EX 4

4.3) $X_1, \dots, X_n \sim U(0, \theta)$ i.i.d

$\theta \in (0, \infty)$

$\hat{\theta}_n = \max(X_1, \dots, X_n)$

- quantiles of $g_n \sim \hat{\theta}_n - \theta$:

$$G_n(q_\alpha) = \alpha$$

$$P(\hat{\theta}_n - \theta < q_\alpha) = \alpha$$

$$P(\max(X_1, \dots, X_n) < \theta + q_\alpha) = \alpha$$

$$P(X_1 < \theta + q_\alpha) \cdots P(X_n < \theta + q_\alpha) = \alpha$$

$$P(X_1 < \theta + q_\alpha)^n = \alpha$$

$$P(X_1 < \theta + q_\alpha) = \alpha^{\frac{1}{n}} \qquad \text{if } -\theta \le q_\alpha \le +0$$

$$\frac{\theta + q_\alpha}{\theta} = \alpha^{\frac{1}{n}}$$

$$q_\alpha = \theta \alpha^{\frac{1}{n}} - \theta$$

- find CI_α:

$$P(\theta \in CI_\alpha) = 1 - \alpha$$

we know: $1 - \alpha = P_\theta\left(q_{\frac{\alpha}{2}} \le \hat{\theta}_n - \theta \le q_{1-\frac{\alpha}{2}}\right) =$

$= P_\theta\left(\theta\left(\frac{\alpha}{2}\right)^{\frac{1}{n}} - \theta \le \hat{\theta}_n - \theta \le \theta\left(1 - \frac{\alpha}{2}\right)^{\frac{1}{n}} - \theta\right) = P_\theta\left(\theta\left(\frac{\alpha}{2}\right)^{\frac{1}{n}} \le \hat{\theta}_n \le \theta\left(1 - \frac{\alpha}{2}\right)^{\frac{1}{n}}\right) =$

$= P_\theta\left(\frac{\left(\frac{\alpha}{2}\right)^{\frac{1}{n}}}{\hat{\theta}_n} \le \frac{1}{\theta} \le \frac{\left(1 - \frac{\alpha}{2}\right)^{\frac{1}{n}}}{\hat{\theta}_n}\right) = P_\theta\left(\hat{\theta}_n\left(\frac{\alpha}{2}\right)^{-\frac{1}{n}} \ge \theta \ge \hat{\theta}_n\left(1 - \frac{\alpha}{2}\right)^{-\frac{1}{n}}\right),$

So $CI_\alpha = \left[\hat{\theta}_n\left(1 - \frac{\alpha}{2}\right)^{-\frac{1}{n}}, \hat{\theta}_n\left(\frac{\alpha}{2}\right)^{-\frac{1}{n}}\right]$

4.2) $X_1, \dots, X_n \sim p_\theta(x)$

$$p_\theta(x) \propto \begin{cases} 0 & , \text{if } x < 0 \\ 1 & , \text{if } 0 \le x \le 1 \\ \dfrac{1}{x^{2(\theta+1)}} & , \text{if } x > 1 \end{cases}$$

$$\int_{\mathbb{R}} p_\theta(x)\,dx = 1 + \int_1^\infty \frac{1}{x^{2\theta+2}}\,dx = 1 + \left[\frac{x^{-2\theta-1}}{-2\theta-1}\right]_1^\infty = \frac{1}{2\theta+1} + 1 =$$

$$= \frac{2\theta+2}{2\theta+1}$$

$$\Downarrow$$

$$p_\theta(x) = \begin{cases} 0, & \text{if } x < 0 \\ \dfrac{2\theta+1}{2\theta+2}, & \text{if } 0 \le x \le 1 \\ \dfrac{2\theta+1}{(2\theta+2)x^{2(\theta+1)}}, & \text{if } x > 1 \end{cases}$$

$$q_\alpha : \quad \alpha = \int_{-\infty}^{q_\alpha} p_\theta(x)$$

if $\alpha \le \dfrac{2\theta+1}{2\theta+2}$:

$$\alpha = \int_0^{q_\alpha} \frac{2\theta+1}{2\theta+2}\,dx = \frac{2\theta+1}{2\theta+2} q_\alpha$$

$$q_\alpha = \frac{2\theta+2}{2\theta+1}\alpha$$

if $\alpha > \dfrac{2\theta+1}{2\theta+2}$:

$$\alpha = \frac{2\theta+1}{2\theta+2} + \int_1^{q_\alpha} \frac{2\theta+1}{2\theta+2} \cdot \frac{1}{x^{2(\theta+1)}}\,dx =$$

$$= \frac{2\theta+1}{2\theta+2} + \frac{1}{2\theta+2}\left(1 - \frac{1}{q_\alpha^{2\theta+1}}\right) \Rightarrow q_\alpha = \sqrt[2\theta+1]{\frac{1}{2(\theta+1)(1-\alpha)}}$$