2.3) b) $\theta \in (0,1)$

n is fixed

$\theta \to 0 \Longrightarrow (1-\theta)^n \to 1 \Longrightarrow P_\theta(\hat{\theta}_n = 0) \to 1 \Longrightarrow$

$\Longrightarrow P_\theta\left(\sqrt{\hat{\theta}_n \cdot (1-\hat{\theta}_n)} = 0\right) = P_\theta(\hat{\theta} = 0) \to 1 \Longrightarrow$

$\Longrightarrow P_\theta(CI_\alpha = [0,0]) \longrightarrow 1 \underset{\text{as } \theta > 0}{\Longrightarrow} P_\theta(\theta \in CI_\alpha) \to 0$

b) $\mathrm{cov}_\theta\left(\hat{\mu}_n, \frac{1}{n-1}\sum_{i=1}^{n}(X_i-\hat{\mu}_n)^2\right) = \frac{n}{n-1}\,\mathrm{cov}_\theta\left(\hat{\mu}_n, X_1^2 - 2\hat{\mu}_n X_1 + \hat{\mu}_n^2\right) = *$

$\mathrm{cov}_\theta(\hat{\mu}_n, X_1^2) = \frac{1}{n}\mathrm{cov}_\theta(X_1, X_1^2) = \frac{1}{n}\mathbb{E}(X_1^3) - \frac{1}{n}\mathbb{E}(X_1)\cdot\mathbb{E}(X_1^2) =$

$= \frac{1}{n}\left(\mathbb{E}((X_1-\mu)^3) + 3\mu\,\mathbb{E}(X_1^2) - 3\mu^2\,\mathbb{E}(X_1) + \mu^3 - \mu(\sigma^2+\mu^2)\right) =$

$= \frac{1}{n}\left(0 + 3\mu(\sigma^2+\mu^2) - \cancel{3\mu^3} + \cancel{\mu^3} - \mu\sigma^2 - \cancel{\mu^3}\right) = \boxed{\frac{2\mu\sigma^2}{n}}$

$\mathrm{cov}_\theta(\hat{\mu}_n, \hat{\mu}_n X_1) = \frac{1}{n^2}\mathrm{cov}_\theta(X_1+\dots+X_n, X_1^2 + X_1X_2 + \dots + X_1X_n) =$

$= \frac{1}{n^2}\left(\mathrm{cov}_\theta(X_1, X_1^2) + (n-1)\,\mathrm{cov}_\theta(X_1, X_1\cdot X_2) + (n-1)\,\mathrm{cov}_\theta(X_2, X_1 X_2)\right) =$

$= \frac{1}{n^2}\left(2\mu\sigma^2 + (2n-2)\cdot\mathrm{cov}_\theta(X_1, X_1X_2)\right) = \boxed{\frac{2\sigma^2\mu}{n}}$

because:

$\mathrm{cov}_\theta(X_1, X_1\cdot X_2) = \mathbb{E}(X_1^2\cdot X_2) - \mathbb{E}(X_1)\cdot\mathbb{E}(X_1\cdot X_2) = \mathbb{E}(X_1^2)\cdot\mathbb{E}(X_1) - \mathbb{E}^3(X_1) =$

$= (\sigma^2+\mu^2)\mu - \mu^3 = \sigma^2\mu$

$\mathrm{cov}_\theta(\hat{\mu}_n, \hat{\mu}_n^2) = \boxed{\frac{2\mu\sigma^2}{n}}$

$* = \frac{n}{n-1}\cdot\left(\frac{2\mu\sigma^2}{n} - 2\cdot\frac{2\mu\sigma^2}{n} + \frac{2\mu\sigma^2}{n}\right) = 0$ ✓

EX 2

1.2.) a) $X_1, \dots, X_n \sim$ i.i.d

$\mathbb{E}(X_i) = \mu$

$Var(X_i) = \mathbb{E}(X_i^2) - \mathbb{E}^2(X_i) = \sigma^2 \Rightarrow \mathbb{E}(X_i^2) = \sigma^2 + \mu^2$

$i \neq j \quad \mathbb{E}(X_i \cdot X_j) = \mathbb{E}(X_i) \cdot \mathbb{E}(X_j) = \mu^2 \Rightarrow$ i.i.d.

$$\mathbb{E}_\theta[\hat{\sigma}_n] = \mathbb{E}_\theta\left[\frac{1}{n-1}\sum_{i=1}^{n}\left(X_i - \frac{1}{n}\sum_{i=1}^{n} X_i\right)^2\right] \overset{\text{i.i.d.}}{=} \frac{n}{n-1}\mathbb{E}_\theta\left[X_1^2 - \frac{2}{n}X_1\sum_{i=1}^{n}X_i + \frac{\left(\sum_{i=1}^{n}X_i\right)^2}{n^2}\right] =$$

$$= \frac{n}{n-1}\left(\mathbb{E}_\theta[X_1^2] - \frac{2}{n}\mathbb{E}_\theta\left[X_1 \cdot \sum_{i=1}^{n}X_i\right] + \frac{1}{n^2}\mathbb{E}_\theta\left[\left(\sum_{i=1}^{n}X_i\right)^2\right]\right) =$$

$$= \frac{n}{n-1}\left(\sigma^2 + \mu^2 - \frac{2}{n}\sigma^2 - 2\mu^2 + \frac{1}{n}\sigma^2 + \frac{1}{n}\mu^2 + \frac{n-1}{n}\mu^2\right) = \underline{\underline{\sigma^2}}$$

because:

$$\mathbb{E}_\theta\left[X_1 \cdot \sum_{i=1}^{n}X_i\right] = \mathbb{E}_\theta\left[X_1^2 + X_1\sum_{i=2}^{n}X_i\right] = \mathbb{E}_\theta[X_1^2] + (n-1) \cdot \mathbb{E}(X_1) \cdot \mathbb{E}(X_1) = \sigma^2 + \mu^2 + (n-1)\mu^2 =$$
$$= \sigma^2 + n\mu^2$$

$$\mathbb{E}_\theta\left[\left(\sum_{i=1}^{n}X_i\right)^2\right] = \mathbb{E}_\theta\left[(X_1 + \dots + X_n)^2\right] = \mathbb{E}_\theta\left[X_1^2 + \dots + X_n^2 + 2X_1X_2 + \dots + 2X_{n-1}X_n\right] =$$

$$= n \cdot \mathbb{E}_\theta[X_1^2] + \binom{n}{2} \cdot 2 \cdot \mathbb{E}[X_1X_2] = n \cdot \sigma^2 + n \cdot \mu^2 + n(n-1) \cdot \mu^2$$

b) $$cov_\theta(\hat{\mu}_n, \hat{\sigma}_n) = cov_\theta\left(\frac{X_1 + \dots + X_n}{n}, \frac{1}{n-1}\sum_{i=1}^{n}\left(X_i - \frac{X_1 + \dots + X_n}{n}\right)^2\right) =$$

$$= \frac{1}{n(n-1)}\left\{\sum_{j=1}^{n} cov_\theta\left(X_j, \sum_{i=1}^{n}\left(X_i - \frac{X_1 + \dots + X_n}{n}\right)^2\right) = \frac{1}{n-1}cov_\theta\left(X_1, \sum_{i=1}^{n}\left(X_i - \frac{X_1 + \dots + X_n}{n}\right)^2\right) =\right.$$

$$= \frac{1}{n-1}cov_\theta\left(X_1, \left(X_1 - \frac{X_1 + \dots + X_n}{n}\right)^2\right) + \frac{1}{n-1} \cdot cov_\theta\left(X_1, \sum_{i=2}^{n}\left(X_i - \frac{X_1 + \dots + X_n}{2}\right)^2\right) =$$

$$= \frac{1}{n-1} \cdot cov_\theta\left(X_1, \left(\frac{n-1}{n}X_1 - \frac{X_2 + \dots + X_n}{n}\right)^2\right) + cov_\theta\left(X_1, \left(\frac{n-1}{n}X_2 - \frac{X_1 + X_3 + \dots + X_n}{n}\right)^2\right) =$$

$=$

, because $\rightarrow$

2.2) $\forall \alpha \in (0,1)$ φ_α is a test of H_0 : $\varphi_\alpha(x) \leq \varphi_{\alpha'}(x)$ $\forall x$

$\cancel{\forall}\, 0 \leq \alpha \leq \alpha' \leq 1$

$$\text{if } \exists\, \theta \in \Theta_0 : P_\theta(\varphi_\alpha = 1) = \alpha \quad \forall \alpha \in (0,1)$$

$$\Longrightarrow$$

$$P_\theta(p \leq u) = u \quad \forall u \in (0,1)$$

1.5 Theorem says that in this case

$$\sup_{\theta \in \Theta_0} P_\theta(p \leq u) \leq u \quad \forall u \in (0,1)$$

$$\Downarrow$$

$$\forall \theta \in \Theta_0\, P_\theta(p \leq u) \leq u \quad \forall u \in (0,1)$$

So for our fixed θ

$$① \quad P_\theta(p \leq u) \leq u \quad \forall u \in (0,1)$$

$$P_\theta(p \leq u) = P_\theta\left(\inf\{\alpha \in (0,1) : \varphi_\alpha(X) = 1\} \leq u\right) =$$

$$\cancel{= P_\theta(\exists\, \alpha \in (0,u] : \varphi_\alpha(X) = 1)}$$

$$= P_\theta\left(\inf\{\ \} \leq u \mid \inf\{\ \} \in \{\ \}\right) + P_\theta\left(\inf\{\ \} \leq u \mid \inf\{\ \} \notin \{\ \}\right) \geq$$

$$\geq P_\theta\left(\inf\{\alpha \in (0,1) : \varphi_\alpha(X) = 1\} \leq u \mid \inf\{\ \} \in \{\ \}\right) =$$

$$= P_\theta\left(\exists\, \alpha \in [0,u] : \varphi_\alpha(X) = 1\right) \underset{\Uparrow}{=} P_\theta(\varphi_u(X) = 1) = u,$$

because if $\alpha' < u$, $\varphi_{\alpha'}(X) = 1 \Rightarrow \varphi_u(X) = 1$

So we obtain ② $P_\theta(p \leq u) \geq u$

$$① + ② : \quad P_\theta(p \leq u) = u$$

$\square$