UNIVERSITÉ DE
NEUCHÂTEL

Faculté des Sciences
Institut de mathématiques
Rue Emile-Argand 11, 2000 Neuchâtel

On computational aspects of hyperbolic reflection groups & two homology theories for graphs

Thèse

présentée à la Faculté des Sciences
pour l'obtention du grade de docteur ès Sciences en mathématiques

par

Rémi Bottinelli

Acceptée sur proposition du jury :

Prof. Aleksandr Kolpakov, Université de Neuchâtel, directeur de thèse
Prof. Alain Valette, Université de Neuchâtel, rapporteur
Prof. Alex Kontorovich, Rutgers University, USA, rapporteur
Prof. Nicolas Monod, EPF Lausanne, rapporteur

Soutenue le 29 septembre 2021

Faculté des Sciences
Secrétariat-décanat de Faculté
Rue Emile-Argand 11
2000 Neuchâtel – Suisse
Tél : + 41 (0)32 718 21 00
E-mail : secretariat.sciences@unine.ch

IMPRIMATUR POUR THESE DE DOCTORAT

La Faculté des sciences de l'Université de Neuchâtel autorise l'impression de la présente thèse soutenue par

Monsieur Rémi BOTTINELLI

Titre:

"On computational aspects of hyperbolic reflection groups and two homology theories for graphs"

sur le rapport des membres du jury composé comme suit:

- Prof. ass. Aleksandr Kolpakov, directeur de thèse, Université de Neuchâtel, Suisse
- Prof. Alain Valette, Université de Neuchâtel, Suisse
- Prof. Nicolas Monod, EPF Lausanne
- Prof. Alex Kontorovich, Rutgers University, USA

Neuchâtel, le 7 octobre 2021

Le Doyen, Prof. A. Bangerter

Remerciements

Sasha for generally being a good guy. People I had the pleasure to collaborate with: Sasha, Laura, Tom, Nikolay, Alexey. Rafael Guglielmetti for his debugging help regarding Vinberg's algorithm. Tommy Hofmann for the "computational number theory hotline". Ege Erdil for his regular explanations and insights. The people of Neuchâtel's mathematics institute for the welcoming atmosphere. My jury members for their kindness and involvement.

Abstract

This thesis consists of two largely independent parts. In the first part, we study three kinds of homologies of metric spaces and graphs.

- The first, magnitude homology, is a categorification of Leinster's magnitude and measures phenomena broadly related to geodesics in the space. After introducing magnitude homology of graphs as a categorification of magnitude, Hepworth and Willerton proved versions of the Künneth formula for products and of the Mayer-Vietoris sequence for certain decompositions. They also introduced the concept of *diagonality*, stating that magnitude homology vanishes for the most part. We verify that the Künneth formula, Mayer-Vietoris sequence and concept of diagonality all hold in the metric context, when appropriately restated. We also show that median graphs (and metric spaces) are diagonal.

- The second, dubbed Roe homology, is a coarse homology theory of locally finite graphs. We study the first homology group, providing a decomposition as a direct sum: the first part measuring the ends of the graph, the second measuring a kind of "coarse cycle structure".

- The third, uniformly finite homology, is a refinement of sorts of Roe homology, defined for uniformly locally finite graphs (graphs with a bound on the degrees of their vertices). We restate some of the previously obtained results on Roe homology in the context of uniformly finite homology. In short, the ends and "coarse cycle structure" also play a part, although in a less satisfying manner. We also define a notion of higher dimensional expansion of simplicial complexes (generalizing usual expansion), and after some tweaks, provide a characterization of the vanishing of the first homology group (with coefficients in $\mathbb{Z}$) for *transitive* graphs in terms of ends, "coarse cycle structure" and expansion.

The second part consists of some comments on the implementation of the so-called *Vinberg algorithm*, which can be used to find a finite-volume fundamental polyhedron for the reflection groups of certain Lorentzian lattices (if such a polyhedron exists). This algorithm has been successfully used in manual computations, and also implemented on computers. Perhaps the best known implementation is that of Guglielmetti; also of note is that of Bogachev and Perepechko. The implementation presented here (written in the Julia programming language) can claim novelty in that it is the first (to my knowledge) able to treat non diagonal lattices over number fields other than $\mathbb{Q}$; this is made possible by the rich ecosystem of mathematical packages available for Julia.

Key words: metric space, graph, homology, coarse geometry, flow, hyperbolic geometry, algorithm, lattice, polyhedron.

Résumé

Ceci est une thèse *pot-pourri*. On commence par étudier trois types d'homologies sur graphes et espaces métriques.

- Premièrement, l'homologie de magnitude, une catégorification de la notion de magnitude due à Leinster, qui mesure certains phénomènes vaguement liés aux géodésiques. Après avoir introduit la notion d'homologie de magnitude sur les graphes, Hepworth et Willerton ont prouvé des versions de la formule de Künneth pour les produits, et de la séquence de Mayer-Vietoris pour certaines décompositions. Ils ont aussi introduit le concept de diagonalité, qui se résume en la nullité d'une majorité des groupes d'homologie. Nous vérifions que la formule de Künneth, la séquence de Mayer-Vietoris et le concept de diagonalité restent valides dans le contexte métrique, une fois correctement rephrasés. Nous prouvons aussi que les graphes et espaces métriques médians sont diagonaux.

- Deuxièmement, l'homologie de Roe, qui est une homologie grossière sur les graphes localement finis. Nous étudions son premier groupe d'homologie, décomposable en une somme directe dont un côté mesure les bouts du graphe et l'autre une certaine forme grossière de «structure de circuits».

- Troisièmement, l'homologie uniformément finie, obtenue en raffinant l'homologie de Roe, est définie pour les graphes uniformément localement finis. Une partie des résultats obtenus pour l'homologie de Roe peuvent être adaptés à l'homologie uniformément finie: les bouts et une forme grossière de «structure de circuits» apparaissent à nouveau, mais de manière moins satisfaisante. Nous définissons une certaine forme d'expansion sur les complexes simpliciaux; celle-ci généralisant la notion usuelle d'expansion. Après quelques adaptations, nous prouvons une caractérisation de la nullité du premier groupe d'homologie uniformément finie (à coefficients dans $\mathbb{Z}$) pour les graphes *transitifs* en termes de bouts, «structure grossière de cicruits», et expansion.

La deuxième partie consiste en quelques commentaires sur l'implémentation de l'algorithme dit *de Vinberg*, qui est utilisé pour trouver un polyhèdre fondamental de volume fini (s'il existe) pour le groupe de réflexions de certains réseaux Lorentziens. Cet algorithme a déjà été utilisé manuellement avec succès, et implémenté sur ordinateur. L'implémentation la plus connue est probablement celle de Guglielmetti; citons aussi celle de Bogachev et Perepechko. L'intérêt de l'implémentation présentée ici (écrite en Julia) est qu'elle est la première (à ce que je sache) à être (théoriquement) capable de traiter des réseaux non diagonaux à coefficients non rationels; chose principalement rendue possible par l'écosystème de bibliothèques mathématiques disponible pour le langage Julia.

Mots clés: espace métrique, graphe, homologie, géométrie grossière, flot, géométrie hyperbolique, algorithme, réseau, polyèdre.

Contents

1 Introduction

Organization. The four main parts are chapters 3, 4, 5, and 7. Inbetween, the necessary definitions and background are introduced as needed. The remainder of chapter 1 will be dedicated to summarizing the content and main results of each of chapters 3, 4, 5, and 7.

Attribution/collaboration disclaimer.

- Chapter 3 is based on joint work with Tom Kaiser [BK21],
- Chapters 4 and 5 are also based on joint work with Tom Kaiser, mostly available at [BK20],
- Chapter 7 is ongoing work with Nikolay Bogachev and Alexander Perepechko.

1.1 Magnitude homology, median spaces and diagonality

Chapter 3 has two somewhat independent purposes, both related to the notion of *magnitude homology*—a categorification of *magnitude*—in short defined as follows. Let X be a metric space. Given two points x, y of X, the interval $[x, y] \subseteq X$ is the set of points $z \in X$ satisfying $d(x, z) + d(z, y) = d(x, y)$. The *magnitude homology* of X is the homology of its *magnitude complex*, which, at height k, is spanned freely by $(k+1)$-tuples of consecutively distinct points in X. Its boundary operator is the alternating sum of the maps ∂_i, each defined as sending the tuple $\langle x_0, \dots, x_i, \dots, x_k\rangle$ to $\langle x_0, \dots, \hat{x}_i, \dots, x_k\rangle$ if $x_i \in [x_{i-1}, x_{i+1}]$, and to 0 otherwise.

Defining the *length* $l(\mathbf{x})$ of a $(k+1)$-tuple $\mathbf{x} := \langle x_0, \dots, x_k\rangle$ as the sum

$$l(\mathbf{x}) := \sum_{i=0}^{k-1} d(x_i, x_{i+1}),$$

we see that the boundary operator preserves length. It follows that magnitude homology is graded (in $\mathbb{R}_{\geq 0}$) by the length.

Magnitude Homology of Median Spaces Our first aim is to analyze the magnitude homology of median metric spaces. Recall that the (metric) space X is called *median* if, for any three distinct points $x, y, z \in X$, the intersection $[x, y] \cap [y, z] \cap [z, x]$ is a singleton. Median spaces are relatively common: examples include trees (more generally $\mathbb{R}$-trees), any product of median spaces with the l^1 metric, and skeleta of CAT(0) cube complexes. For more examples and information on median spaces, see [CDH10]. We call a $(k+1)$-tuple $\langle x_0, \dots, x_k\rangle$ *saturated* if each interval $[x_i, x_{i+1}]$ contains only x_i and x_{i+1}, and, extending the notion of *diagonality* defined in [HW17], say that X is *diagonal* if all its magnitude homology groups are spanned by linear combinations of saturated tuples.

We obtain the following.

Proposition 3.5.3. Median metric spaces are diagonal.

In Section 3.2, we verify that median *graphs* are diagonal. The key tool here is the characterization of median graphs as retracts of hypercubes, due to Bandelt [Ban84]. Then, in Section 3.4, we formally define diagonality and verify that it is stable under some constructions on metric spaces (e.g. retracts, products, and filtrations). Finally, Section 3.5 is devoted to proving diagonality of median spaces, by what amounts to an approximation of a median space by finite median graphs. The argument uses the stability of diagonality mentioned above, along with an equivalence between finite median spaces and finite median graphs, due to Avann [Ava61].

Künneth and Mayer-Vietoris Formulae Section 3.3 is dedicated to the second part of this chapter: verifying that the Künneth and Mayer-Vietoris formulae for graph magnitude homology, proven in [HW17], generalize to the metric setting.

Recall that the l^1 *product* of two metric spaces X, Y has the Cartesian product $X \times Y$ as underlying set, and its distance map is given by

$$d_{X\times Y}((x,y),(x',y')) := d_X(x,x') + d_Y(y,y').$$

The "metrized" variant of the Künneth theorem in [HW17] has the following form.

Proposition 3.3.2 (Künneth theorem—metric case)**.** If X, Y are metric spaces and $X\times Y$ is their l^1 product, then there exists a natural "cross-product" morphism

$$\mathrm{MH}_*(X) \otimes \mathrm{MH}_*(Y) \xrightarrow{\square} \mathrm{MH}_*(X\times Y)$$
$$[f]\otimes[g] \mapsto [f\square g],$$

which fits into a natural short exact sequence

$$0 \to \mathrm{MH}_*(X)\otimes \mathrm{MH}_*(Y) \xrightarrow{\square} \mathrm{MH}_*(X\times Y) \to \mathrm{Tor}(\mathrm{MH}_{*-1}(X), \mathrm{MH}_*(Y)) \to 0.$$

The proof found in [HW17] translates rather easily to the metric setting, as is done in Section 3.3.1.

The Mayer-Vietoris formula in [HW17] uses so-called *projecting decompositions* of graphs. We will use the following metric analogue of a projecting decomposition. A *gated decomposition* of the metric space X is a pair of subspaces Y, Z satisfying $X = Y \cup Z$ and such that:

- $Y \cap Z$ is *gated* relative to Z: given any $z \in Z$, there is a unique element $y_z \in Y\cap Z$, called a *gate*, such that for any $y \in Y \cap Z$, y_z is in $[z,y]$.
- $Y \cap Z$ lies "between" Y and Z: for any $z \in Z$ and $y \in Y$, there exists some $w \in Y\cap Z$ such that w is in $[y,z]$.

The notion of gated subsets in metric spaces is fairly well-studied (see e.g. [DS87]).

Our "metrized" variant of the Mayer-Vietoris theorem of [HW17] has the following form.

Theorem 3.3.12 (Mayer-Vietoris—metric case)**.** If $X = Y \cup Z$ is a gated decomposition of X and $W = Y \cap Z$, then the inclusions

$$j_Y : W \to Y,\ j_Z : W \to Z,\ i_Y : Y \to X,\ i_Z : Z \to X$$

induce a short exact sequence

$$0 \to \mathrm{MH}_*(W) \xrightarrow{\langle (j_Y)_*, -(j_Z)_*\rangle} \mathrm{MH}_*(Y)\oplus \mathrm{MH}_*(Z) \xrightarrow{(i_Y)_*\oplus(i_Z)_*} \mathrm{MH}_*(X) \to 0.$$

The proof requires a bit more care than the Künneth theorem, since the discreteness of graphs allows for some simplifying assumptions. Unlike in [HW17], no naturality property for the Mayer-Vietoris formula is given here, though we do expect a naturality statement similar to the one in [HW17] to hold in the metric case, assuming the right setup.

A remark on betweenness. In a metric space X, we say that a point z is *between* two given points x and y if $z \in [x, y]$, that is, if $d(x, y) = d(x, z) + d(z, y)$. When one discards the grading, the magnitude homology groups only depend on the "betweenness" relation. Thus, many arguments can be worked out without appealing to either notion of length or distance, instead relying only on betweenness. We strove to make this reliance as apparent as possible, while de-emphasizing the length grading. The advantage of this approach appears, for instance, in the proof of Proposition 3.5.3.

1.2 The first Roe homology

In chapter 4, we study the first group of so-called "Roe"[1] homology of locally finite graphs. Fix a locally finite connected graph X (all vertices have finite degree), and consider the increasing Rips complexes $R_r X$ ($r \in \mathbb{N}$) defined over X. On any one of these complexes, one can define a chain complex whose chains are functions from simplices to the coefficient ring A, or "infinite formal sums" of simplices. The inclusion of chains corresponding to a given Rips complex to a larger Rips complex are chain maps, and one can take the inductive limit of those chain complexes. The homology $H_\bullet^\infty(X, A)$ of this inductive limit is what we call *Roe homology*—it is a special case of Roe's coarse homology, defined in greater generality (see e.g. [Roe03]). In particular, $H_\bullet^\infty(X, A)$ is invariant under quasi-isometries [Mos03, Step 3 in the proof of Theorem 12].

We focus only on the *first* group of Roe homology, $H_1^\infty(X, A)$, which enjoys a relatively intuitive combinatorial description. Recall that a (A-valued) flow on a locally finite graph X is an assignment, to each (preliminarily oriented) edge of X, of a value in A, in such a way that, for all vertices v of X, the sum of values of edges pointing to v is equal to the sum of values of edges pointing away from v (i.e., "in=out").

Theorem 4.1.4. If X is a locally finite graph and A a ring, then

$$H_1^\infty(X, A) = \frac{Z_1(X, A)}{\mathscr{C}_\infty(X, A)},$$

where $Z_1(X, A)$ is the space of all (A-valued) flows on X, and $\mathscr{C}_\infty(X, A)$ is the space of all flows which can be decomposed as (potentially infinite) sums of circuits of bounded length.

In a sense, this description tells us that $H_1^\infty(X, A)$ measures the existence of flows which don't arise as "local" phenomena.

In [Dia15], Diana shows that for a tree T with a finite number n of ends, $H_1^\infty(T, A)$ has dimension $n - 1$[2]. If T has infinitely many ends, a similar description can be obtained:

[1] This denomination does not appear to be commonplace.

[2] Diana's result uses uniformly finite homology but for finite-ended trees, Roe and uniformly finite homology agree.

Theorem 4.2.1. If T is a locally finite tree, and $\mathscr{B}$ a carefully chosen[3] set of birays on T which satisfies in particular $|\mathscr{B}| + 1 = |\mathrm{Ends}(T)|$, then

$$H_1^\infty(T, A) \cong A^{\mathscr{B}}.$$

Thus, as can be guessed from theorem 4.1.4, the group $H_1^\infty(T, A)$ is entirely characterized by the ends of T.

We then obtain a decomposition of $H_1^\infty(X, A)$ in the general case.

Theorem 4.3.1. Let X be a locally finite graph and T a carefully chosen[4] subtree of X. Then

$$H_1^\infty(X, A) \cong H_1^\infty(T, A) \oplus \frac{\mathscr{C}(X, A)}{\mathscr{C}_\infty(X, A)},$$

where $\mathscr{C}(X, A)$ is the space of flows on X that can be written as (potentially infinite) sums of circuits.

Note that $\mathscr{C}_\infty(X, A)$ differs from $\mathscr{C}(X, A)$ in that the latter includes sums of circuits of arbitrarily large length, while in the former, each sum only consists of circuits of at most a given length. In other words, the first homology of X is the direct sum of two terms: the first comes from the ends of X, while the second measures the extent to which "sums of circuits" differ from "sums of circuits of bounded length".

From this decomposition, the following are clear:

- If X has at least two ends, its first homology is non trivial.
- If X has circuits of arbitrarily large length that cannot be decomposed into smaller circuits (think: chaining together increasingly long cycle graphs), its first homology is also non trivial. Note that this part is dependent on the choice of coefficient rings!

Finally, in the case where $A = \mathbb{Z}/2\mathbb{Z}$, we can further improve our understanding of $H_1^\infty(X, A)$:

Theorem 4.4.6. If X is a (locally finite) graph and $A = \mathbb{Z}/2\mathbb{Z}$, then $\frac{\mathscr{C}(X,A)}{\mathscr{C}_\infty(X,A)}$ is non-vanishing if and only if it is infinite-dimensional.

The proof of the above uses both compactness/finiteness of $A = \mathbb{Z}/2\mathbb{Z}$, and the fact that A is a field.

1.3 The first uniformly finite homology in $\mathbb{Z}$

In chapter 5, we study the first group of *uniformly finite* homology on uniformly locally finite graphs. Recall that a graph X is said to be *uniformly locally finite* if there is a bound on the degree of all its vertices. If A is a normed coefficient ring (say $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$), we can adapt the construction of Roe homology on X with coefficients in A by restricting the chains to be the maps from simplices to A that are *uniformly bounded* (i.e., elements of ℓ^∞). Using the assumption that X is uniformly locally finite, one gets a well-defined new homology $H_\bullet^{\mathrm{uf}}(X, A)$, called *uniformly finite homology*, due to Block and Weinberger [BW92]. As for Roe homology, uniformly

[3] Such a set always exists.
[4] The requirement is that the inclusion of rays from T to X induces a bijection $\mathrm{Ends}(T) \to \mathrm{Ends}(X)$

finite homology is a quasi-isometry invariant ([Mos03, Step 3 in the proof of Theorem 12]), and [BW92] further proved that $H_0^{\mathrm{uf}}(X, \mathbb{Z})$ vanishes exactly when X is a non-amenable graph. This served as a motivation for our study of $H_1^{\mathrm{uf}}(X, \mathbb{Z})$, our goal being to find a reasonably intuitive characterization for the vanishing of $H_1^{\mathrm{uf}}(X, \mathbb{Z})$.

By simple adaptations of the arguments of chapter 4, we easily get the following.

Theorem 5.1.5. If X is a uniformly locally finite graph, then

$$H_1^{\mathrm{uf}}(X, A) \cong \frac{Z_1^{\mathrm{suf}}(X, A)}{\mathscr{C}_\infty^{\mathrm{suf}}(X, A)},$$

where $Z_1^{\mathrm{suf}}(X, A)$ is the space of all uniformly bounded flows on X, and $\mathscr{C}_\infty^{\mathrm{suf}}(X, A)$ is the space of all flows which can be decomposed as uniformly bounded sums of circuits of bounded length.

Theorem 5.2.1. If T is a uniformly locally finite tree, and $\mathscr{B}$ a carefully chosen[5] set of birays on T, then

$$H_1^{\mathrm{uf}}(T, A) \cong \ell^\infty(\mathscr{B}, A).$$

Thus, similarly to theorem 4.2.1, $H_1^{\mathrm{uf}}(T, A)$ counts the ends of T.

Recall that for Roe homology, Theorem 4.3.1 provides a decomposition of the first homology into a direct sum corresponding to, respectively, the ends of the graph, and its circuit structure. In the case of uniformly finite homology, there is no such decomposition that we know of, but the following can still be said:

Theorem 5.2.2. Let X be a uniformly locally finite graph and T a carefully chosen[6] subtree of X. Then $H_1^{\mathrm{uf}}(T, A)$ embeds in $H_1^{\mathrm{uf}}(X, A)$.

Thus, as soon as X has at least 2 ends, $H_1^{\mathrm{uf}}(X, A)$ does not vanish.

Next, in the case of transitive graph, we find another phenomenon responsible for the non-vanishing of $H_1^{\mathrm{uf}}(X, A)$:

Lemma 5.3.1. Let X be a uniformly locally finite *transitive* graph, and $\mathscr{C}_r^{\mathrm{suf}}(X, A)$ the space of uniformly bounded flows obtained as uniformly bounded sums of circuits of length at most r ($r \in \mathbb{N}$). If

$$\mathscr{C}_r^{\mathrm{suf}}(X, A) \neq \mathscr{C}_\infty^{\mathrm{suf}}(X, A),$$

for all $r \in \mathbb{N}$, then $H_1^{\mathrm{uf}}(X, A) \neq 0$.

The assumption of transitivity can be relaxed slightly, as we will see.

Finally, we define two related notions of n-expansion ($n \in \mathbb{N}$) for simplicial complexes, and "coarsifications" of these; in the case $n = 0$, we recover the usual definition of expansion on graphs. Being relatively opaque (though arguably natural), we postpone the definitions.

The important property of (our variants of) n-expansion is that it allows a characterization of the vanishing of $H_1^{\mathrm{uf}}(X, \mathbb{Z})$ when X is transitive.

Theorem 5.4.15. Let X be a uniformly locally finite transitive graph. Then $H_1^{\mathrm{uf}}(X, \mathbb{Z}) \neq 0$ if and only if either of the three following phenomena occur:

[5] Same condition as in theorem 4.2.1.

[6] Similarly to theorem 4.2.2, the condition is that the inclusion $T \leq X$ induces an injection $\mathrm{Ends}(T) \to \mathrm{Ends}(X)$.

1. X has more than one end;
2. for all r, $\mathscr{C}_r^{\mathrm{suf}}(X, \mathbb{Z}) \neq \mathscr{C}_\infty^{\mathrm{suf}}(X, \mathbb{Z})$;
3. X does not have what we call *pure* H_1^{uf}*-expansion*.

1.4 Implementing Vinberg's algorithm

Chapter 7 contains some comments on mostly practical aspects of the implementation of Vinberg's algorithm on computer.

Given a totally real number field K embedded in $\mathbb{R}$, with ring of integers R, consider a symmetric $(n+1)\times(n+1)$ matrix Q with coefficients in R. This matrix is said to define an admissible Lorentzian quadratic form on $V := \mathbb{R}^{n+1}$ (defined as $x, y \mapsto x^t Q y$) if:

- the signature of Q is $(n, 1)$;
- for any non-trivial embedding σ of K in $\mathbb{R}$, the matrix Q^σ obtained by applying σ to the coefficients of Q is positive definite (i.e., of signature $(n+1, 0)$).

In case $K = \mathbb{Q}$ (and $R = \mathbb{Z}$), being admissible simply means being of signature $(n, 1)$. Let L denote the lattice $(R)^{n+1}$ sitting inside V. Let $\mathscr{O}$ denote the group of linear automorphisms of V that preserve Q and $\mathscr{O}^L$ the subgroup of automorphisms of the lattice L, also preserving Q. Up to taking an index-two subgroup, those two groups can be seen as groups of isometries of hyperbolic space $\mathbb{H}^n$: $\mathscr{O}$ being a double-cover of the full group of isometries $\tilde{\mathscr{O}}$ of $\mathbb{H}^n$, and $\mathscr{O}^L$ a double-cover of a subgroup $\tilde{\mathscr{O}}^L$ thereof. It turns out, by results of Borel and Harish-Chandra, Venkov, and Vinberg, that $\tilde{\mathscr{O}}^L$ is a lattice in the Lie group $\tilde{\mathscr{O}}$, meaning that

- $\tilde{\mathscr{O}}^L$ is a discrete subgroup of $\tilde{\mathscr{O}}$;
- $\tilde{\mathscr{O}}^L$ is of finite covolume in $\tilde{\mathscr{O}}$, i.e., a fundamental polyhedron for $\tilde{\mathscr{O}}^L$ in $\mathbb{H}^n$ is of finite volume.

Being a group of isometries of hyperbolic space, we can consider the subgroup $\tilde{\mathscr{O}}_r^L$ of $\tilde{\mathscr{O}}^L$ generated by the reflections (of hyperbolic space, along hyperplanes) contained in $\tilde{\mathscr{O}}^L$. We will say that the lattice L (by which we actually mean the pair (L, Q)) is reflective if $\tilde{\mathscr{O}}_r^L$ is of finite covolume in $\tilde{\mathscr{O}}$; if that is the case, L is furthermore called anisotropic if $\tilde{\mathscr{O}}_r^L$ is cocompact in $\tilde{\mathscr{O}}$, and isotropic otherwise. It has been shown by Vinberg that in high enough dimensions, reflective lattices don't exist. Vinberg also provided a decomposition of $\tilde{\mathscr{O}}^L$ into a semidirect product

$$\tilde{\mathscr{O}}^L = \tilde{\mathscr{O}}_r^L \rtimes S,$$

where $S := (\operatorname{Sym} P) \cap \tilde{\mathscr{O}}^L$ with $\operatorname{Sym} P$ the group of automorphisms of the fundamental polyhedron P. Thus, reflectivity of $\tilde{\mathscr{O}}^L$ can be restated as finiteness of S.

The (technical) purpose of Vinberg's algorithm is the construction of the fundamental polyhedron P for $\tilde{\mathscr{O}}_r^L$ by enumeration of its sides; if $\tilde{\mathscr{O}}^L$ is reflective, this is a finite process. If $\tilde{\mathscr{O}}^L$ is not reflective, Vinberg's algorithm does not terminate. In that case, though, a strategy devised by Bugaenko allows to detect non reflectivity after a finite (not known in advance) number of steps of Vinberg's algorithm, in some cases; this is usually referred to as the "infinite-order symmetry test". The main idea is that non-reflectivity is equivalent to S being infinite, which is itself equivalent to the intersection of the sets of fixed points of enough elements of S (viewed as isometries of $\mathbb{H}^n$) being empty. The (socio-academical) purpose of Vinberg's algorithm is to find high dimensional hyperbolic finite volume and/or compact Coxeter polyhedra (Coxeter, meaning that the angles between adjacent faces are of the form π/n for $n = 2, 3, \ldots$). Notably, the record such *compact* polyhedron (dimension 8) was found by Bugaenko using this method in 1992. It is somewhat surprising (and personally disappointing) that after almost 30 years and multiple computer implementations of the algorithm, Bugaenko still holds the record!

After some comments on the question of decidability of reflectivity of $\tilde{\mathcal{O}}^L$, we will focus on some aspects of the implementation of this algorithm as an actual computer program.

Apart from manual applications of the algorithm (by Vinberg himself and others) and some ad-hoc implementations for specific purposes (Bugaenko, Mark, McLeod, Allcock, Nikulin), two implementations are commonly known:

- The first one is a C++ program by Guglielmetti [Gug17a]. It has the advantages of being comparatively very efficient, and easy to use. It also contains a version of the infinite-order symmetry test (and quite a few further functionalities), and can thus be (at least theoretically) used to prove both reflectivity and non-reflectivity. Its main drawback is that it only deals with diagonal quadratic forms. Another one is that it is not entirely *field-agnostic*, in that the range of possible number fields usable is restricted to $\mathbb{Q}(\sqrt{d})$ (for the majority of sensible $d \in \{1, \ldots, 100\}$) and $\mathbb{Q}(\cos(\frac{2\pi}{7}))$. This is probably a minor inconvenience, since we know, by results of Vinberg, that reflective lattices are only defined over $\mathbb{Q}(\sqrt{2}), \mathbb{Q}(\sqrt{5}), \mathbb{Q}(\cos(\frac{2\pi}{7}))$ in dimension $n \geq 22$.

- The second is a SageMath/Python script by Bogachev and Perepechko [BP17; BP18]. This implementation only deals with forms over $\mathbb{Z}$, but covers non diagonal ones.

The implementation presented here combines aspects of both [Gug17a; BP17] and is, at least theoretically, applicable to any (totally real) number field K for which R is principal and any admissible form over R. It is written in the Julia programming language [Bez+17], and uses the Hecke number theory package [Fie+17] to cover the necessary number theoretic aspects, along with quadratic forms, linear algebra, etc.

2 Preliminaries and notation: graphs and metric spaces

2.1 Graphs

Recall that a (combinatorial) graph X is the data of a set VX of vertices and a symmetric, anti-reflexive relation $EX \subseteq (VX)^2$, whose elements, up to symmetry, are called edges. Two vertices, u, v are neighbors, or adjacent, written $u \sim v$, if $(u, v) \in EX$ and the valency or degree of a vertex v is the number of its neighbors. The graph X is said to be locally finite if all vertices of X have finite valency; it is uniformly locally finite if there exists some natural number K such that all vertices have valency at most K. A path in X is a sequence of vertices, each neighboring its successor in the sequence; it is called simple if no vertex appears twice. The index set of the sequence can be

- $\{0, \dots, n\}$ for some positive integer n, or
- $\mathbb{N}$, in which case the path is called a ray, or
- $\mathbb{Z}$, in which case it is called a biray.

The length of the path is n in the first case, and infinite in the latter two. A circuit is a path of finite length such that the first vertex in the sequence is equal to the last; it is simple if no vertex, except the first/last, appears twice. The graph X is connected if any pair of its vertices appear in some common path. The induced distance on the connected graph X is the metric on VX obtained by setting

$$d_X(u, v) = \text{length of a shortest path containing both } u \text{ and } v.$$

If X is not connected, d_X is sometimes called a pseudo-distance. If $v \in VX$ is a vertex, the ball $B(v, r)$ of radius $r \in \mathbb{N}$ is the set of vertices at distance at most r from v. Note that if X is locally finite, all balls are finite, while if X is uniformly locally finite with all vertices of valency $\leq K$, all balls of radius r have cardinality at most $\frac{K^{r+1}-1}{K-1}$. From now on, all graphs are assumed to be locally finite and connected. Two rays in X are said to be close if no removal of a finite set of vertices (a compact set) disconnects the rays; being close is an equivalence relation on the set of rays. An end of X is an equivalence class of rays under "being close". Let us write Ends X for the set of ends of X; it is common knowledge that Ends X can also be described as the appropriate limit of the inverse system $\{X - K \mid K \text{ compact}\}$.

An orientation O of X is a subset of EX such that for each $e = (u, v) \in EX$, exactly one of $(u, v), (v, u)$ lies in O, that is, we direct the edges. If O is an orientation of X, a flow on X is a function $f : O \to A$ (for some ring A) satisfying:

$$\sum_{\substack{u \text{ s.t.} \\ (u,v) \in O}} f(u, v) - \sum_{\substack{u \text{ s.t.} \\ (v,u) \in O}} f(v, u) = 0 \quad \forall v \in VX.$$

If $U \subseteq VX$, we let $\partial^e U$ denote the set of edges with exactly one endpoint in U. The value of a flow f on the boundary $\partial^e U$ is defined as

$$\sum_{e \in (\partial^e U) \cap O} f(e).$$

An automorphism of X is a function $\phi : VX \to VX$ satisfying $u \sim v \Leftrightarrow \phi(u) \sim \phi(v)$; X is called (vertex) transitive if for any pair of vertices $v_1, v_2 \in VX$, there exists an automorphism ϕ of X such that $\phi(v_1) = v_2$.

Given two graphs X and Y, a map $\phi : VX \to VY$ is said to be a quasi-isometry (abbreviated *QI*) if there exists a constant $K > 0$ such that:

- If $v, v' \in VX$ are at distance d, then $\phi v, \phi v'$ are at distance at least $K^{-1}d - K$ and at most $Kd + K$.
- Any $v \in VY$ is at distance at most K from some element of $\phi[VX]$.

Two graphs X and Y are said to be *quasi-isometric* if there exists some quasi-isometry from one to the other; this is an equivalence relation. Equivalently, X, Y are quasi-isometric if there exist $\phi : VX \to VY$, $\psi : VY \to VX$ and K such that:

- $d_Y(\phi v, \phi v') \leq K d_X(v, v')$ for all $v, v' \in X$;
- $d_X(\psi u, \psi u') \leq K d_Y(u, u')$ for all $u, u' \in Y$;
- $d_X(\psi\phi v, v) \leq K$ for all $v \in X$;
- $d_Y(\phi\psi u, u) \leq K$ for all $u \in Y$,

in which case ϕ, ψ are called quasi-inverses to each other.

Note that the cardinality of $\mathrm{Ends}(X)$ is quasi-isometry invariant (actually, a quasi-isometry $\phi : X \to Y$ functorially induces a map $\mathrm{Ends}(\phi) : \mathrm{Ends}(X) \to \mathrm{Ends}(Y)$).

Cayley graphs. If G is a group and $S \subseteq G$ a set of generators of G, the graph $C(G, S)$ whose vertices are the elements of G and such that g is adjacent to h if and only if $gh^{-1} \in S \cup S^{-1}$ ($g \neq h$) is called the Cayley graph of G with respect to S. This graph is always transitive, and if S is finite, it is also locally finite, hence uniformly locally finite.

2.2 Simplicial complexes

Recall that a simplicial complex is the data of a set V of vertices, plus a subset X of the set of finite subsets $\mathscr{P}_f(V)$ of V, such that X

- contains all singletons $\{v\}$ for $v \in V$; and
- is closed under subsets: if $\tau \subseteq \sigma \in X$, then $\tau \in X$.

The elements of X are called the simplices of X. Let us write X_n or $X_{(n)}$ for the simplices of cardinality $n + 1$, called n-simplices. The n-skeleton of X is the simplicial complex consisting of all simplices of X of cardinality at most $n + 1$; the 1-skeleton of a simplicial complex is (equivalent to) a graph. X is said to be [uniformly] locally finite if its 1-skeleton is [uniformly] locally finite. The faces of an n-simplex σ are the $(n-1)$-simplices contained in σ; we write $\tau < \sigma$ to say that τ is a face of σ. Finally, an *orientation* of X consists (for our purposes) in a choice of sign $[\tau : \sigma] = \pm$ for each simplex σ and face τ of σ, such that if θ is a common face of τ, τ', both faces of σ, then $[\theta : \tau'] \cdot [\tau' : \sigma] = -[\theta : \tau] \cdot [\tau : \sigma]$. From now on, assume that all simplicial complexes are endowed with a choice of orientation (it does not matter which one). Any graph defines a simplicial complex. Fix $r \in \mathbb{N}^{>0}$ and a graph X; the r-th *Rips complex* $R_r(X)$ is the simplicial complex on vertex set VX, and with simplices:

$$\{\sigma \in \mathscr{P}_f(V) : \; d(u,v) \leq r \, \forall u, v \in \sigma\},$$

where $\mathscr{P}_f(S)$ is the set of finite subsets of S. In other words, the simplices are the finite subsets of vertices of diameter at most r. It is clear that the 1-skeleton of $R_1(X)$ is just X, and that if $r \leq s$, then $R_r(X) \subseteq R_s(X)$. If X is a [uniformly] locally finite graph, then $R_r(X)$ is also [uniformly] locally finite. We will call any simplex in some $R_r(X)$ a *large* simplex, in contrast to the *small* simplices of X.

3 Magnitude homology, median spaces and diagonality

In this chapter, we verify that the Künneth and Mayer-Vietoris formulae for magnitude homology of graphs, proven by Hepworth and Willerton, generalize naturally to the metric setting. Similarly, we extend the notion of diagonality of graphs to metric spaces, and verify its stability under products, retracts, and filtrations. As an application, we show that median spaces are diagonal; in particular any Menger convex median space has vanishing magnitude homology.

3.1 Background

In this section, we provide necessary definitions and settle on notation used in the chapter.

3.1.1 Metric Spaces

Let (X, d) be a metric space. Finite sequences of points in X are written using angle brackets: $\mathbf{x} = \langle x_0, \dots, x_k \rangle$ and identified with maps $[k] \to X$, where $[k] := \{0, \dots, k\}$. If $x_i \neq x_{i+1}$ for all $0 \leq i < k$, we call such a sequence a *k-path*[1]; the set of k-paths in X is denoted by $P_k(X)$, and the set of all paths by $P(X)$. The *length* $l(\mathbf{x})$ of a k-path $\mathbf{x} = \langle x_0, \dots x_k \rangle$ is defined as the sum

$$l(\mathbf{x}) := \sum_{i=0}^{k-1} d(x_i, x_{i+1}).$$

Given two points $x, y \in X$, we say that a third point $z \in X$ lies *between* them if

$$d(x, y) = d(x, z) + d(z, y).$$

In other words, z turns the triangle inequality into an equality. If furthermore $z \neq x$ and $z \neq y$, we say that z lies *strictly between* x and y. We write $[x, y]$ for the set of points between x and y and $]x, y[$ for those strictly between. We call $[x, y]$ and $]x, y[$ *intervals* for obvious reasons. A k-path $\mathbf{x}$ is *saturated* if each strict interval $]x_i, x_{i+1}[$ is empty. A metric space is *Menger convex* if no strict interval between distinct points is empty.

A map $f\colon X \to Y$ between metric spaces is *non-expanding* (or 1-*Lipschitz*) if for all $x, x' \in X$, we have

$$d(fx, fx') \leq d(x, x').$$

We let Met denote the category with objects metric spaces and morphisms given by 1-Lipschitz maps.

A subset A of a metric space X is *convex* if for all $a, b \in A$, the interval $[a, b]$ in X is contained in A; in other words, any point between two points of A is also in A. Note that this definition is stronger than the one found in [HW17] for graphs.

If X is a set and A is a directed set, a *filtration* of X is a family $(U_\alpha)_{\alpha \in A}$ of subsets of X

[1] This definition of *path* supersedes the graph-theoretical one of chapter 2 for the rest of this chapter.

satisfying $\bigcup_\alpha U_\alpha = X$ and such that for any $\alpha \leq \beta$, we have $U_\alpha \subseteq U_\beta$.

If Y is a subspace of X, a *retraction* of X onto Y is a 1-Lipschitz map $f\colon X \to Y$ satisfying $f|Y = \mathrm{Id}_Y$.

Finally, if X and Y are metric spaces, we will always endow the set $X \times Y$ with the l^1 metric, that is:

$$d_{X\times Y}((x,y),(x',y')) := d_X(x,x') + d_Y(y,y').$$

Note that the l^1 product of metric space is *not* the categorical product in Met. Instead, under the interpretation of metric spaces as categories enriched over $[0,\infty]$ (so-called *Lawvere metric spaces*), the l^1 product becomes the *tensor* product in that category (see [Lei13, Section 1.4]).

3.1.2 Graphs

For simplicity's sake, we will assume all graphs to be connected. This assumption makes the pair (V, d_G), where d_G is the induced distance, into a metric space; note that d_G takes values in $\mathbb{N}$, thus has discrete range.

In the case of graphs, the condition for 1-Lipschitz maps can be restated in terms of edges as follows: If $G = (V,E)$ and $G' = (V',E')$ are graphs, then $f\colon V \to V'$ is 1-Lipschitz if and only if, for all $u, v \in V$, $(u,v) \in E$ implies $(fu, fv) \in E'$ or $fu = fv$.

From the data (V, d_G), the set E of edges of G can easily be recovered: it is the set of pairs (u,v) satisfying $d_G(u,v) = 1$. With this fact in mind, we will henceforth view graphs as special cases of metric spaces (given by pairs (V, d_G)) rather than combinatorial objects (given by pairs (V,E)).

A subset U of vertices of a graph $G = (V,E)$ can be understood to define a metric space in two ways. On the one hand, one can look at the graph $G(U) := (U, E \cap U^2)$, and consider the induced distance $d_{G(U)}$; on the other hand, one can simply restrict d_G to $U \times U$. According to our interpretation of graphs as "special" metric spaces, we will only use the second definition.

In the case of graphs, it is easily seen that the graph-theoretic Cartesian product agrees with our definition of the l^1 product of metric spaces. Hence, from now on, we will use the term "Cartesian product" with this definition in mind, and not that of a *categorical* direct product.

Since our definitions differ from the ones in [HW17], some care has to be taken in the "translation" process. The results stated in [HW17] tend to assume a more combinatorial setting, while the generalizations we present here are stated from a metric viewpoint.

3.1.3 Magnitude Homology

Magnitude homology is a $\mathbb{R}_{\geq 0}$-graded homology introduced in [HW17] for graphs, and in [LS17] for arbitrary metric spaces, among other structures. It is a categorification of magnitude, itself an invariant (in the form of a power series) of finite metric spaces introduced in [Lei13]. More precisely, in the case of a finite graph X, the magnitude $\#X$ of X is recovered from its magnitude homology $(\mathrm{MH}_k^l(X))_{k\in\mathbb{N}}^{l\geq 0}$ via the formula (borrowed from [HW17, p.32]):

$$\#X(q) = \sum_{k,l} (-1)^k \operatorname{rank}(\mathrm{MH}_k^l(X)) \cdot q^l,$$

that is, as a "weighted" Euler characteristic. We will focus strictly on magnitude *homology*, but the interested reader can find an up-to-date bibliography of magnitude and magnitude homology maintained by T. Leinster in [Lei].

Let us recall its definition, in the metric case:

Definition 3.1.1 (Magnitude complex)**.** Let X be a metric space. The *magnitude complex* of X is the chain complex $(\mathrm{MC}_*(X), \partial_*)$ defined by:

$$\mathrm{MC}_k(X) \coloneqq \mathbb{Z}[P_k(X)],$$

(the free abelian group on the set of all k-paths in X) with boundary map

$$\partial_k \coloneqq \sum_{i=1}^{k-1} (-1)^i \partial_{k,i} : \mathrm{MC}_k(X) \to \mathrm{MC}_{k-1}(X),$$

where $\partial_{k,i} : \mathrm{MC}_k(X) \to \mathrm{MC}_{k-1}(X)$ is defined by

$$\partial_{k,i}\langle x_0, \dots, x_i, \dots, x_k\rangle \coloneqq \begin{cases} \langle x_0, \dots, \widehat{x_i}, \dots, x_k\rangle & \text{if } x_i \text{ is between } x_{i-1} \text{ and } x_{i+1}; \\ 0 & \text{otherwise.} \end{cases}$$

We will write $\mathrm{MZ}_k(X)$ for $\ker \partial_k$ and $\mathrm{MB}_k(X)$ for $\operatorname{im} \partial_{k+1}$ respectively, so that the k-th *magnitude homology* group of X is

$$\mathrm{MH}_k(X) \coloneqq \mathrm{MZ}_k(X)/\mathrm{MB}_k(X).$$

The magnitude complex enjoys a grading on $\mathbb{R}_{\geq 0}$ by letting $\mathrm{MC}_k^l(X)$ be the subgroup of $\mathrm{MC}_k(X)$ spanned by the k-paths of length l.

A 1-Lipschitz map $f : X \to Y$ induces a morphism of magnitude complexes by letting:

$$\mathrm{MC}_k(f)\langle x_0, \dots, x_k\rangle \coloneqq \begin{cases} \langle f x_0, \dots, f x_k\rangle & \text{if } l\langle f x_0, \dots, f x_k\rangle = l\langle x_0, \dots, x_k\rangle; \\ 0 & \text{otherwise.} \end{cases}$$

Thus, $\mathrm{MC}_*(\bullet)$ defines a functor from the category Met of metric spaces and 1-Lipschitz maps to the category of chain complexes over $\mathbb{Z}$ with $\mathbb{R}_{\geq 0}$-grading.

3.2 Median Graphs

Definition 3.2.1 (Median graphs)**.** A graph X is said to be *median* if, for any three pairwise distinct points $x, y, z \in X$, the intersection $[x, y] \cap [y, z] \cap [x, z]$ consists of a single point, written $m(x, y, z)$.

Median graphs have been thoroughly studied. The two prototypical examples of median graphs are trees and hypercubes. Since, in a Cartesian product of graphs, the interval between two pairs of points is exactly the product of the respective intervals in each coordinate, it is easily seen that a Cartesian product of median graphs is still median. Any convex subgraph of a median graph is median as well, thus yielding many other examples. Finally, it has been shown by Chepoi ([Che00, Theorem 6.1]) that median graphs are exactly the 1-skeleta of CAT(0) cube complexes. The cycle graph of length 3 and the graph obtained by adding a diagonal path of length 2 to a square are both examples of *non* median graphs, the former failing at *existence*, and the latter at *uniqueness* of middle points (that is, vertices in the intersection of the intervals given by three distinct vertices).

In a graph X, all paths have integral length, which implies that the groups $\mathrm{MC}_k^l(X)$ vanish when $l \notin \mathbb{N}$. Furthermore, no k-path has length less than k, so the groups $\mathrm{MC}_k^l(X)$ also vanish when $l < k$. Graphically, this means that, when placed on a quadrant of the $\mathbb{Z}^2$ grid, all magnitude homology groups vanish "above the diagonal".

Definition 3.2.2 (Diagonality ([HW17, Definition 7.1]))**.** A graph X is *diagonal* if, for all $k \neq l \in \mathbb{N}$, the group $\mathrm{MH}_k^l(X)$ vanishes.

In other words, X is diagonal if the magnitude homology groups vanish outside the diagonal. In this section, and as preparation for Sections 3.4 and 3.5, we prove the following:

Proposition 3.2.3. *Median graphs are diagonal.*

We will shortly present a beautiful characterization of median graphs, due to Bandelt, which will be used in the proof of Proposition 3.2.3. Some preliminary definitions are necessary:

Fix a set X, and let Q_X be the graph whose vertices are the *finite* subsets of X, and which has an edge between two subsets if and only if either contains exactly one more element than the other. A graph isomorphic to some Q_X is called a *hypercube* ([BM83]).

If X is a graph, $Y \subseteq X$ a subspace (recall our convention on subspaces of graphs), and $f\colon X \to Y$ a 1-Lipschitz map satisfying $f|_Y = \mathrm{Id}_Y$ *and* such that $d(x, x') = 1$ implies $d(fx, fx') = 1$ for all $x, x' \in X$, then we say that Y is an *edge-preserved* retract of X. Note that edge-preserved retracts are in particular retracts, by definition.

Theorem 3.2.4 ([Ban84, Theorem 2])**.** *Median graphs are precisely the edge-preserved retracts of hypercubes.*

We need three simple properties of diagonality, whose proofs we will not linger on, since generalizations will be given in Section 3.4.

Proposition 3.2.5 ([HW17, Proposition 7.3])**.** *Cartesian products of diagonal graphs are diagonal.*

Proof. By applying the Künneth formula and noting that diagonal graphs have torsion-free homologies. See [HW17, Proposition 7.3] for the full argument. □

Corollary 3.2.6. *Finite hypercubes are diagonal.*

Proof. The complete graph on two vertices K_2 is diagonal, and a finite hypercube is a finite Cartesian product of copies of K_2. □

Proposition 3.2.7. *Retracts of diagonal graphs are diagonal.*

Proof. A retraction $f\colon X \to Y$ has left inverse the inclusion $\iota\colon Y \to X$. Functoriality of MH_k^l implies that $\mathrm{MH}_k^l(f) : \mathrm{MH}_k^l(X) \to \mathrm{MH}_k^l(Y)$ is surjective. Hence, if $\mathrm{MH}_k^l(X)$ vanishes outside the diagonal, so does $\mathrm{MH}_k^l(Y) = 0$. □

Proposition 3.2.8. *Graphs with filtrations by diagonal graphs are diagonal.*

Proof. Let $(U_\alpha)_\alpha$ be a filtration of X, so that $\mathrm{MZ}_k^l(X) = \bigcup_\alpha \mathrm{MZ}_k^l(U_\alpha)$ and $\mathrm{MB}_k^l(X) = \bigcup_\alpha \mathrm{MB}_k^l(U_\alpha)$. If for all α and $k \neq l$, $\mathrm{MZ}_k^l(U_\alpha) = \mathrm{MB}_k^l(U_\alpha)$, then $\mathrm{MZ}_k^l(X) = \mathrm{MB}_k^l(X)$ and the homology vanishes outside the diagonal. □

We can now proceed with the proof of Proposition 3.2.3:

Proof of Proposition 3.2.3. Fix a median graph X and a hypercube Q of which X is a retract. Q has a filtration by finite hypercubes, which are diagonal (Corollary 3.2.6); hence so is Q (Proposition 3.2.8), and thus X (Proposition 3.2.7). □

3.3 The Künneth and Mayer-Vietoris Formulae

In [HW17], Hepworth and Willerton describe versions of the Künneth, excision, and Mayer-Vietoris formulae for magnitude homology of graphs. In [LS17], Leinster and Shulman, extending magnitude homology to metric spaces, asked whether those extend to this new setting. The answer is yes, assuming the right reinterpretations. More precisely:

Hepworth and Willerton's statement and proof of the Künneth theorem ([HW17, Theorem 5.3]) extend verbatim to l^1 products of metric spaces.

Similarly, Hepworth and Willerton's statement and proof of the excision and Mayer-Vietoris formulae ([HW17, Theorem 6.5]) extend to *gated* decompositions of metric spaces (Definition 3.3.10) with minimal changes. Those "minimal changes" are a bit trickier than simple generalizations. In particular, the "metric excision formula" that we define is not strictly a generalization of the graph-theoretic one of [HW17], since the definitions we use are not themselves generalizations of the ones in [HW17].

Since our arguments mainly consist in tweaking the original constructions of Hepworth and Willerton, having a copy of [HW17] at hand will prove useful!

3.3.1 The Künneth Formula

If X, Y are metric spaces, we endow the Cartesian product $X \times Y$ with the l^1 metric:

$$d((x,y),(x',y')) := d(x,x') + d(y,y').$$

This implies that the intervals satisfy the identity

$$[(x,y),(x',y')] = [x,y] \times [x',y'],$$

which explains the l^1 metric appearing in this context. Recall also that the l^1 product reduces to the usual Cartesian product in the case of graphs.

Summary of differences. The arguments in [HW17, Section 8] go through verbatim when proving the Künneth formula in the case of metric spaces, since the main ingredient is the "interval structure", which generalizes directly from graphs to metric spaces. Our downplaying of length as a grading of magnitude homology simplifies some expressions by virtue of getting rid of some $\bigoplus_l$s and $\bigvee_l$s; this is syntactical. Our arguments do not provide any new insight, but merely confirm that the generalization holds.

We now retrace [HW17, Section 8] closely, with the metric case in mind.

Definition 3.3.1 (Interleavings, cross product ([HW17, Definition 5.2])). Fix $n, l \in \mathbb{N}$, let $k = n + l$, and write $[k]$ for the set $\{0, \dots, k\}$.

A map $\sigma = \langle \sigma_h, \sigma_v \rangle : [n+l] \to [n] \times [l]$ satisfying

- $\sigma(0) = (0,0)$ and $\sigma(n+l) = (n,l)$;
- if $\sigma(i) = (a,b)$, then $\sigma(i+1)$ is either $(a+1,b)$ or $(a,b+1)$,

is called a *staircase path*. Write $\llcorner n, l \urcorner$ for the set of (n,l) staircase paths. A staircase path is just a geodesic from $(0,0)$ to (n,l) in the obvious grid. The *sign* $\operatorname{sgn}\sigma$ of σ is $(-1)^s$, where s is the number of squares "below the staircase", i.e.

$$s = |\{(a,b) \in [n] \times [l] \ : \ a = \sigma_h(i) \Rightarrow b < \sigma_v(i)\}|.$$

If $\mathbf{x}$ is an n-path in X, $\mathbf{y}$ an l-path in Y, and $\sigma \in \llcorner n, l \urcorner$ a staircase path, the *interleaving of* $\mathbf{x}$ *and* $\mathbf{y}$ *along* σ is the k-path $\mathbf{x} \overset{\sigma}{\times} \mathbf{y}$ defined by $\mathbf{x} \overset{\sigma}{\times} \mathbf{y} \coloneqq (\mathbf{x} \times \mathbf{y}) \circ \sigma$ (where $\mathbf{x} \times \mathbf{y}$ is identified with the map $[n] \times [l] \to X \times Y$).

The *cross product* is the morphism of chain complexes

$$\square : \mathrm{MC}_*(X) \otimes \mathrm{MC}_*(Y) \to \mathrm{MC}_*(X \times Y),$$

sending a pure tensor $\mathbf{x} \otimes \mathbf{y}$ to the alternating sum of all its interleavings:

$$\mathbf{x} \otimes \mathbf{y} \overset{\square}{\mapsto} \sum_{\sigma \in \llcorner n, l \urcorner} \operatorname{sgn}(\sigma)(\mathbf{x} \overset{\sigma}{\times} \mathbf{y}).$$

Verifying that $\square$ really defines a morphism mostly involves the same arguments as for the cross product map in simplicial homology: we will refer to those arguments as "generic". The non-generic part appears due to the possible vanishing of a "partial boundary" $\partial_i \mathbf{z}$, which happens when $z_i \notin [z_{i-1}, z_{i+1}]$. Thus, to show that $\square$ indeed commutes with the boundaries, we must understand the betweenness relations in a given interleaving $\mathbf{z} = \mathbf{x} \overset{\sigma}{\times} \mathbf{y}$ in terms of the betweenness relations in the paths $\mathbf{x}$ and $\mathbf{y}$, and of σ.

We visualize a staircase path $\sigma = \langle \sigma_h, \sigma_v \rangle$ as an actual (irregular) staircase on the $[n] \times [l]$ grid, going from bottom-left $(0,0)$ to top-right (n, l), with horizontal coordinate given by $\mathbf{x}$ and vertical by $\mathbf{y}$. Any $0 < m < n + l$ defines a point $\sigma(m)$ on the staircase; exactly one of:

A "corner": which means that its predecessor and successor differ in both coordinates. There are two distinct types of corners, looking like $\ulcorner$ and $\lrcorner$ respectively. In that case, $(\mathbf{x} \overset{\sigma}{\times} \mathbf{y})_m$ will necessarily be between its neighbors.

A "flat": which means that $\sigma_v(m+1) = \sigma_v(m) = \sigma_v(m-1)$ and $\sigma_h(m) = \sigma_h(m-1) + 1 = \sigma_h(m+1) - 1$. In that case $(\mathbf{x} \overset{\sigma}{\times} \mathbf{y})_m$ is between its neighbors if and only if $\mathbf{x}_{\sigma_h(m)}$ is between its neighbors, independently of $\mathbf{y}$.

A "wall": which means that $\sigma_h(m+1) = \sigma_h(m) = \sigma_h(m-1)$ and $\sigma_v(m) = \sigma_v(m-1) + 1 = \sigma_v(m+1) - 1$. In that case $(\mathbf{x} \overset{\sigma}{\times} \mathbf{y})_m$ is between its neighbors if and only if $\mathbf{y}_{\sigma_v(m)}$ is between its neighbors, independently of $\mathbf{x}$.

If σ is a staircase path and m is a corner point for σ, then there exists a unique other staircase path σ' with "dual" corner and opposite sign. More precisely, σ' and σ are equal, except at m, where

$$\sigma'(m) = (\sigma_h(m-1) + \sigma_v(m) - \sigma_v(m-1), \sigma_v(m-1) + \sigma_h(m) - \sigma_h(m-1)).$$

When applying ∂_m, the two interleavings $\mathbf{x} \overset{\sigma}{\times} \mathbf{y}$ and $\mathbf{x} \overset{\sigma'}{\times} \mathbf{y}$ will thus cancel out and the betweenness relations in $\mathbf{x}$ and $\mathbf{y}$ do not matter.

If m is a flat of σ, one can delete the column with coordinate $\sigma_h(m)$ and get a new staircase σ' in $\llcorner n-1, l \urcorner$. The sign of σ' differs from that of σ by $(-1)^{\sigma_v(m)}$, and

$$\partial_m(\mathbf{x} \overset{\sigma}{\times} \mathbf{y}) = \partial_{\sigma_h(m)} \mathbf{x} \overset{\sigma'}{\times} \mathbf{y}.$$

A similar description holds for walls. With these identities, it is a simple matter to adapt the "generic arguments" to the case of the magnitude complex.

Proposition 3.3.2 (Künneth theorem ([HW17, Theorem 4.3])). *The cross product map induces a morphism*

$$\begin{aligned} \mathrm{MH}_*(X) \otimes \mathrm{MH}_*(Y) &\xrightarrow{\square} \mathrm{MH}_*(X \times Y) \\ [f] \otimes [g] &\mapsto [f \square g] \end{aligned}$$

which fits into a natural short exact sequence

$$0 \to \mathrm{MH}_*(X) \otimes \mathrm{MH}_*(Y) \xrightarrow{\square} \mathrm{MH}_*(X \times Y) \to \mathrm{Tor}(\mathrm{MH}_{*-1}(X), \mathrm{MH}_*(Y)) \to 0.$$

To prove Proposition 3.3.2 (at the end of the subsection), we first set the stage and verify a few preliminary results.

Definition 3.3.3 ([HW17, Definition 8.1]). If X is a metric space, we define the pointed simplicial set $\mathrm{MS}(X)$ with, as k-simplices, the $(k+1)$-tuples of points $\langle x_0, \dots, x_k \rangle : [k] \to X$ in X, plus basepoint simplices pt_k, along with face and degeneracy maps defined by

$$\mathrm{d}_{k,i}\langle x_0, \dots, x_i, \dots, x_k \rangle := \begin{cases} \langle x_0, \dots, \widehat{x_i}, \dots, x_k \rangle & \text{if } x_i \in [x_{i-1}, x_{i+1}], \\ \mathrm{pt}_{k-1} & \text{otherwise,} \end{cases}$$

and

$$\mathrm{s}_{k,i}\langle x_0, \dots, x_i, \dots, x_k \rangle := \langle x_0, \dots, x_i, x_i, \dots, x_k \rangle,$$

and on basepoints:

$$\begin{aligned} \mathrm{d}_{k,i}\mathrm{pt}_k &:= \mathrm{pt}_{k-1}, \\ \mathrm{s}_{k,i}\mathrm{pt}_k &:= \mathrm{pt}_{k+1}. \end{aligned}$$

In [HW17, Definition 8.1], for a graph G and $l \in \mathbb{N}$, the simplicial set $M_l(G)$ is defined. In our notation, this set corresponds to the sub-simplicial set of $\mathrm{MS}(G)$ obtained by restricting to paths of length l, so that our $\mathrm{MS}(G)$ is equal to their $\bigvee_l M_l(G)$.

Proposition 3.3.4 ([HW17, Proposition 8.2]). *Let X, Y be metric spaces. The morphism of pointed simplicial sets*

$$\begin{aligned} \square : \mathrm{MS}(X) \wedge \mathrm{MS}(Y) &\to \mathrm{MS}(X \times Y) \\ [\mathbf{x}, \mathbf{y}] &\mapsto \langle \mathbf{x}, \mathbf{y} \rangle \end{aligned}$$

is an isomorphism.

Let us clarify the notation: $\mathbf{x}$ and $\mathbf{y}$ are k-simplices of $\mathrm{MS}(X)$ and $\mathrm{MS}(Y)$ respectively, that is, maps $[k] \to X$ and $[k] \to Y$. Thus, the pair $(\mathbf{x}, \mathbf{y})$ is an element of $\mathrm{MS}(X) \times \mathrm{MS}(Y)$, and $[\mathbf{x}, \mathbf{y}]$ an element of $\mathrm{MS}(X) \wedge \mathrm{MS}(Y)$. Finally, $\langle \mathbf{x}, \mathbf{y} \rangle : [k] \to X \times Y$ is the "product" of the given maps, hence an element of $\mathrm{MS}(X \times Y)$.

Proof. Bijectivity and commutation with degeneracy maps is clear. For face maps, one uses the product identity for intervals in the l^1 product, plus the fact that $\mathrm{d}_{k,i}(\mathbf{x}, \mathbf{y}) \neq \mathrm{pt}_{k-1}$ if and only if both $\mathrm{d}_{k,i}\mathbf{x} \neq \mathrm{pt}_{k-1}$ and $\mathrm{d}_{k,i}\mathbf{y} \neq \mathrm{pt}_{k-1}$ hold. □

Still following [HW17], given a pointed simplicial set S, the *normalized reduced* chain complex $\overline{\mathrm{N}}_*(S)$ associated to S is defined by

$$\overline{\mathrm{N}}_k(S) := \mathbb{Z}[\{k\text{-simplices}\}]/\mathbb{Z}[\{\text{degenerate and basepoint simplices}\}],$$

with boundary map induced by

$$\partial_k = \sum_{i=0}^{k} (-1)^i \mathrm{d}_{k,i}.$$

Since a simplex in $\mathrm{MS}(X)$ is degenerate if and only if two consecutive points are equal, the following holds:

Proposition 3.3.5 ([HW17, Lemma 8.3])**.** *$\overline{\mathrm{N}}_k(\mathrm{MS}(X))$ and $\mathrm{MC}_k(X)$ are isomorphic chain complexes.*

Proof. $\mathrm{MC}_k(X)$ is generated by the k-paths in X; that is, the $(k+1)$-tuples of consecutively distinct points in X. $\overline{\mathrm{N}}_k(\mathrm{MS}(X))$ is generated by the non-degenerate non-basepoint simplices of $\mathrm{MS}(X)$, which are exactly the k-paths. Thus, the groups are isomorphic. On $\overline{\mathrm{N}}_k(\mathrm{MS}(X))$, the boundary is defined as $\partial_k = \sum_{i=1}^{k-1} (-1)^i \mathrm{d}_{k,i}$. Since $\mathrm{d}_{k,i}$ sends a simplex $\mathbf{x}$ to a basepoint if and only if $x_i \notin [x_{i-1}, x_{i+1}]$, $\mathrm{d}_{k,i}$ sends $\mathbf{x}$ to zero at the level of chain maps, which shows that the boundary maps agree. □

From now on, we will identify $\overline{\mathrm{N}}_k(\mathrm{MS}(X))$ with $\mathrm{MC}_k(X)$.

Remember that given a simplicial set S, there exists, for any k, a natural bijection

$$s(\bullet) : S_k \leftrightarrow Mor(\Delta[k], S)$$

where $\Delta[k]$ is the canonical k-simplex. The (inverse of this) bijection is obtained by sending a morphism $f : \Delta[k] \to S$ to the image through f of the single non-degenerate k-simplex $\mathrm{Id} : [k] \to [k]$ in $\Delta[k]$. Under the interpretation of simplicial sets as presheaves, this bijection is simply the Yoneda lemma.

If $\sigma = \langle \sigma_h, \sigma_v \rangle$ is a (n,l)-staircase, σ defines a morphism of simplicial complexes

$$\sigma_* : \Delta[n+l] \to \Delta[n] \times \Delta[l],$$

by sending a face $\phi : [m] \to [n+l]$ of $\Delta[n+l]$ to the pair of faces

$$(\sigma_h \circ \phi : [m] \to [n], \sigma_v \circ \phi : [m] \to [l])$$

in $\Delta[n] \times \Delta[l]$. Finally, if $\mathbf{x}$ and $\mathbf{y}$ are simplices in the simplicial sets S and T respectively, they are naturally associated to morphisms $s(\mathbf{x}) : \Delta[n] \to S, s(\mathbf{y}) : \Delta[l] \to T$, so that one has a morphism:

$$s(\mathbf{x}) \times s(\mathbf{y}) : \Delta[n] \times \Delta[l] \to S \times T.$$

Given pointed simplicial sets S, T, we now define the *reduced Eilenberg-Zilber map*

$$\begin{aligned} \nabla^{\overline{N}} : \; & \overline{\mathrm{N}}_*(S) \otimes \overline{\mathrm{N}}_*(T) \to \overline{\mathrm{N}}_*(S \wedge T) \\ \mathbf{x} \otimes \mathbf{y} \in \overline{\mathrm{N}}_n(S) \otimes \overline{\mathrm{N}}_l(T) \mapsto & \sum_{\sigma \in \llcorner n,l \urcorner} [s^{-1}((s(\mathbf{x}) \times s(\mathbf{y})) \circ \sigma_*)], \end{aligned}$$

where $[\bullet] : S \times T \to S \wedge T$ is the collapsing map.

The following abstract property of $\nabla^{\overline{N}}$ is proven in [HW17]:

Proposition 3.3.6 ([HW17, Proposition 8.4])**.** *$\nabla^{\overline{N}}$ is a quasi-isomorphism.*

Let us now concretely describe the map $\nabla^{\overline{N}}$ in the case at hand. Fix generators $\mathbf{x} \in \overline{\mathrm{N}}_n(\mathrm{MS}(X))$ and $\mathbf{y} \in \overline{\mathrm{N}}_l(\mathrm{MS}(Y))$. When $\mathbf{x}$ is seen as a simplex of $\mathrm{MS}(X)$, we have

$$\mathbf{x} : [n] \to X.$$

Through the identification "simplex↔morphism", $\mathbf{x}$ becomes

$$\begin{aligned} s(\mathbf{x}) : \Delta[n] &\to \mathrm{MS}(X) \\ (\phi : [m] \to [n]) \in \Delta[n]_m &\mapsto (\mathbf{x} \circ \phi : [m] \to X) \in \mathrm{MS}(X)_m. \end{aligned}$$

Thus, the composite

$$(s(\mathbf{x}) \times s(\mathbf{y})) \circ \sigma_* : \Delta[n+l] \to \mathrm{MS}(X) \times \mathrm{MS}(Y)$$

is defined as

$$\begin{aligned} (\phi : [m] \to [n+l]) \in \Delta[n+l]_m \mapsto & ((s(\mathbf{x}) \times s(\mathbf{y})) \circ \sigma_*)(\phi) \in \mathrm{MS}(X) \times \mathrm{MS}(Y) \\ &= (\mathbf{x} \circ \sigma_h \circ \phi, \mathbf{y} \circ \sigma_v \circ \phi), \end{aligned}$$

and passing back from morphisms to simplices (evaluating at $\mathrm{Id} : [n+l] \to [n+l]$), the result is simply

$$(\mathbf{x} \circ \sigma_h \circ \mathrm{Id}, \mathbf{y} \circ \sigma_v \circ \mathrm{Id}) = (\mathbf{x} \circ \sigma_h, \mathbf{y} \circ \sigma_v) \in \mathrm{MS}(X)_n \times \mathrm{MS}(Y)_l.$$

Proposition 3.3.7 ([HW17, Proof of Theorem 5.3])**.** *The cross product*

$$\square : \mathrm{MC}_*(X) \otimes \mathrm{MC}_*(Y) \to \mathrm{MC}_*(X \times Y)$$

is a quasi-isomorphism.

Proof. The proof is essentially obtained by forgetting the grading in [HW17, Proof of Theorem 5.3]. More precisely, the map $\square$ is a quasi-isomorphism if and only if it is one at each "level" of the grading, i.e., if, for each $l \geq 0$, it restricts to a quasi-isomorphism:

$$\square_l : \bigoplus_{l_1+l_2=l} \mathrm{MC}_*^{l_1}(X) \otimes \mathrm{MC}_*^{l_2}(Y) \to \mathrm{MC}_*^{l}(X \times Y),$$

which is the content of [HW17, Proof of Theorem 5.3]. □

Proof of Proposition 3.3.2. Applying the algebraic Künneth formula to $\mathrm{MC}_*(X)$ and $\mathrm{MC}_*(Y)$ yields a short exact sequence

$$\begin{aligned} 0 \to \mathrm{MH}_*(X) \otimes \mathrm{MH}_*(Y) \to H_*(\mathrm{MC}_*(X) \otimes \mathrm{MC}_*(Y)) & \\ \to \mathrm{Tor}(\mathrm{MH}_{*-1}(X), \mathrm{MH}_*(Y)) \to 0.& \end{aligned}$$

By Proposition 3.3.7, the middle term is isomorphic, through $H_*(\square)$, to $\mathrm{MH}_*(X\times Y)$. Naturality follows from naturality in the algebraic Künneth formula and that of the cross product map. □

Note that the "length aware" sequence in [HW17] can easily be recovered by fixing l in $H_*(\mathrm{MC}_*(X)\otimes\mathrm{MC}_*(Y))$.

3.3.2 The excision Formula

Definition 3.3.8 (Gated sets)**.** Given a metric space X, a subset A of X is said to be *gated* if for any $x \in X$, there exists some $a_x \in A$ such that a_x is between x and all $a \in A$. The point a_x is called a *gate* between x and A.

Gated sets enjoy the following properties (see [DS87]—more about gated sets in median spaces can be found in [CDH10]).

Proposition 3.3.9 ([DS87, pp. 114, 112, 115 respectively])**.** *Let X be a metric space and A a gated subset of X. Then:*

- *A is convex.*
- *For any $x \in X$, there exists a* unique *gate $a_x \in A$ for x.*
- *The map $x \mapsto a_x$ is non-expanding, and is the identity on A.*

From now on, we write $\pi\colon X \to A$ for the map sending x to its gate a_x. Note that by the above, π is a (1-Lipschitz) retraction from X to A; it follows that $\mathrm{MC}_*(\pi) : \mathrm{MC}_*(X) \to \mathrm{MC}_*(A)$ is a (well-defined) epimorphism.

Summary of differences. As for the Künneth formula, the proofs of excision and Mayer-Vietoris in [HW17] essentially generalize without trouble, yet some more care is needed, mainly because of slight differences in the definitions.

Apart from the differences in definitions, the main obstacle to generalizing [HW17] appears in their [HW17, Lemma 9.5] and [HW17, Proof of Theorem 6.6][2] in which, once a length l is fixed, [HW17] uses the vanishing of the groups $\mathrm{MC}_k^l(X)$ for $k > l$; this does not hold in general for metric spaces.

Finally, unlike [HW17], we have not verified naturality of the Mayer-Vietoris sequence in the metric case.

To conclude, we will (again) follow [HW17, Section 9] very closely and highlight where changes are required.

Definition 3.3.10 (Gated decomposition)**.** Let X be a metric space and Y, Z, W subspaces satisfying $X = Y \cup Z$ and $W = Y \cap Z$. If W is gated with respect to Z, and for each $z \in Z$ and $y \in Y$, the intersection $W \cap [y, z]$ is non-empty, then we say that the triple $(X; Y, Z)$ is a *gated decomposition* of X.

Note that from W being gated in Z and $W \cap [y, z]$ being non-empty, it follows that $\pi(z)$ lies between z and any element of Y.

Following [HW17], we write $\mathrm{MC}_*(Y, Z)$ for the subcomplex of $\mathrm{MC}_*(X)$ spanned by paths entirely contained in either Y or Z. We can now state the excision theorem:

[2] Note that [HW17] contains both a "Proof of Theorem 6.6 assuming Theorem 6.5" (page 50), and a "Proof of Theorem 6.6" (page 52). The latter is actually the proof of Theorem 6.5.

Theorem 3.3.11 (excision—metric setting). *If $(X;Y,Z)$ is a gated decomposition of X, then the inclusion*

$$\mathrm{MC}_*(Y,Z) \hookrightarrow \mathrm{MC}_*(X)$$

is a quasi-isomorphism.

The Mayer-Vietoris theorem follows easily from excision.

Theorem 3.3.12 (Mayer-Vietoris—metric case). *If $(X;Y,Z)$ is a gated decomposition of X and $W = Y \cap Z$, then the inclusions*

$$j_Y\colon\ W \to Y,\ j_Z\colon\ W \to Z,\ i_Y\colon\ Y \to X,\ i_Z\colon\ Z \to X$$

induce a short exact sequence

$$0 \to \mathrm{MH}_*(W) \overset{\langle (j_Y)_*, -(j_Z)_*\rangle}{\longrightarrow} \mathrm{MH}_*(Y) \oplus \mathrm{MH}_*(Z) \overset{(i_Y)_* \oplus (i_Z)_*}{\longrightarrow} \mathrm{MH}_*(X) \to 0.$$

Proof. The proof is obtained from the argument in [HW17, Proof of Theorem 6.6, assuming Theorem 6.5] by dropping the grading (and noting that since we deal with metric spaces, we do not have to worry about path-connected components as they do). In our case theorem 3.3.11 takes on the role of their Theorem 6.5. □

Proof of excision.

For the remainder of the section, let us fix a gated decomposition $(X;Y,Z)$ and let $W \coloneqq Y \cap Z$.

We define, for $a \in Y - Z, b \in Z - Y$ (or vice versa) and $k \geq 0$:

$$A_k(a,b) \coloneqq \mathbb{Z}[\{\langle x_0,\dots,x_k\rangle \mid x_0 = a, x_k = b, x_1,\dots,x_{k-1} \in W\}] \leq \mathrm{MC}_k(X).$$

For $b \in Z - Y$ we define

$$\begin{aligned} B_k(b) &\coloneqq \mathbb{Z}[\{\langle x_0,\dots,x_k\rangle \mid x_k = b, x_0,\dots,x_{k-1} \in Y\}] \leq \mathrm{MC}_k(X),\\ \tilde{B}_k(b) &\coloneqq \mathbb{Z}[\{\langle x_0,\dots,x_k\rangle \mid x_k = b, x_0,\dots,x_{k-1} \in W\}] \leq \mathrm{MC}_k(X), \end{aligned}$$

and for $i \in \mathbb{N}$

$$F_k(b;i) \coloneqq \mathbb{Z}[\{\langle x_0,\dots,x_k\rangle \mid x_k = b, x_0,\dots,x_{i-1} \in Y, x_i,\dots,x_{k-1} \in W\}] \leq \mathrm{MC}_k(X),$$

and symmetrically for $b \in Y - Z$. Finally for $i \in \mathbb{N}$ set

$$G_k(i) \coloneqq \mathbb{Z}[\{\langle x_0,\dots,x_k\rangle \mid x_0,\dots,x_{k-i} \text{ all lie in } Y, \text{ or all lie in } Z\}] \leq \mathrm{MC}_k(X).$$

These all define sub-chain complexes of $\mathrm{MC}_*(X)$; the definitions match the ones found in [HW17, Section 9], except that our last $G_*(i)$s correspond to the F_is found in [HW17, Proof of Theorem 6.6, p. 52].

It is clear that $G_*(0) = \mathrm{MC}_*(Y,Z)$, $G_k(l) = \mathrm{MC}_k(X)$ for all $k \leq l$, and $G_*(l) \leq G_*(l+1)$ for all l. It follows that $\mathrm{MC}_*(X)$ is the direct limit of the system

$$\mathrm{MC}_*(Y,Z) = G_*(0) \leq G_*(1) \leq \cdots \leq G_*(l) \leq G_*(l+1) \leq \dots .$$

Thus, to show that the inclusion $\mathrm{MC}_*(Y,Z) \hookrightarrow \mathrm{MC}_*(X)$ is a quasi-isomorphism, it is enough to do so for each inclusion $G_*(l) \hookrightarrow G_*(l+1)$. Indeed, once this is done, the whole system becomes a chain of isomorphisms after passing to homology, and each inclusion of $G_*(l)$ to the limit $\mathrm{MC}_*(X)$ as well. In particular, this is true for the inclusion $\mathrm{MC}_*(Y,Z) \hookrightarrow \mathrm{MC}_*(X)$.

This is essentially the only thing we have to change from the argument of [HW17].

Let l be fixed from now on, and given a chain complex C_*, write $\Sigma^j C_*$ for the shifted chain complex $(\Sigma^j C_*)_k = C_{k-j}$.

Proposition 3.3.13 ([HW17, Lemma 9.2])**.** *The complex $A_*(a,b)$ is acyclic.*

Proof. The proof of [HW17, Lemma 9.2] applies verbatim, once the grading is dropped.

Recall that gated decompositions are required to satisfy the following: for $y \in Y$ and $z \in Z$, $W \cap [y,z] \neq \emptyset$. This condition is necessary for the case $k = 1$ in the cited proof: if $a \in Y - Z$ and $b \in Z - Y$, it is necessary for a point of W between a and b to exist, so that $\pi(b)$ is well-defined. □

Define the set

$$J_Z(l) := \{\mathbf{x} = \langle x_0, \dots, x_l \rangle \ : \ x_0, \dots x_l \in Y,\ x_l \notin Z\},$$

and define $J_Y(l)$ symmetrically.

Proposition 3.3.14 ([HW17, Lemma 9.5])**.** *For any $b \in Z - Y$, there is an isomorphism*

$$F_*(b, l+1)/F_*(b,l) \cong \bigoplus_{\mathbf{x} \in J_Z(l)} \Sigma^l A_*(x_l, b).$$

In particular, the quotient $F_*(b,l)/F_*(b,0)$ is acyclic. The same holds for $b \in Y - Z$ with $J_Z(l)$ replaced by $J_Y(l)$.

Proof. Apply the proof of [HW17, Lemma 9.5], or more precisely the part showing that each complex F_i/F_{i-1} (in their notation) is acyclic. □

Proposition 3.3.15. *For any $b \in Y \Delta Z$, the quotient $B_*(b)/\tilde{B}_*(b)$ is acyclic.*

Proof. Consider the directed system:

$$\tilde{B}_*(b) = F_*(b,0) \le F_*(b,1) \le \cdots \le F_*(b,l) \le F_*(b,l+1) \le \dots.$$

Since we have inclusions $F_*(b,l) \le B_*(b)$ for all l, and for each $k \le l$,

$$F_k(b,l) = B_k(b),$$

it follows that $B_*(b)$ is the direct limit of the above system. Passing to homology, each inclusion $F_*(b,l) \le F_*(b,l+1)$ becomes an isomorphism by Proposition 3.3.14. It follows then that the inclusion $\tilde{B}_*(b) \le B_*(b)$ also becomes an isomorphism. □

For $k \ge l$, define the set

$$K_k(l) := \{\mathbf{x} = \langle x_{k-l}, \dots, x_k \rangle \ : \ x_{k-l} \in Y \Delta Z\},$$

and if $k < l$, let $K_k(l) := \emptyset$.

Proposition 3.3.16 ([HW17, Proof of Theorem 6.6 p. 52]). *There is an isomorphism of chain complexes*
$$G_k(l+1)/G_k(l) \cong \bigoplus_{\mathbf{x}\in K_k(l)} \left(\Sigma^l B_*(x_{k-l})/\tilde{B}_*(x_{k-l})\right)_k.$$
In particular, the inclusion $G_*(l) \leq G_*(l+1)$ is a quasi-isomorphism.

Proof. Apply the part of [HW17, Proof of Theorem 6.6 p. 52] showing that each quotient F_i/F_{i-1} (in their notation) is acyclic. □

We can now prove the excision formula:

Proof of theorem 3.3.11. Each inclusion in the directed system
$$\mathrm{MC}_*(Y,Z) = G_*(0) \leq \dots \leq G_*(l) \leq G_*(l+1) \leq \dots$$
is a quasi-isomorphism by Proposition 3.3.16, and $\mathrm{MC}_*(X)$ is the direct limit of this system. Thus, the inclusions induce isomorphisms
$$\mathrm{MH}_*(Y,Z) = H(G_*(0)) \cong \dots \cong H(G_*(l)) \cong H(G_*(l+1)) \cong \dots$$
and $\mathrm{MH}_*(X)$ (along with the morphisms induced by inclusions into $\mathrm{MH}_*(X)$) is the limit of this diagram. It follows that each inclusion, in particular $\mathrm{MC}_*(Y,Z) \hookrightarrow \mathrm{MC}_*(X)$ is a quasi-isomorphism. □

3.4 Diagonality

In the first section, we have seen that median graphs are diagonal (in the sense of Hepworth and Willerton). Knowing that median *graphs* are special cases of median *spaces* motivates us to try and find a corresponding description for median spaces. In this section, we introduce the notion of diagonality for metric spaces and verify some of its properties. As hoped, we will see in the next section that median spaces are indeed diagonal.

Recall that a path $\mathbf{x} = \langle x_0, \dots, x_k\rangle$ is saturated if all strict intervals $]x_i, x_{i+1}[$ are empty. In what follows, for any chain $\sigma := \sum_{\mathbf{x}} \lambda_{\mathbf{x}}\mathbf{x}$, we will write σ_S for the "saturated" part of σ, that is, $\sigma_S := \sum_{\mathbf{x} \text{ saturated}} \lambda_{\mathbf{x}}\mathbf{x}$.

Definition 3.4.1 (Diagonality). A space X is said to be *diagonal* if $\mathrm{MH}_k(X)$ is generated by linear combinations of saturated paths, for all $k \in \mathbb{N}$.

Let $S_k(X)$ denote the span of saturated paths, as a submodule of $\mathrm{MC}_k(X)$. Recall that the *support* of a chain $\sum_{\mathbf{x}\in P_k(X)} \lambda_{\mathbf{x}}\mathbf{x}$ is by definition the set $\{\mathbf{x} \in P_k(X) \ : \ \lambda_{\mathbf{x}} \neq 0\}$.

Proposition 3.4.2. *The supports of elements in $S_k(X)$ and $\mathrm{MB}_k(X)$ are mutually disjoint. In particular, $S_k(X) \cap \mathrm{MB}_k(X) = \{0\}$.*

Proof. By linearity, it suffices to verify that for any path $\mathbf{x}$, its boundary $\partial\mathbf{x} = \sum_i (-1)^i \partial_i \mathbf{x}$ has no saturated path in its support. This follows by definition of the boundary operator, since $\partial_i\mathbf{x}$ is non-zero exactly when $x_i \in]x_{i-1}, x_{i+1}[$, which implies that $\langle x_0, \dots, x_{i-1}, x_{i+1}, \dots x_k\rangle$ is not saturated. □

Thus, diagonality can be restated in different ways:

Proposition 3.4.3. *The following are equivalent:*

1. *X is diagonal;*
2. $\mathrm{MZ}_k(X) = (\mathrm{MZ}_k(X) \cap S_k(X)) \oplus \mathrm{MB}_k(X)$*;*
3. *Given a cycle $\sigma = \sigma_S + \sigma' \in \mathrm{MZ}_k(X)$, with σ_S the part corresponding to saturated paths and σ' the rest, we must have $\sigma' \in \mathrm{MB}_k(X)$;*
4. *The "inclusion-then-quotient" morphism $\mathrm{MZ}_k(X) \cap S_k(X) \to \mathrm{MH}_k(X)$ is an isomorphism.*

Proof. We show 1⇒2⇒4⇒1and 1⇔3.

1⇒2 By Proposition 3.4.2, $\mathrm{MZ}_k(X) \cap S_k(X)$ and $\mathrm{MB}_k(X)$ intersect trivially. By diagonality, given $\sigma \in \mathrm{MZ}_k(X)$, there exists $\sigma_s \in \mathrm{MZ}_k(X) \cap S_k(X)$ such that σ and σ_s are equal in homology, i.e., $\sigma - \sigma_s \in \mathrm{MB}_k(X)$. Thus, $\mathrm{MZ}_k(X) \cap S_k(X) + \mathrm{MB}_k(X) = \mathrm{MZ}_k(X)$.

2⇒4 Clear.

4⇒1 Clear.

1⇒3 Fix $\sigma = \sigma_S + \sigma' \in \mathrm{MZ}_k(X)$ as in 3. By diagonality, there exists $\sigma_S' \in \mathrm{MZ}_k(X) \cap S_k(X)$ and $\sigma'' \in \mathrm{MB}_k(X)$ such that $\sigma_S + \sigma' = \sigma_S' + \sigma''$, hence $\sigma_S - \sigma_S' = \sigma'' - \sigma'$. Since the support of $\sigma_S - \sigma_S'$ consists of saturated chains and σ'' has no saturated chains in its support, necessarily $\sigma' = \sigma''$ and $\sigma_S = \sigma_S'$.

3⇒1 Clear.

□

In particular, it follows directly from the last item that:

Corollary 3.4.4. *A diagonal space has torsion-free homology.*

The following proposition supports our choice of terminology:

Proposition 3.4.5. *A graph is diagonal in the sense of Hepworth and Willerton if and only if it is diagonal in the above sense.*

Proof. In a graph, a k-path is saturated if and only if it has length k. Thus, having vanishing homology outside the diagonal and having homology groups generated by (linear combinations of) saturated paths are equivalent conditions. □

Corollary 3.4.6. *A diagonal Menger convex space has vanishing homology (except possibly at $k = 0$).*

Proof. A Menger convex space has no saturated path. □

In the next few propositions, we verify stability of diagonality under some usual constructions.

Proposition 3.4.7. *If $(U_\alpha)_\alpha$ is a filtration of a space X such that each U_α is diagonal, then so is X.*

Proof. Fix $\sigma \in \mathrm{MZ}_k(X)$, and write $\sigma = \sigma_S + \sigma'$. Fix α large enough that U_α contains all points in the support of σ, and, for each non-saturated path $\mathbf{x}$ in the support of σ', also contains a "witness" to non-saturation of the path (that is, a point $p \in]x_i, x_{i+1}[$ for some i). Then, $\sigma \in \mathrm{MZ}_k(U_\alpha)$, σ_S still consists of saturated paths *in* U_α, and σ' of non-saturated paths *in* U_α. Since U_α is diagonal, we conclude that $\sigma' \in \mathrm{MB}_k(U_\alpha) \leq \mathrm{MB}_k(X)$. □

Proposition 3.4.8. *An l^1 product of diagonal spaces is diagonal.*

Proof. By applying the (metric) Künneth formula. Let X, Y be diagonal spaces. For any fixed k, we have a short exact sequence

$$0 \to \bigoplus_{n+l=k} \mathrm{MH}_n(X) \otimes \mathrm{MH}_l(Y) \xrightarrow{\square} \mathrm{MH}_k(X \times Y) \to \mathrm{Tor}(\dots, \dots) \to 0.$$

The torsion part being zero (Corollary 3.4.4), an isomorphism

$$\bigoplus_{n+l=k} \mathrm{MH}_n(X) \otimes \mathrm{MH}_l(Y) \xrightarrow{\square} \mathrm{MH}_k(X \times Y)$$

remains. Since both $\mathrm{MH}_n(X)$ and $\mathrm{MH}_l(Y)$ are generated by (linear combinations of) saturated paths, the whole domain of the isomorphism is generated by pure tensors of such, and $\mathrm{MH}_k(X \times Y)$ by their images. Noting that if $\mathbf{z}$ is an interleaving of two paths $\mathbf{x}, \mathbf{y}$ as in the definition of the map $\square$, then $\mathbf{z}$ is saturated if and only if both $\mathbf{x}, \mathbf{y}$ are, we conclude that $\mathrm{MH}_k(X \times Y)$ is generated by (linear combinations of) saturated paths. □

Proposition 3.4.9. *If $(X; Y, Z)$ is a gated decomposition and Y, Z are convex and diagonal, then X is diagonal.*

Proof. By applying a fragment of the (metric) Mayer-Vietoris sequence, we have an epimorphism

$$\mathrm{MH}_*(Y) \oplus \mathrm{MH}_*(Z) \to \mathrm{MH}_*(X),$$

and the image of saturated paths in either Y or Z are still saturated in X by convexity. □

Definition 3.4.10 (Betweenness preservation, reflection)**.** If X, Y are two metric spaces, and $f\colon X \to Y$ is an injective map, we say that it

- *preserves betweenness* if $y \in [x, z]$ implies $fy \in [fx, fz]$; and
- *reflects betweenness* if $fy \in [fx, fz]$ implies $y \in [x, z]$.

In case f both preserves and reflects betweenness, we say it is a *betweenness embedding*. If it is also surjective, it becomes a *betweenness isomorphism*.

Let us give a few examples of maps that do or do not preserve or reflect betweenness:

- If X is a metric space, $\lambda > 0$ and λX denotes the metric space obtained by rescaling X by λ, then the identity map $X \to \lambda X$ is both preserving and reflecting.
- The composition of preserving (resp. reflecting) maps is still preserving (resp. reflecting). Thus, one sees that being preserving or reflecting is not related to being 1-Lipschitz, since, at least for finite metric spaces, one can always compose a map with a rescaling to be (or not) 1-Lipschitz.

- The projections onto a coordinate $\mathbb{R}^2 \to \mathbb{R}$ are preserving but not reflecting.
- The piecewise-linear map on the unit interval $[0,1] \to [0,1]$ defined by

$$x \mapsto \begin{cases} 2x & \text{if } x \leq \frac{1}{2} \\ 2(1-x) & \text{if } x \geq \frac{1}{2}, \end{cases}$$

  is neither preserving nor reflecting.

Proposition 3.4.11. *If $f\colon X \to Y$ is a betweenness embedding, then f induces a morphism of chain complexes:*

$$\begin{aligned} f_* : \mathrm{MC}_*(X) &\to \mathrm{MC}_*(Y) \\ (x_0, \dots, x_k) &\mapsto (fx_0, \dots, fx_k). \end{aligned}$$

If f is bijective, this turns into an isomorphism.

Note that this induced morphism is not the one given by the magnitude functor, since a betweenness embedding may fail to preserve the length of paths.

Proof. Injectivity plus the betweenness preserving and reflecting imply that f commutes with boundaries. □

Proposition 3.4.12. *If $f\colon X \to Y$ is a betweenness isomorphism between metric spaces and Y is diagonal, then so is X.*

Proof. Since f preserves and reflects betweenness, images of saturated paths are saturated, and vice versa. The same can be said of homological cycles and boundaries. □

Proposition 3.4.13. *Retracts of diagonal spaces are diagonal.*

Proof. Let $f\colon X \to Y$ be a retraction.

If $\mathbf{y}$ is a path in Y, then $\mathbf{y}$ is saturated in Y if and only if it is saturated in X. Indeed, assume first that $\mathbf{y}$ is not saturated in X, so that there exists some $x \in X$ strictly between y_i and y_{i+1}. Then, since f is non-expanding and fixes y_i and y_{i+1}, it follows that $fx \in Y$ is also strictly between y_i and y_{i+1}, so that $\mathbf{y}$ is not saturated in Y. This shows that non-saturation in X implies non-saturation in Y, and the converse is obvious.

Consider now a cycle $\sigma = \sigma_S + \sigma' \in \mathrm{MZ}_k(Y)$. Since $Y \subseteq X$, σ is still a cycle in X, and its decomposition into "saturated+non-saturated" in X is still $\sigma = \sigma_S + \sigma'$. Thus, assuming X is diagonal, $\sigma' \in \mathrm{MB}_k(X)$, that is, there exists some $\tau \in \mathrm{MC}_{k+1}(X)$ with $\sigma' = \partial\tau$. Applying $f_* : \mathrm{MC}_*(X) \to \mathrm{MC}_*(Y)$ to σ', we get $f_*\sigma' = f_*\partial\tau = \partial f_*\tau$. Since σ' has support in Y, $f_*\sigma' = \sigma'$. Thus $\sigma' = \partial f_*\tau \in \mathrm{MB}_k(Y)$. This shows that Y is diagonal. □

3.5 Median Spaces are Diagonal

We will need the following fact due to Avann:

Proposition 3.5.1 ([Ava61]). *If X is a finite median space, then there exists a finite graph $G(X)$ and a betweenness isomorphism $\phi\colon X \to G(X)$. In particular, $G(X)$ is median.*

Another important property of median spaces:

Proposition 3.5.2. *Any median space has a filtration by finite median subspaces.*

Proof. Any finite subset of a median space X has a finite so-called *median hull*, that is, a smallest median subspace of X containing the set in question (see [BC08, p.7]). Taking all such finite median hulls yields a filtration by finite median subspaces. □

It is now easy to conclude with the following:

Proposition 3.5.3. *Median spaces are diagonal.*

Proof. Finite median graphs are diagonal, and by applying Propositions 3.4.12 and 3.5.1, so are finite median spaces. □

Corollary 3.5.4. *Median Menger convex spaces have vanishing homology (except possibly at $k = 0$).*

Menger convexity is already known to be strongly related to vanishing in the magnitude homology:

- In [LS17], it is shown that $\mathrm{MH}_1(X) = 0$ if and only if X is Menger convex, and $\mathrm{MH}_2(X) = 0$ if X is Menger convex and satisfies two more "straightness" conditions (Corollary 4.5 and Theorem 4.21 respectively).
- [Jub18, Corollary 4.9] shows that if $\mathrm{MH}_n(X) = 0$ for some $n \geq 1$, then X is Menger convex. Conversely, and extending the results of [LS17], [Jub18, Corollary 7.3] shows that assuming Menger convexity and the same "straightness" conditions as above, $\mathrm{MH}_n(X) = 0$ for all $n \geq 1$ (this result is also obtained in [KY18]).

Those two straightness conditions are *geodeticy* and *no* 4*-cuts*:

- X is geodetic if for any $x, y, z, w \in X$, it follows from $z, w \in [x, y]$ that either $z \in [x, w]$ or $z \in [w, y]$.
- X has no 4-cuts if for any sequence $x \neq y \neq z \neq w \in X$, it follows from $y \in [x, z]$ and $z \in [y, w]$ that $y, z \in [x, w]$.

Note that median spaces do not necessarily satisfy either geodeticy or having no 4-cuts: the four-cycle graph C_4 being a prime example of a graph satisfying neither. Conversely, Euclidean space is geodetic and has no 4-cuts, but is not median. As such, Corollary 3.5.4 yields a new crop of spaces with vanishing homology.

3.6 What (could be) next

Betweenness. As we have seen already, the proof of diagonality of median spaces required us to step out of the category of metric spaces and 1-Lipschitz maps. This indicates that, perhaps, the metric structure is too rigid for the study of magnitude homology. We can instead restrict our attention to a much more basic notion of "betweenness" space: a set X endowed with a map $[\bullet, \bullet] : X \times X \to \mathscr{P}(X)$ satisfying the following axioms:

1. $x, z \in [x, z]$;
2. $y \in [x, z]$ implies $[x, y] \subseteq [x, z]$ and $[y, z] \subseteq [x, z]$;
3. $w, y \in [x, z]$ implies: $w \in [x, y]$ if and only if $y \in [w, z]$.

With simple adaptations of the definitions, a working notion of magnitude homology appears, and essentially all of the above constructions still work. Complications arise when looking for a suitable notion of morphism that would generalize both that of 1-Lipschitz map and of the *betweenness embedding* defined earlier.

A simpler proof of excision. The proof of excision presented above is very opaque (to me) and quite algebraic in nature. Our approach on generalizing it from graphs to metric spaces consisted essentially in finding where the argument failed for metric spaces, and patching it up. It would be interesting to find an argument more direct and geometric, one that would enlighten us to what really happens.

Coarsifying magnitude. Is there a good quasi-isometry invariant notion of magnitude homology? None that we have found. It seems that magnitude homology is too sensitive to the rigid structure of graphs, to be easily coarsified, and no approach tried so far was satisfying.

4 The first Roe homology

In this chapter, we give a combinatorial description of the first group of Roe homology $H_1^\infty(X, A)$ for a locally finite graph X, in terms of flows and circuits. We prove that when X is a tree, $H_1^\infty(X, A)$ essentially counts ends. In the general case, we decompose $H_1^\infty(X, A)$ into a direct sum $H_1^\infty(X, A) \cong H_1^\infty(T, A) \oplus \mathscr{C}(X, A)/\mathscr{C}_\infty(X, A)$: the first component counts ends, while the second measures the extent to which infinite sums of circuits ($\mathscr{C}(X, A)$) differ from infinite sums of circuits of uniformly bounded length ($\mathscr{C}_\infty(X, A)$)—the choice of ring A matters. Finally, using "bases" for the space $\mathscr{C}(X, A)$ of flows decomposable as infinite sums of circuits, we show that if $A = \mathbb{Z}_2$, then $\mathscr{C}(X, A)/\mathscr{C}_\infty(X, A)$ does not vanish if and only if it is infinite dimensional.

4.1 Flows, circuits, and H_1^∞

Recall that all graphs are assumed to be connected and locally finite, meaning that all vertices are of finite degree; in particular, all graphs are countable. From local finiteness follows that all metric balls $B_r(v) = \{u \in VX \ : \ d(u, v) \leq r\}$ are finite. Furthermore, using connectivity, X is the union of its metric balls centered at any of its vertices, $VX = \bigcup_{r\in\mathbb{N}} B_r(v)\ \forall v \in VX$. Endowing the (vertex set of the) graph X with the discrete topology, the compact sets are just the finite ones, and the above property allows (a simple form of) "exhaustion by compacts".

4.1.1 Local Finiteness

In this chapter, we will often view functions $f : S \to A$ from a set S to a coefficient ring A as infinite formal sums $\sum_{s\in S} f(s) \cdot s$. We might want to apply a function to each $s \in S$ and extend "linearly" to infinite sums, or sum an infinite number of such sums. For all of this to be well-defined, some care is needed; the point of this subsection is to give a general explanation of why all the constructions appearing later are well-defined and behave as expected; it is elementary and may safely be skipped.

Fix any set S and ring A: the support $\operatorname{supp} f$ of a function $f : S \to A$ is the set of elements of S on which f is non-zero. A set F of functions $f : S \to A$ is said to be *locally finite* if any $s \in S$ lies in the support of only finitely many functions f in F. If F is locally finite, one can define the sum

$$\sum F := \sum_{f\in F} f \ : \ S \to A$$

of all elements of F. This sum can either be defined as the limit of its finite sub-sums under the product topology on A^S (A discrete), or, more elementarily, by setting:

$$\left(\sum F\right)(s) := \sum_{\substack{f\in F \\ s\in\operatorname{supp} f}} f(s).$$

We also say that a set of subsets of S is *locally finite* if the corresponding set of indicator functions is itself locally finite. Although overly convoluted, one can rephrase the local finiteness of a graph X using the above terminology, by remarking that X is locally finite if and only if the family $\{\{u, v\} \mid u \sim v\}$ is itself locally finite.

Let $\phi : S \to A^T$ be a map such that $\phi[S] \subseteq A^T$ is locally finite. Given a function $f : S \to A$, one can define

$$\phi f : T \to A; t \mapsto \sum_{s \in S} f(s) \cdot \phi(s)(t),$$

which is well-defined by local finiteness of $\phi[S]$. Moreover, given any locally finite set $F \subseteq A^S$ of functions, the set

$$\{\phi f \mid f \in F\}$$

will still be locally finite.

4.1.2 Roe Homology

Given a locally finite simplicial complex X and a choice of coefficient ring A, one can extend the usual definition of simplicial homology on X to a form of *homology with infinite support* (in a more general context also known as Borel-Moore, or closed, homology). This is the homology induced by the chain complex $(C_\bullet(X, A), d_\bullet(X, A))$ with:

- $C_n(X, A)$ the A-module of functions from X_n to A (generally interpreted as infinite formal sums of n-simplices);
- $d_n(X, A) : C_n(X, A) \to C_{n-1}(X, A)$ obtained by sending an n-simplex to the alternating sum of its faces, and extending "linearly":

$$f = \sum_{\sigma} f(\sigma) \cdot \sigma \quad \mapsto \quad \sum_{\sigma} \underbrace{\left(f(\sigma) \sum_{\tau < \sigma} [\tau : \sigma] \cdot \tau \right)}_{=: d_n(f(\sigma) \cdot \sigma)}.$$

A routine calculation (making essential use of the local finiteness condition (see section 4.1.1)) ensures that this is well-defined and yields a chain complex. With notation

$$Z_n(X, A) \coloneqq \ker d_n(X, A), \qquad B_n(X, A) \coloneqq \operatorname{im} d_n(X, A),$$

we get $H_n(X, A) = Z_n(X, A)/B_n(X, A)$.

For $r \leq s$, we have $C_\bullet(R_r(X), A) \subseteq C_\bullet(R_s(X), A)$ and this inclusion agrees with the boundary operators $d_\bullet$. We can therefore define the *Roe chain complex* $C_\bullet^\infty(X, A)$ by letting:

$$C_\bullet^\infty(X, A) \coloneqq \bigcup_{r > 0} C_\bullet(R_r(X, A)),$$

and defining the boundary operator $d_\bullet^\infty(X, A)$ accordingly. Finally, the *Roe homology* (of X with coefficients in A) is simply the homology of this chain complex. With notation

$$Z_n^\infty(X, A) \coloneqq \ker d_n^\infty(X, A), \qquad B_n^\infty(X, A) \coloneqq \operatorname{im} d_n^\infty(X, A),$$

we get $H_n^\infty(X, A) = Z_n^\infty(X, A)/B_n^\infty(X, A)$. Note that we also have

$$H_n^\infty(X, A) = \frac{\bigcup_r Z_n^\infty(R_r(X, A))}{\bigcup_r B_n^\infty(R_r(X, A))}.$$

If no confusion is possible, we will not index our chain complexes with the choice of base ring and/or of simplicial complex. Furthermore, from now on, A will always denote the chosen coefficient ring; if the choice does not matter, we will usually drop it.

We state the following essential property of Roe homology without proof:

Theorem 4.1.1. *Roe homology is quasi-isometry invariant.*

Proof. Follows from [Mit01, Theorem 3.10]. □

4.1.3 Circuits, birays and flows

Fix a (locally finite) graph X. Let us call any element of $Z_1(X)$ a *flow*. This terminology is motivated by the fact that, given an orientation of X, an element of $Z_1(X)$ can be described as an assignment to each edge e from u to v of some value $f(e) \in A$, interpreted as a quantity of "flow" running through e, from u to v; the prescription $d_1 f = 0$ states the usual condition that the sum of incoming flow on a vertex is equal to the sum of outgoing flow.

We can single out two indecomposable (in an informal sense) units of flow on X; those given by simple circuits and simple birays. More precisely, any circuit and biray can be traversed in two directions (corresponding to inversing the indexing sequence). Choose one representative direction for each: this defines an orientation of the circuits and birays. Let $\mathfrak{S}$ denote the set of (oriented) simple circuits and birays. Now, to any $s \in \mathfrak{S}$, associate the element $\mathscr{D}s \in Z_1(X)$, defined as the sum of edges appearing in s with sign, depending on whether the edge has the same or opposite orientation to that of s. In other words:

$$(\mathscr{D}s)(e) := \begin{cases} 0 & \text{if } e \notin \operatorname{supp} s, \\ 1 & \text{if } e \in \operatorname{supp} s \text{ and } e, s \text{ have agreeing orientation,} \\ -1 & \text{if } e \in \operatorname{supp} s \text{ and } e, s \text{ have disagreeing orientation.} \end{cases}$$

Any flow on X can be decomposed into a sum of simple circuits and birays:

Lemma 4.1.2. *Given any (A-valued) flow f on the graph X, there exists a function $\lambda : \mathfrak{S} \to A$, such that $\{\lambda(s) \cdot \mathscr{D}s \mid s \in \mathfrak{S}\}$ is locally finite, and satisfying*

$$f = \sum_{s \in \mathfrak{S}} \lambda(s) \cdot \mathscr{D}s.$$

If λ is such that $\{\lambda(s) \cdot \mathscr{D}s \mid s \in \mathfrak{S}\}$ is locally finite, we call λ *locally finitely supported.* Given a locally finitely supported function $\lambda : \mathfrak{S} \to A$, let us write $\mathscr{D}\lambda$ for the sum $\sum_{s \in \mathfrak{S}} \lambda(s) \cdot \mathscr{D}s$.

Proof. Fix an enumeration $\{e_i\}_{i=1}^\infty$ of the edges of X. We will construct λ iteratively, starting with the zero function, iterating over edges and adding first circuits, then birays to λ in a locally finite manner.

Start with $\lambda = 0$, and iter over each edge e_i in order:

- If e_i lies in some circuit of $\operatorname{supp}(f - \mathscr{D}\lambda)$, add this circuit with proper weighting to λ, in such a way that now $e_i \notin \operatorname{supp}(f - \mathscr{D}\lambda)$.
- Otherwise, do nothing.

Note that after each step, λ stays locally finite, and that after step i, $(\mathscr{D}\lambda)(e_i)$ stabilizes. In the end (or rather, in the limit) $\operatorname{supp}(f - \mathscr{D}\lambda)$ is a forest. Indeed, assume it contains a circuit and let e_i be the first edge in this circuit; the very construction of λ at step i yields a contradiction.

Since $f - \mathscr{D}\lambda$ is a flow, this forest has no vertex of degree 1, and any path can be extended into a biray. Now, iterate again over each edge e_i in order:

- If e_i lies in $\operatorname{supp}(f - \mathscr{D}\lambda)$, add any biray containing e_i with proper weighting to λ, in such a way that now $e_i \notin \operatorname{supp}(f - \mathscr{D}\lambda)$.
- Otherwise, do nothing.

Note that after each step, λ remains locally finite and that $\operatorname{supp}(f - \mathscr{D}\lambda)$ stays a forest. At the limit all edges have been removed, and thus $\mathscr{D}\lambda = f$, as needed. □

Circuit space. Fix a graph X. Let $\mathfrak{C} = \mathfrak{C}(X)$ denote the set of circuits of X, chosen with some orientation, and $\mathfrak{C}_r = \mathfrak{C}_r(X)$ the subset of $\mathfrak{C}(X)$ consisting of circuits of length at most r. As seen above, to any oriented circuit $c \in \mathfrak{C}$, one can associate a function $\mathscr{D}c : EX \to A$ sending an edge to 0 if it does not lie in the circuit, and to ± 1 if it does, according to orientation; note that $\mathscr{D}c$ lies in $Z_1(X)$. Let $A^{(\mathfrak{C})}$ denote the set of functions from $\mathfrak{C}$ to A that are locally finitely supported. Formally,

$$A^{(\mathfrak{C})} := \left\{ f : \mathfrak{C} \to A \ : \forall e \in EX \, |\{c : f(c) \neq 0 \wedge e \in \operatorname{supp} \mathscr{D}c\}| < \infty \right\}.$$

It is easy to see that $A^{(\mathfrak{C})}$ is an A-module. This association can be extended to a linear map:

$$\mathscr{D} : A^{(\mathfrak{C}(X))} \to Z_1(X),$$

with essential use of the local finiteness condition. For any $r > 0$, we similarly get a module of functions $A^{(\mathfrak{C}_r(X))}$ and a restricted map

$$\mathscr{D}_r : A^{(\mathfrak{C}_r(X))} \to Z_1(X).$$

Actually, local finiteness of an element of $A^{(\mathfrak{C}_r(X))}$ comes for free since the local finiteness of the graph implies that only a finite number of circuits of a given length can contain a given edge. Thus, $A^{(\mathfrak{C}_r(X))}$ is simply the set of functions $\mathfrak{C}_r(X) \to A$.

Finally, let us write

$$\mathscr{C}(X, A) := \operatorname{im} \mathscr{D}, \quad \mathscr{C}_r(X, A) := \operatorname{im} \mathscr{D}_r, \quad \mathscr{C}_\infty(X, A) := \bigcup_r \operatorname{im} \mathscr{D}_r.$$

Under the interpretation of $Z_1(X)$ as the space of all flows on X, the space $\mathscr{C}(X)$ [resp. $\mathscr{C}_r(X)$, $\mathscr{C}_\infty(X)$] corresponds to flows that can be written as any locally finite sum of circuits [resp. circuits of length at most r, circuits of uniformly bounded length]. Some remarks:

- Each $\mathscr{C}_\bullet(X, A)$ and $\mathscr{C}(X, A)$ is an A-module.
- $\mathscr{C}_1(X) \subseteq \mathscr{C}_2(X) \subseteq \ldots \subseteq \mathscr{C}_\infty(X) \subseteq \mathscr{C}(X)$.
- If X is a tree, then all $\mathscr{C}_\bullet(X)$ and $\mathscr{C}(X)$ are trivial.

In the sequel, when no confusion is in sight, we will usually not distinguish between an oriented circuit $c \in \mathfrak{C}$ and the element $\mathscr{D}c \in \mathscr{C}$ it defines (similarly for birays and sums of such).

Let us say that X has *A-large circuits* if the sequence $\mathscr{C}_1(X, A) \subseteq \mathscr{C}_2(X, A) \subseteq \ldots$ does not stabilizes (i.e., if $\mathscr{C}_\infty(X, A)$ is different from $\mathscr{C}_r(X, A)$ for all r).

Proposition 4.1.3. *Both having A-large circuits and the vanishing of $\mathscr{C}(X, A)/\mathscr{C}_\infty(X, A)$ are invariant under quasi-isometries.*

Proof. Let X, Y be two locally finite graphs, and $\phi : VX \to VY$, $\psi : VY \to VX$ quasi-inverses with constant K.

If u, v are two vertices both in X or Y, let $p_{u,v}$ be a geodesic path joining u to v. If p is a path in X of length l, let ϕp denote the path in Y obtained by concatenating the segments $p_{\phi u_i, \phi u_{i+1}}$, for each edge (u_i, u_{i+1}) of p; define similarly ψp if p is a path in Y. Since ϕ is a quasi-isometry, ϕp has length at most lK, and $\psi\phi p$ at most lK^2. If v is a vertex of p, $\psi\phi v$ is at distance at most K from v. It follows that for each edge $e := (u_i, u_{i+1})$ of p, e and $\psi\phi e$ differ by a circuit of length $3K+1$: this circuit consists of the concatenation, in order, of e, then $p_{u_{i+1}, \psi\phi u_{i+1}}$, then the reverse of $\psi\phi e$, then $p_{\psi\phi u_i, u_i}$. By summing over the edges of p, it follows that p and $\psi\phi p$ differ by an element of $\mathscr{C}_L(X, A)$, for $L \geq 3K + 1$.

Assume now that $\mathscr{C}_r(Y, A) = \mathscr{C}_\infty(Y, A)$, and let $p \in \mathscr{C}_s(X, A)$ for some large enough s. Since p is a uniformly bounded sum of circuits of length at most s, ϕp is a uniformly bounded sum of circuits of length at most Ks, i.e., lies in $\mathscr{C}_{Ks}(Y, A)$. For s large enough, $\mathscr{C}_{Ks}(Y, A) = \mathscr{C}_r(Y, A)$, and $\phi p \in \mathscr{C}_r(Y, A)$, so that $\psi\phi p \in \mathscr{C}_{Kr}(X, A)$. Since $\psi\phi p$ and p differ by an element of $\mathscr{C}_L(X, A)$, we conclude that $p \in \mathscr{C}_{\max(Kr,L)}(X, A)$. Thus, $\mathscr{C}_\infty(X, A) = \mathscr{C}_{\max(Kr,L)}(A)$, as was to be shown.

The fact that the vanishing of $\mathscr{C}(X, A)/\mathscr{C}_\infty(X, A)$ is invariant under quasi-isometries follows the same pattern. Fix X, Y, ϕ, ψ, K as above with $\mathscr{C}(Y, A) = \mathscr{C}_\infty(Y, A)$ and $f \in \mathscr{C}(X, A)$. The image ϕf of f through ϕ is an element of $\mathscr{C}(Y, A)$. Indeed, if $f = \sum_{c \in \mathfrak{C}(X)} \lambda_c c$ is locally finite, then $\phi f = \sum_{c \in \mathfrak{C}(X)} \lambda_c \phi c$ is also locally finite, since the existence of an infinite number of circuits ϕc with $\lambda_c \neq 0$ sharing an edge would imply an infinite number of circuits c with $\lambda_c \neq 0$ intersecting on a compact in X, hence also sharing an edge. Now, ϕf is in $\mathscr{C}_\infty(Y, A)$ by assumption, so that $\psi\phi f \in \mathscr{C}_\infty(X, A)$, and $\psi\phi f$ and f differ by an element of $\mathscr{C}_L(X, A)$. We conclude that $f \in \mathscr{C}_\infty(X, A)$ too. □

4.1.4 H_1^∞ in terms of flows.

In this subsection, we provide an equivalent definition of H_1^∞, which can be summarized as "all flows" modulo "all sums of circuits of uniformly bounded lengths".

Fix a locally finite graph X, and choose, for each pair of vertices $x_1, x_2 \in VX$ of X, a geodesic path p_{x_1,x_2} connecting x_1 to x_2.

If $c = (c_1, \ldots, c_n) \in \mathfrak{C}_r$ is a circuit in X of length $\leq r$, associate to c its triangulation:

$$\Delta c = \sum_{i=1}^{n-1} \underbrace{[c_i, c_{i+1}, c_1]}_{\text{triangle in } R_r(X)} \in A^{(R_r(X))_{(2)}}.$$

This assignment is locally finite, and therefore extends to a linear map:

$$\Delta : A^{\mathfrak{C}_r} \to A^{(R_r(X))_{(2)}} = C_2(R_r(X)),$$

which, by construction, satisfies $d_2 \circ \Delta = \mathscr{D}$. In particular, we see that:

$$\mathscr{C}_r(X, A) = \mathscr{D}(A^{\mathfrak{C}_r}) = d \circ \Delta[\mathfrak{C}_r] \subseteq d[C_2(R_r(X), A)] = B_1(R_r(X), A)$$

so that, since $\mathscr{C}_r(X, A) \subseteq Z_1(X, A)$:

$$\mathscr{C}_r(X, A) \subseteq B_1(R_r(X), A) \cap Z_1(X, A). \tag{4.1.1}$$

Conversely, given a triangle $t = [x_1, x_2, x_3] \in R_r(x)_{(2)}$, associate to t the circuit $ct := p_{x_1,x_2} p_{x_2,x_3} p_{x_3 x_1} \in \mathfrak{C}_{3r}$ obtained by concatenating the geodesics. Again, this assignment is locally finite, and extends to a linear map:

$$c : C_2(R_r(X)) = A^{(R_r(X))_{(2)}} \to A^{\mathfrak{C}_{3r}}.$$

Assume now that $f \in B_1(R_r(X), A) \cap Z_1(X, A)$, so that $f = dF$ with $F \in C_2(R_r(X), A)$. From the fact that $f \in Z_1(X, A)$ follows that $dF = \mathscr{D}cF$. This shows that:

$$B_1(R_r(X), A) \cap Z_1(X, A) \subseteq \mathscr{C}_{3r}(X, A) \tag{4.1.2}$$

By combining eq. (4.1.1), eq. (4.1.2) and the fact that $B_1^\infty(X, A) = \bigcup_r B_1^\infty(R_r(X), A)$, we conclude:

$$\mathscr{C}_\infty(X, A) = Z_1(X, A) \cap B_1^\infty(X, A). \tag{4.1.3}$$

Now fix some $r \in \mathbb{N}$ and consider the composite morphism:

$$\Phi_r : Z_1(X, A) \hookrightarrow Z_1(R_r(X), A) \twoheadrightarrow H_1(R_r(X), A).$$

We claim that Φ_r is surjective, and that its kernel satisfies:

$$\mathscr{C}_{2r} \subseteq \ker \Phi_r \subseteq \mathscr{C}_{3r}.$$

Indeed, let $f = \sum_{[u,v] \in R_r(X)_{(2)}} f([u, v]) \cdot [u, v]$ be a representative of an element of $H_1(R_r(X), A)$, and define

$$\tilde{f} := \sum_{[u,v] \in R_r(X)_{(2)}} f([u, v]) p_{u,v}.$$

$\tilde{f}$ is obtained by replacing edges in $R_r(X)$ by geodesics in X. By construction, $\tilde{f}$ and f differ by an appropriate locally finite sum of triangles of $R_r(X)$, hence are equal in homology. This verifies surjectivity. The two inclusions involving the kernel follow from eqs. (4.1.1) and (4.1.2), since $\ker \Phi_r = Z_1(X, A) \cap B_1(R_r(X), A)$.

We now easily obtain the desired desciption of H_1^∞:

Theorem 4.1.4. *If X is a (locally finite) graph, then*

$$H_1^\infty(X, A) \cong \frac{Z_1(X, A)}{\mathscr{C}_\infty(X, A)}.$$

Proof. We consider the composite morphism

$$\Phi_\infty : Z_1(X, A) \hookrightarrow Z_1^\infty(X, A) \to \frac{Z_1^\infty(X, A)}{B_1^\infty(X, A)} = \frac{\bigcup_r Z_1(R_r(X), A)}{\bigcup_r B_1(R_r(X), A)}.$$

Surjectivity of all Φ_r implies surjectivity of Φ. Furthermore, $\ker \Phi_\infty = \mathscr{C}_\infty(X, A)$, since $\ker \Phi_\infty = \bigcup_r \ker \Phi_r$. □

Lemma 4.1.5. *If X is a (locally finite) graph with $\mathscr{C}_r(X, A) = \mathscr{C}_\infty(X, A)$, then, for any large enough s, the map Φ_s descends to an isomorphism:*

$$\tilde{\Phi}_s : \frac{Z_1(X, A)}{\mathscr{C}_r(X, A)} \to H_1(R_s(X), A),$$

as does Φ_∞, to an isomorphism:

$$\tilde{\Phi}_\infty : \frac{Z_1(X, A)}{\mathscr{C}_r(X, A)} \to H_1^\infty(X, A).$$

Furthermore, the following triangle of isomorphisms commutes:

$$\begin{tikzcd} & \frac{Z_1(X,A)}{\mathscr{C}_r(X,A)} \arrow[dl, "\tilde{\Phi}_s"'] \arrow[dr, "\tilde{\Phi}_\infty"] & \\ H_1(R_s(X), A) \arrow[rr, "{[\iota]}"] & & H_1^\infty(X, A). \end{tikzcd}$$

Proof. That we have isomorphisms follows from the fact that $\mathscr{C}_r(X, A) = \mathscr{C}_s(X, A) = \mathscr{C}_\infty(X, A)$ for large enough s, commutativity from the fact that all maps involved are appropriate quotients of inclusions. □

4.2 The case of trees

The goal of this section is to understand the relation between H_1^∞ and trees. In the first part, we will compute $H_1^\infty(T)$, where T is a tree. The second part is devoted to understanding the relation between the homology of a graph X and that of a tree that "represents" the ends of X.

4.2.1 H_1^∞(tree)

We will show (expanding on a result of [Dia15]) that given a tree T, there exists a set $\mathscr{B}$ of birays on T such that $H_1^\infty(T, A)$ can be identified with the set $A^{\mathscr{B}}$ of functions $\mathscr{B} \to A$. We will first briefly explain the properties we need $\mathscr{B}$ to have and illustrate how to construct it. A formal description of its construction and verification of the properties it ought to satisfy is tedious, but straightforward. We choose to provide an illustration instead; see fig. 4.1.

Simplifying assumptions. Let us start with three easy steps. First, from the decomposition $H_1^\infty(T) = Z_1(T)/\mathscr{C}_\infty(T)$ and the fact that $\mathscr{C}_\infty$ is trivial for trees, it follows that $H_1^\infty(T) = Z_1(T)$. Second, all elements of $Z_1(T)$ are entirely supported on edges incident to vertices of degree at least 2, so that all vertices of degree 1 can be discarded from the tree without interfering with $Z_1(T)$. We can therefore safely assume that T has no vertices of degree 1. Finally, by local finiteness and a simple "unrolling" of high valency vertices, one can transform T into a tree T' with only vertices of valency 2 or 3, and such that at least one is of valency 2; it is then a simple matter to verify that $Z_1(T)$ and $Z_1(T')$ are isomorphic. From now on, let us therefore assume that T is a tree with all vertices of degree 2 or 3, and at least one of degree 2.

Comb decomposition. Let T be a tree (with only vertices of degrees 2 and 3 by assumption) and $v_0 \in VT$ be a vertex of degree 2. We will make use of a decomposition of T into a set of rays covering it, dubbed *comb decomposition*, and defined as an $\mathbb{N}$-indexed sequence $(\mathscr{P}_n)_n$ of sets of simple rays on T, with the following properties:

1. Each edge of T is covered by exactly one ray in exactly one $\mathscr{P}_n$.
2. $\mathscr{P}_0$ contains exactly two rays, both starting at v_0.
3. For any $n > k$, the only vertex that a ray in $\mathscr{P}_n$ shares with a ray in $\mathscr{P}_k$ is its starting point, and in that case $k = n - 1$.

Let T_n be the subtree of T covered by all rays in $\bigcup_{k<n} \mathscr{P}_k$.

Let us explain how to construct such a decomposition. First, $\mathscr{P}_0$ consists of two edge-disjoint rays leaving v_0. Then, assuming $\mathscr{P}_0, \dots, \mathscr{P}_n$ have been defined, let V_{n+1} consist of all vertices contained in some ray of some $\mathscr{P}_n$, such that one edge incident to the vertex is not contained in any previously defined ray. Let $\mathscr{P}_{n+1}$ consist of one ray per vertex in V_{n+1}, starting at this vertex and leaving through the free edge in any direction, as long as it does not intersect T_n. This process can end after some n if the set V_{n+1} is empty, or continue indefinitely.

It is easy to see that this construction yields a family satisfying the required properties (see fig. 4.1).

Good set of birays. Given a comb decomposition $(\mathscr{P}_n)_n$ of T, we construct a set $\mathscr{B}$ of birays on T endowed with a well ordering $\leq$ satisfying the following conditions:

1. Each edge is covered by at least 1 and at most 3 birays of $\mathscr{B}$.

For any edge $e \in ET$, let $\mathscr{B}(e)$ be the set of elements of $\mathscr{B}$ covering e. Let us say that $b \in \mathscr{B}$ is *last* on e if $b \in \mathscr{B}(e)$ and for all $b' \in \mathscr{B}(e)$, we have $b' \leq b$. For any $b \in \mathscr{B}$, define the set

$$M_b := \{e \;:\; b \text{ is last on } e\}.$$

2. Each set M_b is non-empty and connected, and $\bigcup_{b\in\mathscr{B}} M_b = ET$.
3. Whenever b is last on e, there exists some other edge e' such that $\mathscr{B}(e) = \mathscr{B}(e') \sqcup \{b\}$.

The construction of the set $\mathscr{B}$ naturally follows from that of a comb decomposition. A preliminary (well-founded but not total) order structure comes naturally with it, and it suffices to extend it carefully to find our well ordering on $\mathscr{B}$. Instead of explaining this construction precisely, let us refer to fig. 4.1: the construction can easily be deduced from this example.

We can now state and prove our main result concerning trees.

Theorem 4.2.1. *If $\mathscr{B}$ is a good set of birays for the tree T, then*

$$H_1^\infty(T, A) \cong A^{\mathscr{B}}.$$

Proof. Consider the morphism

$$\begin{aligned} \mathscr{D} \;:\; A^{\mathscr{B}} &\to Z_1(T, A) = H_1^\infty(T, A) \\ \phi &\mapsto \sum_{b\in\mathscr{B}} \phi(b) \cdot b. \end{aligned}$$

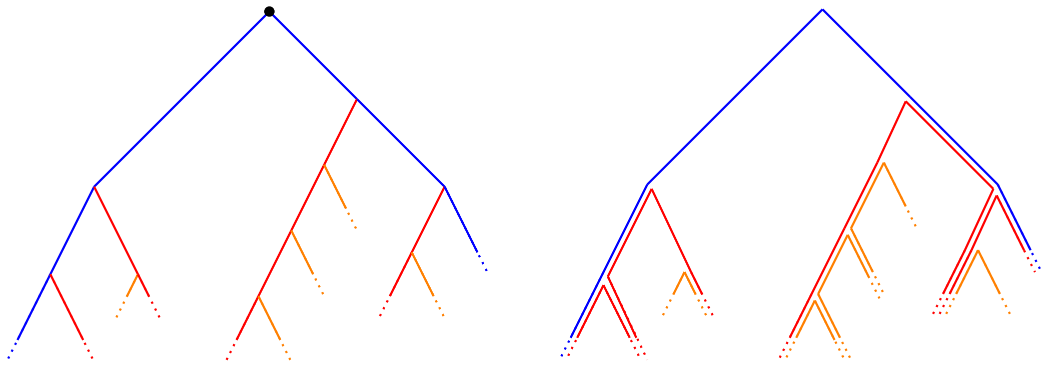

Figure 4.1: Comb decomposition of a tree and the corresponding set of birays $\mathscr{B}$. The black dot is v_0.

Since only a finite number of elements of $\mathscr{B}$ passes through any given edge, this map is well-defined, and A-linearity is clear.

It remains to verify that $\mathscr{D}$ is injective and surjective.

For injectivity, assume that $\phi \in A^{\mathscr{B}}$ is non-zero, and let b the first (w.r.t our order) biray with $\phi(b) \neq 0$. Take some $e \in M_b$ (item 2); by definition of M_b, the value of $\mathscr{D}(\phi)$ on e is equal to $[e : b]\phi(b)$ (where $[e : b]$ corrects the sign if the orientations of e and b disagree), which is non-zero by hypothesis.

For surjectivity, given an element $f \in Z_1(T, A)$, we construct a preimage ϕ of f by inductively defining its values on the rays $b \in \mathscr{B}$, using the well-ordering on $\mathscr{B}$. Fix some $b \in \mathscr{B}$ and assume that for all $b' < b$, $\phi(b')$ is defined and is such that whenever an edge $e \in ET$ is covered only by birays $b' < b$, the sum $\sum_{e \in b'}[e : b']\phi(b')$ is equal to $f(e)$ (this is an obviously necessary condition, since the value of $\mathscr{D}\phi$ on e is entirely defined by the coefficients associated to each $b' \in \mathscr{B}(e)$, and all elements therein have already been given a coefficient). Take any edge $e \in M_b$, and define

$$\phi(b) := f(e) - \sum_{e \in b' < b} [e : b']\phi(b').$$

Let us verify that this definition does not depend on the choice of $e \in M_b$. By item 2, it suffices to compare the values obtained for two adjacent edges e_1, e_2. Let v be the vertex shared by e_1, e_2; it is either of degree 2 or of degree 3, in which case it is incident to a third edge e_3. We treat only the case of degree 3, the other one being similar but simpler. For definiteness, assume that the local configuration around v is as presented in fig. 4.2. In particular, the orientation of edges (thick arrows) and all birays involved (dotted thin arrows) are assumed to be as given by the figure. First, note that since $e_1, e_2 \in M_b$ all birays in $\mathscr{B}$ traversing e_1, e_2 come before b, which implies that this is also the case for e_3: indeed, any biray traversing e_3 must then either traverse e_1 or e_2 (otherwise it would get stuck). A few remarks (with index arithmetic mod 3):

- $\mathscr{B}(e_i) \cap \mathscr{B}(e_{i+1}) \cap \mathscr{B}(e_{i+2}) = \emptyset$ and $\mathscr{B}(e_i) \subseteq \mathscr{B}(e_{i+1}) \cup \mathscr{B}(e_{i+2})$.
- Any $b' \in \mathscr{B}(e_3)$ comes before b, so that $f(e_3) = \sum_{b' \in \mathscr{B}(e_3)}[e : b']\phi(b')$ by the induction hypothesis.
- For $i = 1, 2$ and any $b' \in \mathscr{B}(e_i)$, $[e_i : b'] = +$. If $b' \in \mathscr{B}(e_3)$, then $[e_3, b']$ is $+$ if $b' \in \mathscr{B}(e_1)$ and $-$ otherwise. This is a consequence of our choice of orientations.

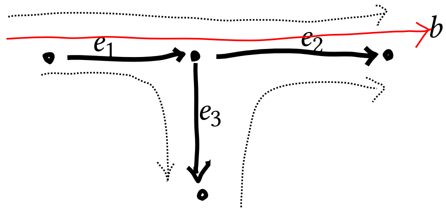

Figure 4.2: Local configuration in the inductive construction of $\mathscr{D}$.

We then get (with $\mathscr{B}_i := \mathscr{B}(e_i)$ and $\mathscr{B}_{ij} := \mathscr{B}(e_i) \cap \mathscr{B}(e_j)$)

$$\begin{aligned}
f(e_1) - f(e_2) = f(e_3) & & (f \text{ is a flow}) \\
&= \sum_{b' \in \mathscr{B}_3} [e_3 : b']\phi(b') & \text{(second observation)} \\
&= \sum_{b' \in \mathscr{B}_{13}} \phi(b') - \sum_{b' \in \mathscr{B}_{23}} \phi(b') & \text{(decomposing \& third observation)} \\
&= \sum_{b' \in \mathscr{B}_{13}} \phi(b') + \sum_{b' \in \mathscr{B}_{12} - \{b\}} \phi(b') & \\
&\quad - \sum_{b' \in \mathscr{B}_{12} - \{b\}} \phi(b') - \sum_{b' \in \mathscr{B}_{23}} \phi(b') & \text{(adding zero)} \\
&= \sum_{b' \in \mathscr{B}_1 - \{b\}} \phi(b') - \sum_{b' \in \mathscr{B}_2 - \{b\}} \phi(b') & \text{(grouping)}
\end{aligned}$$

and a rebalancing of the sum yields the desired equality. By construction, the definition of ϕ on b satisfies the inductive hypothesis, and letting ϕ be the resulting map $\mathscr{B} \to A$, it is clear that $\mathscr{D}\phi = f$; surjectivity is verified. □

4.2.2 Subtrees

Fix a graph X, some vertex $v_0 \in VX$ and a tree T contained in X and containing v_0. The set of ends of T is in natural bijection with the set of simple rays starting at v_0; recall that each such ray also defines an end of X. Consider the map

$$\operatorname{Ends} T \to \operatorname{Ends} X,$$

obtained by sending a simple ray starting at v_0 in T to the end of X it represents. We say that T is *end-respecting* if this map is injective, and *end-defining* if it is bijective. End-defining trees always exist.

Theorem 4.2.2. *Let X be a graph and T an end-respecting subtree of X. Then*

$$Z_1(T) \cap \mathscr{C} = \{0\}.$$

In particular, the natural map $H_1^\infty(T) \to H_1^\infty(X)$ is injective.

Before proving this, we need a small technical lemma.

Lemma 4.2.3. *Let X be a graph, T an end-respecting subtree of X and e an edge of T. Let $\{\mathscr{E}_+, \mathscr{E}_-\}$ be the partition of* Ends T *induced by the removal in T of the edge e. Then, there exists some compact K of X such that the partition of* Ends T *(seen as a subset of* Ends X*) induced by the removal of K in X is as fine as the partition $\{\mathscr{E}_+, \mathscr{E}_-\}$. In other words, there exists some K such that whenever two ends of T are separated by e in T, then these same ends, seen as ends of X, are now separated by K in X.*

Proof. Assume this is not the case, and fix some vertex $v_0 \in VT$. Then, for each compact ball $B(v_0, r)$ (with r large enough), there exist rays e_+^r and e_-^r representing ends in $\mathscr{E}_+$ and $\mathscr{E}_-$ respectively and such that e_+^r and e_-^r are not separated by $B(v_0, r)$. Since the ends represented by e_+^r, e_-^r are ends of T, we can assume that e_+^r, e_-^r are actually rays of T. Take respective accumulations points e_+ and e_- of the sequences $(e_+^r)_r$ and $(e_-^r)_r$. We then have

- e_+ and e_- define ends of X, since they are (simple) rays of T.
- e_+ represents an element in $\mathscr{E}_+$, by its construction as an accumulation point of rays in $\mathscr{E}_+$; the same holds for $e_-, \mathscr{E}_-$.
- e_+, e_- are close in X, since given any compact K, one can choose long enough approximations e_+^r, e_-^r of (e_+, e_- respectively) that are not separated by K, by assumption.

The second and third items are contradictory, since two distinct ends cannot be represented by close elements. This proves the lemma. □

We can now proceed with proving the theorem.

Proof of Theorem 4.2.2. The second part of the theorem follows from the first, using $H_1^\infty(T) = Z_1(T)$ (T being a tree), and the characterization of $H_1^\infty(X)$ as $\frac{Z_1(X)}{\mathscr{C}_\infty}$.

To prove the first part, assume towards a contradiction that it does not hold. Then, there exists some non-zero element f of $Z_1(T)$ that also lies in $\mathscr{C}$: f can be written as some sum $\sum_c \lambda_c \cdot c$ of circuits of X. For notational convenience, let us discard the coefficients λ_c and let C be the set of circuits appearing in the sum, so that $f = \sum_{c \in C} c$. Fix some edge e_0 of T such that $f(e_0) \neq 0$; e_0 partitions the ends of T into exactly two parts $\mathscr{E}_+, \mathscr{E}_-$, and since T is end-respecting, these parts correspond to two disjoint parts of ends of X. Using lemma 4.2.3, take a compact $K \subseteq X$ that

- contains e_0, and
- separates the ends of T as finely as e_0 does.

By extending K if needed, one can assume that all connected components of $X - K$ are actually infinite. Let L be the union of all connected components that contain ends of $\mathscr{E}_+$, and L' the union of all remaining connected components (both L and L' may contain ends of X that are not ends of T; the important point is that all ends in $\mathscr{E}_+$ are in L and all ends in $\mathscr{E}_-$ are in L').

We make a few remarks:

- Both $\partial^e L$ and $\partial^e L'$ are *cuts* of the graph X, in the sense that removing either of those sets of edges disconnects the graph into exactly two connected components. This is a direct consequence of the definitions of L and L'.

- The value of f on the cut $\partial^e L$, defined as $\sum_{e\in\partial^e L} f(e)$, is equal to $f(e_0)$. Indeed, this can be seen by decomposing f into a sum of birays in T: each biray in this decomposition contributes to $f(e_0)$ in the same amount as it contributes to the value of f on the cut $\partial^e L$ (by construction of L).

- For any circuit c and coefficient λ, the value $\sum_{e\in\partial^e L} \lambda c(e)$ of λc on $\partial^e L$ is zero. Indeed, any time an edge of $\partial^e L$ is crossed by c, another one is crossed in the opposite direction.

Let C_K be the set of all circuits in C that contain some vertex of K. It is clear that $f - \sum C_K$ is zero on all of K and also on $\partial^e L$ (for the latter, note that for any edge in $\partial^e L$, all the circuits of C containing this edge must necessarily lie in C_K). Since the set C_K is finite, the third of our remarks implies that $f - \sum C_K$ has non-zero value on $\partial^e L$, a contradiction. □

4.3 Tree & Circuits decomposition

By the result of the last section, we know that given a graph X and an end-defining tree T in X, the (first Roe) homology of T embeds in that of X. The following shows that this embedding is in fact a summand in a direct decomposition of $H_1^\infty(X)$—the other summand being $\mathscr{C}/\mathscr{C}_\infty$.

Theorem 4.3.1. *Let X be a (locally finite) graph and T an end-defining subtree of X. Then*

$$H_1^\infty(X) \cong H_1^\infty(T) \oplus \frac{\mathscr{C}(X)}{\mathscr{C}_\infty(X)}.$$

Proof. We first show that $Z_1(T) + \mathscr{C}(X) = Z_1(X)$, i.e., that any $f \in Z_1(X)$ can be written as a sum $f = f_T + f_{\mathscr{C}}$ with $f_T \in Z_1(T)$ and $f_{\mathscr{C}} \in \mathscr{C}$.

Intuitively, we will use circuits (elements of $\mathscr{C}$) to "push" the flow f onto the tree T. First, decompose f into a sum of circuits and birays, using lemma 4.1.2. We can then write:

$$f = \sum_{e\in\mathrm{Ends}\,X} \sum_{b\in\mathfrak{B}(e,e)} \lambda_b \mathscr{D} b \quad + \quad \sum_{e\neq e'\in\mathrm{Ends}\,X} \sum_{b\in\mathfrak{B}(e,e')} \lambda_b \mathscr{D} b \quad + \quad \sum_{c\in\mathfrak{C}} \lambda_c \mathscr{D} c,$$

where $\mathfrak{B}(e,e')$ is the set of birays coming from end e and going to end e'. Fix a ball $B(v_0,r)$ and consider all birays that appear in the decomposition of f and intersect the ball.

- If $b \in \mathfrak{B}(e,e)$ is a biray coming from the same end it is going to, there must exist a path outside of the ball connecting the two remaining strands of b. The concatenation of this path with the piece of b intersecting the ball yields a circuit, whose removal pushes b outside of the ball.

- If $b \in \mathfrak{B}(e,e')$ is a biray coming from a different end it is going to, there exists a biray b_T in the tree T with the same ends, and there exist two paths outside of the ball, connecting the matching strands of b and b_T. The concatenation of the two connecting paths with the appropriate pieces of b and b_T yields a circuit, whose removal pushes b outside of $B(v_0,r) - T$.

Thus, in each case, one can remove a circuit from a biray to push it outside of $B(v_0,r) - T$. To build the function $f_{\mathscr{C}}$, proceed then as follows. Start with some ball $B(v_0,r)$, and connect the birays intersecting the ball as explained above. Then, iteratively, choose a ball containing all connecting paths of the previous step, and find new connecting paths. This process will result in one infinite ladder for each biray b in some $\mathfrak{B}(e,e)$ (sides: two rays defined by b, rungs: the connecting paths), and one bi-infinite ladder for each biray b in some $\mathfrak{B}(e,e')$ (sides: b and the corresponding biray of T, rungs: the connecting paths). Each such ladder is decomposed into a sum of circuits in the obvious way. Let then

$$f_{\mathscr{C}} = \sum_b \lambda_b(\text{circuits in the ladder of } b) + \sum_{c\in\mathfrak{C}} \lambda_c c.$$

To see that this function is well-defined as a locally finite sum, note that:

- The part $\sum_{c\in\mathfrak{C}} \lambda_c c$ is locally finite to begin with, so that it suffices to ensure that the other one is locally finite.
- At each step of the process, only a finite number of circuits are used, since only a finite number of birays intersect any given finite ball (the original decomposition of f was locally finite to begin with). Furthermore, any edge can only be in the support of circuits in two consecutive steps of the construction (each new step adds circuits in a given "annulus").

This shows that, indeed, $f_{\mathscr{C}}$ is locally finite and such that $f - f_{\mathscr{C}}$ is supported on T.

Combining the above with theorem 4.2.2, we conclude that

$$Z_1(T) \oplus \mathscr{C}(X) = Z_1(X),$$

and taking a quotient by $\mathscr{C}_\infty(X)$ yields

$$Z_1(T) \oplus \frac{\mathscr{C}(X)}{\mathscr{C}_\infty(X)} = \frac{Z_1(X)}{\mathscr{C}_\infty(X)} \cong H_1^\infty(X),$$

as was to be shown. □

4.4 More on circuits

As seen above, given a graph X, and any end-defining subtree T of X, we know that

$$H_1^\infty(X,A) \cong H_1^\infty(T,A) \oplus \mathscr{C}(X,A)/\mathscr{C}_\infty(X,A).$$

In section 4.2.1, we saw that the group $H_1^\infty(T,A)$ can be naturally described as the function space $A^{\mathscr{B}}$, for some well chosen set of birays $\mathscr{B}$ on T—we have found a basis of sorts for the left summand in the direct sum decomposition of $H_1^\infty(X,A)$. We will not reach such a simple description of the quotient $\mathscr{C}/\mathscr{C}_\infty$. Nevertheless, by an infinite Gaussian elimination process, we will find nested bases for the spaces $\mathscr{C}_r$ and $\mathscr{C}$ (but not $\mathscr{C}_\infty$), and see that as soon as $\mathscr{C}_\infty \neq \mathscr{C}$, the quotient space $\mathscr{C}/\mathscr{C}_\infty$ is infinite dimensional (when A is a field, a necessary condition for Gaussian elimination to work in general). Thus, in the rest of this section, A will always denote a field.

4.4.1 A basis for $\mathscr{C}$

Let $\mathfrak{C}^s$ denote the countable set of simple circuits in X, chosen with orientation. Remark that the spaces $A^{(\mathfrak{C}^s)}$ and $A^{(\mathfrak{C})}$ have the same image through $\mathscr{D}$, namely $\mathscr{C}$.

From now on, given a fixed coefficient ring A, we call a subset $\mathscr{B}$ of $\mathscr{C}$ a *basis* if it is locally finite and is such that the map

$$A^{\mathscr{B}} \to \mathscr{C}; \quad \phi \mapsto \Sigma\phi := \Sigma_{b\in\mathscr{B}}\phi(b)\cdot b$$

is bijective. Note that this map is well-defined by virtue of the local finiteness of $\mathscr{B}$. Surjectivity of this map amounts to saying that the set $\mathscr{B}$ spans $\mathscr{C}$ under infinite sums; injectivity to the fact that no non-trivial infinite sum vanishes. We make the same definition for the subspaces $\mathscr{C}_r$ ($r < \infty$).

Theorem 4.4.1. *There exists a basis for $\mathscr{C}$.*

Proof. First, enumerate the edges of X, say $EX = \{e_i\}_{i\in\mathbb{N}}$. If $f \in \mathscr{C}$, let $l(f)$ denote the *first* edge (w.r.t. our enumeration) in the support of f and $v(f) := f(l(f))$; by convention, let also $l(0) = \infty$, and if $f \in \mathscr{C}$, $f(\infty) = 0$. Then, enumerate the elements of $\mathscr{D}[\mathfrak{C}^s]$, say $\mathscr{D}[\mathfrak{C}^s] = \{c_j\}_{j\in\mathbb{N}}$.

We now define the following function inductively

$$\begin{aligned} F : \mathbb{N}\times\mathbb{N} &\to A^{EX} \\ F_{0,j} &:= c_j \quad \forall j, \\ F_{t+1,j} &:= \begin{cases} F_{t,j} & \text{if } j \le t, \\ F_{t,j} - \frac{F_{t,j}(l(F_{t,t}))}{v(F_{t,t})}\cdot F_{t,t} & \text{if } j \ge t+1. \end{cases} \end{aligned}$$

with the convention that if $F_{t,t} = 0$, the fraction is just zero. This function is nothing more than the "history" of applying (column-wise) Gaussian elimination to the infinite matrix whose rows are the simple circuits: t denotes the iteration number, so that at $t = 0$, the matrix has not yet changed, and j is the row number. Two elementary remarks are in order:

1. For all $t \ge j$, we have $F_{t,j} = F_{j,j}$. Let us thus write $g_j := F_{j,j}$.
2. For all $i > j$, we then have $g_i(l(g_j)) = 0$.

If i is of the form $l(g_j)$ for some j, we call e_i a *leading* edge.

We claim that the set $\mathscr{B} := \{g_j \mid j \in \mathbb{N}, g_j \neq 0\}$ satisfies the requirements of the theorem. We must therefore show that it is locally finite, and that the assignment

$$\Sigma : A^{\mathscr{B}} \to \mathscr{C}; \quad \phi \mapsto \sum_{b\in\mathscr{B}} \phi(b)\cdot b$$

is a bijection.

Note first that by the second remark made above, all non-zero g_js are distinct.

<u>Local finiteness.</u> Assume $\mathscr{B}$ is not locally finite. Then, there exists some edge e_i lying in the support of infinitely many g_js. This edge cannot be a leading edge, by the second remark above. Assume then that e_i is not a leading edge, and that it is the first edge supported on infinitely many g_js. Then, for all $i' < i$, $e_{i'}$ is in the support of finitely many g_js, so that there exists some large enough j satisfying:

- $g_j(e_{i'}) = 0$ for all $i' < i$;
- $g_j(e_i) \neq 0$.

But this means that $i = l(g_j)$, a contradiction.

Injectivity of Σ. Since Σ is linear, it suffices to show that $\Sigma(\phi) = 0$ implies $\phi = 0$. Fix ϕ non-zero, and take the least i with $\phi(g_i) \neq 0$. By construction, $\Sigma(\phi)(l(g_i)) = \phi(g_i) \neq 0$, thus proving injectivity.

Surjectivity of Σ. Fix some $f \in \mathscr{C}$, so that, without loss of generality $f = \sum_j f_j \cdot c_j$. By hypothesis, the family $\{f_j \cdot c_j\}_j$ is locally finite. By "reversing" the Gaussian elimination process, we get

$$c_j = g_j + \sum_{t=0}^{j-1} \lambda_{t,j} g_t,$$

for some coefficients $\lambda_{t,j}$. Thus,

$$f = \sum_j (f_j \cdot g_j + \sum_{t=0}^{j-1} f_j \lambda_{t,j} \cdot g_t). \tag{4.4.4}$$

For any t, consider the set $J_t := \{j > t \ : \ f_j \lambda_{t,j} \neq 0\}$. We claim that J_t is finite for all t. Indeed, assuming this is not the case, there exists a least t with J_t infinite. Let e_i be the leading edge of g_t; we have, for all $j > t$

$$c_j(e_i) = g_j(e_i) + \sum_{s=0}^{j-1} \lambda_{s,j} g_s(e) = 0 + \sum_{s=0}^{t} \lambda_{s,j} g_s(e)$$

because e_i, being the leading edge of g_t, lies in the support of no g_s with $s > t$. It follows that

$$f_j c_j(e_i) = \sum_{s=0}^{t} f_j \lambda_{s,j} g_s(e). \tag{4.4.5}$$

By assumption, J_s is finite for all $s < t$, so that $J_t - \bigcup_{s<t} J_s$ is itself infinite. For any j in this set, the sum in eq. (4.4.5) becomes

$$f_j c_j(e_i) = f_j \lambda_{t,j} g_t(e) \neq 0,$$

which is in contradiction with the local finiteness assumption on f, and thus proves that all sets J_t are finite. With this fact at hand, we can permute the sums in eq. (4.4.4) and get

$$f = \sum_t \left(f_t g_t + \sum_{j>t} f_j \lambda_{t,j} \cdot g_t \right).$$

Hence f lies in $\Sigma[\mathscr{B}]$, thus showing surjectivity.

$\square$

Note that in this process, we have chosen to start the Gaussian elimination with simple circuits, but could have chosen any other set that generates $\mathscr{C}$ ("generates" in the sense that any element of $\mathscr{C}$ can be written as a locally finite sum of simple circuits). Similarly, had we added all birays to this initial set, the same process would have given a basis of the space $Z_1(X)$ in one stroke, since we know already that the set $\mathfrak{S}$ generates $Z_1(X)$.

4.4.2 Nested bases for the spaces $\mathscr{C}_r$s

By theorem 4.4.1, we have found a basis $\mathscr{B}$ of elements of $\mathscr{C}$. The goal here is to refine this basis into a nested sequence:

$$\mathscr{B}'_1 \subseteq \mathscr{B}'_2 \subseteq \cdots \subseteq \mathscr{B}'_n \subseteq \mathscr{B}'_{n+1} \subseteq \ldots \subseteq \mathscr{B}',$$

in such a way that each $\mathscr{B}'_i$ [resp. $\mathscr{B}'$] is a basis of the space $\mathscr{C}_i$ [resp. $\mathscr{C}$].

Choose first a well-ordering $\{g_\alpha | \alpha < \beta\}$ of $\mathscr{C}$ and an increasing sequence of ordinals $\beta_1 < \beta_2 < \ldots < \beta_i < \beta_{i+1} < \ldots < \beta$ $(i \in \mathbb{N})$ such that $\mathscr{C}_i = \{g_\alpha | \alpha < \beta_i\}$ for all $i \in \mathbb{N}$.

Let $\mathscr{B}$ be the basis of $\mathscr{C}$ provided by theorem 4.4.1. The set $\mathscr{B}$ is countable; let us choose an enumeration $\{b_i | \, i \in \mathbb{N}\}$ of $\mathscr{B}$. By theorem 4.4.1, there is an isomorphism $A^{\mathscr{B}} \leftrightarrow \mathscr{C}$. Any element g of $\mathscr{C}$ thus (uniquely) defines a function $\mathbb{N} \to A$ by means of our chosen enumeration of $\mathscr{B}$. From now on, we identify $\mathscr{C}$ and $(\mathbb{N} \to A) = A^{\mathbb{N}}$: if $g \in \mathscr{C}$, $g(i)$ is the coefficient associated to $b_i \in \mathscr{B}$ in the unique decomposition of g as a (infinite) sum of elements of $\mathscr{B}$.

Let

$$l(g) := \min\{i \,:\, g(i) \neq 0\}.$$

By convention, let also $l(0) = \infty$, and for any $g \in \mathscr{C}$, $g(\infty) = 0$.

Lemma 4.4.2. *Let X be a (locally finite) graph. There exists a nested sequence of locally finite subsets of $\mathscr{C}$*

$$\mathscr{B}'_1 \subseteq \mathscr{B}'_2 \subseteq \cdots \subseteq \mathscr{B}'_n \subseteq \mathscr{B}'_{n+1} \subseteq \ldots \subseteq \mathscr{B}',$$

such that each $\mathscr{B}'_i$ is a basis of $\mathscr{C}_i$ and $\mathscr{B}'$ is a basis of $\mathscr{C}$.

The proof follows the same pattern as that of theorem 4.4.1.

Proof. Recall that we have chosen a well-ordering $\{g_\alpha | \, \alpha < \beta\}$ of the set $\mathscr{C}$, and ordinals $\beta_1 < \beta_2 < \ldots$ such that $\mathscr{C}_i = \{g_\alpha | \, \alpha < \beta_i\}$ for all i, and that any element of $\mathscr{C}$ is silently identified with a map $\mathbb{N} \to A$ by means of theorem 4.4.1 and an enumeration $\{b_i\}_{i \in \mathbb{N}}$ of $\mathscr{B}$.

We define the "Gaussian elimination" map

$$v \,:\, \beta \times \beta \times \mathbb{N} \to A$$

inductively over its first parameter as follows:

$$\begin{aligned} v_{0,\alpha} &= g_\alpha, \\ v_{\tau,\alpha} &= \begin{cases} \lim_{\sigma<\tau} v_{\sigma,\alpha} & \text{if } \tau \text{ is a limit ordinal,} \\ v_{\alpha,\alpha} & \text{otherwise and } \tau > \alpha, \\ v_{\tau-1,\alpha} - \frac{v_{\tau-1,\alpha}(l(v_{\tau-1,\tau-1}))}{v_{\tau-1,\tau-1}(l(v_{\tau-1,\tau-1}))} \cdot v_{\tau-1,\tau-1} & \text{otherwise.,} \end{cases} \end{aligned} \tag{4.4.6}$$

where the limit is a pointwise limit (meaning that at any coordinate $r \in \mathbb{N}$, the transfinite sequence $(v_{\sigma,\alpha}(r))_{\sigma<\tau}$, is eventually constant), and whenever $v_{\tau-1,\tau-1}$ is zero, the fraction in 4.4.6 is set to zero, by convention.

From now on, we will write v_α for the diagonal element $v_{\alpha,\alpha}$, and $\lambda_{\tau,\alpha}$ for the fraction $\frac{v_{\tau-1,\alpha}(l(v_{\tau-1,\tau-1}))}{v_{\tau-1,\tau-1}(l(v_{\tau-1,\tau-1}))}$ when $\tau \leq \alpha$ and τ is successor (again set to zero by convention when $v_{\tau-1,\tau-1} = 0$), or 0 for any other τ.

Let us briefly explain our interpretation of the map v as an infinite Gaussian elimination. An element of $\mathscr{C}$ is uniquely determined by a map $\mathbb{N} \to A$, as explained above, which we view as an infinite row. The sequence $(g_\alpha)_{\alpha<\beta}$ of elements of $\mathscr{C}$ is viewed as a list of rows: a matrix $(v_\alpha(n))_{\alpha<\beta,n\in\mathbb{N}}$. Finally, the matrix is modified by iterating over each row and subtracting (a suitable multiple of) it from the next rows; we thus get a "stack" of matrices: $(v_{\tau,\alpha}(n))_{\alpha,\tau<\beta,n\in\mathbb{N}}$.

We start with the matrix whose rows are the elements of $\mathscr{C}$, ordered along the ordinal β. At time $\tau = 0$, the matrix has not been modified. At time $\tau = \sigma + 1$, we pivot around the element $v_{\sigma,\sigma}$. Note that if the row α lies before or is the pivot (i.e., $\alpha \leq \sigma$), it is not modified. If row α lies after the pivot ($\alpha \geq \tau$), we must subtract the pivot from row α. In case τ is a limit ordinal, we just accumulate all changes that happened up until time τ.

We start by showing by induction on τ that:

(P1) For any $n \in \mathbb{N}$, the number of ordinals $\sigma \leq \tau$ such that $v_\sigma(n) \neq 0$ is finite.

(P2) The limit in Equation (4.4.6) is actually well-defined.

(P3) For any α and $\sigma < \min(\tau, \alpha)$, we have $v_{\tau,\alpha}(l(v_\sigma)) = 0$.

(P4) The value of $v_{\tau,\alpha}$ is

$$v_{\tau,\alpha} = v_{0,\alpha} - \sum_{\sigma<\min\{\alpha,\tau\}} \lambda_{\sigma+1,\alpha} \cdot v_\sigma. \tag{4.4.7}$$

Indeed, assume the above holds for any $\theta < \tau$.

If $\tau = 0$: Then *(P1)*,*(P2)*,*(P3)* and *(P4)* hold trivially.

If τ is successor: *(P1)* is easy: by the induction hypothesis, $\{\sigma \leq \tau - 1 \ : \ v_\sigma(n) \neq 0\}$ is finite, and $\{\sigma \leq \tau \ : \ v_\sigma(n) \neq 0\}$ can contain at most one more element.

(P2) does not apply to successor ordinals.

For *(P3)*, we see that if $\tau > \alpha$, then (by definition of v)

$$v_{\tau,\alpha}(l(v_\sigma)) = v_\alpha(l(v_\sigma))$$

which is zero by the induction hypothesis (i.e. *(P3)* holds at $\tau = \alpha$). If $\tau \leq \alpha$, then

$$v_{\tau,\alpha} = v_{\tau-1,\alpha} - \lambda_{\tau,\alpha} \cdot v_{\tau-1,\tau-1}, \tag{4.4.8}$$

which, evaluated at $l(v_{\tau-1,\tau-1})$, is zero by construction. If $\sigma < \tau - 1$, both $v_{\tau-1,\alpha}(l(v_\sigma))$ and $v_{\tau-1,\tau-1}(l(v_\sigma))$ are zero, hence so is $v_{\tau,\alpha}$.

Finally, *(P4)*. If $\tau \leq \alpha$, then $\min\{\tau, \alpha\} = \tau$, and:

$$\begin{aligned} v_{\tau,\alpha} &= v_{\tau-1,\alpha} & &-\lambda_{\tau,\alpha} \cdot v_{\tau-1,\tau-1} \\ &= v_{0,\alpha} - \sum_{\sigma<\tau-1} \lambda_{\sigma+1,\alpha} \cdot v_\sigma & &-\lambda_{\tau,\alpha} \cdot v_{\tau-1,\tau-1}, \\ &= v_{0,\alpha} - \sum_{\sigma<\min(\tau,\alpha)} \lambda_{\sigma+1,\alpha} \cdot v_\sigma \end{aligned}$$

by definition of $v_{\tau,\alpha}$. If $\tau > \alpha$, we get $\min\{\tau,\alpha\} = \alpha$, and

$$\begin{aligned} v_{\tau,\alpha} = v_{\tau-1,\alpha} = v_{0,\alpha} &+ \sum_{\sigma<\min\{\alpha,\tau-1\}} \lambda_{\sigma+1,\alpha} \cdot v_\sigma \\ = v_{0,\alpha} &+ \sum_{\sigma<\min\{\alpha,\tau\}} \lambda_{\sigma+1,\alpha} \cdot v_\sigma. \end{aligned}$$

<u>If τ is limit:</u> We first check *(P1)*. It suffices to show that for any n, the number of ordinals $\sigma < \tau$ with $v_\sigma(n) \neq 0$ is finite. If $n = l(v_\sigma)$ for some $\sigma < \tau$, then the number of $\theta < \sigma$ for which $v_\theta(n) \neq 0$ is finite by the induction hypothesis (*(P1)*), and for any $\tau > \theta > \sigma$, $v_\theta(n) = 0$, also by the induction hypothesis (*(P3)*). Assume then that there exists some n which is not a leading coefficient (i.e., of the form $l(v_\sigma)$ for some $\sigma < \tau$) *and* in the support of infinitely many v_σ ($\sigma < \tau$). We can take n to be the least such integer, and for σ large enough (similarly to the local finiteness part in the proof of theorem 4.4.1), we will have:

$$v_\sigma(n) \neq 0, \quad v_\sigma(k) = 0, \ \forall k < n,$$

so that n is actually the leading coefficient of v_σ, a contradiction.

We now check *(P2)*. Let α be arbitrary, then

$$v_{\tau,\alpha} = \lim_{\sigma<\tau} \left(v_{0,\alpha} - \sum_{\sigma'<\min\{\alpha,\sigma\}} \lambda_{\sigma'+1,\alpha} \cdot v_{\sigma'} \right)$$

which we need to verify to be well-defined. In other words, we need to check that for any n, the sequence

$$\sigma \mapsto v_{0,\alpha}(n) + \sum_{\sigma'<\min\{\alpha,\sigma\}} \lambda_{\sigma'+1,\alpha} \cdot v_{\sigma'}(n)$$

stabilizes starting at some ordinal. But by *(P1)*, $v_{\sigma'}(n)$ is zero for σ' large enough, so that the sequence indeed stabilizes. Furthermore, it stabilizes to:

$$v_{0,\alpha}(n) + \sum_{\sigma<\min\{\alpha,\tau\}} \lambda_{\sigma+1,\alpha} \cdot v_\sigma(n),$$

which also proves *(P4)*.

It remains to check *(P3)*. If $\sigma < \tau, \alpha$, then

$$v_{\tau,\alpha}(l(v_\sigma)) = \lim_{\sigma'<\tau} v_{\sigma',\alpha}(l(v_\sigma)) = \lim_{\sigma<\sigma'<\tau} v_{\sigma',\alpha}(l(v_\sigma)) = \lim 0 = 0.$$

This closes the inductive verification of ??.

Let now

$$\mathscr{B}' := \{v_\alpha | \alpha < \beta, v_\alpha \neq 0\}, \quad \mathscr{B}'_i = \{v_\alpha | \alpha < \alpha_{i+1}, v_\alpha \neq 0\}.$$

Note that by *(P1)*, each non-zero v_α has a well-defined leading index $l(v_\alpha)$, and $v_\alpha \neq v_\beta$ implies $l(v_\alpha) \neq l(v_\beta)$. It follows that $\mathscr{B}'$ is at most countable. We can now check the following consequences of this process:

1. $\mathscr{B}'_i$ spans $\mathscr{C}_i$ and $\mathscr{B}'$ spans $\mathscr{C}$ (i.e., the maps $A^{\mathscr{B}'_i} \to \mathscr{C}_i$ and $A^{\mathscr{B}'} \to \mathscr{C}$ are surjective).

Indeed, 4.4.7 yields (by setting $\alpha = \tau$)

$$v_\tau = v_{0,\tau} - \sum_{\sigma<\tau} \lambda_{\sigma+1,\tau} \cdot v_\sigma,$$

so that we get

$$v_{0,\tau} = v_\tau + \sum_{\sigma<\tau} \lambda_{\sigma+1,\tau} \cdot v_\sigma,$$

so that any element of $\mathscr{C}$ (resp. $\mathscr{C}_i$) is a locally finite sum of elements of $\mathscr{B}'$ (resp. $\mathscr{B}'_i$).

2. $\mathscr{B}'$ is locally finite, as a subset of $A^{\mathscr{B}}$, by the same argument as the proof of *(P1)* for τ limit.

3. $\mathscr{B}'$ is locally finite, as a subset of A^{EX}. Indeed, every $e \in EX$ appears in the support of a finite number of $b \in \mathscr{B}$, and every $b \in \mathscr{B}$ appears in the support of a finite number of elements of $\mathscr{B}'$ (by the point above).

4. The set $\mathscr{B}'$ is independent (i.e., the map $A^{\mathscr{B}} \to \mathscr{C}$ is injective). Suppose not. Then, there exists a subset $\tilde{\beta} \subset \beta$ and non-zero coefficients $(\mu_\sigma)_{\sigma\in\tilde{\beta}}$ such that $\sum_{\sigma\in\tilde{\beta}} \mu_\sigma \cdot v_\sigma = 0$. Let $\tau_0 := \min \tilde{\beta}$. Then $\sum_{\sigma\in\tilde{\beta}} v_\sigma(l(v_{\tau_0})) = \mu_\sigma$, a contradiction.

□

4.4.3 Infinite dimensionality of $\mathscr{C}/\mathscr{C}_\infty$

Using lemma 4.4.2, we get a sufficient condition for the infinite dimensionality of the quotient $\mathscr{C}/\mathscr{C}_\infty$, as an A-vector space.

Lemma 4.4.3. *Let X be a locally finite graph. If $\mathscr{C}_\infty \neq \mathscr{C}_i$ for all $i \in \mathbb{N}$, then $\mathscr{C}/\mathscr{C}_\infty$ is infinite-dimensional as a vector space.*

Proof. The condition $\forall i \in \mathbb{N}\ \mathscr{C}_\infty \neq \mathscr{C}_i$ means that for all i, there exists $i' > i$ with $\mathscr{C}_i \subsetneq \mathscr{C}_{i'}$; let $\{i_j\}_j$ be a sequence of indices with $\mathscr{C}_{i_j} \subsetneq \mathscr{C}_{i_{j+1}}$ for all j, and choose an element $b_j \in \mathscr{B}'_{i_j} - \mathscr{B}'_{i_{j-1}}$ for each j. Let $\{J_n\}_{n\in\mathbb{N}}$ be a partition of $\mathbb{N}$ into $\mathbb{N}$ subsets of cardinality $\mathbb{N}$, and

$$f_n := \sum_{j\in J_n} b_j, \quad n \in \mathbb{N}.$$

We claim that the family $\{f_n\}_n$ is linearly independent in the quotient, i.e., that no non-trivial sum thereof lies in $\mathscr{C}_\infty$. Let us verify the claim. Such a sum has the form

$$S = \sum_n \lambda_n f_n = \sum_n \sum_{j\in J_n} \lambda_n b_j,$$

(which is well-defined since the b_js form a locally finite set). Assuming the sum lies in $\mathscr{C}_\infty$, hence in some $\mathscr{C}_i$, and we must thus get

$$S = \sum_{b\in\mathscr{B}'_i} \mu_b b,$$

so that

$$0 = S - S = \sum_n \sum_{j\in J_n} \lambda_n b_j - \sum_{b\in\mathscr{B}'_i} \mu_b b.$$

Grouping the coefficients belonging to a given $b \in \mathscr{B}'$ yields a sum

$$0 = \sum_{b \in \mathscr{B}'} \kappa_b b,$$

and each κ_b must then be zero, since $\mathscr{B}'$ is itself a basis. But, by construction of the f_ns, there are non-zero coefficients λ_n belonging to elements $b \in \mathscr{B}_i'$ for arbitrary high i, while the μ_b belong only to elements $b \in \mathscr{B}_i'$. This is a contradiction, and concludes the proof. □

4.4.4 More about $\mathbb{Z}/2\mathbb{Z}$

The results obtained so far can be strengthened in the case of $A = \mathbb{Z}/2\mathbb{Z}$ by using compactness.

Lemma 4.4.4. *Let X be a locally finite graph, and $A = \mathbb{Z}/2\mathbb{Z}$. If there exists r such that any circuit of X lies in $\mathscr{C}_r$, then $\mathscr{C} = \mathscr{C}_r$.*

Proof. Endow both A^{EX} and $A^{\mathfrak{C}}$ with the product topology, with A discrete; both spaces are compact. The set $A^{\mathfrak{C}_r}$ is closed in $A^{\mathfrak{C}}$, hence compact, and the restriction of $\mathscr{D} : A^{\mathfrak{C}} \to A^{EX}$ to $A^{\mathfrak{C}_r}$ is continuous. By assumption, for any circuit $c \in \mathfrak{C}$, there exists some $\phi_c \in A^{\mathfrak{C}_r}$ with $\mathscr{D}\phi_c = \mathscr{D}c$. Fix any element of $\mathscr{C}$ and any $\psi \in A^{\mathfrak{C}}$ with $\mathscr{D}\psi$ equal to this element. Let $\{c_i\}_{i \in \mathbb{N}}$ be an enumeration of the elements of $\mathfrak{C}$ appearing in ψ. Then, by construction,

$$\mathscr{D}\psi = \sum_i \mathscr{D}c_i = \lim_n \sum_{i=1}^{n} \mathscr{D}c_i.$$

One can then replace $\mathscr{D}c_i$ by $\mathscr{D}\phi_i$ and use linearity of $\mathscr{D}$:

$$= \lim_n \sum_{i=1}^{n} \mathscr{D}\phi_{c_i} = \lim_n \mathscr{D}\left(\sum_{i=1}^{n} \phi_{c_i}\right).$$

By compactness of $A^{\mathfrak{C}}$ and closure of $A^{\mathfrak{C}_r}$ therein, one may choose a subsequence (indexed by $(n_m)_m$) of $(\sum_{i=1}^{n} \phi_{c_i})_n$ converging to some element ϕ of $A^{\mathfrak{C}_r}$. Then:

$$= \lim_m \mathscr{D}\left(\sum_{i=1}^{n_m} \phi_{c_i}\right) = \mathscr{D}\left(\lim_m \sum_{i=1}^{n_m} \phi_{c_i}\right) = \mathscr{D}\phi,$$

now using continuity of $\mathscr{D}|_{A^{\mathscr{D}_r}}$. □

We easily deduce the following.

Corollary 4.4.5. *Let X be a locally finite graph, and $A = \mathbb{Z}/2\mathbb{Z}$. Then, the following are equivalent:*

1. *There exists some r such that $\mathscr{C} = \mathscr{C}_r$.*
2. *There exists some r such that $\mathscr{C}_\infty = \mathscr{C}_r$.*
3. *There exists some r such that for all $r' \geq r$ $\mathscr{C}_{r'} = \mathscr{C}_r$.*
4. *There exists some r such that any circuit of X lies in $\mathscr{C}_r$.*

Proof. Indeed, each downward implication is obvious, and lemma 4.4.4 allows closing the chain. □

Finally, putting together what we have so far:

Theorem 4.4.6. *Let X be a locally finite graph and $A = \mathbb{Z}/2\mathbb{Z}$. If the quotient $\mathscr{C}/\mathscr{C}_\infty$ does not vanish, then it is infinite dimensional.*

Proof. $\mathscr{C}/\mathscr{C}_\infty \neq 0$ implies that $\mathscr{C} \neq \mathscr{C}_\infty$, and hence that none of the equivalent conditions of corollary 4.4.5 hold. It then suffices to apply lemma 4.4.3. □

4.5 What (could be) next

$\mathbb{Z}$-large circuits. In the case of $A = \mathbb{Z}_2$, all notions of "having large circuits" are equivalent. This is a priori also the case for any finite field $\mathbb{Z}_p$, but not otherwise. One might ask the following question: assuming that any circuit of length $\leq s$ lies in $\mathscr{C}_r(X, \mathbb{Z})$, does it hold that $\mathscr{C}_s(X, \mathbb{Z}) \subseteq \mathscr{C}_r(X, \mathbb{Z})$? Without compactness, mimicking the argument for $A = \mathbb{Z}_2$ does not work. On the other hand, we didn't manage to find a counter-example to this question.

Large circuits and finitely presented groups. It is relatively clear that a finitely presented group has $\mathbb{Z}_2$-small circuits, since any circuit defines a relation, and all relations are made of a finite number of small relations. Does the converse hold ? (I guess not).

5 The first uniformly finite homology in $\mathbb{Z}$

This is a continuation of chapter 4. We give a combinatorial description of the first group of uniformly finite homology $H_1^{\mathrm{uf}}(X, A)$ for a uniformly locally finite graph X, in terms of bounded flows and circuits. We prove that when X is a tree, $H_1^{\mathrm{uf}}(X, A)$ essentially counts ends. In the general case, we verify that $H_1^{\mathrm{uf}}(X, A)$ contains a copy of $H_1^{\mathrm{uf}}(T, A)$, when $T \leq X$ is a subtree of X such that the ends of T embed in the ends of X. We define a rather ad-hoc definition of higher dimensional expansion for simplicial complexes, and show that for transitive locally finite graphs, the non-vanishing of $H_1^{\mathrm{uf}}(X, \mathbb{Z})$ is equivalent to the disjunction of three phenomena: 1. the existence of more than one end; 2. a sufficiently massaged notion of expansion; 3. the existence of circuits that cannot be decomposed into smaller circuits (in a suitably coarse sense).

In this chapter, all graphs considered will be *uniformly locally finite* and connected. The definition of uniformly finite homology is obtained from that of Roe homology by restricting the functions in the chain complexes to be uniformly bounded (that is, by replacing A^S by $\ell^\infty(S, A)$). It follows that uniformly finite and Roe homology share some properties; we will rely on the constructions of chapter 4, and point out the differences as they appear.

In [BW92], Block and Weinberger show that the vanishing of $H_0^{\mathrm{uf}}(X, \mathbb{Z})$ is equivalent to non-amenability of X. The main motivation for this chapter is the search for a characterization for the vanishing of $H_1^{\mathrm{uf}}(X, \mathbb{Z})$.

5.1 Flows, circuits, and H_1^{uf}

From now on, let A be a normed ring.

5.1.1 Uniformly finite homology

Recall that given a locally finite simplicial complex X, $C_\bullet(X, A)$ is the chain complex whose chains are the functions $X_{(n)} \to A$. We define *simplicial uniformly finite* homology as the homology of the sub-chain complex $C_\bullet^{\mathrm{suf}}(X, A)$ of $C_\bullet(X, A)$ whose chains are the functions $X_{(n)} \to A$ of finite ℓ^∞ norm, i.e., the elements of $\ell^\infty(X_{(n)}, A)$. Similarly, we define *uniformly finite* homology as the homology of the inductive limit $C^{\mathrm{uf}}{}_\bullet(X, A)$ of the chain complexes $C_\bullet^{\mathrm{suf}}(X, A)$.

We state the following essential property of uniformly finite homology without proof:

Theorem 5.1.1 (Uniformly homology is QI invariant). *Uniformly finite homology is quasi-isometry invariant.*

Proof. [BW92], or by adapting the argument of quasi-isometry invariance of Roe homology. □

5.1.2 Circuits, birays and flows

If S is an arbitrary set and $F \subseteq \ell^\infty(S, A)$, call F *uniformly locally finite* if each F is locally finite (section 4.1.1) and $\sum F \in \ell^\infty(S, A)$. Recall (lemma 4.1.2) that any flow on X can be decomposed into a sum of circuits and birays. A similar statement holds for *uniformly bounded* flows:

Lemma 5.1.2. *Given any uniformly bounded flow f on the uniformly locally finite graph X, there exists a function $\lambda \in \mathfrak{S} \to A$, such that $\{\lambda(s) \cdot \mathscr{D}s \,|s \in \mathfrak{S}\}$ is uniformly locally finite, and satisfying:*

$$f = \sum_{s \in \mathfrak{S}} \lambda(s) \cdot \mathscr{D}s.$$

Proof. Follows from the proof of lemma 4.1.2. □

When $A = \mathbb{Z}$, we can be a bit more precise. If f, g are $\mathbb{Z}$-valued flows on X, let us write $f \le g$ to mean that for all $e \in EX$, $f(e) \neq 0$ implies $0 \le f(e) \le g(e)$ or $g(e) \le f(e) \le 0$; we say that f is below g. Visually, this means that f is a "smaller" flow than g.

Lemma 5.1.3. *Given any uniformly bounded $\mathbb{Z}$-valued flow f on the uniformly locally finite graph X, there exists a function $\lambda \in \mathfrak{S} \to \mathbb{Z}$, such that $\{\lambda(s) \cdot \mathscr{D}s \,|s \in \mathfrak{S}\}$ is uniformly locally finite, satisfies*

$$f = \sum_{s \in \mathfrak{S}} \lambda(s) \cdot \mathscr{D}s,$$

and such that $\lambda(s) \cdot \mathscr{D}s \le f$ for all s.

Proof. First, note that given any non-zero $\mathbb{Z}$-valued flow f, there exists some simple biray or circuit s such that $\pm\mathscr{D}s \le f$ (with sign depending on orientation).

The argument then follows the proof of lemma 4.1.2: we iterate over the edges of X, and at each edge, remove a circuit $\pm s$ with $\pm\mathscr{D}s \le f$ as long as such a circuit exists. At the limit, no circuit $\pm s$ with $\pm\mathscr{D}s \le f$ remains, and we iterate over all edges again, now subtracting birays $\pm s$ with $\pm\mathscr{D}s \le f$, in the same manner. □

Fix a (uniformly locally finite) graph X. Recall that $A^{(\mathfrak{C}(X))}$ denotes the set of functions from $\mathfrak{C}(X)$ to A that are locally finitely supported, and that

$$\mathscr{D} \;:\; A^{(\mathfrak{C}(X))} \to Z_1(X, A),$$

maps such a function to its representation as a flow on X. Let us write

$$\mathscr{C}_r^{\mathrm{suf}}(X, A) := \mathscr{D}[\ell^\infty(\mathfrak{C}_r(X), A)],$$
$$\mathscr{C}_\infty^{\mathrm{suf}}(X, A) := \bigcup_r \mathscr{C}_r^{\mathrm{suf}}(X, A).$$

The same remarks as for Roe homology apply here.

Lemma 5.1.4. *The stabilization of the sequence $(\mathscr{C}_r^{\mathrm{suf}}(X, A))_r$ is quasi-isometry invariant.*

Proof. By adapting the argument of proposition 4.1.3 (note that if X, Y, ϕ, ψ, K are as given in proposition 4.1.3, and $f \in Z_1^{\mathrm{suf}}(X, A)$, then f and $\psi\phi f$ differ by an element of $C_\infty^{\mathrm{suf}}(X, A)$ by uniform local finiteness). □

5.1.3 H_1^{uf} in terms of flows.

Recall that the first Roe homology can be described as $H_1^\infty(X) = \frac{Z_1(X)}{\mathscr{C}_\infty(X)}$. A similar description holds for uniformly finite homology:

Theorem 5.1.5. *If X is a (uniformly locally finite) graph, then*

$$H_1^{\mathrm{uf}}(X) = \frac{Z_1^{\mathrm{suf}}(X)}{\mathscr{C}_\infty^{\mathrm{suf}}(X)}.$$

If $(\mathscr{C}_r^{\mathrm{suf}}(X))_r$ stabilizes, then $H_1^{\mathrm{uf}}(X) = H_1^{\mathrm{suf}}(R_r X)$ for large enough r.

Proof. Follows from the proofs of theorem 4.1.4 and lemma 4.1.5 by using uniform local finiteness and the fact that all functions at play are uniformly bounded. □

If X is a uniformly finite graph, there is an obvious map $z : Z_1^{\mathrm{suf}}(X, \mathbb{Z}) \to Z_1(X, \mathbb{Z}/2\mathbb{Z})$ mapping f to $q \circ f$, where $q : \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ is the quotient map. Furthermore, $z[\mathscr{C}_\infty^{\mathrm{suf}}(X, \mathbb{Z})] \subseteq \mathscr{C}_\infty(X, \mathbb{Z}/2\mathbb{Z})$, so that z passes to a morphism $H_1^{\mathrm{uf}}(X, \mathbb{Z}) \to H_1^\infty(X, \mathbb{Z}/2\mathbb{Z})$.

Lemma 5.1.6. *Let X be a uniformly locally finite graph. The morphism*

$$H_1^{\mathrm{uf}}(X, \mathbb{Z}) \to H_1^\infty(X, \mathbb{Z}/2\mathbb{Z})$$

is surjective.

Proof. Fix a representative $f \in Z_1(X, \mathbb{Z}/2\mathbb{Z})$ of an element of $H_1^\infty(X, \mathbb{Z})$. Take the decomposition of f into a sum of circuits and birays (theorem 4.3.1). Since f takes value in $\mathbb{Z}/2\mathbb{Z}$, this decomposition satisfies the property that the circuits and birays involved do not intersect on edges. For each of those, choose an orientation, thus defining an element $\tilde{f} \in Z_1(X, \mathbb{Z})$ with values in $\{-1, 0, 1\}$. Clearly $\tilde{f}$ is uniformly bounded, and a preimage to f. □

Thus, by the above, $H_1^{\mathrm{uf}}(X, \mathbb{Z})$ cannot vanish if $H_1^\infty(X, \mathbb{Z}/2\mathbb{Z})$ doesn't.

5.2 Trees

This section, following section 4.2.1, is devoted to the analysis of $H_1^{\mathrm{uf}}(X, A)$ in relation to trees. We first verify that $H_1^{\mathrm{uf}}(X, A)$ essentially counts the number of ends of X if the latter is a tree. Furthermore, we show that whenever X is a nicely embedded subtree of some Y, then the first uniformly finite homology of X embeds in that of Y.

5.2.1 H_1^{uf}(tree)

Let T be a (uniformly locally finite) connected tree. By the same arguments as section 4.2.1, one can assume without loss of generality, up to quasi-isometry, that T is composed of vertices of degree 2 and 3 exclusively.

Furthermore, from the decomposition

$$H_1^{\mathrm{uf}}(T) \cong Z_1^{\mathrm{suf}}(T)/\mathscr{C}_\infty^{\mathrm{suf}}(T),$$

it is clear that $H_1^{\mathrm{uf}}(T) \cong Z_1^{\mathrm{suf}}(T)$.

An important property of the construction $\mathscr{B}$ of a good set of birays is that: any edge of T is contained in at most 3 birays of $\mathscr{B}$.

Using this property, theorem 4.2.1 translates easily to a similar characterization of the first uniformly finite homology of T.

Theorem 5.2.1. *If $\mathscr{B}$ is a good set of birays for the tree T, then*

$$H_1^{\mathrm{uf}}(T, A) \cong \ell^\infty(\mathscr{B}, A).$$

Proof. Recall the isomorphism

$$\begin{aligned} \mathscr{D} : A^{\mathscr{B}} &\to Z_1(T, A) = H_1^\infty(T, A) \\ \phi &\mapsto \sum_{b \in \mathscr{B}} \phi(b) \cdot b, \end{aligned}$$

from the proof of theorem 4.2.1. It suffices to verify that $\mathscr{D}$ restricts to an isomorphism from $\ell^\infty(\mathscr{B}, A)$ to $Z_1^{\mathrm{suf}}(T, A) = H_1^{\mathrm{uf}}(T, A)$.

$\underline{\mathscr{D} \text{ maps } \ell^\infty(\mathscr{B}, A) \text{ into } Z_1^{\mathrm{suf}}(T, A)}$: since any edge has at most three elements of $\mathscr{B}$ lying over it, so that $\|D\phi\|_\infty \le 3\|\phi\|_\infty$.

$\underline{\mathscr{D}|_{\ell^\infty(\mathscr{B},A)} \text{ is surjective onto } Z_1^{\mathrm{suf}}(T, A)}$: Indeed, recall that given $f \in Z_1(T, A)$, a preimage $\phi \in A^{\mathscr{B}}$ of f is constructed inductively on $\mathscr{B}$ by setting

$$\phi(b) := f(e) - \sum_{e \in b' < b} [e : b']\phi(b'),$$

where e is an edge such that b is the last element of $\mathscr{B}$ containing e. Recall also that by 3 (in the list of properties that the set $\mathscr{B}$ satisfies), there exists some edge e' such that

$$\{b' \in \mathscr{B} : e' \in b'\} = \{b' \in \mathscr{B} : e \in b'\} - \{b\},$$

which means that

$$\left| \sum_{e \in b' < b} [e : b']\phi(b') \right|_\infty = \left| \sum_{e' \in b'} [e' : b']\phi(b') \right|_\infty = \|f(e')\|_\infty \le \|f\|_\infty,$$

where the first and second equalities follow from the inductive definition of ϕ. It follows that

$$\|\phi(b)\| \le 2\|f\|_\infty,$$

and thus, $\|\phi\|_\infty \le 2\|f\|_\infty$.

Note that when restricted to $\ell^\infty(\mathscr{B}, A) \to Z_1^{\mathrm{suf}}(T, A)$, $\mathscr{D}$ becomes a homeomorphism with respect to the ℓ^∞ norm.

□

5.2.2 Subtrees

By the decomposition of Roe homology, we know that a graph with more than one end has non-vanishing first Roe homology. This is also the case for uniformly finite homology.

Theorem 5.2.2. *Let X be a graph and T an end-respecting subtree of X. Then*

$$Z_1^{\mathrm{suf}}(T) \cap \mathscr{C}_\infty^{\mathrm{suf}} = \{0\}.$$

In particular, the natural map $H_1^{\mathrm{uf}}(T) \to H_1^{\mathrm{uf}}(X)$ is injective.

Proof. The proof of theorem 4.2.1 applies. □

5.3 Large circuits

Let us say that a uniformly locally finite graph X has A-large circuits for uniformly finite homology if

$$\forall r > 0\ \mathscr{C}_r^{\mathrm{suf}}(X, A) \neq \mathscr{C}_\infty^{\mathrm{suf}}(X, A).$$

From now on, let us only consider the case $A = \mathbb{Z}$.

Lemma 5.3.1. *If X is vertex-transitive and has $\mathbb{Z}$-large circuits for uniformly finite homology, then $H_1^{\mathrm{uf}}(X, \mathbb{Z}) \neq 0$.*

Proof. First, fix a vertex $v_0 \in VX$ of X. By transitivity, one can choose a family $\{\phi_{i,r}\}_{i,j,r\in\mathbb{N}}$ of automorphisms of X such that the sets $\phi_{i,r}[B(v_0, i)]$ are all pairwise disjoints.

Second, given $g \in \ell^\infty(\mathfrak{C}_r, \mathbb{Z})$, we claim that one can always decompose g as a sum $g = \sum_{i=1}^n g_i$ such that each $g_i \in \ell^\infty(\mathfrak{C}_r, \mathbb{Z})$ is of norm 1 and such that the circuits in the support of g_i are all edge-disjoint. Indeed, one can first write $g = g_+ - g_-$, with $g_+, g_- \geq 0$, and treat them separately, so that without loss of generality $g \geq 0$. Let us then consider a maximal set C_g of pairwise non-edge-intersecting circuits in the support of g (Z o r n); subtract $\sum_{c\in C_g} c$ from g and iterate. This process must end after a finite number of steps. Indeed, if c is a circuit (of length $\leq r$) in the support of g, there is a uniformly bounded number of other circuits of length $\leq r$ intersecting g, and each can be subtracted at most $\|g\|_\infty$ times, so that c will be subtracted (by maximality) after a uniformly bounded number of steps. The sums $\sum_{c\in C_g} c$ of each step define the desired g_is.

Let us now proceed with the bulk of the proof. Assume that $(\mathscr{C}_s^{\mathrm{suf}}(X, \mathbb{Z}))_s$ does not stabilize, but $H_1^{\mathrm{uf}}(X, \mathbb{Z}) = 0$. For each s, fix some $f_s \in \mathscr{C}_\infty^{\mathrm{suf}}(X, \mathbb{Z}) - \mathscr{C}_s^{\mathrm{suf}}(X, \mathbb{Z})$; by construction, $f_s \in \mathscr{C}_r^{\mathrm{suf}}(X, \mathbb{Z})$ for some r, and $f_s = \mathscr{D} g_s$ for some $g_s \in \ell^\infty(\mathfrak{C}_r, \mathbb{Z})$. By the argument above, one may decompose g_s as a finite sum $g_s = \sum_i g_i$, so that $f_s = \sum_i \mathscr{D} g_i$, which implies that at least one $\mathscr{D} g_i$ lies in $\mathscr{C}_\infty^{\mathrm{suf}}(X, \mathbb{Z}) - \mathscr{C}_s^{\mathrm{suf}}(X, \mathbb{Z})$. By construction, $\mathscr{D} g_i$ has norm 1. We can thus (up to replacing f_s by $\mathscr{D} g_i$) assume that $f_s = \mathscr{D} g_s$ has norm 1 and g_s is an edge-disjoint sum of circuits. For each $i \in \mathbb{N}$, let g_s^i be the restriction of g_s to circuits entirely contained in $B(v_0, i)$, and let $f_s^i = \mathscr{D} g_s^i$. It is clear that $g_s^i \xrightarrow{i\to\infty} g_s$ pointwise, and similarly $f_s^i \to f_s$.

Let $f = \sum_{i,r} f_r^i \circ \phi_{i,r}^{-1}$. By construction, the supports of each $f_r^i \circ \phi_{i,r}^{-1}$ are pairwise disjoint, so that $f \in Z_1^{\mathrm{suf}}(X, \mathbb{Z})$ and $\|f\|_\infty = 1$. By the hypothesis that $H_1^{\mathrm{uf}}(X, \mathbb{Z}) = 0$, there exists some $s \in \mathbb{N}$ and $k \in \ell^\infty(\mathfrak{C}_s, \mathbb{Z})$ such that $\mathscr{D} k = f$. Consider the family $(k \circ \phi_{i,r})_i$, for $r = s + 1$. By construction, $\mathscr{D}(k \circ \phi_{i,r})$ is equal to f_r on increasingly large balls around v_0, when $i \to \infty$. Since the sequence $(k \circ \phi_{i,r})_i$ is uniformly bounded, it has an accumulation point, say $\phi \in \ell^\infty(\mathfrak{C}_s, \mathbb{Z})$, which implies that $\mathscr{D}\phi$ is an accumulation point of $(f_r^i)_i$. Since $(f_r^i)_i$ converges to f_r, we conclude that $f_r = \mathscr{D}\phi \in \mathscr{C}_s(X, \mathbb{Z})$, a contradiction to the assumption that $f_r \notin \mathscr{C}_r(X, \mathbb{Z})$. □

5.4 Expansion

Recall that a (uniformly locally finite) graph X is said to be amenable if it satisfies the following condition:

$$\forall \epsilon > 0\ \exists U \subseteq_{\mathrm{f}} VX\ :\ |\partial^e U| < \varepsilon |U|.$$

If X is non-amenable, then

$$\varepsilon := \inf_{\emptyset \neq U \subseteq_{\mathrm{f}} VX} \frac{|\partial^e U|}{|U|}$$

is called the Cheeger constant of X, and X is sometimes called an expander.

One of the selling points of uniformly finite homology, proved by Block and Weinberger, is the following.

Theorem 5.4.1 (Part of [BW92, Theorem 3.1])**.** *A uniformly locally finite graph X is an expander if and only if $H_0^{\mathrm{uf}}(X, \mathbb{Z}) = 0$.*

Motivated by the above, our goal in this section is to capture the geometric idea behind expansion into a higher dimensional analogue.

5.4.1 Basics

Let X be a uniformly locally finite simplicial complex. If $U \subseteq VX$ is a set of vertices, we also use U for the induced simplicial subcomplex.

Definition 5.4.2 (H^{suf}-Expansion)**.** We say that X has H_n^{suf}-expansion if there exists a function $K : \mathbb{N} \to \mathbb{N}$ satisfying the following: for any $f \in Z_n^{\mathrm{suf}}(X, \mathbb{Z})$ and $U \subseteq_{\mathrm{f}} VX$, there exists $g \in C_{n+1}^{\mathrm{suf}}(X, \mathbb{Z})$ such that $d^{\mathrm{suf}} g|_U = f|_U$ and $\|g\|_\infty \leq K(\|f\|_\infty)$.

We say that X has *pure* H_n^{suf}-expansion if there exists a function $K : \mathbb{N} \to \mathbb{N}$ satisfying the following: for any $f \in B_n^{\mathrm{suf}}(X, \mathbb{Z})$ and $U \subseteq_{\mathrm{f}} VX$, there exists $g \in C_{n+1}^{\mathrm{suf}}(X, \mathbb{Z})$ such that $d^{\mathrm{suf}} g|_U = f|_U$ and $\|g\|_\infty \leq K(\|f\|_\infty)$.

Intuitively, expansion can be interpreted as homological cycles locally being homological boundaries, with control. In other words, one can always kill “pieces of cycles” with “pieces of boundaries” in a bounded way. We will see that this property always implies that simplicial uniformly finite homology vanishes, at the corresponding dimension. Note that H_n^{suf}-expansion is stronger than pure H_n^{suf}-expansion. Note also that the function $K : \mathbb{N} \to \mathbb{N}$ in the definition of [pure] expansion can be taken monotonous.

Lemma 5.4.3. *If X has H_n^{suf}-expansion, then $H_n^{\mathrm{suf}}(X, \mathbb{Z}) = 0$.*

Proof. Assume that $K : \mathbb{N} \to \mathbb{N}$ is as given by H_n^{suf}-expansion. Fix $f \in Z_n^{\mathrm{suf}}(X, \mathbb{Z})$, and $(U_i \subseteq VX)_i$ a nested sequence of finite subsets exhausting VX. For each i, let $g_i \in C_{n+1}^{\mathrm{suf}}(X, \mathbb{Z})$ satisfying $d^{\mathrm{suf}} g_i|_{U_i} = f|_{U_i}$ and $\|g_i\| \leq K(\|f\|)$, as given by H_n^{suf}-expansion.

Then the sequence $(g_i)_i$ has an accumulation point, say g, since its elements are uniformly bounded (by $K(\|f\|)$). Since $d^{\mathrm{suf}} g$ is also an accumulation point of $(d^{\mathrm{suf}} g_i)_i$, which converges to f, we conclude that $d^{\mathrm{suf}} g = f$.

Thus, f is in $B_n^{\mathrm{suf}}(X, \mathbb{Z})$ and $H_n^{\mathrm{suf}}(X, \mathbb{Z}) = 0$. □

Remark 5.4.4. Consider the following condition on X: there exists some constant N such that for any $f \in Z_n^{\mathrm{suf}}(X, \mathbb{Z})$, one can write $f = \sum_{i=1}^{N\|f\|} f_i$, with each f_i of norm 1 and in $Z_n^{\mathrm{suf}}(X, \mathbb{Z})$. If this condition held, we could always take the function $K : \mathbb{N} \to \mathbb{N}$ in the definition of [pure] H_n^{suf}-expansion to be linear. We do not know whether this holds, even in dimension 1.

5.4.2 Dimension 0

For $n = 0$, we have an equivalence between the following three properties

1. X is an expander;
2. $H_0^{\mathrm{uf}}(X, \mathbb{Z}) = 0$;
3. X has H_0^{suf}-expansion.

The equivalence between the first two items is already known. We will reproduce one direction of this equivalence here, and (re) prove the other by way of H_0^{suf}-expansion. Note that $H_0^{\mathrm{suf}}(X, \mathbb{Z}) = H_0^{\mathrm{uf}}(X, \mathbb{Z})$, since $C_0^{\mathrm{suf}}(X, \mathbb{Z}) = Z_0^{\mathrm{suf}}(X, \mathbb{Z}) = Z_0^{\mathrm{uf}}(X, \mathbb{Z})$ and any edge in a Rips complex can be approximated by a path between its end points.

Proposition 5.4.5 ([BW92, Proof of Theorem 3.1, p.912])**.** *Let X be a uniformly locally finite graph. If $H_0^{\mathrm{uf}}(X, \mathbb{Z}) = 0$, then X is an expander.*

We follow [BW92, Proof of Theorem 3.1, p.912], with essentially cosmetic modifications.

Proof. Assume $H_0^{\mathrm{uf}}(X, \mathbb{Z}) = 0$, so that any bounded function on the vertices of X is the boundary of a bounded function on its edges. Consider the constant function $\mathbb{1} : VX \to \mathbb{Z}$. By assumption, there exists some $\phi \in C_1^{\mathrm{suf}}(X, \mathbb{Z})$ with $d^{\mathrm{suf}}\phi = \mathbb{1}$. Fix some $U \subseteq_{\mathrm{f}} VX$. Then

$$\begin{aligned} |U| &= \sum_{v \in U} \mathbb{1}(v) \\ &= \sum_{v \in U} (d^{\mathrm{suf}}\phi)(v), \end{aligned}$$

now, using the fact that $(d^{\mathrm{suf}}\phi)(v) = \sum_{v \in e}[v : e]\phi(e)$

$$= \sum_{v \in U} \sum_{v \in e} [v : e]\phi(e),$$

and if e is an edge with both endpoints in U, it will appear twice in the above sum, with opposite signs, and cancel out, while if e has both endpoints outside of U, it will not appear, so that

$$\begin{aligned} &= \sum_{e \in \partial^e U} [v : e]\phi(e) \\ &\leq \sum_{e \in \partial^e U} |\phi(e)| \leq \|\phi\|_\infty \cdot |\partial^e U|. \end{aligned}$$

Thus, $|\partial^e U| \geq \|\phi\|_\infty^{-1} |U|$, and $\|\phi\|_\infty^{-1}$ is a lower bound to the Cheeger constant. □

This shows the implication (2) ⇒ (1). We know already that (3) ⇒ (2) (lemma 5.4.3), and it remains to show (1) ⇒ (3).

Lemma 5.4.6. *Let X be a uniformly locally finite graph. If X is an expander, then X has H_0^{suf}-expansion.*

When $n = 0$, the definition of H_n^{suf}-expansion simplifies greatly. First, any uniformly bounded function $f : VX \to \mathbb{Z}$ can be decomposed as $f_+ - f_-$, with $f_+, f_- \geq 0$. Then, each of f_+, f_- can be decomposed into a finite sum of $0/1$-valued functions (as many as their respective norms). The definition of H_0^{suf}-expansion can thus be restated as the existence of a constant K such that for any 0/1-valued function f and finite set U, there exists some $g \in \ell^\infty(EX, \mathbb{Z})$ with $(d^{\mathrm{suf}} g)|_U = f|_U$ and $\|g\|_\infty \leq K$. Interpreting f as an indicator function, we get: For any $U \subseteq_f VX$ and $W \subseteq U$ there exists some $g \in \ell^\infty(EX, \mathbb{Z})$ with $(d^{\mathrm{suf}} g)|_U = \chi_W$ and $\|g\|_\infty \leq K$. We will use this restatement, along with an argument of Benjamini and Schramm based on the "max flow/min cut" theorem to prove lemma 5.4.6:

Proof of lemma 5.4.6. Assume that X is non-amenable with Cheeger constant $\varepsilon > 0$. Given $U \subseteq_f VX$ and $W \subseteq U$, the argument in [BS97, Proof of lemma 2.1] constructs a flow g on U. In their argument, the flux at vertices in W, (which corresponds exactly to $d^{\mathrm{suf}} g$) is a rational $\varepsilon > 0$ and 0 everywhere else, while the value of g on edges is at most 1. Scaling by ε^{-1} yields a bound of $K = \varepsilon^{-1}$ for g. □

5.4.3 Dimension 1

In dimension 1 (the one we are interested in), we do not know whether H_1^{suf}-expansion is equivalent to the vanishing of $H_1^{\mathrm{suf}}(X, \mathbb{Z})$ in general, though it is true in the transitive case.

Before showing this, we need a technical lemma:

Lemma 5.4.7. *Assume X is uniformly locally finite and one-ended. Let $f \in Z_1^{\mathrm{suf}}(X, \mathbb{Z})$ and $U \subseteq_f VX$. Then, there exists $f' \in Z_1^{\mathrm{suf}}(X, \mathbb{Z})$ with finite support, $f'|_U = f|_U$ and such that $\|f'\| \leq \|f\|$.*

Proof. Fix f and U as in the statement. Consider the decomposition of f as a sum of circuits and birays given by lemma 5.1.3. Let g be the sum of circuits and birays of this decomposition (with same weights) that intersect U; by construction $\|g\| \leq \|f\|$. Take some $V \subseteq VX$ large enough that all circuits in g are contained in V. We can write $d^{\mathrm{suf}}(g|_V) = \sum_v n_v v$ for some coefficients n_v, in such a way that each v with $n_v \neq 0$ corresponds to a vertex in V from which n_v rays start, counted with sign and multiplicity. Since $\sum_v n_v = 0$, there are as many rays with positive sign as negative sign (with multiplicity). Since X is one-ended, all rays in g' can be connected outside of V. Furthermore one can always connect two rays with opposite sign without intersecting any other ray: if we start at ray v_1 (positive) and a path to v_n (negative) intersects $v_2, \ldots, v_{n-1}$, there will necessarily be a pair v_i, v_{i+1} with opposite sign, and one can connect those with the restriction of our original path from v_1 to v_n. Then, we modify g by connecting two rays of opposite signs (making them into a circuit) without touching anything else, and iterate the process (i.e., take some big ball V containing all circuits, and connect two rays) until we reach some flow f', consisting only of a finite number of circuits, and satisfying $\|f'\| \leq \|f\|$ and $f'|_U = f|_U$. □

Theorem 5.4.8. *If X is an infinite, vertex-transitive uniformly locally finite simplicial complex and $H_1^{\mathrm{suf}}(X, \mathbb{Z}) = 0$, then X has H_1^{suf}-expansion.*

Proof. First, note that $H_1^{\mathrm{suf}}(X, \mathbb{Z}) = 0$ implies that X is one-ended (theorem 5.2.2).

Suppose X does not have H_1^{suf}-expansion but $H_1^{\mathrm{suf}}(X, \mathbb{Z}) = 0$. Then, there exist sequences of functions $(f_i \in Z_1^{\mathrm{suf}}(X, \mathbb{Z}))_i$ and of finite sets $(U_i \subseteq_f VX)_i$ as in the definition, with $\|f_i\| = C$ for some C, but such that if $d^{\mathrm{suf}} F_i|_{U_i} = f_i|_{U_i}$, then $\|F_i\| \geq K_i$, with $K_i \to \infty$.

For all i, let f_i' be as given by lemma 5.4.7. Since X is infinite and transitive, one can assume that the supports of all f_i's are pairwise disjoint.

Let $f := \sum_i f_i'$. Then, $f \in Z_1^{\mathrm{suf}}(X, \mathbb{Z})$ (of norm $\leq C$) implies the existence of some $F \in C_2^{\mathrm{suf}}(X, \mathbb{Z})$ with $d^{\mathrm{suf}} F = f$. In particular, $d^{\mathrm{suf}} F|_{U_i} = f_i'$, and since $\|F\| \leq K_i$ for large enough i, we get a contradiction. □

Note that we do not actually need as much as transitivity, but merely being able to move finite pieces of the graphs sufficiently far.

Note also that if lemma 5.4.7 held in dimension higher than 1, then so would theorem 5.4.8.

5.4.4 H_1^{uf}-expansion

The notion of H_n^{suf}-expansion is clearly not a coarse one. Our goal being to characterize the vanishing of $H_1^{\mathrm{uf}}(X, \mathbb{Z})$, we need to make it quasi-isometry invariant.

Let us first isolate a basic but useful property of flows on Rips complexes that will be used later on:

Lemma 5.4.9. *Let X be a uniformly locally finite graph. For any $r \in \mathbb{N}$, there exists some $K \in \mathbb{N}$ such that for all $f \in Z_1^{\mathrm{suf}}(R_r X, \mathbb{Z})$, there exists $G \in C_2^{\mathrm{suf}}(R_r X, \mathbb{Z})$ satisfying $f - d^{\mathrm{suf}} G \in Z_1^{\mathrm{suf}}(X, \mathbb{Z})$ and $\|G\| \leq K\|f\|$; in particular, $\|f - d^{\mathrm{suf}} G\| \leq K'\|f\|$ for some K' depending only on r and X.*

In other words, flows on Rips complexes can be pushed to flows on the original graph with “fixed cost”.

Proof. For any pair of vertices $u, v \in VX$, choose a geodesic path $p_{u,v}$ connecting u to v. If $d_X(u, v) \leq r$, then the edge $[u, v] \in E(R_r X)$ is equal to $p_{u,v}$, modulo a sum of triangles in $R_r X$. By uniform local finiteness, the number of triangles involved is uniformly bounded (for fixed r), and by summing over the edges in $f \in Z_1^{\mathrm{suf}}(R_r X, \mathbb{Z})$, we get the desired element $G \in C_2^{\mathrm{suf}}(R_r X, \mathbb{Z})$. Uniform local finiteness also implies that $d^{\mathrm{suf}} G$ has norm bounded by a constant (depending on r) times the norm of f, whence the last statement. □

Definition 5.4.10 (H_n^{uf}-expansion)**.** Let X be a uniformly locally finite graph. We say that X has H_n^{uf}-expansion [respectively pure H_n^{uf}-expansion] if there exists $K : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ and $u : \mathbb{N} \to \mathbb{N}$ such that for any $r \in \mathbb{N}$ and $f \in Z_n^{\mathrm{suf}}(R_r X, \mathbb{Z})$ [respectively $f \in B_n^{\mathrm{suf}}(R_r X, \mathbb{Z})$] and $U \subseteq_{\mathrm{f}} VX$, there exists $F \in C_{n+1}^{\mathrm{suf}}(R_{u(r)} X, \mathbb{Z})$ with $\|F\| \leq K(\|f\|, r)$ and $(d^{\mathrm{suf}} F)|_U = f|_U$.

The definition of [pure] H_n^{suf}-expansion on a simplicial complex states that any cycle [boundary] f can be killed locally at a fixed cost depending on $\|f\|$. Similarly [pure] H_n^{uf}-expansion states that at a Rips complex of any coarseness r, any cycle [boundary] f can be killed locally at a fixed cost depending on $\|f\|$ and r, with a possible change of coarseness $u(r)$.

Unrolling the definition of [pure] expansion yields the formula

$$\forall r \, \exists u \, \forall N \, \exists K \; : \quad \forall U \subseteq_{\mathrm{f}} VX \, \forall f \in W_n^{suf}(R_r X, \mathbb{Z}) \text{ with } \|f\| = N \; : \\ \exists F \in C_{n+1}^{\mathrm{suf}}(R_u X, \mathbb{Z}) \text{ with } (d^{\mathrm{suf}} F)|_U = f|_U \text{ and } \|F\| \leq K, \quad (5.4.1)$$

with $W = Z$ for expansion and $W = B$ for pure expansion. Negating this formula yields:

$$\exists r \, \forall u \, \exists N \, \forall K \; : \quad \exists U \subseteq_{\mathrm{f}} VX \, \exists f \in W_n^{suf}(R_r X, \mathbb{Z}) \text{ with } \|f\| = N \; : \\ \forall F \in C_{n+1}^{\mathrm{suf}}(R_u X, \mathbb{Z}) \text{ with } (d^{\mathrm{suf}} F)|_U = f|_U \text{ we have } \|F\| > K. \quad (5.4.2)$$

Lemma 5.4.11. *If X is a uniformly locally finite graph with H_n^{uf}-expansion, then $H_n^{\mathrm{uf}}(X, \mathbb{Z}) = 0$.*

Proof. By the same argument as lemma 5.4.3. If $f \in Z_n^{\mathrm{uf}}(X,\mathbb{Z})$ then $f \in Z_n^{\mathrm{suf}}(R_rX,\mathbb{Z})$ for some r. Then, larger and larger pieces of f can be killed by boundaries of elements of $C_{n+1}^{\mathrm{suf}}(R_{u(r)}X,\mathbb{Z})$ of uniformly bounded norms. It suffices then to take an accumulation point of these elements. □

Lemma 5.4.12. *If X is a uniformly locally finite graph and R_sX eventually has [pure] H_n^{suf}-expansion, then X has [pure] H_n^{uf}-expansion.*

Proof. If R_sX has [pure] H_n^{suf}-expansion, then there exists $K_s : \mathbb{N} \to \mathbb{N}$ as in the definition. Assume that R_sX has [pure] H_n^{suf}-expansion for all $s \geq r$. Letting $u(s) := \max(s,r)$ and $K(N,s) := K_{u(s)}(N)$, it follows that X has [pure] H_n^{uf}-expansion. □

Lemma 5.4.13. *H_n^{uf}-expansion and pure H_n^{uf}-expansion are quasi-isometry invariants.*

Proof. Assume Y has [pure] H_n^{uf}-expansion (with functions $K \in \mathbb{N}^{\mathbb{N}\times\mathbb{N}}, u \in \mathbb{N}^{\mathbb{N}}$), and $\phi : X \to Y, \psi : Y \to X$ are quasi-isometry inverses with constant N. Fix $f \in Z_n^{\mathrm{suf}}(R_{r_0}X,\mathbb{Z})$ [resp. $B_n^{\mathrm{suf}}(R_{r_0}X,\mathbb{Z})$] and U finite. Take $s(r_0)$ large enough that all simplices in $R_{r_0}X$ are mapped to simplices in $R_{s(r_0)}Y$ via ϕ; let $g \in Z_n^{\mathrm{suf}}(R_{s(r_0)}Y,\mathbb{Z})$ [resp. $B_n^{\mathrm{suf}}(R_{s(r_0)}Y,\mathbb{Z})$] be the image of f via ϕ, and $V := \phi[U]$. There is some L depending only on r_0 and N such that $\|g\| \leq L\|f\|$. Since Y has [pure] H_n^{suf}-expansion, there exists some $G \in C_{n+1}^{\mathrm{suf}}(R_{u(s(r_0))}Y,\mathbb{Z})$ with $\|G\| \leq K(L\|f\|, s(r_0))$ and $(d^{\mathrm{suf}}G)|_V = g|_V$. Let $\tilde{F}$ be the image of G via ψ, in a sufficiently large Rips complex $R_{r(u(s(r_0)))}X$. The function $r_0 \mapsto r(u(s(r_0)))$ does not depend on f, but only on N and $u \in \mathbb{N}^{\mathbb{N}}$. Similarly the norm of $\tilde{F}$ is bounded by $L'\|G\| \leq L'K(L\|f\|, s(r))$ for some constant L' depending only on s and N. Now, $d^{\mathrm{suf}}\tilde{F}$ can be homotoped to f in $R_{r(u(s(r_0)))}X$, with a cost depending only on $\|f\|$ and r_0, s, N. This shows that X has [pure] H_n^{uf}-expansion. □

Lemma 5.4.14. *If X is such that $\mathscr{C}_r^{\mathrm{suf}}(X,\mathbb{Z}) = \mathscr{C}_\infty^{\mathrm{suf}}(X,\mathbb{Z})$ for some r and is one-ended, then $H_1^{\mathrm{uf}}(X,\mathbb{Z}) \neq 0$ implies that X does not have pure H_1^{uf}-expansion.*

In other words, if $H_1^{\mathrm{uf}}(X,\mathbb{Z}) \neq 0$ then at least one of the three phenomena (ends, large circuits, pure expansion) is responsible for it.

Proof. Assume that $H_1^{\mathrm{uf}}(X,\mathbb{Z}) \neq 0$, X is one-ended and, for some r_0, $\mathscr{C}_{r_0}^{\mathrm{suf}}(X,\mathbb{Z}) = \mathscr{C}_\infty^{\mathrm{suf}}(X,\mathbb{Z})$. By lemma 5.4.11, X does not have H_1^{uf}-expansion, and it remains to show that it does not have *pure* H_1^{uf}-expansion.

Since X does not have H_1^{uf}-expansion, and by eq. (5.4.2), after some quantified r,u,N,K, we can find $f \in Z_1^{\mathrm{suf}}(R_rX,\mathbb{Z})$ and $U \subseteq_f VX$ such that $\|f\| = N$ and for any $F \in C_1^{\mathrm{suf}}(R_rX,\mathbb{Z})$ satisfying $(d^{\mathrm{suf}}F)|_U = f|_U$, we have $\|F\| > K$. First, note that r can be assumed to be arbitrarily large, since if $f \in Z_1^{\mathrm{suf}}(R_rX,\mathbb{Z})$ then $f \in Z_1^{\mathrm{suf}}(R_{r'}X,\mathbb{Z})$ for any $r' \geq r$. Since R_rX is one-ended and by lemma 5.4.7, f can be replaced by some f' of finite support with same norm and equal to f on U; we may therefore assume that f has finite support. By lemma 5.4.9, there exists some $\Delta_f \in C_2^{\mathrm{suf}}(R_rX,\mathbb{Z})$ also of finite support, such that $g := f - d^{\mathrm{suf}}\Delta_f \in Z_1^{\mathrm{suf}}(X,\mathbb{Z})$. Now, g is a flow of finite support on X, and can thus be decomposed as a finite sum of circuits. By the assumption $\mathscr{C}_{r_0}^{\mathrm{suf}}(X,\mathbb{Z}) = \mathscr{C}_\infty^{\mathrm{suf}}(X,\mathbb{Z})$, g can be written as a (uniformly bounded) sum of circuits of length $\leq r_0$, and by triangulating those, $g \in B_1^{\mathrm{suf}}(R_{r_0}X,\mathbb{Z})$. Thus, $f = g + d^{\mathrm{suf}}\Delta_f$, with both summands lying in $B_1^{\mathrm{suf}}(R_rX,\mathbb{Z})$ (since r was assumed big enough that $r \geq r_0$), so that $f \in B_1^{\mathrm{suf}}(R_rX,\mathbb{Z})$ too. Thus, in context of the quantifiers r,u,N,K, f can be chosen in $B_1^{\mathrm{suf}}(R_rX,\mathbb{Z})$, which shows that X has pure H_1^{uf}-expansion. □

Theorem 5.4.15. *Let X be a uniformly locally finite transitive graph, and consider the following three conditions:*

1. *X has more than one end.*
2. *X has $\mathbb{Z}$-large circuits; that is, $\mathscr{C}_r^{\mathrm{suf}}(X,\mathbb{Z}) \neq \mathscr{C}_\infty^{\mathrm{suf}}(X,\mathbb{Z})$ for any r.*
3. *X does not have pure H_1^{uf}-expansion.*

Then $H_1^{\mathrm{uf}}(X,\mathbb{Z}) \neq 0$ if and only if either of the above holds.

Proof. Items 1 and 2 imply $H_1^{\mathrm{uf}}(X,\mathbb{Z}) \neq 0$ by the content of section 5.2.1 and by lemma 5.3.1, respectively. Let us then assume that neither of items 1 and 2 hold, but item 3 does. Towards a contradiction, assume furthermore that $H_1^{\mathrm{uf}}(X,\mathbb{Z}) = 0$. By theorem 5.1.5, $H_1^{\mathrm{suf}}(R_sX,\mathbb{Z}) \cong H_1^{\mathrm{uf}}(X,\mathbb{Z}) = 0$ for s large enough. By transitivity and theorem 5.4.8, R_sX eventually has H_1^{suf}-expansion, and in particular pure H_1^{suf}-expansion. This implies in particular that X has pure H_1^{uf}-expansion .

Conversely, if $H_1^{\mathrm{uf}}(X,\mathbb{Z}) \neq 0$, but X is one-ended and $(\mathscr{C}_r^{\mathrm{suf}}(X,\mathbb{Z}))_r$ stabilizes, we can apply lemma 5.4.14, and X does not have pure H_1^{uf}-expansion. □

5.5 What (could be) next

Non-transitive graphs. After reading theorem 5.4.15, the first question coming to mind is probably: what about the non transitive case? As remarked before, transitivity is a comfortable assumption, but can be weakened: all we really use is the ability to move finite sets around, so that e.g., each vertex having an infinite orbit under $\operatorname{Aut} X$ is strong enough. This is obviously still far from general, but we could not find suitable alternatives to theorem 5.4.8 and lemma 5.3.1.

Higher dimensions. We know that H_n^{uf}-expansion is quasi-isometry invariant and implies the vanishing of H_n^{uf}. When does the converse hold? Can some arguments of the case $n = 1$ be generalized to arbitrary n?

A better understanding of the vanishing of H_1^{uf}. In a sense, the decomposition of theorem 5.4.15 is quite artificial, and it is not clear that each of the three phenomena is independent of the others. Perhaps this could be refined.

6 Preliminaries and notation: Lorentzian lattices

6.1 Hyperbolic Space

We mainly work with the hyperboloid model of hyperbolic space $\mathbb{H}^n$, whose construction is sketched below. Fix an $(n+1)$-dimensional real vector space V, endowed with a non-degenerate symmetric bilinear form q of signature $(n, 1)$. If $x, y \in V$, we will usually write $xy := q(x, y)$, and $x^2 := q(x, x)$, which we call the length of x. Say that x is positive [resp. negative, isotropic] if its length is positive [resp. negative, zero]. By Sylvester's law of inertia, there exists a basis $B := \{v_0, \dots, v_n\}$ of V such that q can be written

$$q\left(\sum_i x_i v_i, \sum_i y_i v_i\right) = -x_0 y_0 + \sum_{i\geq 1} x_i y_i,$$

or, in other words, that the Gram matrix $(q(v_i, v_j))_{i,j}$ is diagonal with diagonal entries $-1, 1, \dots, 1$. From now on, we fix this basis B of V, and if x is an element of V, x_i represents its i-th coordinate with respect to B. Let

$$C_V := \{x \in V \ :\ x^2 < 0\}.$$

This is a convex cone, consisting of two connected components (also convex cones) C_V^+ and $C_V^- = -C_V^+$ with, without loss of generality, $v_0 \in C_V^+$; in other words, C_V^+ consists of the elements $x \in V$ satisfying $x^2 < 0$ and $x_0 > 0$. Note that the action of $\mathbb{R}^+$ on V preserves the sign of the length of elements ($(\lambda x)^2 = \lambda^2 x^2$) and preserves each cone $C_V^\pm$. Hyperbolic n-space is then defined as the quotient:

$$\mathbb{H}^n := C_V^+/\mathbb{R}^+.$$

Since any $x \in V$ of negative length can be normalized to have length -1, the elements of $C_V^+/\mathbb{R}^+$ are uniquely represented by elements x with $x_0 > 0$ and of length -1; this set is usually called the upper sheet of the hyperboloid.

From now on, we will usually not distinguish elements $x \in V$ of negative length from their equivalence class in $\mathbb{H}^n$.

The cone $Z_V := \{x \in V \ :\ x^2 = 0\} - \{0\}$ consists of two connected components $Z_V^+, Z_V^- = -Z_V^+$ (with, again, $Z_V^+ = \{x \in Z_V \ :\ x_0 > 0\}$), and $\mathbb{R}^+$ acts, again, on Z_V^+. Let us define

$$\partial\mathbb{H}^n := Z_V^+/\mathbb{R}^+,$$

which is, by definition, the set of ideal points of hyperbolic space, and $\overline{\mathbb{H}^n} := \mathbb{H}^n \sqcup \partial\mathbb{H}^n$. Let V^+ be the set of elements $x \in V$ with $x_0 > 0$. The sets $\mathbb{H}^n, \partial\mathbb{H}^n$ are topologized as subspaces of the quotient $V^+/\mathbb{R}^+$. Inside $V^+/\mathbb{R}^+$, the topological closure of $\mathbb{H}^n$ is $\overline{\mathbb{H}^n}$; this set is compact.

A distance can be defined on $\mathbb{H}^n$ by setting (assuming $x^2 = y^2 = -1$)

$$d_{\mathbb{H}^n}(x, y) := \cosh^{-1}(-xy),$$

and it might be verified that the topology induced by this distance is the correct one.

Let $\mathcal{O}$ be the group of linear automorphisms of V preserving the form q, and $\tilde{\mathcal{O}} < \mathcal{O}$ the index-2 subgroup of $\mathcal{O}$ of elements preserving C_V^+. The action of $\tilde{\mathcal{O}}$ on C_V^+ descends to an action on $\mathbb{H}^n$ which can be seen to preserve $d_{\mathbb{H}^n}$. The group $\tilde{\mathcal{O}}$ is exactly the group of isometries of $\mathbb{H}^n$ with respect to $d_{\mathbb{H}^n}$. Note that $\tilde{\mathcal{O}}$ also naturally acts on $\partial\mathbb{H}^n$.

If W is a $(k+1)$-dimensional linear subspace of V intersecting C_V^+, the quotient $[W] := (W \cap C_V^+)/\mathbb{R}^+$ is, by definition, a k-dimensional subspace of W. The correspondence $W \mapsto [W]$ is injective. Furthermore, since $W \cap C_V^+ \neq \emptyset$, the restriction of q to W is of signature $(k,1)$, so that $[W]$ is isometric to $\mathbb{H}^k$. 1-dimensional subspaces are called lines, and are exactly the geodesic lines with respect to $d_{\mathbb{H}^n}$ (i.e., isometric embeddings of $\mathbb{R}$ in $\mathbb{H}^n$). $(n-1)$-dimensional subspaces are called hyperplanes; the complement of a hyperplane consists of two connected components, whose closures are called half-spaces.

If r is an element of V of positive length, then

$$r^\perp := \{x \in V \ : \ rx = 0\}$$

is an n-dimensional vector subspace of V interesting C_V^+, hence defines a hyperplane H_r of $\mathbb{H}^n$. Any hyperplane is of the form H_r for a unique r of positive length, up to multiplication by an element of $\mathbb{R}^*$. The (closures of) the two connected components of $\mathbb{H}^n - H_r$ correspond exactly to the elements v of $\mathbb{H}^n$ such that $vr \geq 0$ and $vr \leq 0$, respectively, and are written H_r^+ and H_r^- respectively (passing from r to $-r$ doesn't change the hyperplane H_r but swaps the half-spaces). Thus, hyperplanes are in bijection with vectors of positive length, up to multiplication by an element of $\mathbb{R}^*$, and half-spaces are in bijection with vectors of positive length, up to multiplication by an element of $\mathbb{R}^+$. Let us call oriented hyperplane the data of a hyperplane H_r and choice of corresponding half-space (equivalently, of a vector of positive length, up to multiplication by an element of $\mathbb{R}^+$). The vector r (of positive length) defines an isometry $\rho_r \in \tilde{\mathcal{O}}$ via

$$\begin{aligned} \rho_r \ : \ & V \to V \\ & v \mapsto v - 2\frac{vr}{r^2}r. \end{aligned}$$

The map ρ_r is called a reflection. Note that $\rho_{\lambda r} = \rho_r$ for any $\lambda \in \mathbb{R}^*$; r is called root of the reflection. The fixed-point set of ρ_r is $r^\perp$, and the hyperplane H_r is called the mirror of the reflection.

The distance between a point $v \in \mathbb{H}^n$ and a hyperplane H_r is

$$d(v, H_r) = \sinh^{-1}\sqrt{-\frac{(vr)^2}{v^2r^2}}.$$

If H_r and H_s are two oriented hyperplanes, with $r^2 = s^2 = 1$, then:

- they intersect iff $rs < 1$, in which case the dihedral angle between H_r, H_s is $\cos^{-1}(-rs)$;
- if $rs = 1$, the hyperplanes don't intersect, but their closure in $\overline{\mathbb{H}^n}$ do, and the distance between H_r, H_s is zero;
- if $rs > 1$, the hyperplanes don't intersect, neither do their closures, and the distance between H_r, H_s is $\cosh^{-1}(-rs)$; this distance is realized by a unique geodesic segment.

6.1.1 Polyhedra

The definition of subspaces of $\mathbb{H}^n$ as (quotients of non trivial) intersections of vector subspaces of V with C_V^+, allows, assuming some care, a back and forth between convex geometry in $\mathbb{H}^n$ (even $\overline{\mathbb{H}^n}$) and cone-convex geometry in V. In particular, the convex hull of a set of points in $\overline{\mathbb{H}^n}$ can be obtained via the cone-convex hull of the corresponding rays in V.

We call a locally finite intersection of half-spaces $P := \bigcap_i H_{r_i}^-$ with non-empty interior a (convex) polyhedron (locally finite, meaning that any compact subset of $\mathbb{H}^n$ intersects finitely many hyperplanes H_{r_i}). Given any polyhedron P, there is a unique minimal set of half-spaces $\{H_{r_i}^-\}$ satisfying $P = \bigcap_i H_{r_i}^-$; the (oriented) hyperplanes H_{r_i} are said to bound P. From now on, when writing $P := \bigcap_i H_{r_i}^-$, it is silently assumed that all H_{r_i} bound P. The (necessarily nonempty) intersection of P with the hyperplane H_{r_i} is called a facet of P, and if $i \neq j$ and $H_i \cap H_j \cap P$ is nonempty, this intersection is called a ridge. An ideal vertex of P is an element of $\partial\mathbb{H}^n$ that is vertex of the closure of P in $\overline{\mathbb{H}^n}$.

We have the following criteria for compactness and finiteness of volume of P:

- P is compact iff it is the convex hull of a finite number of vertices.
- P has finite volume iff it is the convex hull of a finite number of potentially ideal vertices.

The Gram matrix $G(P)$ of $P = \bigcap_i H_{r_i}^-$ is the matrix $(r_i r_j)_{i,j}$ (assuming $r_i^2 = r_j^2 = 1$). The polyhedron is said to be acute-angled if all its bounding hyperplanes either don't intersect, or do so at acute or right angle ($\leq \pi/2$). By a result of Andreev, two intersecting hyperplanes in an acute-angled polyhedron will necessary do so at a ridge. The polyhedron is said to be Coxeter if (it is acute-angled and) all angles are sub-multiples of π (i.e., of the form π/n ($n = 2, 3, \ldots$)).

6.1.2 Discrete groups

We call a subgroup $\Gamma < \tilde{\mathcal{O}}$ discrete if given any point $v \in \mathbb{H}^n$, the orbit $\Gamma v = \{\gamma v |\ \gamma \in \Gamma\}$ is a discrete subset of $\mathbb{H}^n$ (this is equivalent to Γ being discrete as a topological subgroup of $\tilde{\mathcal{O}}$ under the compact open topology ([Rat06, Exercise 5.3.10, p. 165])).

If Γ is discrete, and $v \in \mathbb{H}^n$ is fixed by no element of Γ, the Dirichlet domain of Γ at v is the set:

$$D_\Gamma(v) := \{u \in \mathbb{H}^n \ :\ d(u, v) \leq d(u, \gamma v) \forall \gamma\},$$

that is, the set of points closer to v than to any of its images under Γ. It is a polyhedron and a fundamental domain for Γ. The (discrete) group Γ is said to be cocompact if $D_\Gamma(v)$ is compact, and of finite covolume if $D_\Gamma(v)$ is of finite volume.

The group Γ is infinite iff it has an element of infinite order ([Rat19, Lemma 1, p. 601]), and an element of Γ has infinite order iff its fixed point set is empty (in $\mathbb{H}^n$).

Lemma 6.1.1. *If $\Gamma < \tilde{\mathcal{O}}$ is discrete and preserves a proper subspace $U \leq V$, then Γ is of infinite covolume.*

Proof. Follows from [Fur76, Theorem, p. 211] (Borel density theorem). □

6.1.3 Reflection groups

Fix a symmetric $k \times k$ matrix $(m_{i,j})_{i,j}$ with non-diagonal entries in the set $\{0, 2, 3, \ldots\}$ and ones on the diagonal. A Coxeter group is a group given by a presentation of the form

$$\Gamma := \left\langle g_1, \ldots, g_k \mid\ (g_i g_j)^{m_i}\ \forall i, j \right\rangle$$

(with $(g_i g_j)^0$ interpreted as the empty word, i.e., no relation). The matrix $(m_{i,j})_{i,j}$ is called the Coxeter matrix of Γ. The graph whose vertices are the generators g_i and with an edge between g_i and g_j, labeled $m_{i,j}$, iff $m_{i,j} \neq 2$ is called the Coxeter diagram of Γ. A Coxeter diagram is called irreducible if its underlying graph is connected. A subdiagram of a Coxeter diagram is the graph defined by any subset of vertices.

A discrete group Γ generated by reflections in $\mathbb{H}^n$ (or Euclidean or spherical space) is called a reflection group. Let M_Γ be the set of mirrors of reflections of Γ; any (closure of a) connected component of $\mathbb{H}^n - \bigcup M_\Gamma$ is called a cell. Any cell is the Dirichlet domain of any of its interior points, and is furthermore a Coxeter polyhedron. Let $H_{r_1}, \ldots, H_{r_k}$ be the (oriented) hyperplanes bounding a cell D, and let $\pi/m_{i,j}$ be the dihedral angle between H_{r_i} and H_{r_j}, or $m_{i,j} = 0$ if the hyperplanes don't intersect. By Poincaré's polyhedron theorem, the Coxeter group with matrix $(m_{i,j})_{i,j}$ is isomorphic to Γ [EP94].

A Coxeter diagram is called hyperbolic [resp. Euclidean, spherical] iff it is the Coxeter diagram of a hyperbolic [resp. Euclidean, spherical] reflection group. The rank of a spherical diagram is the number of its vertices; the rank of an Euclidean diagram is the number of its vertices, minus the number of its maximal irreducible subdiagrams. A subdiagram of a Coxeter diagram is called Euclidean [resp. spherical] if it itself is a Euclidean [resp. spherical] diagram.

The face structure of the (hyperbolic) reflection group Γ can be read off the Coxeter diagram defined by a cell P of Γ:

- k-dimensional faces of P correspond to spherical subdiagrams of rank $n - k$.
- Ideal vertices of P correspond to Euclidean subdiagrams of rank $n - 1$.

Compactness and volume-finiteness of P can also be read off as follows ([Bug92]):

- P is compact iff each spherical subdiagram of rank $n - 1$ (an edge of P) can be extended in exactly two ways into a spherical subdiagram of rank n (a vertex of P).
- P has finite volume iff each spherical subdiagram of rank $n-1$ can be extended in exactly two ways into either a spherical subdiagram of rank n or an Euclidean subdiagram of rank $n - 1$ (an ideal vertex of P).

6.1.4 Reflection subgroups

Let $\Gamma < \tilde{\mathscr{O}}$ be a discrete group, and Γ_r the subgroup of Γ generated by reflections (in Γ). The group Γ is said to be reflective if $[\Gamma_r : \Gamma]$ is finite; in particular, if Γ is of finite covolume in $\tilde{\mathscr{O}}$, then Γ is reflective iff Γ_r is of finite covolume. Let P be a cell of Γ_r. The group Γ decomposes as a semi-direct product

$$\Gamma = \Gamma_r \rtimes \mathrm{Sym}_\Gamma P,$$

where $\mathrm{Sym}_\Gamma P$ is the group of automorphisms of P contained in Γ ([Vin72, Proposition 3, p. 27]). In particular,

- Γ is reflective iff $\mathrm{Sym}_\Gamma P$ is finite, iff all elements of $\mathrm{Sym}_\Gamma P$ have a fixed point (in $\mathbb{H}^n$);
- Γ acts transitively on the set of cells of Γ_r.

6.2 Number fields

A number field K is a finite field extension of $\mathbb{Q}$, i.e., of the form $K = \mathbb{Q}(\alpha_1, \ldots, \alpha_n)$ for algebraic $\alpha_1, \ldots \alpha_n$. We only care about totally real number fields, that is, number fields K such that all embeddings of K in $\mathbb{C}$ are contained in $\mathbb{R}$. If K is a number field, we let R denote the ring of algebraic integers of K, i.e., those elements of K that are roots of some monic polynomial over $\mathbb{Z}$. We only care about number fields whose ring of integers are principal ideal domains. In that case, R is a free $\mathbb{Z}$-module of same rank as the degree of K over $\mathbb{Q}$ ([Sam67, Corollary, p. 48]). By Dirichlet's unit theorem, the group of units of R is finitely generated.

Write

$$\begin{aligned} t_2 \,:\, K &\to \mathbb{Q} \\ x &\mapsto \mathrm{tr}(x^2) = \sum_\sigma (\sigma x)^2 \end{aligned}$$

where σ runs through all the (necessarily real) embeddings of K in $\mathbb{C}$. Note that the trace of $x \in K$ is equal to one of the coefficients of the minimal polynomial of x over $\mathbb{Q}$, which ensures that $t_2(x) \in \mathbb{Q}$ for all $x \in K$; furthermore, it implies that if $x \in R$, then $t_2(x) \in \mathbb{Z}$ (indeed, $t_2(x) = \mathrm{tr}(x^2)$, and since $x \in R$, $x^2 \in R$ too, which means that its minimal polynomial has coefficients in $\mathbb{Z}$). We call $t_2(x)$ the t_2 norm of x, by abus; this is clearly just the quadratic form associated to the bilinear form $x, y \mapsto \mathrm{tr}(xy)$. The square root of t_2 is a norm on K, seen as a vector space over $\mathbb{Q}$; it is actually the ℓ^2 norm on $\mathbb{R}^m$ applied to the embedding $K \hookrightarrow \mathbb{R}^m$ sending x to the tuple $(\sigma x)_\sigma$, with σ running through the m embeddings of K in $\mathbb{C}$. The embedding $K \hookrightarrow \mathbb{R}^m$ maps R into a lattice of $\mathbb{R}^m$ ([Sam67, Proposition 1, p. 68]). In particular, for any $r \in \mathbb{N}$, only finitely many elements of R have t_2 norm bounded above by r. This is important for us: although the embeddings of R in $\mathbb{R}$ are not discrete, the embedding in $\mathbb{R}^m$ is.

6.3 Lorentzian lattices

Fix a totally real number field K with R principal. From now on, assume that an embedding $\mathrm{Id} : K \hookrightarrow \mathbb{R}$ has been chosen (i.e., K is understood as a subfield of $\mathbb{R}$). A free finitely generated R-module L of rank $n+1$ endowed with a symmetric bilinear form $q : L \times L \to R$ of signature $(n, 1)$ with values in R is called a Lorentzian lattice. If an R-basis $\{v_1, \ldots, v_{n+1}\}$ of L is chosen, the condition that q takes values in R is equivalent to the Gram matrix $Q \coloneqq (q(v_i, v_j))_{i,j}$ having all coefficients in R. We say that the form q is admissible if for all non trivial field embeddings $\sigma : K \hookrightarrow \mathbb{R}$, the form $q^\sigma = \sigma \circ q$ is positive definite (i.e., of signature $(n+1, 0)$). We only care about admissible forms.

By extension of coefficients, the form q defines a quadratic form of signature $(n, 1)$ over the $\mathbb{R}$-vector space $V \coloneqq \mathbb{R} \otimes L$. As in section 6.1, let $\mathcal{O}$ be the group of linear automorphisms of V preserving q, and $\tilde{\mathcal{O}}$ the index-2 subgroup of $\mathcal{O}$ preserving the connected components of C_V. An element τ of $\tilde{\mathcal{O}}$ preserves the lattice iff its matrix representation T w.r.t. the basis $\{v_i\}_i$ has entries in R, and preserves the form iff $T'QT = Q$ (where T' is the adjoint of T). By taking determinants, it follows that $(\det T)^2 = 1$, so that $\det T \in R$ is invertible in R, and in particular, τ^{-1} has coefficients in R. Let $\tilde{\mathcal{O}}^L$ be the subgroup of $\tilde{\mathcal{O}}$ of elements preserving the lattice L (equivalently, with matrix coefficients in R).

By results of Borel and Harish-Chandra [BH62] the group $\tilde{\mathcal{O}}^L$ is of finite covolume in $\tilde{\mathcal{O}}$ when $n \geq 2$, and admissibility of q makes it discrete. If $n \geq 4$, $K = \mathbb{Q}$, and $\tilde{\mathcal{O}}^L$ is reflective, then the fundamental polyhedron P for $\tilde{\mathcal{O}}_r^L$ is necessarily non-compact (thus has at least one vertex at infinity). Conversely, if $K \neq \mathbb{Q}$, a vertex at infinity corresponds to a vector $v \neq 0$ satisfying $q(v, v) = 0$; applying a non-trivial embedding σ to this equation yields $q^\sigma(\sigma v, \sigma v) = 0$ with $\sigma v \neq 0$, which is impossible since q^σ is positive definite; this shows that if $K \neq \mathbb{Q}$, a finite volume polyhedron P for $\tilde{\mathcal{O}}_r^L$ is necessarily compact.

From now on call the lattice L reflective if the group $\tilde{\mathcal{O}}^L$ is.

Consider a reflection $\rho_r \in \tilde{\mathcal{O}}$ with respect to the vector $r \in V$. For ρ_r to preserve the lattice, it is necessary and sufficient that $\rho_r(v_i) \in L$ for all basis vectors v_i, i.e., that

$$v_i - 2\frac{v_i r}{r^2} r \in L \quad \forall i,$$

or equivalently that

$$2(v_i r) r_j \in r^2 R \quad \forall i \forall j. \tag{6.3.1}$$

This is usually called the crystallographic condition. By rescaling r, one can assume without loss of generality that some coefficient of r, say r_1, is equal to 1; in particular, eq. (6.3.1) ($j = 1$) implies that $2\frac{v_i r}{r^2} \in R$ for all i. It follows that all coefficients r_j must lie in K; indeed, there exists at least one i such that $2\frac{v_i r}{r^2}$ is not zero, so that multiplying

$$\underbrace{2\frac{v_i r}{r^2}}_{\in R} r_j \in R,$$

by the inverse of $2\frac{v_i r}{r^2}$ implies $r_j \in K$. Consequently, up to rescaling, one can always assume that r has coordinates in K, and by multiplying by an appropriately big common multiple of the denominators, that $r \in L$. Since R is principal, hence a UFD, one can furthermore assume that the gcd of (the coefficients) of r is a unit (of R). Let us call a lattice vector with unit gcd primitive. The crystallographic condition for primitive vectors can be simplified to:

$$2(v_i r) \in r^2 R \quad \forall i.$$

By a result of Vinberg ([Vin84, Proposition 24, p. 98]), the length of a primitive vector satisfying the crystallographic condition necessarily divides twice the last invariant factor F_q of q. Two primitive and colinear vectors r, s necessarily differ by a unit, so that r^2 and s^2 differ by the square of a unit. Since the group of units R^* of R is (abelian) finitely generated, the quotient $R^*/(R^*)^2$ is finite; let U be a set of representatives in R^* of this quotient. Let D be a set of representatives of divisors of $2F_q$, up to units, and UD the set of products ud with $u \in U$ and $d \in D$; this set is finite. If r is a primitive root satisfying the crystallographic condition, there is a unique unit u (up to sign) such that $(ur)^2 \in UD$. Indeed, if u_1, u_2 are such that $(u_1 r)^2, (u_2 r)^2 \in UD$, we get $u_i^2 r^2 = w_i d_i$ for $i = 1, 2$ and $w_i \in U, d_i \in D$, and rearranging the equalities shows that d_1, d_2 differ by a unit, hence must be equal, and w_1, w_2 differ by the square of a unit, hence must also be equal; it follows that $u_1^2 = u_2^2$, and $u_1 = \pm u_2$ (showing unicity). Furthermore, if r is primitive satisfying the crystallographic condition, r^2 divides $2F_q$, so that there exists a unit w with $wr^2 \in D$ and a unit u with $u^2 w^{-1} \in U$, so that $(ur)^2 = u^2 r^2 = u^2 w^{-1} w r^2 \in UD$ (showing existence).

The conclusion of this discussion is that there is a bijective correspondence between reflections preserving L and primitive lattice vectors satisfying the crystallographic condition of length contained in the (finite) set UD, up to sign.

7 Implementing Vinberg's algorithm

This chapter contains some comments on practical aspects of Vinberg's algorithm. In order: decidability questions, implementation details, and some examples (not necessarily new) of reflective and non-reflective lattices found by means of the algorithm. The implementation discussed—hereafter called *Vinny*—is available at [BB21].

7.1 Non-reflective groups

Fix a discrete group $\Gamma < \tilde{\mathscr{O}}$ of finite covolume, and let Γ_r denote the subgroup generated by reflections of Γ. Let P be a cell of Γ_r. If P is of finite volume, it is necessarily finite-sided ([Rat19, Theorem 12.4.10, p. 639]). If P is not of finite volume (i.e., Γ is not reflective), two cases must be distinguished.

Lemma 7.1.1. *Γ_r is either trivial or infinite.*

Proof. Note that Γ_r being a reflection group, it is finite iff it has a finite number of reflections[1].

Towards a contradiction, assume that Γ_r is non-trivial but finite. By the decomposition $\Gamma = \Gamma_r \rtimes \mathrm{Sym}_\Gamma P$ and finiteness of Γ_r, for Γ to be of finite covolume, so must be $\mathrm{Sym}_\Gamma P$.

Since Γ_r is finite, it has a finite number of mirrors, and P is finite-sided. Let $r_1, \ldots, r_k$ be the respective roots of the bounding hyperplanes of P, and let $m := \sum_i r_i$. Since $\mathrm{Sym}_\Gamma P$ sends a bounding hyperplane to a bounding hyperplane, $\mathrm{Sym}_\Gamma P$ permutes the corresponding set of roots, and therefore fixes m. By lemma 6.1.1, $\mathrm{Sym}_\Gamma P$ has infinite covolume, a contradiction. □

Lemma 7.1.2. *If Γ is not reflective, and Γ_r is not trivial, then P is infinite-sided.*

Proof. The polyhedron P has at least one bounding hyperplane since Γ_r is non-trivial. Assume that P is finite-sided, with bounding hyperplanes $H_{r_1}, \ldots, H_{r_k}$. Let $I := \bigcap_i r_i^\perp \leq V$ be the intersection of (the subspaces induced by) all bounding hyperplanes of P. Note that:

- Γ_r preserves I since each reflection along bounding hyperplanes of P does; $\mathrm{Sym}_\Gamma P$ preserves I since its elements permute the bounding hyperplanes of P; hence $\Gamma = \Gamma_r \rtimes \mathrm{Sym}_\Gamma P$ preserves I;
- I is empty iff there exist two ultraparallel faces of P;
- given two ultraparallel faces F_1, F_2 of P, there exists a unique point $m(F_1, F_2)$ at the center of the unique geodesic joining the hyperbolic subspaces spanned by F_1 and F_2, and orthogonal to both.

Now, if I is non-empty, applying lemma 6.1.1 shows that Γ is not of finite covolume, a contradiction. On the other hand, if I is empty, let m be the center of mass of all points of the form $m(F_1, F_2)$ for all pairs of ultraparallel faces F_1, F_2 of P. $\mathrm{Sym}_\Gamma P$ fixes m, hence must be finite, contradicting the non-reflectivity of Γ_r.

In conclusion, for Γ to be non-reflective and Γ_r non-trivial, P must be infinite-sided. □

[1] Finitely many reflections implies finitely many mirrors, implies finitely many cells, hence a finite group.

By the above, we now know that P is of infinite volume if and only if it is either infinite-sided or the whole space $\mathbb{H}^n$.

Lemma 7.1.3. *If Γ_r is not trivial, the set of roots $\{r_i\}_i$ of bounding hyperplanes of P spans V.*

Proof. Let $I := \bigcap_i r_i^\perp \leq V$ be the intersection of (the subspaces induced by) all bounding hyperplanes of P. Since I is preserved by Γ, it must be trivial (lemma 6.1.1), hence $\operatorname{span}\{r_i\}_i = V$. □

In particular, there necessarily exists a set of $n+1$ linearly independent roots (this will be used later on).

7.1.1 Trivial reflection subgroup

Gaël Collinet constructed[2] a simple example of an integer Lorentzian lattice with no roots. It is given by the matrix

$$\begin{pmatrix} 0 & 0 & 49 \\ 0 & 49 & 7 \\ 49 & 7 & 3 \end{pmatrix}.$$

Thus, in this case, the group $\tilde{\mathcal{O}}^L$ has a trivial reflection subgroup $\tilde{\mathcal{O}}_r^L$, whose fundamental polyhedron is the whole of $\mathbb{H}^2$, and Vinberg's algorithm does not apply, since there are no bounding hyperplanes.

7.2 Abstract description of Vinberg's algorithm

Fix a discrete group $\Gamma < \tilde{\mathcal{O}}$ of finite covolume, and let Γ_r denote the subgroup generated by reflections of Γ. Vinberg's algorithm [Vin72] is a procedure enabling the construction of a cell P of Γ_r, by listing its bounding hyperplanes by increasing distance to a given basepoint $p_0 \in \mathbb{H}^n$. If P is of finite volume, it is necessarily finite-sided ([Rat19, Theorem 12.4.10, p. 639]) and the algorithm halts, while if P is of infinite volume, the algorithm diverges.

If $p_0 \in \mathbb{H}^n$ is a given point, let $\Gamma_r(p_0)$ denote the subgroup of Γ_r generated by reflections fixing p_0 ($\Gamma_r(p_0)$ can be trivial). Let P_0 be any choice of cell for $\Gamma_r(p_0)$; it is a cone at p_0 and there is a unique cell P for Γ_r equal to P_0 in a small enough neighborhood of p_0. The algorithm runs as follows:

1. Fix a basepoint $p_0 \in \mathbb{H}^n$.
2. Let $\{H_0, \ldots H_k\}$ be the set of oriented bounding hyperplanes for a cell P_0 of $\Gamma_r(p_0)$.
3. Assume now that the oriented hyperplanes $H_0, \ldots, H_l$ ($l \geq k$) have been found by the algorithm, by increasing distance $d(p_0, H_i)$ and with $r := d(p_0, H_l)$. Assume furthermore that the polyhedron $P_l := \bigcap_{i=0}^{l} H_i^-$ is not of finite volume. Consider the set $\mathscr{H}_l$ of oriented hyperplanes H such that:
   - $p_0 \in H^-$, and p_0 is at distance $\geq r$ from H,
   - H is the mirror of a reflection of Γ,
   - H does not intersect either of $H_0, \ldots, H_l$ with an obtuse angle.

[2] In a talk given in Tunis in October 2013, titled *Sur les réseaux lorentziens et leurs groupes d'automorphismes.*

Choose $H_{l+1} \in \mathscr{H}_l$ minimizing its distance to p_0.

4. Repeat the previous step as long as the polyhedron P_l is not of finite volume.

Note that by discreteness, there can only be a finite number of reflection mirrors intersecting any given ball around p_0, so that there always exists some H_{l+1} minimizing the distance.

Thus, the algorithm constructs a sequence $(H_i)_{i=0}^N$ ($N \in \mathbb{N} \sqcup \{\infty\}$) of oriented hyperplanes. The sequence is finite if P_l is found to be of finite volume, in which case $N = l$.

Let us briefly explain why the sequence $(H_i)_{i=0}^N$ enumerates all bounding hyperplanes of P, and only those. We will need the following property of bounding hyperplanes:

Lemma 7.2.1 ([Vin72, Proof of Proposition 4 (implicit), p. 28])**.** *An oriented hyperplane H_e at non-zero distance from p_0 bounds P if and only if for each bounding hyperplane $H_{e'}$ strictly closer to p_0 than H_e, we have $ee' \leq 0$ (i.e., H_e and $H_{e'}$ do not intersect with an obtuse angle).*

Now, assume for a moment that the algorithm does not explicitly stop when P_l is first found to be of finite volume. We claim that the sequence of enumerated hyperplanes consists exactly of those bounding P.

First, let us assume that some H_l does *not* bound P, and let l be the first such index. Since the cone hyperplanes $H_0, \dots, H_k$ are chosen to bound P_0 (hence P) by construction, H_l does not contain p_0. Let H_e be any bounding hyperplane closer to p_0 than H_l. H_e has non-obtuse angles with all hyperplanes $H_0, \dots, H_{l-1}$, since those are assumed to be bounding (and P is acute-angled). Thus, H_e satisfies the conditions of step 3. of the algorithm, and must have been enumerated before H_l, from which follows that H_l has non-obtuse angle with H_e too. Thus H_l has non-obtuse angles with all bounding hyperplanes closer to p_0 than H_l is, and by the lemma, H_l is bounding, a contradiction.

Conversely, let us show that all hyperplanes bounding P are enumerated. If H_e is such a hyperplane, it necessarily has non-obtuse angles with all other bounding hyperplanes, and in particular with all enumerated hyperplanes (since they have been verified to be bounding). By discreteness, there are only a finite number of bounding hyperplanes (a fortiori enumerated hyperplanes) closer to p_0 than H_e, or at the same distance, and after having enumerated those, H_e is necessarily the next chosen hyperplane at step 3. of the algorithm. It follows that H_e must be enumerated.

Thus, putting aside the "halt on finite volume" part, the algorithm enumerates exactly the bounding hyperplanes of P. The importance of this halting condition is that without it, the algorithm never knows to stop. The following lemma ensures that with this condition added back, the algorithm still enumerates all bounding hyperplanes.

Lemma 7.2.2 ([Vin72, Proposition 5])**.** *If P_l is of finite volume, then $P = P_l$.*

In particular, if P_l is of finite volume, P and P_l have the same bounding hyperplanes, and the algorithm enumerates them. Conversely, by [Rat19, Theorem 12.4.10, p. 639], if P is of finite volume, it has finitely many bounding hyperplanes, which are enumerated by the algorithm. In conclusion, the algorithm halts iff P has finite volume, and outputs its finite number of bounding hyperplanes.

7.3 Infinite order symmetry test

Fix, as before, a discrete group $\Gamma < \tilde{\mathcal{O}}$ of finite covolume, and $P = \bigcap_{i=0}^{N} H_{r_i}^-$ a cell of Γ_r such that the sequence of oriented hyperplanes H_{r_i} is as given by Vinberg's algorithm, and $N \in \mathbb{N} \cup \{\infty\}$ (depending on whether P is finite-sided or not). Assume furthermore that Γ_r is non-trivial. From the decomposition $\Gamma = \Gamma_r \rtimes \mathrm{Sym}_\Gamma P$, it follows that Γ is reflective iff $\mathrm{Sym}_\Gamma P$ is finite, which is equivalent to all elements of $\mathrm{Sym}_\Gamma P$ being of finite order (equivalently, having a fixed point in $\mathbb{H}^n$).

Fix a set of $n+1$ linearly independent roots $S := \{r_{i_0}, \ldots, r_{i_n}\}$ defining bounding hyperplanes of P (lemma 7.1.3). Any $\gamma \in \Gamma$ is obviously entirely defined by its action on S. Furthermore, any $\gamma \in \mathrm{Sym}_\Gamma P$ must send bounding hyperplanes of P to bounding hyperplanes of P, hence roots thereof to roots thereof. Therefore, given $\gamma \in \mathrm{Sym}_\Gamma P$, there exist $n+1$ linearly independent roots $T := \{r_{j_0}, \ldots, r_{j_n}\}$ (defining bounding hyperplanes of P) such that $\gamma r_{i_k} = r_{j_k}$ for all $k \in \{0, \ldots, n\}$.

Conversely, assume that P has at least one vertex, and fix

- two sets of $n+1$ linearly independent roots defining bounding hyperplanes, say $S := \{r_{i_0}, \ldots, r_{i_n}\}$ and $T := \{r_{j_0}, \ldots, r_{j_n}\}$, and
- two sets of roots, say V_S and V_T, defining the bounding hyperplanes at vertices v_S, v_T of P, respectively.

If the assignment $r_{i_k} \mapsto r_{j_k}$ defines an element γ of Γ, and if γ sends V_S to V_T, then γ necessarily lies in $\mathrm{Sym}_\Gamma P$. Indeed, since $\gamma[V_S] = V_T$, γ sends the polyhedral angle $P_S := \bigcap_{r \in V_S} H_r^-$ at vertex v_S to the polyhedral angle $P_T := \bigcap_{r \in V_T} H_r^-$ at vertex v_T, and since P is the only cell equal to P_S [resp. P_T] in a small neighborhood of v_S [resp. v_T], it follows that γ preserves P, and thus $\gamma \in \mathrm{Sym}_\Gamma P$.

Now, fix any element $\gamma \in \mathrm{Sym}_\Gamma P$, and sets S, T, V_S, V_T as above, with $\gamma[S] = T$ and $\gamma[V_S] = V_T$. Since Vinberg's algorithm enumerates all bounding hyperplanes of P, all roots in $S \cup T \cup V_S \cup V_T$ will have been enumerated after finitely many steps, and the symmetry γ can be found by looking at all possible pairs S, T and testing whether the mapping $S \to T$ extends to an isometry in Γ and sends V_S to V_T. In particular, all elements of $\mathrm{Sym}_\Gamma P$ of infinite order can be enumerated. Finally, since the non-reflectivity of Γ is equivalent to the existence of such an element of $\mathrm{Sym}_\Gamma P$, it can be decided, assuming P has a vertex, after a finite number of steps of Vinberg's algorithm. This argument is essentially an argument of Guglielmetti [Gug17b, 6.2.6.2 p.116], itself a variation of [Bug92, Lemmas 3.1-3.3, pp. 44-45]. Note that the argument only works if P has a vertex; I don't know if this is always the case.

7.4 Decidability

Fix, as before, a discrete group $\Gamma < \tilde{\mathcal{O}}$ of finite covolume, and let P be a cell of Γ_r.

Lemma 7.4.1. *If Γ_r is non-trivial and P has at least one vertex, then reflectivity of Γ is decidable.*

Proof. If Γ is reflective, then P is finite-sided and Vinberg's algorithm halts with an enumeration of its bounding hyperplanes. On the other hand, assume that Γ is non-reflective (but Γ_r is non-trivial and P has at least one vertex). Then

- $\mathrm{Sym}_\Gamma P$ contains some element γ of infinite order;

- there exists a set R of $n+1$ linearly independent roots defining bounding hyperplanes of P;
- there exists a set V_R of roots defining a vertex of P.

After a finite number of steps of Vinberg's algorithm, all roots r and $\gamma(r)$ ($r \in R \cup V_R$) have been enumerated by the algorithm, and applying the infinite symmetry test, the isometry $\gamma \in \mathrm{Sym}_\Gamma P$ is found, thus witnessing non-reflectivity.

In both cases, an answer is obtained in finite time. □

Thus, the first obstacle to the general decidability of the reflectivity of Γ is the question of existence of at least one reflection in Γ. In case $\Gamma = \tilde{\mathcal{O}}^L$, Γ_r is non-trivial iff there exists a lattice element $r \in L$ that satisfies the crystallographic condition, i.e., such that:

$$2\frac{re_i}{r^2}r_j \in R \quad \forall i, \forall j,$$

with $e_0, \ldots, e_n$ the standard basis of L. This can be rephrased as the existence of a vector $k \in L$ satisfying:

$$2\,(r'Qe_i)\,r_j = k_j\,(r'Qr) \quad \forall i, \forall j,$$

where $'$ denotes transposition. Thus, reflectivity of $\Gamma = \tilde{\mathcal{O}}^L$ amounts to solving a set of diophantine equations of degree 3 in R. I don't know if this type of diophantine equation system is restricted enough to be decidable in general.

The second obstacle to decidability of the reflectivity of Γ appears when applying the infinite-order symmetry test: we need to ensure that any candidate symmetry γ really does preserve the cell P. The strategy highlighted above consists in checking that γ sends the polyhedral angle at some vertex of P to the polyhedral angle at some other vertex. Thus, P needs to have at least one vertex; I don't know if this is always the case. It might also be the case that the constructed γ necessarily preserves P, by virtue of sending $n+1$ linearly independent roots (defining bounding hyperplanes) to $n+1$ linearly indepedent roots (defining bounding hyperplanes). In short, it is possible that this second obstacle really isn't one.

7.5 Implementation

This section explains in some detail the practical aspects of implementing Vinberg's algorithm, as presented in section 7.2, when the discrete group at hand is of the form $\tilde{\mathcal{O}}^L$. This implementation (*Vinny*) is strongly inspired by [Gug17a] and [BP17], although it evolved to be relatively different from both.

The input to the algorithm is given as an $(n+1) \times (n+1)$ R-matrix Q, with the standing assumption that K is totally real, and R principal. Let L be the R-lattice spanned by a basis $\{e_0, \ldots, e_n\}$, $V := L \otimes \mathbb{R}$ the associated real vector space and q the quadratic form on L and V whose matrix with respect to the basis $\{e_i\}_i$ is Q. In what follows, we assume that the primitives needed to work with K and R are already implemented (in practice, this is covered by the Hecke library [Fie+17]).

7.5.1 Setup

Admissibility. We first need to check that q is an admissible Lorentzian form, i.e., that Q is of signature $(n, 1)$ and that under all non-identity (real) embeddings σ, Q^σ is positive definite.

This is most easily done by diagonalizing the quadratic form: by a process very similar to Gaussian reduction[3], one can construct an R-matrix P, invertible in K, such that $D := P'QP$ is diagonal. Clearly, $D^\sigma = (P^\sigma)'Q^\sigma P^\sigma$, and by counting the number of occurrences of positive (negative, zero) entries in D and in the varying D^σ, one can verify that Q defines an admissible form.

Diagonalization. To facilitate the enumeration of roots and related computations, we mainly work using a diagonalization of the form. Let, as above, P and D be R-matrices with P invertible, D diagonal and $P'QP = D$. Let $d_0, \ldots, d_n$ denote the diagonal entries of D, and assume without loss of generality that $d_0 < 0$ and that $d_i > 0$ for all $i \geq 1$. Let $p_0, \ldots, p_n$ be the columns of P. By construction, $\{p_i\}_i$ forms an orthogonal basis of $L \otimes K$, satisfying $q(p_i, p_i) = d_i$. However, $\{p_i\}_i$ does not necessarily form a basis of the lattice L: its R-span can be a strict subgroup of L. Nevertheless, there exist coefficients $s_0, \ldots, s_n \in R$ such that

$$\operatorname{span}_R \{p_i\}_i \leq L \leq \operatorname{span}_R \left\{\frac{p_i}{s_i}\right\}_i .$$

These can be obtained as the respective least common multiples of each row of the matrix P^{-1}. From now on, the bulk of root enumeration and lattice computations is done using the diagonal basis $\{p_i/s_i\}_i$, keeping in mind that not all R-combinations thereof are lattice elements.

Root lengths. Recall, by section 6.3, that enumerating the reflection mirrors of $\tilde{\mathcal{O}}^L$ amounts to enumerating the primitive roots of L of length contained in the set UD, up to sign, where:

- U is a set of representatives for $R^*/(R^*)^2$;
- D is a set of representatives of the divisors of $2L$ up to units, where L is the last invariant factor of Q.

Since R^* is finitely generated and abelian, U is finite, and can be computed. Similarly, L can be computed by applying Smith normal form to Q, and since R is principal, the set D is finite, and can also be computed.

Note finally that not all elements of UD are valid root lengths: If $r^2 = l > 0$, then $(\sigma r)^2 = \sigma l > 0$, for all non-trivial embeddings σ, since q^σ is positive-definite. Therefore, the set of root lengths we need to consider consists of those elements of UD that have only strictly positive Galois conjugates.

Choice of basepoint. In theory, the algorithm works for any given choice of basepoint v_0 in $\mathbb{H}^n$ (or even $\partial\mathbb{H}^n$, modulo some adaptations). We will only treat the case where $v_0 \in \mathbb{H}^n$ is p_0, which was assumed to have negative length.

7.5.2 Enumerating roots

As seen above, enumerating the roots reduces to enumerating the roots of given length l, for a finite number of possible ls. Recall that the distance between the hyperplane defined by a root r and a point defined by v (with $v^2 < 0$) is:

$$\sinh^{-1} \sqrt{-\frac{(rv)^2}{r^2 v^2}}.$$

[3] https://math.stackexchange.com/questions/1388421

By monotonocity of $\sinh^{-1}$ and $\sqrt{\cdot}$, minimizing the distance amounts to minimizing the quantity $-\frac{(rv)^2}{r^2v^2}$, and, if v is fixed, to minimizing $\frac{(rv)^2}{r^2}$ (noting that $v^2 < 0$, thus canceling the negative sign in front of the fraction). If r has length l and coordinates $(k_0, \ldots, k_n)$ (with $k_i \in s_i^{-1}R$) with respect to the diagonal basis $\{p_i\}_i$, and $v = v_0 = p_0$, this latter quantity is simply

$$\frac{(d_0k_0)^2}{l},$$

which is obviously minimized simultaneously with $\frac{k_0^2}{l}$.

Note in particular that the roots corresponding to hyperplanes containing v_0 are exactly those with $k_0 = 0$. We will now explain the procedure of enumeration of roots with prescribed length l and first diagonal coordinate k_0.

Roots for a given pair (k_0, l)**.** Let r be a root of length l with coordinates $(k_0, \ldots, k_n)$ (with $k_i \in s_i^{-1}R$) with respect to the diagonal basis $\{p_i\}_i$. By applying all embeddings $\sigma : K \to \mathbb{R}$ ($\sigma = \mathrm{Id}$ included) to the equality $r^2 = l$, we get a collection of equalities

$$\sum_i \sigma(d_i)(\sigma(k_i))^2 = \sigma(l) \quad \forall \sigma.$$

By admissibility of Q, $\sigma(d_i) > 0$ for all $i \geq 1$ and σ, which implies, for $i_0 \geq 1$, that

$$\sigma(k_{i_0})^2 = \frac{\sigma(l) - \sum_{i \neq i_0} \sigma(d_i)\sigma(k_i)^2}{\sigma(d_{i_0})} \leq \frac{\sigma(l) - \sum_{i<i_0} \sigma(d_i)\sigma(k_i)^2}{\sigma(d_{i_0})} \quad \forall \sigma.$$

By summing over the embeddings σ, we get:

$$t_2(k_{i_0}) = \sum_\sigma \sigma(k_{i_0}^2) \leq \sum_\sigma \frac{\sigma(l) - \sum_{i<i_0} \sigma(d_i)\sigma(k_i)^2}{\sigma(d_{i_0})},$$

which implies that, if the length l and the prefix $(k_0, \ldots, k_{i_0-1})$ of diagonal coordinates of a root r are given, the possible values for k_{i_0} have bounded t_2-norm, and are therefore in finite number (and computable).

Recall that to be able to cover all lattice elements in our search for roots, the diagonal coordinates $(k_0, \ldots, k_n)$, cannot be simply taken in R, but that k_i must be taken in $s_i^{-1}R$. Writing $k_i := s_i^{-1}k_i'$, the inequalities above can easily be rewritten, yielding:

$$t_2(k_{i_0}') \leq \sum_\sigma \frac{\sigma(l) - \sum_{i<i_0} \sigma(d_i)\sigma(k_i)^2}{\sigma(d_{i_0})\sigma(s_{i_0})^2}. \tag{7.5.1}$$

This essentially provides our approach to finding all roots of length l with first diagonal coordinate k_0: assuming a prefix $(k_0, \ldots, k_i)$ has been fixed, enumerate all possible k_{i+1}' with appropriately bounded t_2 norm, and repeat the procedure for all possible prefixes $(k_0, \ldots, k_{i+1})$. After applying the base change matrix P to a completed diagonal coordinate vector $(k_0, \ldots, k_n)$, thus getting a vector r in standard coordinates, it remains to verify that:

- r **is an element of** L **(i.e., has entries in** R**).** Indeed, not all elements of the form $\sum k_i p_i$ with $k_i \in s_i^{-1}R$ are contained in L.

- r **is primitive.** Indeed, we know that it is enough to enumerate primitive roots. Verifying primitivity means computing a gcd.
- $r^2 = l$ **holds.** Because using eq. (7.5.1) at each step only ensures bounds on the t_2 norms of each diagonal coordinate k_i.
- r **satisfies the crystallographic condition (i.e., the reflection along** r **preserves the lattice).** This means verifying that l divides $2(re_i)$ for all basis vectors e_i.
- r **has non-obtuse angles with all roots already found by the algorithm.** This amounts to computing a finite amount of products of the form $r' \cdot Q \cdot r$ and verifying that the results are non-strictly negative.

Early exit. The root enumeration strategy described above is not optimal, in that, given a prefix $(k_0, \ldots, k_i)$, it may already be known that there is no way to extend it to a valid primitive root. Let us now explain a few ways in which we can filter out bad prefixes earlier. Fix a prefix $(k_0, \ldots, k_i)$.

- **Integrality.** Assume that $r_i := \sum_{j \leq i} k_j p_j$ has a non-integral m-th coordinate with respect to the canonical basis $\{e_0, \ldots, e_{n+1}\}$ and that for all $j > i$, the vector p_j has zero m-th coordinate with respect to the canonical basis. No matter how $k_{i+1}, \ldots, k_n$ are chosen, the resulting vector $r = \sum_j k_j p_j$ cannot be in L. Thus, the prefix $(k_0, \ldots, k_i)$ can be dropped as soon as such an m exists.
- **Norm bounds under conjugates.** Since $\sigma(d_i)$ $(i > 1)$ is positive for all σ, and $r^2 = l$, it follows that $\sum_{j \leq i} \sigma(d_j)\sigma(k_j)^2 \leq \sigma(l)$ must hold for all σ. If this is not the case, the prefix can be dropped.
- **Crystallographic condition.** Recall that the crystallographic condition states that the reflection along r preserves the lattice. To verify that this holds, it suffices to check that the reflection sends each basis vector e_i to an element of the lattice. A weaker (necessary) condition is that the reflection sends each vector p_i to an element of the lattice. This latter condition can be rephrased as $l \mid 2k_i d_i$, for all i. Consequently, if any k_i does not satisfy this relation, the prefix can be dropped.
- **Acute angles.** If a set of previous roots $r_1, \ldots, r_j$ found by the algorithm are given, the condition that r has non-obtuse angles with $r_1, \ldots, r_j$ can be stated as a set of linear constraints on r. It may be known, given only the prefix $(k_0, \ldots, k_i)$ that r cannot satisfy those constraints, and hence be dropped.

Finally, note that if the prefix $(k_0, \ldots, k_{n-1})$ of a root of length l is known, then k_n is necessarily a square root of $\frac{l - \sum_{i=1}^{n} d_i k_i^2}{d_{n+1}}$; this doesn't seem to useful in practice.

7.5.3 Fundamental cone

Recall that a fundamental cone P_0 at the point v_0 is a fundamental chamber of the group $\tilde{\mathcal{O}}_r^L(v_0)$ of (lattice preserving) reflections fixing v_0. $\tilde{\mathcal{O}}_r^L(v_0)$ is a discrete subgroup of the stabilizer of v_0, a compact group, and is therefore finite.

By section 7.5.2, and the fact that reflections fixing v_0 are exactly those along roots with first diagonal coordinate $k_0 = 0$, we know how to enumerate the roots of reflections in $\tilde{\mathcal{O}}_r^L(v_0)$.

Assume that those roots are $r_1, \ldots, r_m$. To construct the set of roots defining the bounding hyperplanes B of P_0, the algorithm proceeds as follows:

1. Start with B empty;
2. Iterate over the roots $r_1, \ldots, r_m$ in no particular order. Assuming the intersection $P_B := \bigcap_{r\in B} H_r^-$ is non-degenerate (contains an open set) and r_i is the next root in the iteration:
   - If $H_{r_i}^- \cap P_B$ is degenerate, skip r_i.
   - Otherwise, add r_i to B.
3. After having iterated over all roots, remove those that do not bound P_B from the set B.

The resulting cone P_B is equal to P_0. Both testing degeneration of $H_{r_i}^- \cap P_B$ and the removal of non-bounding hyperplanes amount to feasibility of a finite set of linear inequalities, and can thus be solved by linear programming methods.

Note that the usual linear programming libraries available in Julia do not handle exact computations in R. In *Vinny*, to compute the fundamental cone P_0, the coefficients of linear inequalities in R are therefore first approximated to a given precision, and then fed to the methods used to check feasibility. In theory, this could yield incorrect results, but it works well enough in practice. To make this computation exact, it would probably be easiest to reimplement any feasibility algorithm "by hand" on R.

7.5.4 Remaining roots

Recall that the main bulk of Vinberg's algorithm is the enumeration of hyperplanes of reflections in $\tilde{\mathscr{O}}^L$, by increasing distance to v_0. Assume that roots $r_1, \ldots r_k$ have been found, defining hyperplanes $H_1, \ldots, H_k$, with, in particular:

- $r_i r_j \leq 0$ for all i, j (acute angles);
- $r_i v_0 \leq 0$ for all i ($v_0 \in H_i^-$);
- the distance from v_0 to H_i are increasing.

The next step is to find a root r_{k+1} defining a closest hyperplane H_{k+1} to v_0, verifying the above conditions.

If $(k_0, \ldots, k_n)$ is the diagonal representation of the root r of length l defining hyperplane H_r, the condition $rv_0 \leq 0$ simply means that $k_0 \geq 0$ (since $v_0 = p_0$, so that $rv_0 = d_0 k_0$). Furthermore, the distance between H_r and v_0 is monotonically increasing with the fraction k_0^2/l. By section 7.5.2, we know how to enumerate all roots with given length l and first diagonal coordinate k_0, and how to filter out only those satisfying the acute angle conditions. It therefore remains to enumerate all pairs (k_0, l) by increasing value of k_0^2/l.

By admissibility of q, for all $\sigma \neq \mathrm{Id}$, we have $\sigma(d_0) > 0$. Thus, given a root r of length l and first diagonal coordinate k_0, we have $\sigma(k_0)^2 \leq \sigma(l)/\sigma(d_0)$, so that

$$t_2(k_0) = k_0^2 + \sum_{\sigma\neq\mathrm{Id}} \sigma(k_0)^2 \leq k_0^2 + \sum_{\sigma\neq\mathrm{Id}} \sigma(l)/\sigma(d_0).$$

In particular, given some $M > 0$, the set possible values for $k_0 \leq M$ have bounded t_2-norm, and are finite in number. Given a fixed length l, one can therefore fix some arbitrary M, enumerate all possible $k_0 \leq M$, and order them to find the smallest possible ratio k_0^2/l. By increasing M as needed, one can continue and list the of possible k_0 by increasing order. Since the number of possible lengths l is finite, this allows iterating over the pairs (k_0, l) by increasing distance, and thus iterating over hyperplanes/roots.

7.5.5 Checking finiteness

Assume the algorithm found roots $r_1, \ldots, r_l$, defining polyhedron $P_l := \bigcap_{i=1}^{l} H_{r_i}^-$, and let D_l be the Coxeter diagram associated to P_l. By section 6.1.3, we know that P_l has finite volume iff all spherical subdiagrams of D_l of rank $n-1$ extend in exactly two ways to either a spherical diagram of rank n or an Euclidean diagram of rank $n-1$. Checking volume finiteness thus reduces to a problem of enumeration of Euclidean and spherical subdiagrams and both types are classified into a finite number of easily recognizable families.

7.5.6 Infinite-order symmetry test

Assume the algorithm found roots $r_0, \ldots, r_l$, defining polyhedron $P_l := \bigcap_{i=0}^{l} H_{r_i}^-$. To apply the symmetry test, we first list all $(n+1)$-elements subsets $S := \{r_{i_0}, \ldots, r_{i_n}\}$ of $\{r_0, \ldots, r_l\}$ and keep only those S that are linearly independent (which amounts to computing the rank of a matrix). We also list all subsets V_S of $\{r_0, \ldots, r_l\}$ defining vertices of P_l. For each pair of such subsets $(S := \{r_{i_0}, \ldots, r_{i_n}\}, V_S)$ and $(T := \{r_{j_0}, \ldots, r_{j_n}\}, V_T)$, we check that the assignment $r_{i_k} \mapsto s_{j_k}$ $(k = 0, \ldots, n)$ extends to an element of $\tilde{\mathcal{O}}^L$, sending V_S to V_T, and if it does, we check that it has no fixed point in $\mathbb{H}^n$. In that case, we conclude that $\mathrm{Sym}_{\tilde{\mathcal{O}}^L} P$ is infinite, and that $\tilde{\mathcal{O}}^L$ is non-reflective. In more details, the following conditions need to be verified in order:

1. The Gram matrices of S and T agree (at that point, the assignment $r_{i_k} \mapsto s_{j_k}$ defines a q-preserving linear isomorphism ϕ of V).
2. ϕ preserves the upper sheet (now $\phi \in \tilde{\mathcal{O}}$).
3. The matrix of ϕ (obtained as $s^{-1}t$, where s, t are the matrices whose columns are the elements of S and T, respectively) has coefficients in R ($\phi \in \tilde{\mathcal{O}}^L$).
4. $\phi[V_S] = V_T$ ($\phi \in \mathrm{Sym}_{\tilde{\mathcal{O}}^L} P$).
5. ϕ has no fixed point in $\mathbb{H}^n$, which is equivalent to checking that $\ker(\phi - \mathrm{Id})$ has a positive semi-definite basis (i.e., that $\ker(\phi - \mathrm{Id})$ does not intersect C_V).

7.5.7 Main libraries used

The implementation of Vinberg's algorithm in *Vinny* made heavy use of the rich ecosystem of libraries available for Julia. The entire list is relatively long, so let us only list the key ones:

`Hecke`, `Nemo`, & `AbstractAlgebra` [Fie+17] are general computer algebra and number theory libraries. They provide essentially all number field related primitives (based on the FLINT/ANTIC libraries [HJP13]) used in *Vinny*. Some linear algebra primitives (computation of kernels, normal forms of matrices, etc) are also used.

`Tulip` [TAL21] is a linear programming library used for the construction of the fundamental cone P_0 (section 7.5.3).

`IntervalArithmetics` **[San+21]** is a library providing primitives for safe floating point arithmetic, using intervals. This library is used for the management of linear inequalities on roots stemming from the acute angle condition (section 7.5.2). Order comparison for field elements is a relatively expensive operation, even more so in comparison to floating point based interval arithmetic. Consequently, by first approximating field elements by floating point intervals, and computing with those, a majority of "acute-angle" induced computations can be made faster.

`LightGraphs` **[Bc17]** is used to check Coxeter diagram isomorphisms. Since `LightGraphs` can be used to find isomorphisms of edge- and vertex-labeled graphs, it is also used in the infinite symmetry test (for which we actually require isomorphisms of Gram matrices).

7.6 Some (non)-reflective lattices

This section is devoted to some examples of reflective and non-reflective lattices. Only some of them are new, but all have been checked using *Vinny*.

If M, N are square matrices, let us denote the block-diagonal matrix $\begin{pmatrix} M & 0 \\ 0 & N \end{pmatrix}$ by $M \oplus N$; if M is a ring element, it is to be understood as a 1×1 matrix in the expression $M \oplus N$. Fix a totally real number field K and its ring of algebraic integers R. Clearly, if M is of signature (m_+, m_-) and N of signature (n_+, n_-), then $M \oplus N$ is of signature $(m_+ + n_+, m_- + n_-)$, and for any embedding $\sigma : K \to \mathbb{R}$, $(M \oplus N)^\sigma = M^\sigma \oplus N^\sigma$. Therefore, if M defines an admissible quadratic form, and each N^σ is positive-definite, then $M \oplus N$ also defines an admissible quadratic form. This leads to the usual approach to building admissible quadratic form: let M be a positive element of R all of whose conjugates are negative, and N a positive-definite $\mathbb{Z}$-matrix. The matrix $M \oplus N$ is necessarily admissible.

In the remainder of this section, we will identify a (symmetric admissible) matrix Q with the Lorentzian lattice L it defines (the number field K and ring of integers R being clear in context), and with the (any) cell P of the group $\tilde{\mathscr{O}}_r^L$.

7.6.1 Bugaenko's dimension 7 and 8 compact polyhedra

Let E_6, E_7, E_8 respectively denote the following positive-definite matrices:

$$\begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}.$$

Let also $K = \mathbb{Q}(\sqrt{5})$, so that $R = \mathbb{Z}[\phi]$, with $\phi := \frac{1+\sqrt{5}}{2}$. In dimension 8, exactly one compact Coxeter polyhedron is known, it is the cell P_8 of the reflective lattice $-2\phi \oplus E_8$, as found by Bugaenko [Bug92]. In dimension 7, two compact Coxeter polyhedra are known: the first, P_7, is the cell of the reflective lattice $-2\phi \oplus E_7$ and the second, P_7', is the cell of the reflection centralizer of $\tilde{\mathscr{O}}_r^L$, for $L = -2\phi \oplus E_8$; P_7 is due to Bugaenko, and P_7' to Allcock [All13]. It turns out that P_7' is actually also a cell of the lattice $-\phi \oplus 2E_7$ (note the 2 moving around). Furthermore, P_8 is also a cell of $-\phi \oplus 2E_8$, and neither $-2\phi \oplus E_6$ nor $-\phi \oplus 2E_6$ is reflective. In summary :

lattice	$-2\phi \oplus E_6$	$-\phi \oplus E_6$	$-2\phi \oplus E_6$	$-\phi \oplus 2E_7$	$-2\phi \oplus E_8$	$-\phi \oplus 2E_8$
polyhedron	∞	∞	P_7	P_7'	P_8	P_8

with ∞ meaning that the polyhedron has infinite volume.

7.6.2 Over $\mathbb{Q}(\cos(\frac{2\pi}{7}))$

Let $K := \mathbb{Q}(\cos(2\pi/7))$, so that $R = \mathbb{Z}[2\phi]$, with $\phi = \cos(2\pi/7)$. Let furthermore

$$\phi_0 := 2\phi, \quad \phi_1 := 2\cos(4\pi/7), \quad \phi_2 := 2\cos(6\pi/7).$$

Consider the lattice:

$$\begin{pmatrix} -\phi_0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1\phi_0 - 5\phi_1 - 7\phi_2 \end{pmatrix}.$$

This is an example of Guglielmetti [Gug17b, p.132]. This lattice is reflective, and, given the basepoint $p_0 := (1, 0, 0)$, *Vinny* finds the following roots:

$$\begin{aligned}
&(0, -1, 0), \\
&(0, 0, 3\phi_0 + 1\phi_1 + 2\phi_2), \\
&(2\phi_0 + \phi_2, \phi_0, 0), \\
&(-3\phi_0 - \phi_1 - 3\phi_2, 0, -4\phi_0, -\phi_1, -3\phi_2),
\end{aligned}$$

the first two of which are cone roots. This example is worth mentioning for two reasons. First, it is a correction of [Gug17b], which lists 5 roots; after investigation, it seems our fourth root is missed by `AlVin`. Second, the dihedral angle between the hyperplanes of the third and fourth roots is $\pi/14$, which is relatively high: a rare sight in practice.

7.6.3 Extending two families of Bogachev & Perepechko

Let $U := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Let also A_k denote the $k \times k$ matrix with 2s on the diagonal and -1s on the upper and lower diagonals, and I_k the identity $k \times k$ matrix. The matrix U is of signature $(1, 1)$, and both I_k and A_k are positive definite (A_k is the Gram matrix of the irreducible spherical root system of type A_k). It follows that each matrix $M_k := -2 \oplus A_2 \oplus I_k$ is admissible over the field $\mathbb{Q}$.

In [BP18], Bogachev and Perepechko prove that M_k is reflective for $k = 0, \ldots, 4$. In fact, it turns out that M_k is reflective for $k \leq 8$, and not reflective for $k = 9$. By a result of Bugaenko [Bug92], non-reflectivity of M_k implies non-reflectivity of $M_l = M_k \oplus I(l-k)$ for all $l \geq k$, thus, reflectivity of the family M_k is entirely classified. A fundamental cell of M_8 has 20 sides, and after listing the 16 first hyperplanes of a cell of M_9, an infinite order symmetry can be found.

Another family of lattices investigated in [BP18] is $-k \oplus A_3$ for small values of k. The resulting number of sides (∞ being equivalent to non-reflectivity) is given below. All non-reflective entries, and those with $k \geq 16$ have been computed using *Vinny*. For the non-reflective case, no more than the 20 first hyperplanes were necessary.

k	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
#sides	4	5	5	4	6	6	∞	7	9	12	∞	5	∞	∞	12	5	∞	12

7.7 What (could be) next

Decidability. Given an admissible Lorentzian R-lattice L, the first obstacle to deciding reflectivity of $\tilde{\mathcal{O}}^L$ amounts to solving a system of cubic diophantine equations in R. On one hand, it is known that *arbitrary* systems of quadratic diophantine equations are undecidable. On the other, the equations appearing in our context are restricted enough that they may be decidable. The second obstacle is the potential absence of a vertex of the fundamental cell of $\tilde{\mathcal{O}}_r^L$. We should either prove this to never be the case, or find an example.

Non-principal number rings. In the implementation *Vinny*, the fact that R is a PID (hence a unique factorization domain) is relatively important, since it allows the choice of unique representatives for roots of hyperplanes. It is probable that, with some care, the algorithm could also be adapted to work over non-principal rings.

New high dimensional reflective [cocompact] lattices. The original motivation behind *Vinny* was the search for a new compact Coxeter polyhedron in dimension $7, 8$ or even 9. Quite a few lattices of a similar type to those used by Bugaenko have been tried, but no reflective one has been found so far.

Performance. *Vinny* is not especially performant, and is in practice useless for matrices that are either of high dimension, far from diagonal, or defined over "complicated" number rings. I believe that with some architectural changes to the code, relatively large speed-ups can be achieved. This may or may not be useful in practice.

Bibliography

[All13] Daniel Allcock. "Reflection centralizers in Coxeter groups". In: *Transform. Groups* 18.3 (2013), pp. 599–613. ISSN: 1083-4362. DOI: 10.1007/s00031-013-9236-7. URL: https://doi.org/10.1007/s00031-013-9236-7.

[Ava61] S. P. Avann. "Metric ternary distributive semi-lattices". In: *Proc. Amer. Math. Soc.* 12 (1961), pp. 407–414. ISSN: 0002-9939. DOI: 10.2307/2034206. URL: https://doi.org/10.2307/2034206.

[Ban84] H.-J. Bandelt. "Retracts of hypercubes". In: *J. Graph Theory* 8.4 (1984), pp. 501–510. ISSN: 0364-9024. DOI: 10.1002/jgt.3190080407. URL: https://doi.org/10.1002/jgt.3190080407.

[BB21] N. V. Bogachev and R. Bottinelli. *Vinny*. https://github.com/bottine/VinbergsAlgorithmNF/. 2021.

[BC08] Hans-Jürgen Bandelt and Victor Chepoi. "Metric graph theory and geometry: a survey". In: *Surveys on discrete and computational geometry*. Vol. 453. Contemp. Math. Amer. Math. Soc., Providence, RI, 2008, pp. 49–86. DOI: 10.1090/conm/453/08795. URL: https://doi.org/10.1090/conm/453/08795.

[Bc17] Seth Bromberger and other contributors. *JuliaGraphs/LightGraphs.jl*. Version v1.1.0. Sept. 2017. DOI: 10.5281/zenodo.1412141. URL: https://doi.org/10.5281/zenodo.1412141.

[Bez+17] Jeff Bezanson et al. "Julia: A fresh approach to numerical computing". In: *SIAM review* 59.1 (2017), pp. 65–98. URL: https://doi.org/10.1137/141000671.

[BH62] Armand Borel and Harish-Chandra. "Arithmetic subgroups of algebraic groups". In: *Ann. of Math. (2)* 75 (1962), pp. 485–535. ISSN: 0003-486X. DOI: 10.2307/1970210. URL: https://doi.org/10.2307/1970210.

[BK20] Rémi Bottinelli and Tom Kaiser. *The first uniformly finite homology group with coefficients in* $\mathbb{Z}$ *and a characterisation of its vanishing in the transitive case*. 2020. arXiv: 2001.04857 [math.CO].

[BK21] Rémi Bottinelli and Tom Kaiser. "Magnitude Homology, Diagonality, and Median Spaces". In: *Homology Homotopy Appl.* 23.2 (2021), pp. 121–140. URL: http://dx.doi.org/10.4310/HHA.2021.v23.n2.a7.

[BM83] Hans-J. Bandelt and Henry Martyn Mulder. "Infinite median graphs, (0, 2)-graphs, and hypercubes". In: *J. Graph Theory* 7.4 (1983), pp. 487–497. ISSN: 0364-9024. DOI: 10.1002/jgt.3190070415. URL: https://doi.org/10.1002/jgt.3190070415.

[BP17] N. V. Bogachev and A. Y. Perepechko. *VinAl*. https://github.com/aperep/vinal. 2017.

[BP18] N. V. Bogachev and A. Yu. Perepechko. "Vinberg's algorithm for hyperbolic lattices". In: *Mat. Zametki* 103.5 (2018), pp. 769–773. ISSN: 0025-567X. DOI: 10.4213/mzm11889. URL: https://doi.org/10.4213/mzm11889.

[BS97] I. Benjamini and O. Schramm. "Every graph with a positive Cheeger constant contains a tree with a positive Cheeger constant". In: *Geom. Funct. Anal.* 7.3 (1997), pp. 403–419. ISSN: 1016-443X. DOI: 10.1007/PL00001625. URL: https://doi.org/10.1007/PL00001625.

[Bug92] V. O. Bugaenko. "Arithmetic crystallographic groups generated by reflections, and reflective hyperbolic lattices". In: *Lie groups, their discrete subgroups, and invariant theory.* Vol. 8. Adv. Soviet Math. Amer. Math. Soc., Providence, RI, 1992, pp. 33–55. DOI: 10.1007/bf00883263. URL: https://doi.org/10.1007/bf00883263.

[BW92] Jonathan Block and Shmuel Weinberger. "Aperiodic tilings, positive scalar curvature and amenability of spaces". In: *J. Amer. Math. Soc.* 5.4 (1992), pp. 907–918. ISSN: 0894-0347. DOI: 10.2307/2152713. URL: https://doi.org/10.2307/2152713.

[CDH10] Indira Chatterji, Cornelia Druţu, and Frédé ric Haglund. "Kazhdan and Haagerup properties from the median viewpoint". In: *Adv. Math.* 225.2 (2010), pp. 882–921. ISSN: 0001-8708. DOI: 10.1016/j.aim.2010.03.012. URL: https://doi.org/10.1016/j.aim.2010.03.012.

[Che00] Victor Chepoi. "Graphs of some CAT(0) complexes". In: *Adv. in Appl. Math.* 24.2 (2000), pp. 125–179. ISSN: 0196-8858. DOI: 10.1006/aama.1999.0677. URL: https://doi.org/10.1006/aama.1999.0677.

[Dia15] Francesca Diana. *Aspects of uniformly finite homology.* Jan. 2015. URL: https://epub.uni-regensburg.de/31224/.

[DS87] Andreas W. M. Dress and Rudolf Scharlau. "Gated sets in metric spaces". In: *Aequationes Math.* 34.1 (1987), pp. 112–120. ISSN: 0001-9054. DOI: 10.1007/BF01840131. URL: https://doi.org/10.1007/BF01840131.

[EP94] David B. A. Epstein and Carlo Petronio. "An exposition of Poincaré's polyhedron theorem". In: *Enseign. Math. (2)* 40.1-2 (1994), pp. 113–170. ISSN: 0013-8584.

[Fie+17] Claus Fieker et al. "Nemo/Hecke: Computer Algebra and Number Theory Packages for the Julia Programming Language". In: *Proceedings of the 2017 ACM on International Symposium on Symbolic and Algebraic Computation.* ISSAC '17. New York, NY, USA: ACM, 2017, pp. 157–164. DOI: 10.1145/3087604.3087611. URL: https://doi.acm.org/10.1145/3087604.3087611.

[Fur76] Harry Furstenberg. "A note on Borel's density theorem". In: *Proc. Amer. Math. Soc.* 55.1 (1976), pp. 209–212. ISSN: 0002-9939. DOI: 10.2307/2041874. URL: https://doi.org/10.2307/2041874.

[Gug17a] R. Guglielmetti. *AlVin.* https://github.com/rgugliel/alvin. 2017.

[Gug17b] Rafael Guglielmetti. "Hyperbolic isometries in (in-) finite dimensions and discrete reflection groups: theory and computations". PhD thesis. é, 2017.

[HJP13] W. Hart, F. Johansson, and S. Pancratz. *FLINT: Fast Library for Number Theory.* Version 2.4.0, http://flintlib.org. 2013.

[HW17] Richard Hepworth and Simon Willerton. "Categorifying the magnitude of a graph". In: *Homology Homotopy Appl.* 19.2 (2017), pp. 31–60. ISSN: 1532-0073. DOI: 10.4310/HHA.2017.v19.n2.a3. URL: https://doi.org/10.4310/HHA.2017.v19.n2.a3.

[Jub18] Benoit Jubin. *On the magnitude homology of metric spaces.* 2018. arXiv: arXiv:1803.05062 [math.MG].

[KY18] Ryuki Kaneta and Masahiko Yoshinaga. *Magnitude homology of metric spaces and order complexes.* 2018. arXiv: arXiv:1803.04247 [math.AT].

[Lei] Tom Leinster. *Magnitude: a bibliography.* https://www.maths.ed.ac.uk/~tl/magbib/. [Accessed October 1, 2020].

[Lei13] Tom Leinster. "The magnitude of metric spaces". In: *Doc. Math.* 18 (2013), pp. 857–905. ISSN: 1431-0635. DOI: 10.1007/s00500-013-1105-8. URL: https://doi.org/10.1007/s00500-013-1105-8.

[LS17] Tom Leinster and Michael Shulman. *Magnitude homology of enriched categories and metric spaces.* 2017. arXiv: arXiv:1711.00802 [math.AT].

[Mit01] Paul D Mitchener. "Coarse homology theories". In: *Algebraic & Geometric Topology* 1.1 (2001), pp. 271–297.

[Mos03] Lee Mosher. *Homology and dynamics in quasi-isometric rigidity of once-punctured mapping class groups.* 2003. arXiv: math/0308065 [math.GR].

[Rat06] John G. Ratcliffe. *Foundations of hyperbolic manifolds.* Second. Vol. 149. Graduate Texts in Mathematics. Springer, New York, 2006, pp. xii+779. ISBN: 978-0387-33197-3; 0-387-33197-2.

[Rat19] John G. Ratcliffe. *Foundations of hyperbolic manifolds.* Third. Vol. 149. Graduate Texts in Mathematics. Springer, Cham, [2019] ©2019, p. 800. ISBN: 978-3-030-31597-9; 978-3-030-31596-2. DOI: 10.1007/978-3-030-31597-9. URL: https://doi.org/10.1007/978-3-030-31597-9.

[Roe03] John Roe. *Lectures on coarse geometry.* 31. American Mathematical Soc., 2003.

[Sam67] Pierre Samuel. *Théorie algébrique des nombres.* Hermann, Paris, 1967, p. 130.

[San+21] David P. Sanders et al. *JuliaIntervals/IntervalArithmetic.jl: v0.18.2.* Version v0.18.2. May 2021. DOI: 10.5281/zenodo.4739394. URL: https://doi.org/10.5281/zenodo.4739394.

[TAL21] Mathieu Tanneau, Miguel F. Anjos, and Andrea Lodi. "Design and implementation of a modular interior-point solver for linear optimization". en. In: *Mathematical Programming Computation* (Feb. 2021). ISSN: 1867-2957. DOI: 10.1007/s12532-020-00200-8. URL: https://doi.org/10.1007/s12532-020-00200-8 (visited on 03/07/2021).

[Vin72] È. B. Vinberg. "The groups of units of certain quadratic forms". In: *Mat. Sb. (N.S.)* 87(129) (1972), pp. 18–36.

[Vin84] È. B. Vinberg. "Absence of crystallographic groups of reflections in Lobachevskĭ ı spaces of large dimension". In: *Trudy Moskov. Mat. Obshch.* 47 (1984), pp. 68–102, 246. ISSN: 0134-8663.