Ex3

3.3) $p(x|\theta) = \prod_{i=1}^{n} \theta^{x_i} (1-\theta)^{1-x_i}$, $x = (x_1, \dots, x_n) \in \{0,1\}^n$

$\theta \in (0,1)$

non-informative uniform prior:

$\pi(\theta) = 1$, $\theta \in (0,1)$

posterior:

$$p(\theta|x) \propto p(x|\theta) \cdot \pi(\theta) = \prod_{i=1}^{n} \theta^{x_i} \cdot (1-\theta)^{1-x_i} \cdot 1 =$$

$$= \theta^{\sum_{i=1}^{n} x_i} \cdot (1-\theta)^{n - \sum_{i=1}^{n} x_i}$$

$$c(x) = \left(\int_0^1 \theta^{\sum_{i=1}^{n} x_i} \cdot (1-\theta)^{n - \sum_{i=1}^{n} x_i} \right)^{-1}$$

this a Beta distribution with parameters $(n\bar{x}+1, n-n\bar{x}-1)$

$$\int_0^1 \theta^{n\bar{x}} (1-\theta)^{n-n\bar{x}} \, d\theta$$

$$p(\theta|x) = f_{Beta}(\theta, n\bar{x}-1, n-n\bar{x}-1) = \frac{\theta^{n\bar{x}} (1-\theta)^{n-n\bar{x}}}{B(n\bar{x}+1, n-n\bar{x}+1)}$$

3.2) a) $f(x,y) = c \cdot \sin^2(xy) \cdot \cos^2(-xy) \cdot e^{-8x^2-2|y|^3} = c \cdot h(x,y)$

$g(x,y) = \frac{1}{2\pi} \cdot e^{-\frac{1}{2}(x^2+y^2)}$

we need M s.t. $h(x,y) \leq M \cdot g(x,y)$ $\forall x,y \in \mathbb{R}$

$$h(x,y) \leq e^{-8x^2-2|y|^3} \overset{?}{\leq} M' \cdot e^{-\frac{1}{2}(x^2+y^2)} \qquad M = M' \cdot \frac{1}{2\pi}$$

$$-8x^2 - 2|y|^3 \leq \ln M' - \frac{1}{2}(x^2+y^2)$$

$$-8x^2 \leq -\frac{1}{2}x^2 \ \checkmark$$

$$-2|y|^3 \leq \ln M' - \frac{1}{2}y^2$$

if $y \geq 0$: $\quad -2y^3 \overset{?}{\leq} \ln M - \frac{1}{2}y^2$

$$0 \overset{?}{\leq} \ln M + \underbrace{y^2\left(2y - \frac{1}{2}\right)}_{\text{min of this function is } -\frac{1}{216}}$$

So we need:

$$\frac{1}{216} \leq \ln\left(M \cdot \frac{1}{2\pi}\right)$$

$M = 3\pi$ is good enough

b) From the plots we can see that x, y are in $[-1, 1]$ most of the time.

So $g(x, y) = \dfrac{1}{(1-(-1))(1-(-1))} = \dfrac{1}{4}$

$h(x, y) \leq 1 \quad \forall x, y$, so $M := 4$

3.1) $X_1, \dots, X_n$ i.i.d.

$F(x)$ invertible

$\forall \alpha \in (0,1)$:

$$\hat{F}_n^{+}(\alpha) \xrightarrow[n\to\infty]{i.p.} F^{-1}(\alpha) \iff \forall \varepsilon > 0 \quad P\left(\left|\hat{F}_n^{+}(\alpha) - F^{-1}(\alpha)\right| > \varepsilon\right) \xrightarrow{n\to\infty} 0$$

$$P\left(\left|\hat{F}_n^{+}(\alpha) - F^{-1}(\alpha)\right| > \varepsilon\right) = P\left(\hat{F}_n^{+}(\alpha) - F^{-1}(\alpha) > \varepsilon \text{ or } \hat{F}_n^{+}(\alpha) - F^{-1}(\alpha) < -\varepsilon\right) =$$

$$= P\left(\hat{F}_n^{+}(\alpha) > F^{-1}(\alpha) + \varepsilon\right) + P\left(\hat{F}_n^{+}(\alpha) < F^{-1}(\alpha) - \varepsilon\right) =$$

$$= P\left(\cancel{\hat{F}_n^{+}(\alpha)}\ \hat{F}_n\left(\hat{F}_n^{+}(\alpha)\right) > \hat{F}_n\left(F^{-1}(\alpha) + \varepsilon\right)\right) + P\left(\hat{F}_n\left(\hat{F}_n^{+}(\alpha)\right) \overset{<}{\cancel{>}} \hat{F}_n\left(F^{-1}(\alpha) - \varepsilon\right)\right) \leq$$

because of $\{x \in \mathbb{R} : \hat{F}_n(x) \geq \alpha\} = [\hat{F}_n^{+}(\alpha), \infty)$ $\longrightarrow$ $\alpha \leq \hat{F}_n\left(\hat{F}_n^{+}(\alpha)\right) \leq \alpha + \frac{1}{n}$

$$\leq P\left(\alpha + \tfrac{1}{n} > \hat{F}_n\left(F^{-1}(\alpha) + \varepsilon\right)\right) + P\left(\alpha < \hat{F}_n\left(F^{-1}(\alpha) \cancel{+} - \varepsilon\right)\right)$$

$\Longrightarrow$

$$\xRightarrow{\lim_{n\to\infty}} P\left(\alpha + \tfrac{1}{n} - F > F^{-1} - F\right) = P\left(\alpha + \lim_{n\to\infty}\tfrac{1}{n} - F > \lim_{n\to\infty} F^{-1} - F\right) = P(\alpha > F)$$

because of the law of large numbers *

$n \longrightarrow \infty$:

$$= P\left(\alpha > F\left(F^{-1}(\alpha) + \varepsilon\right)\right) + P\left(\alpha < F\left(F^{-1}(\alpha) - \varepsilon\right)\right) =$$

$= 0 + 0 = 0$, because $F(F^{-1}(\alpha)) = \alpha$ + F is strictly increasing,

so $F(F^{-1}(\alpha) + \varepsilon) > \alpha$

and

$F(F^{-1}(\alpha) - \varepsilon) < \alpha$

$$* \quad \hat{F}_n\left(F^{-1}(\alpha) + \varepsilon\right) = \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}_{(-\infty, F^{-1}(\alpha)+\varepsilon)} X_i \xrightarrow{n\to\infty} \frac{1}{n} \cdot n \cdot \mathbb{E}\left[\mathbb{1}_{(-\infty, F^{-1}(\alpha)+\varepsilon)} X_i\right] =$$

$$= \int_{-\infty}^{F^{-1}(\alpha)+\varepsilon} f(x)\, dx =$$

$$= P\left(X < F^{-1}(\alpha) + \varepsilon\right) =$$

$$= F\left(F^{-1}(\alpha) + \varepsilon\right)$$

<u>again:</u>

$$\lim_{n\to\infty} P\left(\left|\hat{F}_n^{+}(\alpha) - F^{-1}(\alpha)\right| > \varepsilon\right) = \lim_{n\to\infty} P\left(\hat{F}_n^{+}(\alpha) - F^{-1}(\alpha) > \varepsilon \text{ or } \hat{F}_n^{+}(\alpha) - F^{-1}(\alpha) < -\varepsilon\right) =$$

$$= \lim_{n\to\infty}\left(P\left(\hat{F}_n^{+}(\alpha) - F^{-1}(\alpha) > \varepsilon\right) + P\left(\hat{F}_n^{+}(\alpha) - F^{-1}(\alpha) < -\varepsilon\right)\right) = \text{ because } \hat{F}_n \text{ is non-dec.} =$$

$$= \lim_{n\to\infty}\left(P\left(\hat{F}_n\left(\hat{F}_n^{+}(\alpha)\right) > \hat{F}_n\left(F^{-1}(\alpha) + \varepsilon\right)\right) + P\left(\hat{F}_n\left(\hat{F}_n^{+}(\alpha)\right) < \hat{F}_n\left(F^{-1}(\alpha) - \varepsilon\right)\right)\right) \leq$$

because $\{x \in \mathbb{R} : \hat{F}_n(x) \geq \alpha\} = [\hat{F}_n^{+}(\alpha), \infty)$, thus $\alpha \leq \hat{F}_n(\hat{F}_n^{+}(\alpha)) \leq \alpha + \frac{1}{n}$

$$\leq \lim_{n\to\infty}\left(P\left(\alpha + \tfrac{1}{n} > \hat{F}_n\left(F^{-1}(\alpha) + \varepsilon\right)\right) + P\left(\alpha < \hat{F}_n\left(F^{-1}(\alpha) - \varepsilon\right)\right)\right) =$$

$$= P\left(\alpha + \lim_{n\to\infty}\tfrac{1}{n} - F\left(F^{-1}(\alpha) + \varepsilon\right) > \lim_{n\to\infty} \hat{F}_n\left(F^{-1}(\alpha) + \varepsilon\right) - F\left(F^{-1}(\alpha) + \varepsilon\right)\right) + \ldots =$$

because $\lim_{n\to\infty} \hat{F}_n(x) = \mathbb{E}\left(\mathbb{1}_{\{-\infty, x]} \cdot X_i\right) = \int_{-\infty}^{x} f(x)dx = P(X_i < x) = F(x)$ $\quad F^{-1}$ is strictly i.

$$= P\left(\alpha > F\left(F^{-1}(\alpha) + \varepsilon\right)\right) + P\left(\alpha < F\left(F^{-1}(\alpha) - \varepsilon\right)\right) =$$

$$P\left(F^{-1}(\alpha) > F^{-1}(\alpha) + \varepsilon\right) + P\left(F^{-1}(\alpha) < F^{-1}(\alpha) - \varepsilon\right) =$$

$= 2P(\varepsilon < 0) = 0$, because $\varepsilon > 0$