4 Exercises (due Wednesday 29.11.2023, 23:59pm)

There is a total of 7 points to achieve on this third exercise sheet. Please upload your solutions in typed or readable (!) handwritten form (scans, photographs), and also your code, on Moodle and don't forget to flag all the problems you were able to solve. Good luck!

Exercise 4.1 (2P). *Write a Metropolis-Hastings algorithm that generates bivariate random vectors from the pdf* $f(x,y) \propto [\sin(xy)]^2[\cos(-xy)]^2 e^{-8x^2-2|y|^3}$. *Try different proposal densities. Visualize the evolution of the Markov chain in* $2D$. *Visualize the marginal distributions of* X *and* Y *using histograms.*

Hint: *Try a proposal density* $q(z|x)$ *that is only a function of the difference* $z - x$.

Exercise 4.2 (3P). *Consider iid data* $X_1, \ldots, X_n$ *from pdf*

$$p_\theta(x) \propto \begin{cases} 0, & \text{if } x < 0, \\ 1, & \text{if } 0 \leq x \leq 1, \\ \frac{1}{x^{2(\theta+1)}}, & \text{if } x > 1, \end{cases}$$

with $\theta \in \Theta = [0, \infty)$. *Let* $\psi(\theta) = F_\theta^{-1}(\alpha)$ *be the* α *quantile of the corresponding distribution. Conduct a simulation study to evaluate the validity (coverage probability) and accuracy (average length) of the two bootstrap approaches discussed in the lecture for constructing confidence intervals for* $\psi(\theta)$. *Investigate the effect of the parameters* n, B *and* $\alpha \in \{0.5, 0.8, 0.9, 0.99\}$.

Exercise 4.3 (2P). *Consider the model* $X_1, \ldots, X_n \overset{iid}{\sim} Unif[0, \theta]$, $\theta \in \Theta = (0, \infty)$. *Derive the exact theoretical quantiles of the sampling distribution of* $\hat{\theta}_n - \theta$, *where* $\hat{\theta}_n = \max\{X_1, \ldots, X_n\}$, *and use them to derive an exact* $1 - \alpha$ *confidence interval for* θ.

Hint: *An exact CI is one that satisfies* $P_\theta(\theta \in CI_\alpha) = 1 - \alpha$ *for all* $\theta \in \Theta$.