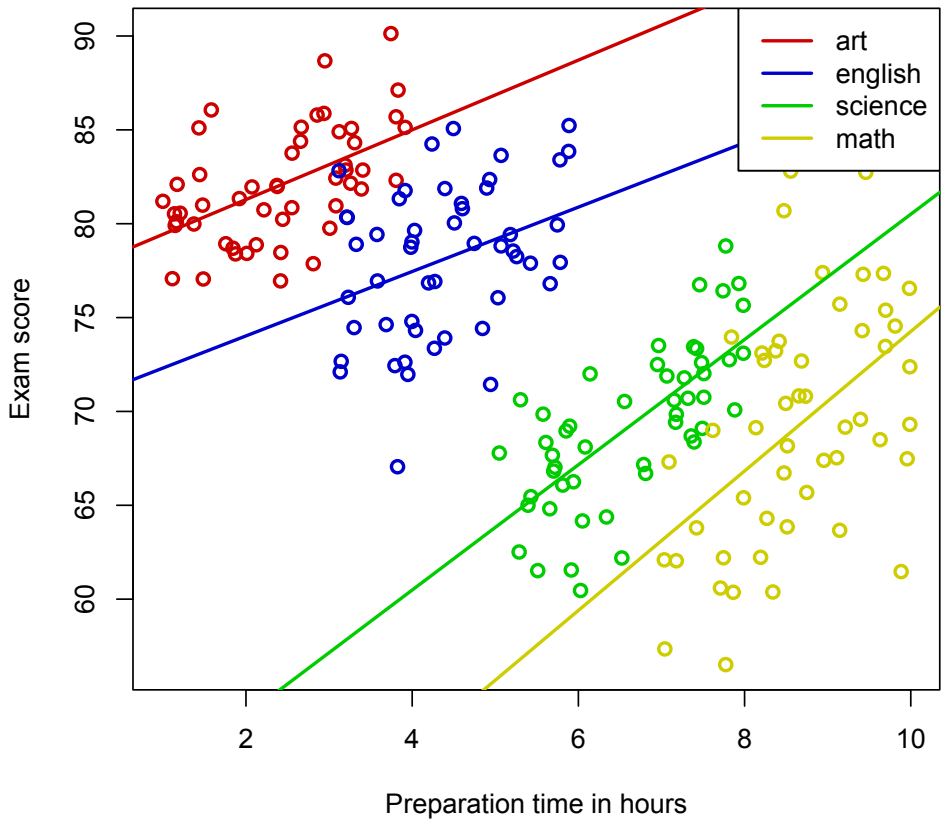


Categorical variables aka factors



Are the four slopes actually different?

Here 'subject' is a categorical variable (*factor*) with four *levels*. How to model that, i.e., how to construct X?

Categorical variables aka factors

How to model a factor variable?

1.) Use a code: art=1, english=2, science=3, math=4.

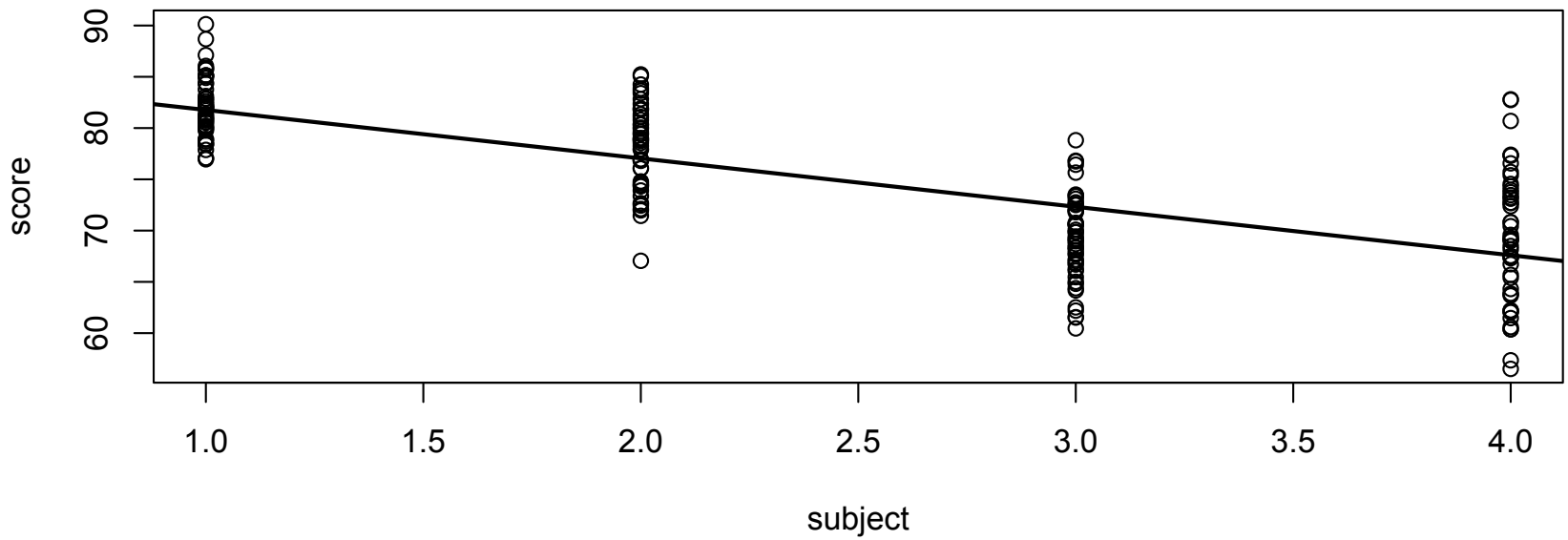
$$X = \begin{pmatrix} 1 & 2.5 & 1 \\ 1 & 3.5 & 2 \\ 1 & 6 & 4 \\ 1 & 7.5 & 3 \\ 1 & 5 & 3 \\ 1 & 1 & 1 \\ 1 & 1.5 & 2 \\ & \vdots & \\ 1 & 9 & 3 \end{pmatrix}$$

time

subject

Categorical variables aka factors

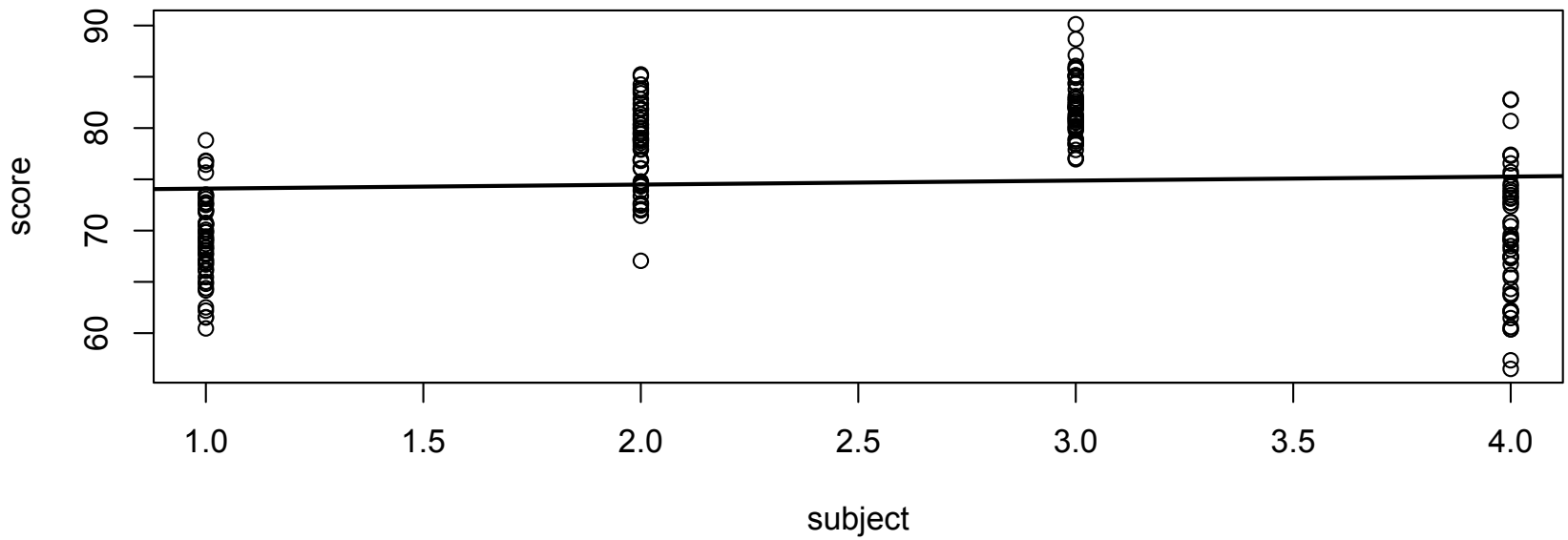
1a.) Use a code: art=1, english=2, science=3, math=4.




```
             Estimate Std. Error t value Pr(>|t|)
(Intercept)  86.4877     0.8512   101.6   <2e-16
subj         -4.7225     0.3108   -15.2   <2e-16
```

Categorical variables aka factors

1b.) Use a different code: art=3, english=2, science=1, math=4.




```
             Estimate Std. Error t value Pr(>|t|)
(Intercept)  73.7040     1.2504  58.946   <2e-16
subj          0.3911     0.4566   0.856    0.393
```

CATEGORICAL VARIABLES AKA FACTORS

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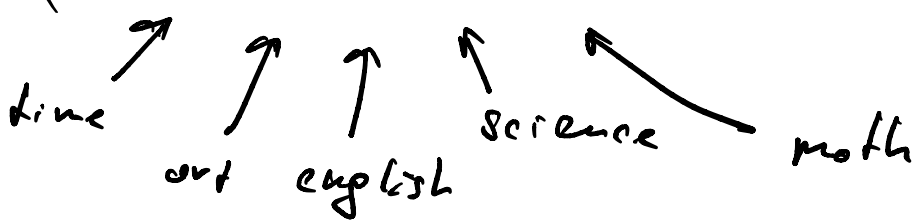
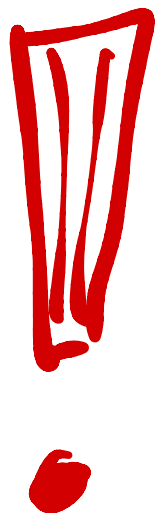
2.) Introduce 'dummy variables'.

$X_{\cdot 7} + X_{\cdot 4} + X_{\cdot 5} + X_{\cdot 6} = X_{\cdot 1}$

$$X = \begin{pmatrix} 1 & 2.5 & 1 & 0 & 0 & 0 & 2.5 & 0 & 0 & 0 \\ 1 & 3.5 & 0 & 1 & 0 & 0 & 0 & 3.5 & 0 & 0 \\ 1 & 6 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 6 \\ 1 & 7.5 & 0 & 0 & 1 & 0 & 0 & 0 & 7.5 & 0 \\ 1 & 5 & 0 & 0 & 1 & 0 & 0 & 0 & 5 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1.5 & 0 & 1 & 0 & 0 & 0 & 1.5 & 0 & 0 \\ & \vdots & & & & & & \vdots & & \\ 1 & 9 & 0 & 0 & 1 & 0 & 0 & 0 & 9 & 0 \end{pmatrix}$$





Test if the slopes are all the same, i.e. $H_0 : \beta_7 = \beta_8 = \beta_9 = \beta_{10}$

Categorical variables aka factors

3a.) Avoiding the *dummy variable trap*:

$$X = \begin{pmatrix} 1 & 2.5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 3.5 & 1 & 0 & 0 & 3.5 & 0 & 0 \\ 1 & 6 & 0 & 0 & 1 & 0 & 0 & 6 \\ 1 & 7.5 & 0 & 1 & 0 & 0 & 7.5 & 0 \\ 1 & 5 & 0 & 1 & 0 & 0 & 5 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1.5 & 1 & 0 & 0 & 1.5 & 0 & 0 \\ & \vdots & & & & & \vdots & \\ 1 & 9 & 0 & 1 & 0 & 0 & 9 & 0 \end{pmatrix}$$

english, science, math

$$\text{all}: \mathbb{E}(Y_i) = \beta_1 + X_{i2}\beta_2$$

$$\text{engl.}: \mathbb{E}(Y_i) = \beta_1 + X_{i2}\beta_2 + \beta_3 + X_{i2}\beta_6 = \beta_1 + \beta_3 + X_{i2}(\beta_2 + \beta_6)$$

$$\text{math}: \mathbb{E}(Y_i) = \beta_1 + X_{i2}\beta_2 + \beta_5 + X_{i2}\beta_8 = \beta_1 + \beta_5 + X_{i2}(\beta_2 + \beta_8)$$

Test if the slopes are all the same, i.e., $H_0 : \beta_6 = \beta_7 = \beta_8 = 0$.

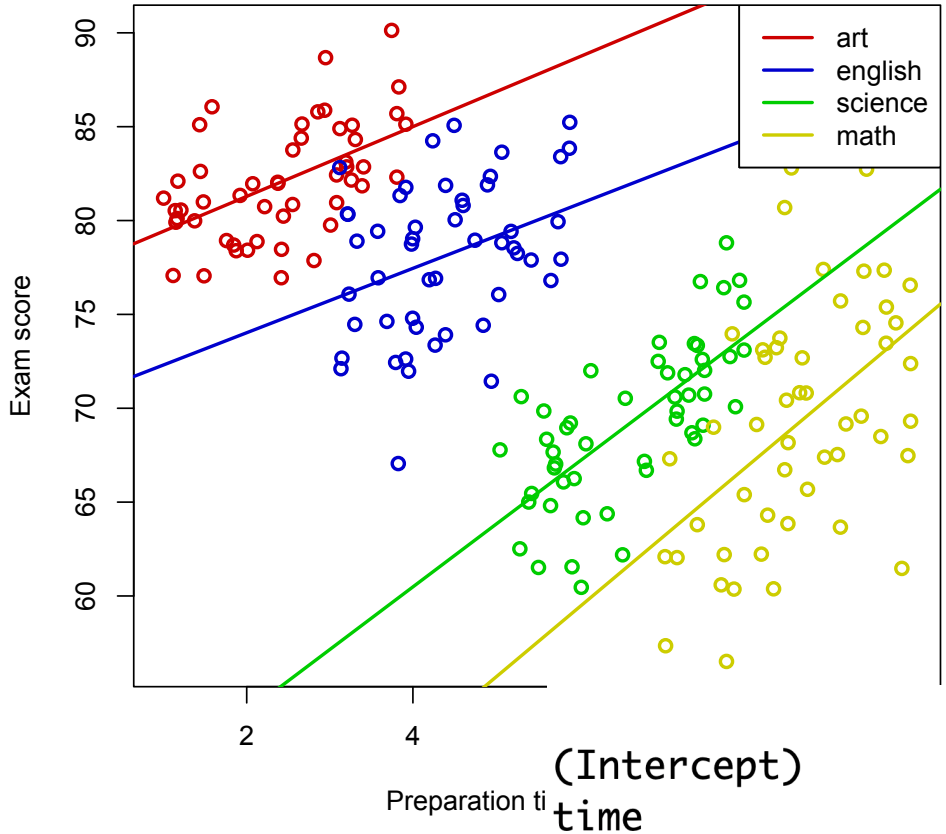
Categorical variables aka factors

3b.) Avoiding the *dummy variable trap*:

$$X = \begin{pmatrix} 1 & 0 & 0 & 0 & 2.5 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 3.5 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 6 \\ 0 & 0 & 1 & 0 & 0 & 0 & 7.5 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 5 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1.5 & 0 & 0 \\ & & \vdots & & & \vdots & & \\ 0 & 0 & 1 & 0 & 0 & 0 & 9 & 0 \end{pmatrix}$$

Test if the slopes are all the same, i.e., $H_0 : \beta_5 = \beta_6 = \beta_7 = \beta_8$.

Categorical variables aka factors



Are the four slopes actually different?

$H_0: \beta_6 = \beta_7 = \beta_8 = 0$

$\alpha = 0.05$

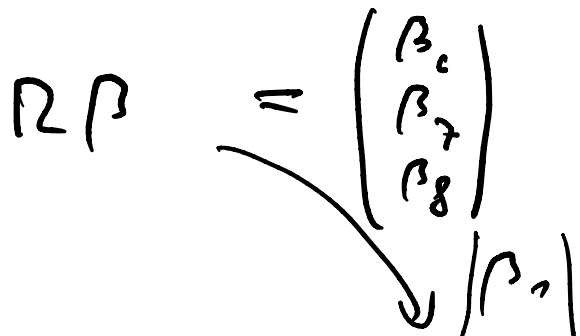
		Estimate	Std. Error	t value	Pr(>|t|)
β_1	(Intercept)	77.5733	1.6652	46.585	< 2e-16
β_2	time	1.8569	0.6487	2.863	0.00467
	subjectenglish	-6.9800	3.4766	-2.008	0.04608
	subjectmath	-40.4671	5.9648	-6.784	1.41e-10
	subjectscience	-30.4375	4.5206	-6.733	1.87e-10
β_6	time:subjectenglish	-0.1419	0.9432	-0.150	0.88053
β_7	time:subjectmath	1.8583	0.9243	2.010	0.04579
β_8	time:subjectscience	1.4801	0.9026	1.640	0.10270

Can we test this only by looking at the standard output? no!

We want to test a general linear hypothesis:

$$H_0 : R\beta = r$$

where $R \in \mathbb{R}^{q \times p}$, $q \leq p$, $\operatorname{rank} R = q$ and $r \in \mathbb{R}^q$.



For instance, with

$$R = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad r = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

we have

$$H_0 : R\beta = r \quad \Longleftrightarrow \quad H_0 : \beta_6 = 0, \beta_7 = 0, \beta_8 = 0$$

We want to test a general linear hypothesis:

$$H_0 : R\beta = r$$

where $R \in \mathbb{R}^{q \times p}$, $q \leq p$, $\operatorname{rank} R = q$ and $r \in \mathbb{R}^q$.

Or, with $R\beta = \begin{pmatrix} \beta_5 - \beta_6 \\ \beta_6 - \beta_7 \\ \beta_7 - \beta_8 \end{pmatrix}$

$$R = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix} \quad \text{and} \quad r = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

we have

$$H_0 : R\beta = r \quad \Longleftrightarrow \quad H_0 : \beta_5 = \beta_6, \beta_6 = \beta_7, \beta_7 = \beta_8$$

$\beta_5 = \beta_6 = \beta_7 = \beta_8$

$$H_0 : R\beta = r$$

Notice:

$$\hat{\beta} \sim N(\beta, \sigma^2(X'X)^{-1}) \quad \Rightarrow \quad R\hat{\beta} - r \sim N(R\beta - r, \sigma^2 R(X'X)^{-1}R')$$

$R\beta - r = 0$ under H_0

Under H_0, we therefore have

$$\frac{[R(X'X)^{-1}R']^{-1/2}(R\hat{\beta} - r)}{\sigma} \sim N(0, I_q)$$

and

$$\frac{(R\hat{\beta} - r)'[R(X'X)^{-1}R']^{-1}(R\hat{\beta} - r)}{\sigma^2} \sim \chi^2_q.$$

$$H_0 : R\beta = r$$

$$\frac{(R\hat{\beta} - r)'[R(X'X)^{-1}R']^{-1}(R\hat{\beta} - r)}{\sigma^2} \sim \chi^2_q.$$

Recall, $\hat{\sigma}^2 := \frac{1}{n-p}\|Y - X\hat{\beta}\|_2^2$ satisfies $\frac{\hat{\sigma}^2(n-p)}{\sigma^2} \sim \chi^2_{n-p}$ independent of $\hat{\beta}_n$.

Definition:
If $S_1 \sim \chi^2_{d_1}$ independent of $S_2 \sim \chi^2_{d_2}$, then $\frac{S_1/d_1}{S_2/d_2} \sim F_{d_1,d_2}$ follows an F-distribution with d_1 and d_2 degrees of freedom.

Hence, under H_0,

$$F := \frac{(R\hat{\beta} - r)'[R(X'X)^{-1}R']^{-1}(R\hat{\beta} - r)/q}{\hat{\sigma}^2} \sim F_{q,n-p}$$

$$H_0 : R\beta = r$$

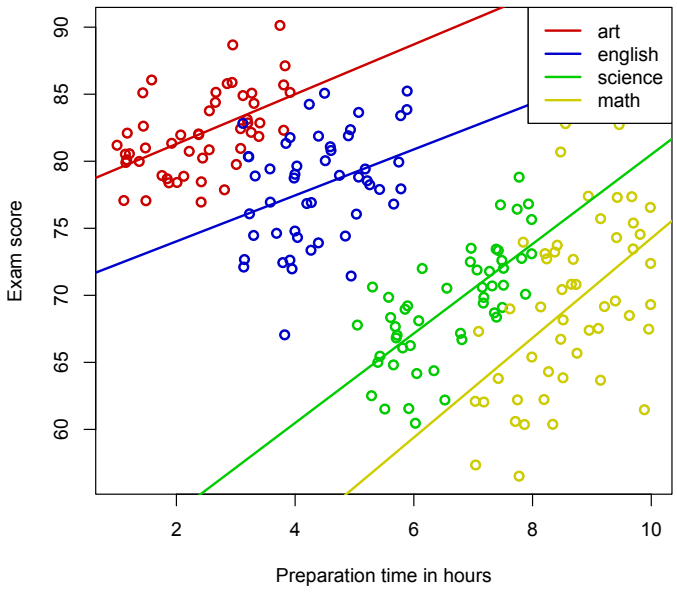
Under H_0,

$$F := \frac{(R\hat{\beta} - r)'[R(X'X)^{-1}R']^{-1}(R\hat{\beta} - r)/q}{\hat{\sigma}^2} \sim F_{q,n-p}$$

Test: Reject H_0 if $F > c_\alpha := q_{1-\alpha}^{(F_{q,n-p})}$.

Hence, $P_{H_0}(F > c_\alpha) = 1 - P_{H_0}(F \leq c_\alpha) = 1 - (1 - \alpha) = \alpha$.

Testing general linear hypotheses



Are the four slopes actually different?

$$H_0 : \beta_6 = \beta_7 = \beta_8 = 0 \iff H_0 : R\beta = r$$

$$R = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, r = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\alpha = 0.05$

q $\quad$ $n-p$

```
F = 2.3877, df1 =   3, df2 = 192, p-value = 0.07029
```

Is there a contradiction with the marginal t-tests?

What if we used another R matrix?

HW

	Estimate	Std. Error	t value	Pr(>|t|)
(Intercept)	77.5733	1.6652	46.585	< 2e-16
time	1.8569	0.6487	2.863	0.00467
subjectenglish	-6.9800	3.4766	-2.008	0.04608
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time:subjectmath	1.8583	0.9243	2.010	0.04579
time:subjectscience	1.4801	0.9026	1.640	0.10270

$H_0: \beta_6 = 0$

Test $H_0 : \beta_6 = \beta_7 = \beta_8 = 0$ at level $\alpha = 0.05$.

```
F = 2.3877, df1 =  3, df2 = 192, p-value = 0.07029
```

	Estimate	Std. Error	t value	Pr(>|t|)
(Intercept)	77.5733	1.6652	46.585	< 2e-16
time	1.8569	0.6487	2.863	0.00467
subjectenglish	-6.9800	3.4766	-2.008	0.04608
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time:subjectenglish	-0.1419	0.9432	-0.150	0.88053
time:subjectmath	1.8583	0.9243	2.010	0.04579
time:subjectscience	1.4801	0.9026	1.640	0.10270

$\leq \alpha$?

$\leq \frac{\alpha}{3}$?

Is there a contradiction with the marginal t-tests?

$\alpha = 0.05 \rightarrow \frac{\alpha}{3} = 0.0166$

F-TEST VS. MARGINAL T-TESTS

$H_0: \beta_6 = \beta_7 = \beta_8 = 0$

$$\varphi_\alpha(Y) := \begin{cases} 1, & \text{if } \varphi_\alpha^{(k)}(Y) = 1 \text{ for some } k \in \{6,7,8\} \\ 0, & \text{else} \end{cases}$$

where $\varphi_\alpha^{(k)}(Y) = \begin{cases} 1, & \text{if } |T_k(Y)| > q_{1-\frac{\alpha}{2}}^{(t_{n-p})} \\ 0, & \text{else} \end{cases}$

is the t-test for $H_0^{(k)}: \beta_k = 0$ which satisfies

$$P_{H_0^{(k)}}(\varphi_\alpha^{(k)} = 1) = \alpha.$$

Is φ_α a level-α test for H_0?

$$P_{H_0}(\varphi_\alpha = 1) \leq P_{H_0}\Big(\varphi_\alpha^{(6)} = 1 \cup \varphi_\alpha^{(7)} = 1 \cup \varphi_\alpha^{(8)} = 1\Big)$$

$$\leq \sum_{k=6}^{8} \underbrace{P_{H_0}(\varphi_\alpha^{(k)} = 1)}_{= \alpha \text{ because } H_0 \subseteq H_0^{(k)}} = 3 \cdot \alpha$$

For many X matrices we can have

$$P_{H_0}(\varphi_\alpha = 1) > \alpha .$$

note: $P_{H_0}(\varphi_{\alpha/3} = 1) \leq 3 \cdot \frac{\alpha}{3} = \alpha$

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Theorem (Bonferroni correction)

Let $(\mathcal{X}, \Theta, \{f_\theta : \theta \in \Theta\})$ be a statistical model, $\alpha \in (0,1)$ and $\Theta_0^{(1)}, \ldots, \Theta_0^{(K)} \subseteq \Theta$ be hypotheses with corresponding level-α tests $\varphi_\alpha^{(1)}, \ldots \varphi_\alpha^{(K)}$, i.e., for every $k \in \{1, \ldots, K\}$, we have $P_\theta(\varphi_\alpha^{(k)} = 1) \leq \alpha$ for all $\theta \in \Theta_0^{(k)}$. Then the test

$$\varphi_\alpha(x) := \begin{cases} 1, & \text{if } \exists k \in \{1, \ldots, K\} : \varphi_\alpha^{(k)}(x) = 1, \\ 0, & \text{else,} \end{cases}$$

satisfies

$$P_\theta(\varphi_{\frac{\alpha}{K}} = 1) \leq \alpha \quad \forall \theta \in \Theta_0 := \bigcap_{k=1}^{K} \Theta_0^{(k)}.$$

Multiple testing

(Intercept)	2.299346	1.307025	1.759	0.08010
Xpopulation	-3.477014	2.323128	-1.497	0.13608
Xhouseholdsize	1.118523	0.598059	1.870	0.06294
Xracepctblack	-0.042317	0.205327	-0.206	0.83693
XracePctWhite	0.022103	0.290556	0.076	0.93944
XracePctAsian	-0.084169	0.139117	-0.605	0.54587
XracePctHisp	0.151692	0.204243	0.743	0.45855
XagePct12t21	-0.239641	0.598155	-0.401	0.68913
XagePct12t29	-0.870986	0.773690	-1.126	0.26165
XagePct16t24	0.381891	0.895346	0.427	0.67019
XagePct65up	0.199093	0.585702	0.340	0.73428
XnumbUrban	3.802921	2.348860	1.619	0.10704
XpctUrban	-0.199798	0.167798	-1.191	0.23521
XmedIncome	-0.386196	0.864797	-0.447	0.65568
XpctWWage	-0.596946	0.551386	-1.083	0.28030
XpctWFarmSelf	-0.308686	0.148121	-2.084	0.03846
XpctWInvInc	-0.751876	0.291920	-2.576	0.01074
XpctWSocSec	-0.721007	0.574259	-1.256	0.21078
XpctWPubAsst	0.238305	0.214464	1.111	0.26786
XpctWRetire	0.064647	0.194944	0.332	0.74053
XmedFamInc	0.871977	0.641222	1.360	0.17543
XperCapInc	-0.876057	0.690411	-1.269	0.20598
XwhitePerCap	0.355919	0.418602	0.850	0.39622
XblackPerCap	-0.302640	0.226430	-1.337	0.18291
XindianPerCap	0.016152	0.120405	0.134	0.89342
XAsianPerCap	-0.147430	0.123883	-1.190	0.23546
XOtherPerCap	0.137916	0.128604	1.072	0.28486
XHispPerCap	-0.072551	0.146598	-0.495	0.62123
XNumUnderPov	-0.392185	0.314491	-1.247	0.21387
XPctPopUnderPov	0.525168	0.363595	1.444	0.15023
XPctLess9thGrade	-0.110777	0.317642	-0.349	0.72765
XPctNotHSGrad	-0.696948	0.439727	-1.585	0.11459
XPctBSorMore	-0.376918	0.385047	-0.979	0.32884

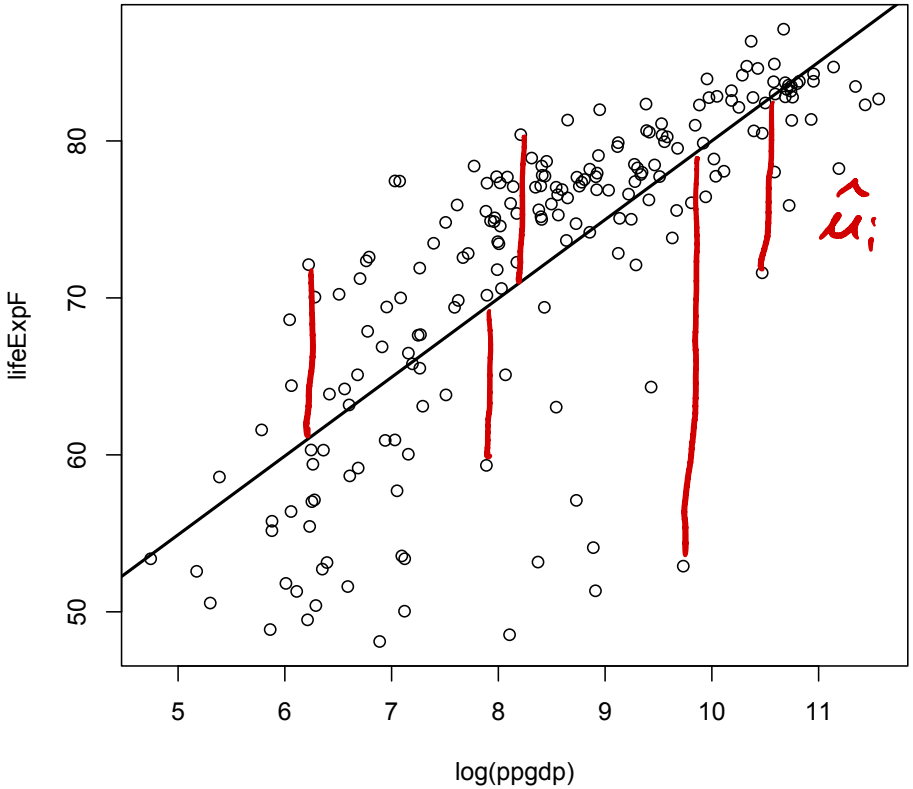
Looking at an output like that and asking, *is there something significant*, is not a good idea!

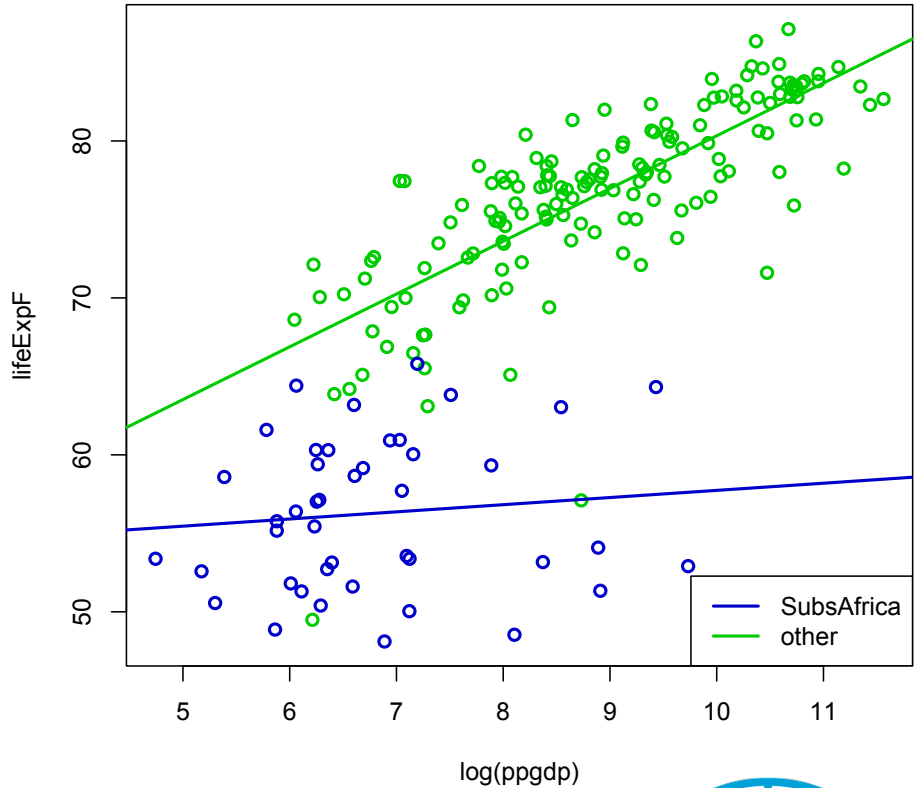
$$H_0 : \forall k : \beta_k = 0$$
$$H_1 : \exists k : \beta_k \neq 0$$

Whats the probability of finding something even though there is nothing?

Model diagnostic plots

Gaussian linear Model: $Y \sim N(X\beta, \sigma^2 I_n)$





What if the model is too complex to visualize it like that? Can we still get a graphical representation of model fit?

Model diagnostic plots

Gaussian linear Model:

$$Y \sim N(X\beta, \sigma^2 I_n)$$

or, equivalently

$$Y_i = X_{i\cdot}\beta + u_i, \quad \text{with} \quad u = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} \sim N(0, \sigma^2 I_n)$$

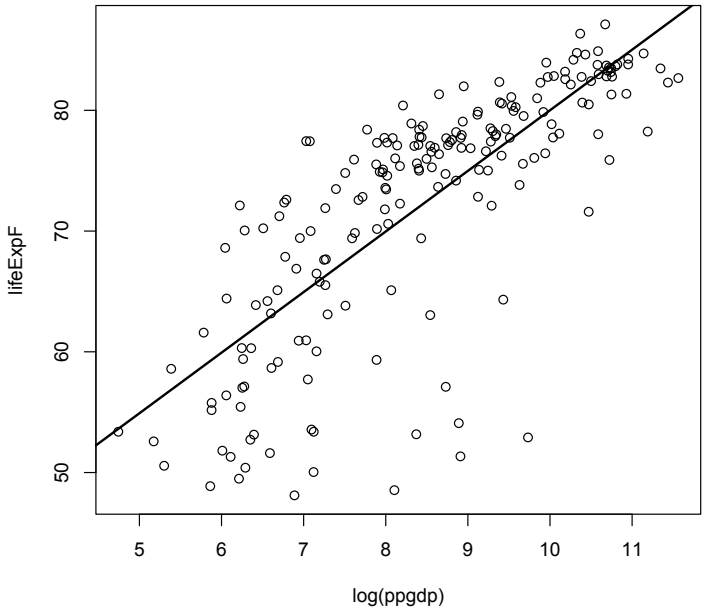
Recall: OLS
$\hat{\beta} = \text{argmin}_{b\in\mathbb{R}^p} \|Y - Xb\|_2^2 = \text{argmin}_{b\in\mathbb{R}^p} \sum_{i=1}^n (Y_i - X_{i\cdot}b)^2$

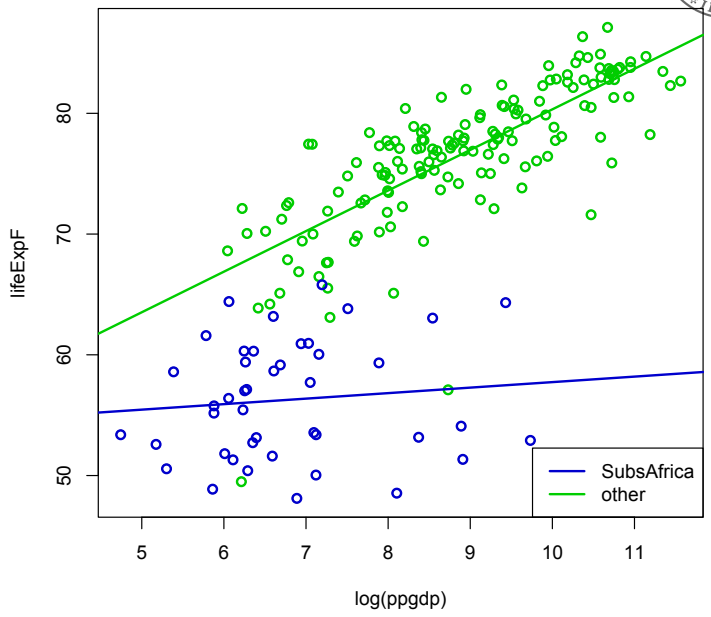
Idea: The *residuals* $\hat{u}_i = Y_i - X_{i\cdot}\hat{\beta}$ should be (approximately) iid Gaussian and evenly distributed around the regression line (independent of their 'location').

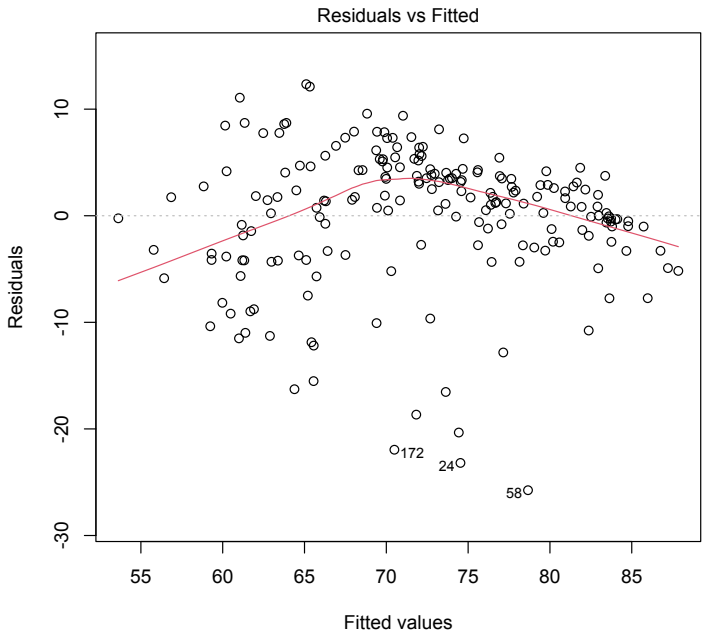
$\hat{Y}_i = X_{i\cdot}\hat{\beta}$ are the *fitted* or *predicted* values 'on' the regression line.

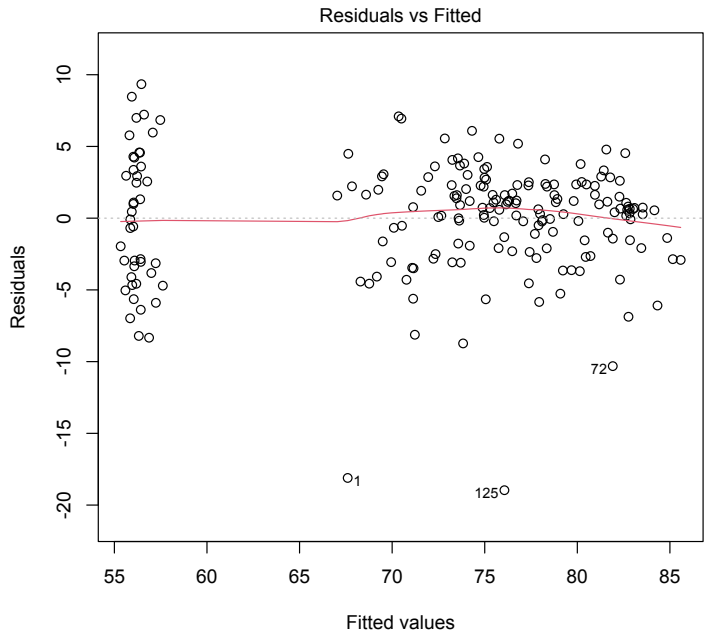
1.) plot $\hat{Y}_i$ against $\hat{u}_i$ 2.) check $\hat{u}_i$ for Gaussianity

RESIDUAL PLOT









1 = Afghanistan
72 = Greenland
125 = Nauru