Unitarizability

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Contents

I Notes

Introduction

This essay compiles some results on the theme of group *unitarizability*, which is the property that any uniformly bounded representation of a given group can be made unitary, via conjugation by an invertible operator.

The first section introduces the concept and the behaviour of unitarizability with respect to some basic group constructions (subgroup, extension and directed union), and shows that amenable groups are unitarizable (proved in [NT51] by adapting an argument of [Sz 47], which only showed this fact for $\mathbb{Z}$).

The second section, following chapter two of [Pis01], defines the two function spaces $B(G)$ and $T_p(G)$, for a group G (and a real number $p \in [1, \infty)$); the spaces are verified to be Banach under suitable norms, and inclusions with the spaces $l^p(G)$ are shown.

The third section, still following chapter two of [Pis01], uses those spaces to describe the following condition on unitarizability: "If G is unitarizable, then $T_1(G) \subseteq B(G)$", and culminates with the fact that any group containing a free subgroup of rank ≥ 2 is not unitarizable. *En passant*, the space $T_p(G)$ is seen to be related to amenability, in a direct way: "G is amenable if and only if $T_p(G) \subseteq l^p(G)$ for all $p \in [1, \infty)$".

The fourth section, mostly based on [Oza11], expands on a specific form of group representation, similar to one constructed in the third section, which allows the strengthening of our condition on unitarizability to: "If G is unitarizable, then $T_1(G) \subseteq l^2(G)$".

The fifth section is mostly independent; it relates unitarizability with group actions on metric spaces of Hilbert operators, as done in [Sch15].

Finally, the last section first introduces an invariant of countable groups, in the form of an element of $[0, \infty]$: the *cost*. The section starts by setting the stage, and verifying some simple properties of the cost (e.g. finitely generated groups have finite cost) and showing a "multiplicativity" property of the cost on finite index subgroups. Then, following [EM09] and using the above condition on unitarizability, a last condition is derived: "If G is a finitely generated, residually finite, unitarizable group, its cost is at most 1".

Note on the appendix

The appendix essentially serves two purposes. The first is to regroup what I deemed to be non obvious/standard results on the various subjects used in the essay (Banach and Hilbert spaces, measure theory, etc.). Some of those will be proved, and some only provided with suitable references. The second is as a reference of sorts for (some of) the terminology and concepts used throughout the essay. In any case, the appendix shouldn't be read linearly.

Notation & Conventions *en Vrac*

All vector spaces are over the base field $\mathbb{C}$.

For normed vector spaces X, Y, let $B(X,Y)$ denote the normed space of bounded linear maps from X to Y, and $B^I(X,Y)$ the subset of $B(X,Y)$ of invertible such maps with a bounded inverse. If X, Y are Banach spaces, then $B^I(X,Y)$ is just the set of bijective elements of $B(X,Y)$, by the Open Mapping Theorem (Proposition A.1). If $X = Y$, $B(X,X)$ (resp. $B^I(X,X)$) will be abbreviated $B(X)$ (resp. $B^I(X)$). An element $T \in B(X,Y)$ is an *isometry* if $\|Tx\|_Y = \|x\|_X$ for all $x \in X$.

If X and Y are Hilbert spaces, then $T \in B(X,Y)$ is *unitary* if it is an isometry and bijective. By the polarisation identities (Proposition B.1), being an isometry is equivalent to preserving the inner product.

Given a (always discrete) group G and a Hilbert space X, a *representation* of G in X is a morphism of groups $\pi : G \to B^I(X)$. Note that given a map $\pi : G \to B(X)$ such that $\pi(e_G) = I_X$ and $\pi(g)\pi(h) = \pi(gh)$ $\forall g, h \in G$, then π has range in $B^I(X)$ and is a representation.

A representation $\pi : G \to B^I(X)$ is said to be *uniformly bounded* if there exists some constant C with

$$\|\pi(g)\| \leq C, \quad \forall g \in G,$$

in which case $|\pi|$ is to be taken as $\sup_g \|\pi(g)\|$. The representation is *unitary* if $\pi(g)$ is unitary for each g.

For any set I the subspace of $\mathbb{C}^I$ of maps with *finite* support is written $\mathbb{C}^{(I)}$, or sometimes $\mathbb{C}[I]$, when I forms a group.

If $f : X \to Y$ is any map, we use square brackets to denote the induced map on the powersets, as follows: $f[A] := \{f(x) \mid x \in A\}$, and $f^{-1}[B] := \{x \in X \;:\; f(x) \in B\}$. The graph of f, written $\Gamma(f)$, is the set $\{(x, f(x)) \mid x \in X\} \subseteq X \times Y$.

If $f : X \to \mathbb{C}$ has domain $\mathbb{C}$, we write $f \geq 0$ or say that f is positive, to mean that $f(x) \in \mathbb{R}^{\geq 0}$ for all $x \in X$.

If X is any set, $A \subseteq_f X$ means that A is a finite subset of X.

Mirroring the notation $A \subseteq_f X$ for finite subsets, we write $\mathcal{P}_f(X)$ for the set of finite subsets of X.

If $f : X \to Y$ and $g : X \to Z$ are two maps, we use $\langle f, g \rangle : X \to Y \times Z$ to denote the map sending $x \in X$ to $(f(x), g(x)) \in Y \times Z$, and if $g : W \to Z$, we use $f \times g : X \times W \to Y \times Z$ for the map sending (x, w) to $(f(x), g(w))$.

For any set X, Δ_X is the diagonal in X^2, that is, the set $\{(x, x) \mid x \in X\} \subseteq X^2$.

1 Unitarizability

This section first introduces the notion of *unitarizability* for operators and representations. After verifying the equivalence of two formulations of unitarizability, amenable groups are shown to be unitarizable. The section ends with "positive" properties of unitarizability: subgroups of unitarizable groups are unitarizable, as are extensions of amenable groups by unitarizable groups, and directed unions of unitarizable groups, when a specific uniformity constraint holds.

The proof that amenable groups are unitarizable follows [NT51] and [Sz 47], but is also done in [Pis01]. The fact that unitarizability passes to subgroups is from [Pis01, Theorem 2.8*], and the two "extension" properties of unitarizability are from [Pis05].

1.1 Basics

Definition 1.1 (Unitarizability)**.** Let H be a Hilbert space, and $T \in B(H)$. The operator T is said to be *unitarizable* if there exists some $Q \in B^I(H)$ such that QTQ^{-1} is unitary.

Note that in this definition, T must lie in $B^I(H)$, because if QTQ^{-1} is unitary, it is invertible, and so is $T = Q^{-1}(QTQ^{-1})Q$. Also, we can require of Q to be self-adjoint, because any $Q \in B^I(H)$ has a so-called *polar* decomposition as $Q = UP$ with U unitary and P self-adjoint (Proposition B.12). Then, P is also invertible (of inverse $Q^{-1}U$) and

$$PTP^{-1} = U^{-1}(UPTP^{-1}U^{-1})U = U^{-1}(QTQ^{-1})U$$

is a composition of unitary maps, hence is itself unitary.

Lemma 1.2. *Let H be a Hilbert space, and $T \in B(H)$. The operator T is unitarizable if and only if there exists an inner product $\rho : H \times H \to \mathbb{C}$ making T unitary and such that ρ and $\langle \cdot, \cdot \rangle$ (the original inner product) induce compatible norms on H.*

Two norms $\| \cdot \|_1, \| \cdot \|_2$ are said to be compatible if there exist $0 < k, K < \infty$ with $k\|x\|_1 \leq \|x\|_2 \leq K\|x\|_1$ for all x. This implies that the induced topologies are the same, and is in fact equivalent.

Proof. Assume T is unitarizable, and let $Q \in B^I(H)$ such that QTQ^{-1} is unitary. Define $\rho : H \times H \to \mathbb{C}$ by

$$\rho(x, y) := \langle Qx, Qy \rangle.$$

Then, for all $x, x', y \in H$ and $\alpha \in \mathbb{C}$:

- $\rho(x, x) = \langle Qx, Qx \rangle \in \mathbb{R}^{\geq 0}$ and $\rho(x, x) = \langle Qx, Qx \rangle = 0$ if and only if $x = 0$ (using the fact that Q is injective);
- $\rho(\alpha x + x', y) = \langle Q(\alpha x + x'), Qy \rangle = \langle \alpha Qx + Qx', Qy \rangle = \alpha \langle Qx, Qy \rangle + \langle Qx', Qy \rangle = \alpha\rho(x, y) + \rho(x', y)$;
- $\rho(y, x) = \langle Qy, Qx \rangle = \overline{\langle Qx, Qy \rangle} = \overline{\rho(x, y)}$;

so that ρ really is an inner product. Furthermore, fix $x, y \in H$:

$$\rho(Tx, Ty) = \langle QTx, QTy \rangle = \langle QTQ^{-1}(Qx), QTQ^{-1}(Qy) \rangle \overset{(*)}{=} \langle Qx, Qy \rangle = \rho(x, y),$$

where $(*)$ uses unitariness of QTQ^{-1}; this shows T to be an isometry relative to ρ. As $T = Q^{-1}(QTQ^{-1})Q$ is a composition of bijective maps, it itself is, and we conclude that T is unitary, with respect to ρ. Finally, let $\| \cdot \|_\rho$ denote the ρ-norm. We have, for all x,

$$\|x\|_\rho = \rho(x, x)^{\frac{1}{2}} = \langle Qx, Qx \rangle^{\frac{1}{2}} \leq (\|Q\|\|Q\|\langle x, x \rangle)^{\frac{1}{2}} = \|Q\|\|x\|,$$

and

$$\|x\| = \|Q^{-1}Qx\| \leq \|Q^{-1}\|\|Qx\| = \|Q^{-1}\|\langle Qx, Qx \rangle^{\frac{1}{2}} = \|Q^{-1}\|\rho(x, x)^{\frac{1}{2}} = \|Q^{-1}\|\|x\|_\rho,$$

so that, for all x:

$$\frac{1}{\|Q^{-1}\|}\|x\| \leq \|x\|_\rho \leq \|Q\|\|x\|,$$

and the norms are compatible.

Now, assume there is a compatible inner product ρ on H making T unitary. Let us denote by H_ρ the Hilbert space defined by ρ. The spaces H and H_ρ have orthonormal bases of same cardinality by Proposition B.7, so that there exists a unitary map (by Proposition B.6):

$$Q : H_\rho \to H.$$

But then, $QTQ^{-1} : H \to H$ is a composition of unitary maps, hence is itself unitary.

More precisely, let $\tilde{Q} : H \to H$ denote the operator $H \to H$ induced by Q (it is the same map, but viewed as an operator on different spaces). The map $\tilde{Q}$ is then not necessarily unitary, but still continuous and linear, since ρ and $\langle \cdot, \cdot \rangle$ induce compatible norms. Similarly, let $\tilde{T} : H_\rho \to H_\rho$ be the operator induced by T. We know that $\tilde{T}$ and Q are unitary, so that $Q\tilde{T}Q^{-1}$ is unitary. But $\tilde{Q}T\tilde{Q}^{-1} = Q\tilde{T}Q^{-1}$, and $\tilde{Q}$ makes T unitary. ☐

Definition 1.3. A representation $\pi : G \to B^I(H)$ is said to be *unitarizable*, if there exists some invertible operator $S \in B^I(H)$ such that $S\pi(g)S^{-1}$ is unitary, for all $g \in G$, that is, such that $S\pi(\cdot)S^{-1}$ is a unitary representation.

Note that any unitarizable representation is uniformly bounded. Indeed, if $\pi : G \to B^I(H)$ is a representation and $S \in B^I(H)$ is such that $S\pi(\cdot)S^{-1}$ is unitary, then

$$\|\pi(g)\| = \|S^{-1}S\pi(g)S^{-1}S\| \leq \|S^{-1}\|\|S\pi(\cdot)S^{-1}\|\|S\| \leq \|S\|\|S^{-1}\|, \qquad \forall g \in G.$$

Lemma 1.4. *A representation $\pi : G \to B^I(H)$ is unitarizable if and only if there exists an inner product ρ on H, inducing a compatible norm to the original one, and making $\pi(g)$ unitary, for all g.*

Proof. If $S \in B^I(H)$ is such that $S\pi(g)S^{-1}$ is unitary for all g, consider the inner product defined by $\rho(x, y) := \langle Sx, Sy \rangle$, and argue as in Lemma 1.2. If ρ is a compatible inner product making π unitary, consider the operator $Q : H_\rho \to H$ given by a bijection on the orthonormal bases of H_ρ and H, and argue, again, as in Lemma 1.2. ☐

The following fact comes from [Pis07, p. 3].

Lemma 1.5. *Let $\pi : G \to B^I(H)$ be a representation. Then π is unitary if and only if $|\pi| = 1$.*

Proof. If π is unitary, then by definition $\pi(t)$ is unitary for all t, and in particular $\|\pi(t)\| = 1$ for all t, which implies $|\pi| = 1$. If $|\pi| = 1$, then for all t, $\|\pi(t)\| \leq 1$, and

$$\begin{aligned} \|\pi(t)x\| &\leq \|\pi(t)\|\|x\| \leq \|x\|, \\ \|x\| &= \|\pi(t^{-1})\pi(t)x\| \leq \|\pi(t^{-1})\|\|\pi(t)x\| \leq \|\pi(t)x\| \end{aligned}$$

so that $\|\pi(t)x\| = \|x\|$ for all t and x. This shows that $\pi(t)$ is an isometry for all t, and being bijective by construction, unitary. ☐

1.2 Amenability

Theorem 1.6 ([Pis01, Theorem 0.6]; Day, Dixmier, Nakamura–Takeda)**.** *If G is an amenable group, and π a uniformly bounded representation of G in H, of bound $|\pi| = K < \infty$, then π is unitarizable.*

Proof. Fix an amenable group G, and a uniformly bounded representation $\pi : G \to B^I(H)$, with $|\pi| = K < \infty$. For any pair x, y of elements of H, define:

$$\begin{aligned} \varphi_{xy} : G &\to \mathbb{C} \\ g &\mapsto \langle \pi(g^{-1})x, \pi(g^{-1})y \rangle, \end{aligned}$$

The map $\varphi_{x,y}$ is bounded (that is, an element of $l^\infty(G)$):

$$\sup_{g\in G} |\varphi_{xy}(g)| = \sup_{g\in G} |\langle \pi(g^{-1})x, \pi(g^{-1})y \rangle| \leq \sup_{g\in G} \|\pi(g^{-1})\|\|\pi(g^{-1})\|\|x\|\|y\| \leq \sup_{g\in G} K^2\|x\|\|y\| = K^2\|x\|\|y\|.$$

Let m be a left-invariant mean on G, as given by amenability. Define

$$\begin{aligned} \rho : H \times H &\to \mathbb{C} \\ (x, y) &\mapsto m(\varphi_{xy}) \end{aligned}$$

and note that this is well-defined, because φ_{xy} is in the domain of m, being bounded.

From here, there are two slightly different possible paths, corresponding to the two equivalent formulations of unitarizability of Lemma 1.4; we follow both:

ρ as an inner product. We argue that ρ is an inner product compatible with the original one on H, and that it makes each $\pi(g)$ unitary.

- First, note that $\varphi_{x,x}(g) \in \mathbb{R}^{\geq 0}$ for all g, because so is $\langle \pi(g^{-1})x, \pi(g^{-1})x \rangle$; this implies that $\rho(x,x) = m(\varphi_{x,x}) \in \mathbb{R}^{\geq 0}$ (by positivity of m, Definition C.3(4)). If $x = 0$, then $\varphi_{x,x} \equiv 0$ and $\rho(x,x) = 0$. Conversely, given x and g, we have

$$\langle x, x \rangle = \langle \pi(g)\pi(g^{-1})x, \pi(g)\pi(g^{-1})x \rangle \leq \|\pi(g)\|^2 \langle \pi(g^{-1})x, \pi(g^{-1})x \rangle \leq K^2 \langle \pi(g^{-1})x, \pi(g^{-1})x \rangle.$$

  This implies that if $x \neq 0$, then $\langle \pi(g^{-1})x, \pi(g^{-1})x \rangle \geq K^{-2}\langle x, x \rangle$ for all g, and by monotonicity of m (Definition C.3(3) and Definition C.3(4)),

$$\rho(x,x) = m(\varphi_{x,x}) \geq m(\langle x, x \rangle / K^2) = \langle x, x \rangle / K^2 > 0.$$

- For linearity in the first component, we have:

$$\begin{aligned}
\rho(\alpha x + x', y) = m(\varphi_{\alpha x + x', y}) &= m\left(g \mapsto \langle \pi(g^{-1})(\alpha x + x'), y \rangle\right) \\
&= m\left(g \mapsto \langle \alpha \pi(g^{-1})x + \pi(g^{-1})x', y \rangle\right) \\
&= m\left(g \mapsto \alpha \langle \pi(g^{-1})x, y \rangle + \langle \pi(g^{-1})x', y \rangle\right) \\
&= m\left(\alpha \varphi_{x,y} + \varphi_{x',y}\right) \\
&= \alpha m(\varphi_{x,y}) + m(\varphi_{x',y}) = \alpha \rho(x,y) + \rho(x',y).
\end{aligned}$$

- And finally for conjugate symmetry, we have:

$$\begin{aligned}
\rho(y,x) &= m(g \mapsto \langle \pi(g^{-1})y, \pi(g^{-1})x \rangle) = m(g \mapsto \overline{\langle \pi(g^{-1})x, \pi(g^{-1}y \rangle}))) \\
&= m(\overline{g \mapsto \langle \pi(g^{-1})x, \pi(g^{-1}y \rangle})) = \overline{m(g \mapsto \langle \pi(g^{-1})x, \pi(g^{-1}y \rangle))} = \overline{\rho(x,y)}.
\end{aligned}$$

And ρ makes T unitary:

$$\begin{aligned}
\rho(\pi(h)x, \pi(h)y) = m(\varphi_{\pi(h)x, \pi(h)y}) &= m\left(g \mapsto \langle \pi(g^{-1})\pi(h)x, \pi(g^{-1})\pi(h)y \rangle\right) \\
&= m\left(g \mapsto \langle \pi(g^{-1}h)x, \pi(g^{-1}h)y \rangle\right) \\
&= m\left(g \mapsto h^{-1}g \mapsto \langle \pi((h^{-1}g)^{-1})x, \pi((h^{-1}g)^{-1})y \rangle\right) \\
&= m\left(\delta_h * \varphi_{x,y}\right) \\
&= m(\varphi_{x,y}) = \rho(x,y),
\end{aligned}$$

where the penultimate equality is due to shift invariance (Definition C.3(5)). This shows that $\pi(h)$ is an isometry for each h, and since it is bijective by definition, we conclude that it is unitary with respect to ρ. We now verify that ρ induces the same topology as $\langle \cdot, \cdot \rangle$. We have

$$\langle \pi(g^{-1})x, \pi(g^{-1})x \rangle \leq \|\pi(g^{-1})\|^2 \langle x, x \rangle \leq K^2 \langle x, x \rangle,$$

and

$$\langle x, x \rangle = \langle \pi(g)\pi(g^{-1})x, \pi(g)\pi(g^{-1})x \rangle \leq \|\pi(g)\|^2 \langle \pi(g^{-1})x, \pi(g^{-1})x \rangle \leq K^2 \langle \pi(g^{-1})x, \pi(g^{-1})x \rangle,$$

for all g and x, so that

$$K^{-2}\langle x, x \rangle \leq \langle \pi(g^{-1})x, \pi(g^{-1})x \rangle \leq K^2 \langle x, x \rangle,$$

for all g and x. Applying m, we get, by monotonicity,

$$m(g \mapsto K^{-2}\langle x, x \rangle) \leq m(g \mapsto \langle \pi(g^{-1})x, \pi(g^{-1})x \rangle) \leq m(g \mapsto K^2 \langle x, x \rangle), \quad \forall x,$$

and since m evaluates constants to themselves, and by definition of ρ,

$$K^{-2}\langle x, x \rangle \leq \rho(x,x) \leq K^2 \langle x, x \rangle, \quad \forall x, \tag{1}$$

which shows that the norms induced by ρ and $\langle \cdot, \cdot \rangle$ are compatible.

Since ρ is an inner product on H, compatible with $\langle \cdot, \cdot \rangle$ and making $\pi(g)$ unitary for all g, we conclude that π is unitarizable.

ρ as inducing an operator. We show that ρ is sesquilinear and bounded, which implies the existence of some $S \in B(H)$ with $\rho(x,y) = \langle Sx, y\rangle$ for all x, y. From S, an operator Q with $Q\pi(\cdot)Q^{-1}$ unitary is constructed.

We first verify that ρ is bounded:

$$\begin{aligned}
\sup_{\|x\|=1=\|y\|} |\rho(x,y)| = \sup_{\|x\|=1=\|y\|} |m(\varphi_{xy})| &\leq \sup_{\|x\|=1=\|y\|} \|\varphi_{xy}\| \\
&= \sup_{\|x\|=1=\|y\|} \sup_{g\in G} \|\langle \pi(g^{-1})x, \pi(g^{-1})y\rangle\| \\
&\leq \sup_{\|x\|=1=\|y\|} \sup_{g\in G} \|\pi(g^{-1})\|^2 \|x\|\|y\| \\
&\leq \sup_{\|x\|=1=\|y\|} \sup_{g\in G} K^2 = K^2,
\end{aligned}$$

where the first inequality is due to m having operator norm 1. For sesquilinearity, refer to the first part of the proof. As ρ is sesquilinear and bounded, there exists a unique $S \in B(H)$ with $\rho(x,y) = \langle Sx, y\rangle$ for all x, y (Proposition B.2). Also, $\rho(y,x) = \overline{\rho(x,y)}$ (by the other part), so that

$$\langle x, Sy\rangle = \overline{\langle Sy, x\rangle} = \overline{\rho(y,x)} = \overline{\overline{\rho(x,y)}} = \langle Sx, y\rangle$$

and S is self-adjoint (actually, S is positive, since $\rho(x,x) \in \mathbb{R}^{\geq 0}$ for all x).

Recall that Equation (1) tells us

$$K^{-2}\langle x, x\rangle \leq \rho(x,x) \leq K^2\langle x, x\rangle, \qquad \forall x,$$

that is,

$$K^{-2}\langle x, x\rangle \leq \langle Sx, x\rangle \leq K^2\langle x, x\rangle, \qquad \forall x,$$

and in particular S is positive and invertible (Proposition B.14). Now, we can take the positive self-adjoint root Q of S, which will satisfy (Proposition B.15)

$$K^{-1}\langle x, x\rangle \leq \langle Qx, x\rangle \leq K\langle x, x\rangle, \qquad \forall x,$$

and in particular be invertible.

It remains to show that $Q\pi(\cdot)Q^{-1}$ is unitary: First, note that

$$\begin{aligned}
\langle \pi(g)^* S\pi(g)x, y\rangle = \langle S\pi(g)x, \pi(g)y\rangle = \rho(\pi(g)x, \pi(g)y) &= m\left(t \mapsto \langle \pi(t^{-1})\pi(g)x, \pi(t^{-1})\pi(g)y\rangle\right) \\
&= m\left(t \mapsto \langle \pi((g^{-1}t)^{-1})x, \pi((g^{-1}t)^{-1})y\right) \\
&= m\left(\delta_g * \varphi_{x,y}\right) = m(\varphi_{x,y}) = \rho(x,y) = \langle Sx, y\rangle,
\end{aligned}$$

for all x, y, so that $\pi(g)^* S\pi(g) = S$ for all g. Then,

$$(Q\pi(g)Q^{-1})^*(Q\pi(g)Q^{-1}) = {Q^{-1}}^*\pi(g)^* Q^* Q\pi(g)Q^{-1} = Q^{-1}\pi(g)^* QQ\pi(g)Q^{-1} = Q^{-1}\pi(g)^* S\pi(g)Q^{-1} = Q^{-1}SQ^{-1} = I,$$

using the fact that Q and Q^{-1} are self-adjoint and that Q^{-1} is the square root of S^{-1}. This shows that $Q\pi(g)Q^{-1}$ has inverse its adjoint, and being bijective, it is unitary, for any g. □

Note that since the operator Q satisfies $\langle Qx, x\rangle \leq K\langle x, x\rangle$ for all x, it has norm at most K, and the same can be said of Q^{-1} (Proposition B.14, Proposition B.18). This implies that any uniformly bounded representation of an amenable group can be made unitary, in a bounded way.

Definition 1.7 (Group unitarizability)**.** A group is *unitarizable* if any uniformly bounded representation of said group is itself unitarizable.

The above theorem can be reformulated concisely as:

Theorem 1.8. *Amenable groups are unitarizable.*

1.3 "Positive" Behaviour of Unitarizability

We know that amenability passes to subgroups, quotients, directed unions and extensions (as in, if $1 \to N \to G \to Q \to 1$ is a SES and N and Q are amenable, then so is G). The following propositions are partial equivalents for unitarizability.

Lemma 1.9. *The property of being unitarizable passes to subgroups.*

Proof. Using the next lemma (Lemma 1.10). If π is a uniformly bounded representation of F, there exists a uniformly bounded representation $\widehat{\pi}$ of G (extending π) which is unitarizable, by unitarizability of G, and then so is π. $\square$

Lemma 1.10 ([Pis01, Theorem 2.8*])**.** *Let G be a group, $F \leq G$ a subgroup and $\pi : F \to B^I(H)$ a uniformly bounded representation of F in H. There exists a Hilbert space $\widehat{H}$, an isometrical embedding $\iota : H \to \widehat{H}$, and a representation $\widehat{\pi} : G \to B^I(\widehat{H})$, satisfying:*

$$\forall t \in F: \quad \widehat{\pi}(t)[\iota[H]] \subseteq \iota[H], \quad \textit{and} \quad \iota \circ \pi(t) = \widehat{\pi}(t)|_{\iota(H)} \circ \iota,$$

and $|\widehat{\pi}| = |\pi|$. If $\widehat{\pi}$ is unitarizable, then so is π.

Proof. Fix G, $F \leq G$, H, and $\pi : F \to B^I(H)$ uniformly bounded. Choose $\{s_j : j \in J\}$ a system of representatives for the left cosets of F in G, that is, such that:

$$G = \bigsqcup_{j \in J} s_j F,$$

and assume without loss of generality that $s_0 = e$ (the neutral element of G).

Define the space $\widehat{H} := \bigoplus_{j \in J} H_j$, with $H_j := H$ for all $j \in J$, and consider the embedding $\iota = \iota_0 : H \to \widehat{H}$ mapping H to H_0 (which is clearly an isometry).

Note that for any $x \in G$ and $i \in J$, there exists a unique $j = j(x,i) \in J$ with $s_i^{-1} x s_j \in F$. Indeed, this requirement is equivalent to $s_j F = x^{-1} s_i F$ and the elements s_k being representatives of the left cosets, there exists a unique j with $s_j F = x^{-1} s_i F$. Note also that $j(x,\cdot) : J \to J$ is a bijection: if $j(x,i) = j(x,i')$, then $x^{-1} s_i F = s_j F = x^{-1} s_{i'} F$, so that $s_i F = s_{i'} F$ and $i = i'$ (injectivity); given a fixed j_0, there exists a unique i_0 with $s_{i_0} F = x s_{j_0} F$, i.e. $x^{-1} s_{i_0} F = s_{j_0} F$, and $j_0 = j(x, i_0)$ (surjectivity).

Define $\widehat{\pi} : G \to B^I(\widehat{H})$ by

$$\begin{aligned} \widehat{\pi}(x) : \widehat{H} &\to \widehat{H} \\ \langle f_i \rangle_i &\mapsto \langle \pi(s_i^{-1} x s_{j(x,i)})(f_{j(x,i)}) \rangle_i, \end{aligned}$$

for all $x \in G$.

$\widehat{\pi}$ is well-defined. Note here that $\pi(s_i^{-1} x s_{j(x,i)})$ is well-defined exactly because $j(x,i)$ makes the product $s_i^{-1} x s_{j(x,i)}$ and element of F, so that we can apply π to it. We first show that $\widehat{\pi}(x)(f)$ really is an element of $\widehat{H}$, for all $f \in \widehat{H}$:

$$\begin{aligned} \|\widehat{\pi}(x) f\|^2 = \sum_i \|\widehat{\pi}(x)(f)_i\|^2 &= \sum_i \|\pi(s_i^{-1} x s_{j(x,i)})(f_{j(x,i)})\|^2 \\ &\leq \sum_i \|\pi(s_i^{-1} x s_{j(x,i)})\|^2 \|f_{j(x,i)}\|^2 \\ &\leq \sum_i |\pi|^2 \|f_{j(x,i)}\|^2 \\ &= |\pi|^2 \sum_i \|f_{j(x,i)}\|^2 \\ &= |\pi|^2 \sum_i \|f_i\|^2 \leq |\pi|^2 \|f\|^2, \end{aligned}$$

which shows that $\widehat{\pi}(x)(f)$ is an element of $\widehat{H}$, of norm at most $|\pi| \|f\|$, and actually:

$$\|\widehat{\pi}(x)\| = \sup_{\|f\|=1} \|\widehat{\pi}(x)(f)\| \leq \sup_{\|f\|=1} |\pi| \|f\| \leq |\pi|$$

for all x, which shows that $\widehat{\pi}(x)$ is bounded, for all x. Taking a step further:

$$\sup_x \|\widehat{\pi}(x)\| \leq |\pi|$$

so that $\widehat{\pi}$ is uniformly bounded itself, of norm at most $|\pi|$ (i.e. $|\widehat{\pi}| \leq |\pi|$).

Now for the linearity of $\widehat{\pi}(x)$ (with $j := j(x,i)$ for conciseness):

$$(\widehat{\pi}(x)(\alpha f + g))_i = \pi(s_i^{-1} x s_j)(\alpha f_j + g_j) = \alpha \pi(s_i^{-1} x s_{j(x,i)})(f_j) + \pi(s_i^{-1} x s_{j(x,i)})(g_j) = \alpha \left(\widehat{\pi}(x) f\right)_i + \left(\widehat{\pi}(x) g\right)_i,$$

by linearity of π.

We conclude that $\widehat{\pi}(x) \in B(\widehat{H})$ for all $x \in G$. To show that $\widehat{\pi}$ is well-defined, it only remains to show that $\widehat{\pi}(x) \in B^I(\widehat{H})$ $\forall x$, i.e. that $\widehat{\pi}(x)$ is bijective, but that will be covered by the fact that $\widehat{\pi}$ is a representation:

$\widehat{\pi}$ **is a representation.** Note that if $x := e_G$, for any given i, $j(x,i) = i$: indeed, $s_i^{-1}e_G s_i = s_i^{-1}s_i = e_G \in F$. Therefore, for any $f \in \widehat{H}$:

$$\widehat{\pi}(e_G)(f)_i = \pi(s_i^{-1}e_G s_i)(f_i) = \pi(e_G)(f_i) = \pi(e_F)(f_i) = f_i,$$

so that $\widehat{\pi}(e_G) = I_{\widehat{H}}$. Also, for any $x, y \in G$, and $f \in \widehat{H}$:

$$\left(\widehat{\pi}(xy)(f)\right)_i = \pi(s_i^{-1}xys_j)(f_j), \qquad \text{where } s_i^{-1}xys_j \in F,$$

and

$$\begin{aligned}\left(\widehat{\pi}(x)\left(\widehat{\pi}(y)(f)\right)\right)_i &= \pi(s_i^{-1}xs_k)(\widehat{\pi}(y)(f)_k), && \text{where } s_i^{-1}xs_k \in F\\ &= \pi(s_i^{-1}xs_k)(\pi(s_k^{-1}ys_l)(f_l)), && \text{where } s_k^{-1}xs_l \in F\\ &= \pi(s_i^{-1}xs_ks_k^{-1}ys_l)(f_l)\\ &= \pi(s_i^{-1}xys_l)(f_l),\end{aligned}$$

and it suffices to show $l = j$ to conclude that $\widehat{\pi}(xy) = \widehat{\pi}(x)\widehat{\pi}(y)$. To see this, observe that

$$s_i^{-1}xys_l = (s_i^{-1}xs_k)(s_k^{-1}ys_l)$$

is a product of two elements of F, hence is itself in F. As j is the *unique* index with $s_i^{-1}xys_j \in F$ and l satisfies this condition, we have $j = l$.

As $\widehat{\pi}$ sends e_G to the identity and respects the group operation, we conclude that in particular, $\widehat{\pi}$ has range in $B^I(H)$, and, finally, that $\widehat{\pi}$ really is a representation of G in H, uniformly bounded by $|\pi|$.

$\widehat{\pi}$ **extends** π**.** Fix $t \in F$, and $f \in H$; we have:

$$\widehat{\pi}(t)(\iota f)_i = \pi(s_i^{-1}ts_j)((\iota f)_j), \quad \text{where } s_i^{-1}ts_j \in F.$$

If $i = 0$, then $s_i^{-1}ts_j = e_G ts_j \in F$ implies $s_j \in t^{-1}F = F$ (because $t \in F$), so that $j = 0$, and

$$\widehat{\pi}(t)(\iota f)_0 = \pi(t)\left((\iota f)_0\right) = \pi(t)f$$

by definition of ι. Conversely, if $\widehat{\pi}(t)(\iota f)_i \neq 0$, then $(\iota f)_j \neq 0$, and $j = 0$, so that, as before, $s_i^{-1}ts_j = s_i^{-1}t \in F$ implies $s_i^{-1} \in F$ and $i = 0$. This shows that $\widehat{\pi}(t)(\iota f)$ is zero at all indices, except at $i = 0$, in which case it is $\pi(t)f$, which means that $\widehat{\pi}(t)(\iota f) = \iota(\pi(t)f)$ for all $t \in F$ and $f \in H$, and we conclude:

$$\forall t \in F: \quad \widehat{\pi}(t)[\iota[H]] \subseteq \iota[H], \quad \text{and} \quad \iota \circ \pi(t) = \widehat{\pi}(t)|_{\iota(H)} \circ \iota.$$

Same norm. We already know that $|\widehat{\pi}| \leq |\pi|$. Recall the definition:

$$|\pi| = \sup_{t\in F}\|\pi(t)\| = \sup_{t\in F}\sup_{f\in H, \|f\|=1}\|\pi(t)f\|.$$

Fix some $t \in F$ and $f \in H$ of norm 1; we have

$$\|\pi(t)(f)\| = \|\iota\left(\pi(t)f\right)\| = \|\widehat{\pi}(t)(\iota f)\| \leq \|\widehat{\pi}(t)\|\|\iota f\| \leq |\widehat{\pi}|\|f\| = |\widehat{\pi}|,$$

(where we used the fact that $\iota : H \to \widehat{H}$ is an isometry, and that $\iota \circ \pi(t) = \pi'(t) \circ \iota$) so that taking the supremum over $t \in F$ and $f \in H$ of norm 1, we can conclude $|\pi| \leq |\widehat{\pi}|$.

Unitarizability. It remains to show that if $\widehat{\pi}$ is unitarizable, then so is π.

Assuming $\widehat{\pi}$ is unitarizable, there exists an inner product ρ on $\widehat{H}$, compatible with the original inner product, and making $\widehat{\pi}$ unitary; more precisely, making $\widehat{\pi}(t)$ an isometry for each $t \in G$. Define $\rho' : H \times H \to \mathbb{C}$ by $\rho'(f, g) := \rho(\iota f, \iota g)$. We show that ρ' is an inner product on H making π unitary and compatible with the original inner product on H. First, for any $f, f', g \in H$ and $\alpha \in \mathbb{C}$:

- $\rho'(f, f) = \rho(\iota f, \iota f) \geq 0$, and $\rho'(f, f) = 0$ if and only if $f = 0$ because ι is an isometry;
- $\rho'(\alpha f + f', g) = \rho(\iota(\alpha f + f'), \iota g) = \rho(\alpha\iota f + \iota f', \iota g) = \alpha\rho(\iota f, g) + \rho(\iota f', g) = \alpha\rho'(f, g) + \rho'(f', g)$;
- $\rho'(g, f) = \rho(\iota g, \iota f) = \overline{\rho(\iota f, \iota g)} = \overline{\rho'(f, g)}$,

which makes ρ' an inner product. Second,

$$\rho'(\pi(t)f, \pi(t)g) = \rho(\iota(\pi(t)f), \iota(\pi(t)g)) = \rho(\widehat{\pi}(t)(\iota f), \widehat{\pi}(t)(\iota g)) \overset{(*)}{=} \rho(\iota f, \iota g) = \rho'(f, g),$$

(where $(*)$ uses the unitariness of $\widehat{\pi}$ with respect to ρ), so that ρ' makes π unitary. Finally, we know that there exist $0 < k \leq K < \infty$ with

$$\rho(f, f) \in [k\|f\|^2, K\|f\|^2],$$

for all $f \in \hat{H}$ (that is, ρ induces a norm compatible with the original one on $\hat{H}$). Then, for all $f \in H$:

$$\rho'(f, f) = \rho(\iota f, \iota f) \in [k\|\iota f\|^2, K\|\iota f\|^2] = [k\|f\|^2, K\|f\|^2],$$

because ι is an isometry, and we are done. ☐

Lemma 1.11. *The property of being unitarizable passes to quotients.*

Proof. Let G be a unitarizable group and $Q := G/N$ for some normal subgroup N, with $q : G \to Q$ the quotient map. Fix a uniformly bounded representation $\pi : Q \to B^I(H)$ of Q. Then $\pi q : G \to B^I(H)$ is also uniformly bounded, hence unitarizable, and so is π. ☐

Lemma 1.12. *Given a group G and a normal subgroup $N \leq G$ of G, if N is unitarizable and G/N is amenable, then G is unitarizable.*

Proof. Fix a uniformly bounded representation $\pi : G \to B^I(H)$ of G on some Hilbert space H. If $\iota : N \hookrightarrow G$ is the inclusion, we know that $\pi\iota$ is unitarizable, and there exists some inner product $\rho_0 : H \times H \to \mathbb{C}$, topologically compatible with the original one and making $\pi\iota$ unitary. Similarly to Theorem 1.6, we construct a new inner product on H. Let $m : l^\infty(G/N) \to \mathbb{C}$ be a left-invariant mean on G/N, as given by amenability, and define, for all $x, y \in H$:

$$\begin{aligned} f_{x,y} : G &\to \mathbb{C} \\ g &\mapsto \rho_0(\pi(g^{-1})x, \pi(g^{-1})y) \end{aligned} \qquad \text{and} \qquad \begin{aligned} \rho : H \times H &\to \mathbb{C} \\ (x, y) &\mapsto m\left([g] \mapsto f_{x,y}(g)\right). \end{aligned}$$

The map ρ is well-defined, because if $g' = gn$, with $n \in N$, we have

$$\begin{aligned} f_{xy}(g') = \rho_0(\pi(g'^{-1})x, \pi(g'^{-1})y) &= \rho_0(\pi((gn)^{-1})x, \pi((gn)^{-1})y) \\ &= \rho_0(\pi(n^{-1})\pi(g^{-1})x, \pi(n^{-1})\pi(g^{-1})y) \\ &= \rho_0(\pi(g^{-1})x, \pi(g^{-1})y) = f_{xy}(g) \end{aligned}$$

by unitariness of $\pi\iota$ with respect to ρ_0.

It suffices now to show that ρ defines an inner product, compatible with ρ_0 and making π unitary, and we are done. The rest of the argument can be adapted from Theorem 1.6.

Let $\|\cdot\|_0$ denote the norm induced by ρ_0, and $|\pi|_0 < \infty$ the corresponding uniform bound. Then:

- For all x and g, $\rho_0(\pi(g^{-1})x, \pi(g^{-1})x) \geq 0$, hence $\rho(x, x) \geq 0$, by positivity of m. If $x = 0$, then $\rho(x, x) = 0$, and if $x \neq 0$, $\rho_0(x, x) > 0$. Then, for all g,

  $$|\pi|_0^2 \rho_0(\pi(g^{-1})x, \pi(g^{-1})x) \geq \rho_0(x, x) > 0$$

  by Equation (2) below, and by monotonicity of m, it follows that

  $$\rho(x, x) \geq |\pi|_0^{-2} \rho_0(x, x) > 0.$$

- Linearity in the first component: given $\alpha \in \mathbb{C}$, x and y, we have:

  $$\begin{aligned} \rho(\alpha x_1 + x_2, y) = m\left(f_{\alpha x_1 + x_2, y}\right) &= m\left(\alpha f_{x_1,y} + f_{x_2,y}\right) \\ &= \alpha m(f_{x_1,y}) + m(f_{x_2,y}) \\ &= \alpha\rho(x_1, y) + \rho(x_2, y). \end{aligned}$$

- And conjugate symmetry: for any x, y,

  $$\begin{aligned} \rho(y, x) &= m(f_{y,x}) \\ &= m([g] \mapsto \rho_0(\pi(g^{-1})y, \pi(g^{-1})x)) \\ &= m([g] \mapsto \overline{\rho_0(\pi(g^{-1})x, \pi(g^{-1})y)}) \\ &= m(\overline{f_{x,y}}) = \overline{m(f_{x,y})} = \overline{\rho(x, y)}. \end{aligned}$$

We conclude that ρ is an inner product on H.

To see that ρ makes π unitary:

$$\begin{aligned}\rho(\pi(g)x,\pi(g)y) &= m\left(f_{\pi(g)x,\pi(g)y}\right)\\ &= m\left([h]\mapsto \rho_0(\pi(h^{-1})\pi(g)x,\pi(h^{-1})\pi(g)y)\right)\\ &= m\left([h]\mapsto \rho_0(\pi(h^{-1}g)x,\pi(h^{-1}g)y)\right)\\ &= m\left([h]\mapsto [g^{-1}][h]\mapsto \rho_0(\pi(h^{-1}g)x,\pi(h^{-1}g)y)\right)\\ &= m\left(\delta_{[g]}*f_{x,y}\right)\overset{(*)}{=} m\left(f_{x,y}\right)=\rho(x,y),\end{aligned}$$

where $(*)$ uses left shift-invariance of m.

Finally, ρ is compatible with ρ_0, which is equivalent to being compatible with $\langle\cdot,\cdot\rangle$: indeed, we have

$$\rho_0(\pi(h^{-1})x,\pi(h^{-1})x)\le \|\pi(h^{-1})\|_0^2\rho_0(x,x)\le |\pi|_0^2\rho_0(x,x),\qquad \forall h\in G,\ \forall x\in H$$

and

$$\rho_0(x,x)=\rho_0(\pi(h)\pi(h^{-1})x,\pi(h)\pi(h^{-1})x)\le |\pi|_0^2\rho_0(\pi(h^{-1})x,\pi(h^{-1})x),\qquad \forall h\in G,\ \forall x\in H. \tag{2}$$

This implies

$$|\pi|_0^{-2}\rho_0(x,x)\le \rho_0(\pi(h^{-1})x,\pi(h^{-1})x)\le |\pi|_0^2\rho_0(x,x),\qquad \forall h\in G,\ \forall x\in H,$$

and applying m, and by monotonicity of m:

$$m\left([h]\mapsto |\pi|_0^{-2}\rho_0(x,x)\right)\le m\left([h]\mapsto \rho_0(\pi(h^{-1})x,\pi(h^{-1})x)\right)\le m\left([h]\mapsto |\pi|_0^2\rho_0(x,x)\right),\qquad \forall x\in H$$

so that, by definition of ρ, and taking the square-root:

$$|\pi|_0^{-1}\|x\|_0\le \rho(x,x)^{\frac12}\le |\pi|_0\|x\|_0,\qquad \forall x\in H,$$

and we conclude that the norms are compatible. □

Lemma 1.13. *Let $\langle G_i\rangle_{i\in I}$ be a directed family of groups (that is: for all i_1,i_2, there exists some i such that $G_{i_1}\cup G_{i_2}\subseteq G_i$). Assume that this family is "uniformly unitarizable", as follows: there exists a non-decreasing map $F:\mathbb{R}^{\ge 0}\to\mathbb{R}^{\ge 0}$ such that, for all $i\in I$, Hilbert space H and uniformly bounded representation $\pi:G_i\to B^I(H)$, there exists an operator $S\in B^I(H)$ such that $S\pi(\cdot)S^{-1}$ is unitary, and $\|S\|\|S^{-1}\|\le F(|\pi|)$.*
Then, the directed union $G:=\bigcup_i G_i$ is unitarizable.

Proof. First, note that we can assume that the operator $S\in B^I(H)$ actually satisfies $\|S\|,\|S^{-1}\|\le\sqrt{F(|\pi|)}$. Indeed, given S with $S\pi(\cdot)S^{-1}$ unitary and $\|S\|\|S^{-1}\|\le F(|\pi|)$, define $T:=\frac{\sqrt{F(|\pi|)}}{\|S\|}S$. Then $T\pi(\cdot)T^{-1}$ is still unitary and $\|T\|,\|T^{-1}\|\le\sqrt{F(|\pi|)}$.

Fix $\pi:G\to B^I(H)$ a uniformly bounded representation. For each $i\in I$, let $\pi_i:=\pi|_{G_i}:G_i\to B^I(H)$ the restriction of π to G_i; then $|\pi_i|\le|\pi|$ and there exists $S_i\in B^I(H)$ such that $S_i\pi_i(\cdot)S_i^{-1}$ is unitary and (without loss of generality) with $\|S_i\|,\|S_i^{-1}\|\le\sqrt{F(|\pi_i|)}\le\sqrt{F(|\pi|)}$.

Let $\rho_i:H\times H\to\mathbb{C}$ be the inner product defined by $\rho_i(x,y):=\langle S_ix,S_iy\rangle$, and $\|\cdot\|_i$ the induced norm ($\|x\|_i=\sqrt{\rho_i(x,x)}$); following the argument in Lemma 1.2 tells us that ρ_i really is an inner product, that it makes π_i unitary and that the norm $\|\cdot\|_i$ satisfies:

$$\frac{1}{\|S_i^{-1}\|}\|x\|\le\|x\|_i\le\|S_i\|\|x\|,$$

so that, for any i and x:

$$\frac{1}{\sqrt{F(|\pi|)}}\|x\|\le\|x\|_i\le\sqrt{F(|\pi|)}\|x\|.$$

Consider the topological space

$$X:=\prod_{(x,y)\in H\times H}\overline{B}(0,F(|\pi|)\|x\|\|y\|)\subseteq\mathbb{C}^{H\times H}$$

which is compact, as a product of compact spaces (in the product topology).

Each ρ_i is an element of X, because for any x,y (by Cauchy-Schwarz):

$$|\rho_i(x,y)|\le\|x\|_i\|y\|_i\le\|x\|\|y\|\sqrt{F(|\pi|)}\sqrt{F(|\pi|)}.$$

By ordering the index set I as $i \leq j$ if and only if $G_i \subseteq G_j$, we get a directed set and a net $\langle \rho_i \rangle_{i \in I} \in X$. Lying in a compact space, this net must have an accumulation point, say ρ. We now show that ρ is actually an inner product for H, compatible with $\langle \cdot, \cdot \rangle$, and that it makes π unitary.

Fix first a subnet $\langle \rho_{i_\lambda} \rangle_\lambda$ of $\langle \rho_i \rangle_i$ that converges to ρ.

ρ is an inner product. For all x, $\rho(x,x)$ is an accumulation point of $\langle \rho_i(x,x) \rangle_i$, and for each i and x,

$$\rho_i(x,x) \in [\frac{1}{F(|\pi|)}\|x\|^2, F(|\pi|)\|x\|^2], \tag{3}$$

which is a closed subset of $\overline{B}(0, F(|\pi|)\|x\|^2)$. This implies that

$$\rho(x,x) \in [\frac{1}{F(|\pi|)}\|x\|^2, F(|\pi|)\|x\|^2], \tag{4}$$

too. This implies in particular that $\rho(x,x)$ is in $\mathbb{R}^{\geq 0}$ for all x, and 0 only at $x = 0$.

Also,

$$\begin{aligned}\rho(\alpha x + x', y) &= \lim_\lambda \rho_{i_\lambda}(\alpha x + x', y) = \lim_\lambda \alpha \rho_{i_\lambda}(x,y) + \rho_{i_\lambda}(x', y) \\ &= \alpha \lim_\lambda \rho_{i_\lambda}(x,y) + \lim_\lambda \rho_{i_\lambda}(x', y) = \alpha\rho(x,y) + \rho(x', y)\end{aligned}$$

and

$$\rho(y,x) = \lim_\lambda \rho_{i_\lambda}(y,x) = \lim_\lambda \overline{\rho_{i_\lambda}(x,y)} = \overline{\lim_\lambda \rho_{i_\lambda}(x,y)} = \overline{\rho(x,y)},$$

and we conclude that ρ really is an inner product.

ρ is compatible with $\langle \cdot, \cdot \rangle$. This is direct from Equation (4).

π is ρ-unitary. Fix x, y and $t \in G$; let $x_t = \pi(t)x$ and $y_t = \pi(t)y$. Fix i with $t \in G_i$ and let $\lambda(i)$ be such that $\forall \lambda \geq \lambda(i)$, $i_\lambda \geq i$. Then

$$\rho(x,y) = \lim_\lambda \rho_{i_\lambda}(x,y) = \lim_{\lambda \geq \lambda(i)} \rho_{i_\lambda}(x,y) = \lim_{\lambda \geq \lambda(i)} \rho_{i_\lambda}(x_t, y_t) = \lim_\lambda \rho_{i_\lambda}(x_t, y_t) = \rho(x_t, y_t),$$

because whenever $\lambda \geq \lambda(i)$, $i_\lambda \geq i$ and $t \in G_{i_\lambda}$, so that

$$\rho_{i_\lambda}(x_t, y_t) = \rho_{i_\lambda}(\pi(t)x, \pi(t)_y) = \rho_{i_\lambda}(\pi_{i_\lambda}(t)x, \pi_{i_\lambda}(t)y) = \rho_{i_\lambda}(x,y).$$

$\square$

2 Two Function Spaces

This section introduces the two function spaces $B(G)$ and $T_p(G)$ (for $1 \leq p < \infty$), following [Pis01, Chapter 2]. The spaces are shown to be normed vector spaces, and the norms verified to make them into Banach spaces. An alternative description of T_p (from [Wys88]) is presented. Finally, the last part of the section verifies that those spaces only depend on groups up to isomorphism, as was expected, and that restricting the domains of their elements to a subgroup makes the resulting maps into elements of the corresponding subgroup space.

If G is a group and $\pi_i : G \to B^I(H_i)$ $(i \in J)$ a family of unitary group representations, we can define a unitary representation

$$\begin{aligned} \bigoplus_i \pi_i : G &\to B^I(\bigoplus_i H_i) \\ g &\mapsto \bigoplus_i \pi_i(g), \qquad \text{with} \qquad \bigoplus_i \pi_i(g) : \langle f_i \rangle_i \mapsto \langle \pi_i(g) f_i \rangle_i. \end{aligned}$$

For all g, the operator $\bigoplus_i \pi_i(g)$ is well-defined, since

$$\|\langle \pi_i(g) f_i \rangle_i\|^2 = \sum_i \|\pi_i(g) f_i\|_i^2 = \sum_i \|f_i\|_i^2 = \|\langle f_i \rangle_i\|^2,$$

which shows that it is actually an isometry, and $\bigoplus_i \pi_i$ being a homomorphism follows from the fact that each π_i is.

2.1 Coefficients of Unitary Representations: $B(G)$

Definition 2.1 ($B(G)$)**.** Let G be a (discrete) group. Define $B(G)$ as the set of maps $f : G \to \mathbb{C}$ such that there exists a Hilbert space H, a unitary representation π of G in H, and two elements x, y of H such that

$$f(t) = \langle \pi(t)x, y \rangle, \qquad \forall t \in G. \tag{5}$$

For the sake of readability, define $W_B(f)$ (as in "witnesses") to be the set of tuples (H, π, x, y) that satisfy the condition above. With this notation, $f \in B(G)$ if and only if $W_B(f) \neq \emptyset$. We can define a norm on $B(G)$ as follows:

$$\|f\|_{B(G)} := \inf\{\|x\|\|y\| \mid (H, \pi, x, y) \in W_B(f)\}. \tag{6}$$

We now verify that this really is a norm, and actually makes $B(G)$ into a complete normed vector space.

Proposition 2.2. *The set $B(G)$ is a sub-vector space of $\mathbb{C}^G$.*

Proof. Assume $f \in B(G)$ and $\alpha \in \mathbb{C}$. Then, $W_B(f)$ is non-empty, and there exists H, π, x, y such that

$$f(t) = \langle \pi(t)x, y \rangle, \qquad \forall t \in G,$$

so that

$$(\alpha f)(t) = \alpha f(t) = \alpha \langle \pi(t)x, y \rangle = \langle \pi(t)(\alpha x), y \rangle, \qquad \forall t \in G,$$

and $(H, \pi, \alpha x, y) \in W_B(\alpha f)$, so that $\alpha f \in B(G)$.

Assume $f_1, f_2 \in B(G)$ and $(H_i, \pi_i, x_i, y_i) \in W_B(f_i)$ for $i = 1, 2$. Then,

$$f_1(t) + f_2(t) = \langle \pi_1(t)x_1, y_1 \rangle + \langle \pi_2(t)x_2, y_2 \rangle = \langle (\pi_1 \oplus \pi_2)(t)(x_1 \oplus x_2), y_1 \oplus y_2 \rangle$$

and as $\pi_1 \oplus \pi_2$ is unitary on the space $H_1 \oplus H_2$, we have $(H_1 \oplus H_2, \pi_1 \oplus \pi_2, x_1 \oplus x_2, y_1 \oplus y_2) \in W_B(f_1 + f_2)$. ☐

Proposition 2.3. *The map $\|\cdot\|_{B(G)}$ is a norm on $B(G)$.*

Proof. First, it is obvious that $\|f\| \geq 0$ because every element in the infimum is non-negative. If $f = 0$, we can take any space H, $\pi(t) = I_H$ and $x, y = 0$, so that $\|f\| = 0$. If $f \neq 0$, there exists some t with $|f(t)| > 0$, but for any $(H, \pi, x, y) \in W_B(f)$, we have

$$0 < |f(t)| = |\langle \pi(t)x, y \rangle| \leq \|x\|\|y\|$$

so that $\|f\| \geq |f(t)| > 0$.

Fix some $f \in B(G)$ and $\alpha \in \mathbb{C}$; if $\alpha = 0$, then $|\alpha|\|f\| = \|\alpha f\|$ is obviously satisfied, and we can now assume $\alpha \neq 0$. Note that $(H, \pi, x, y) \in W_B(\alpha f)$ if and only if $(H, \pi, \alpha^{-1}x, y) \in W_B(f)$. We have:

$$\begin{aligned}
\|\alpha f\| &= \inf\{\|x\|\|y\| : (H, \pi, x, y) \in W_B(\alpha f)\} \\
&= \inf\{\|\alpha x\|\|y\| : (H, \pi, x, y) \in W_B(f)\} \\
&= \inf\{|\alpha|\|x\|\|y\| : (H, \pi, x, y) \in W_B(f)\} \\
&= |\alpha| \inf\{\|x\|\|y\| : (H, \pi, x, y) \in W_B(f)\} = |\alpha|\|f\|.
\end{aligned}$$

Finally, fix $f_1, f_2 \in B(G)$. To show the triangle inequality, we show that for any $\varepsilon > 0$, we have:

$$\|f_1 + f_2\| \leq \|f_1\| + \|f_2\| + \varepsilon.$$

First, fix $(H_i, \pi_i, x_i, y_i) \in W_B(f_i)$ with

$$\|x_i\|\|y_i\| \leq \|f_i\| + \varepsilon/2,$$

for $i = 1, 2$. Note that we can choose $\|x_i\| = \|y_i\|$ (by scaling x_i by $\frac{\sqrt{\|x_i\|\|y_i\|}}{\|x_i\|}$ and y_i by $\frac{\sqrt{\|x_i\|\|y_i\|}}{\|y_i\|}$); so, assume without loss of generality that $\|x_i\| = \|y_i\|$. This implies in particular that $\|x_i\|^2 = \|y_i\|^2 \leq \|f_i\| + \varepsilon/2$. Then, we know that $(H_1 \oplus H_2, \pi_1 \oplus \pi_2, x_1 \oplus x_2, y_1 \oplus y_2) \in W_B(f_1 + f_2)$, and we have:

$$\begin{aligned}
\|f_1 + f_2\| &\leq \|x_1 \oplus x_2\|\|y_1 \oplus y_2\| \\
&= (\|x_1\|^2 + \|x_2\|^2)^{\frac{1}{2}}(\|y_1\|^2 + \|y_2\|^2)^{\frac{1}{2}} \\
&\leq ((\|f_1\| + \varepsilon/2) + (\|f_2\| + \varepsilon/2))^{\frac{1}{2}}((\|f_1\| + \varepsilon/2) + (\|f_2\| + \varepsilon/2))^{\frac{1}{2}} \\
&= ((\|f_1\| + \varepsilon/2) + (\|f_2\| + \varepsilon/2)) \leq \|f_1\| + \|f_2\| + \varepsilon.
\end{aligned}$$

□

Now for some inclusions relative to $B(G)$: we have norm-decreasing inclusions:

$$l^2(G) \subseteq B(G) \subseteq l^\infty(G).$$

Indeed, first note that whenever $f \in B(G)$ and $(H, \pi, x, y) \in W_B(f)$, we have

$$|f(t)| = |\langle \pi(t)x, y\rangle| \leq \|x\|\|y\|, \qquad \forall t \in G$$

so that taking the infimum over x, y and the supremum over t, we get $\|f\|_\infty \leq \|f\|_{B(G)}$, and in particular $B(G) \subseteq l^\infty(G)$ continuously.

Also, recall that $l^2(G)$ is a Hilbert space with $\langle \varphi, \psi\rangle := \sum_i \varphi_i \overline{\psi_i}$; then, for any $f \in l^2(G)$,

$$f(t) = \langle f, \delta_t\rangle = \langle \delta_t, \overline{f}\rangle = \langle \lambda(t)\delta_e, \overline{f}\rangle$$

so that $(l^2(G), \lambda, \delta_e, \overline{f}) \in W_B(f)$ and $\|f\|_{B(G)} \leq \|\delta\|_2\|\overline{f}\|_2 = 1\|f\|_2$ ($\overline{f}$ is simply defined as $\overline{f}(x) = \overline{f(x)}$). This shows in particular that $l^2(G) \subseteq B(G)$ continuously.

With this norm, $B(G)$ is actually a Banach space, and it remains to show completeness.

Proposition 2.4. *The space $B(G)$ is complete.*

Proof. Let $\langle f_n\rangle_{n\in\mathbb{N}}$ be a Cauchy sequence in $B(G)$, we can furthermore assume that we actually have

$$\|f_n - f_{n-1}\|_{B(G)} < 2^{-n}, \qquad \forall n \geq 1.$$

(See Proposition C.1 and Proposition C.2; we can select such a subsequence, and if it converges, then so will the original sequence).

This means that for all $n \geq 1$, there exist H_n, π_n, x_n, y_n with

$$f_n(t) - f_{n-1}(t) = \langle \pi_n(t)x_n, y_n\rangle$$

and

$$\|x_n\|\|y_n\| < 2^{-n}.$$

As before, we assume without loss of generality that $\|x_n\| = \|y_n\|$.

Let $f_{-1} :\equiv 0$ and choose some $(H_0, \pi_0, x_0, y_0) \in W_B(f_0)$. Using the fact that $\|\cdot\|_\infty \leq \|\cdot\|_{B(G)}$, we know that $\langle f_n \rangle$ is still Cauchy with respect to $\|\cdot\|_\infty$, thus has a pointwise limit f. Then, for all t, f satisfies:

$$f(t) = \lim_n f_n(t) = \lim_N \sum_{n=0}^{N} f_n(t) - f_{n-1}(t) = \lim_N \sum_{n=0}^{N} \langle \pi_n(t) x_n, y_n \rangle = \sum_{n=0}^{\infty} \langle \pi_n(t) x_n, y_n \rangle.$$

Define the Hilbert space $H := \bigoplus_n H_n$, and $\pi = \bigoplus_n \pi_n$, $x = \bigoplus_n x_n$ and $y = \bigoplus_n y_n$. First, note that x, y really are elements of H, because

$$\sum_n \|x_n\|^2 = \|x_0\|^2 + \sum_{n\geq 1} \|x_n\|^2 \leq \|x_0\|^2 + \sum_{n \geq 1} 2^{-n} < \infty,$$

and similarly for y. As explained at the beginning of the section, recall also that $\bigoplus_n \pi_n$ is unitary. Now, by definition, we have

$$f(t) = \sum_n \langle \pi_n(t) x_n, y_n \rangle = \langle \pi(t) x, y \rangle, \qquad \forall t,$$

so that $f \in B(G)$. It remains to show that $f_n \to f$ in $B(G)$.

We have to show that

$$\forall \varepsilon > 0\ \exists N\ \forall n \geq N\ :\ \|f - f_n\| < \varepsilon,$$

but that is equivalent to showing that

$$\begin{aligned} \forall \varepsilon > 0\ \exists N\ \forall n \geq N\ \exists (H', \pi', x', y') \quad \text{such that} \quad & f(t) - f_n(t) = \langle \pi'(t) x', y' \rangle \\ \text{and} \quad & \|x'\| \|y'\| < \varepsilon. \end{aligned}$$

Define $x^n \in H$ as $(x^n)_i = 0$ for $i \leq n$ and $(x^n)_i = x_i$ for $i > n$; that is, x^n is x zeroed on its first n places. Define y^n similarly. We then have

$$\begin{aligned} f(t) - f_N(t) &= \sum_{n=0}^{\infty} (f_n(t) - f_{n-1}(t)) - \sum_{n=0}^{N} (f_n(t) - f_{n-1}(t)) \\ &= \sum_{n=N+1}^{\infty} (f_n(t) - f_{n-1}(t)) \\ &= \sum_{n=N+1}^{\infty} \langle \pi_n(t) x_n, y_n \rangle = \langle \pi(t) x^N, y^N \rangle. \end{aligned}$$

But as $\|x^N\| \overset{N\to\infty}{\to} 0$, and similarly for y^N, convergence follows, by choosing $H' = H, \pi' = \pi$ and $x' = x^N, y' = y^N$ for N sufficiently large. (To see that $\|x^N\| \to 0$, note that $\|x^N\|^2 = \sum_{n=N+1}^{\infty} \|x_n\|^2 \leq \sum_{n=N+1}^{\infty} 2^{-n} \leq 2^{-N}$.) $\square$

2.2 Littlewood Functions: $T_p(G)$

Definition 2.5 ($T_p(G)$)**.** Let G be a (discrete) group, and fix $1 \leq p < \infty$. Define $T_p(G)$ as the set of maps $f : G \to \mathbb{C}$ such that there exists $a, b : G \times G \to \mathbb{C}$ with

$$f(st) = a(s,t) + b(s,t), \qquad \forall s, t \in G. \tag{7}$$

and

$$\sup_s \|a(s,\cdot)\|_p < \infty, \qquad \sup_t \|b(\cdot,t)\|_p < \infty. \tag{8}$$

For the sake of readability, define $W_{T_p}(f)$ (as in "witnesses") to be the set of pairs (a,b) that satisfy the condition above. With this notation, $f \in T_p(G)$ if and only if $W_{T_p}(f) \neq \emptyset$.

We can define a norm on $T_p(G)$ as follows:

$$\|f\|_{T_p(G)} := \inf\{\sup_s \|a(s,\cdot)\|_p + \sup_t \|b(\cdot,t)\|_p \mid (a,b) \in W_{T_p}(f)\}. \tag{9}$$

Remark 2.6. Note that $T_p(G)$ could as well be defined as the set of maps $f : G \to \mathbb{C}$ such that there exists $a, b : G \times G \to \mathbb{C}$ with:

$$f(s^{-1}t) = a(s,t) + b(s,t), \quad \forall s, t,$$

and

$$\sup_s \|a(s,\cdot)\|_p < \infty, \qquad \sup_t \|b(\cdot,t)\|_p < \infty,$$

with the norm defined as the infimum over those pairs. Indeed, for any $(a,b) \in W_{T_p}(G)$, we can define $a', b' : G \times G \to \mathbb{C}$ by

$$a'(s,t) := a(s^{-1},t), \quad b'(s,t) := b(s^{-1},t), \qquad \forall s,t.$$

Then,

$$f(s^{-1}t) = a(s^{-1},t) + b(s^{-1},t) = a'(s,t) + b(s,t), \qquad \forall s,t,$$

and since

$$\sup_s \|a'(s,\cdot)\|_p = \sup_s \|a(s^{-1},\cdot)\|_p = \sup_s \|a(s,\cdot)\|_p,$$

and

$$\sup_t \|b'(\cdot,t)\|_p = \sup_t \left(\sum_s |b'(s,t)|^p\right)^{1/p} = \sup_t \left(\sum_s |b(s^{-1},t)|^p\right)^{1/p} = \sup_t \left(\sum_s |b(s,t)|^p\right)^{1/p} = \sup_t \|b(\cdot,t)\|_p,$$

the "weight" of the pair (a',b') is the same as that of (a,b). Conversely, given a pair (a',b'), one can recover a pair (a,b) of same "weight", and we conclude that the two definitions are equivalent.

Let us write $W_{T_p^{-1}}(f)$ for those pairs (a,b), bounded as needed, and such that $f(s^{-1}t) = a(s,t) + b(s,t)$.

We now verify that $T_p(G)$ is a vector space and that $\|\cdot\|_{T_p(G)}$ really is a norm, making $T_p(G)$ into a Banach space.

Proposition 2.7. *The set $T_p(G)$ is a sub-vector space of $\mathbb{C}^G$.*

Proof. If $f \in T_p(G)$ with $(a,b) \in W_{T_p}(f)$, and $\alpha \in \mathbb{C}$, then

$$(\alpha f)(st) = \alpha f(st) = \alpha a(s,t) + \alpha b(s,t) = (\alpha a)(s,t) + (\alpha b)(s,t)$$

and

$$\sup_s \|\alpha a(s,\cdot)\|_p = \sup_s |\alpha| \|a(s,\cdot)\|_p = |\alpha| \sup_s \|a(s,\cdot)\|_p < \infty$$

and similarly for αb, so that $(\alpha a, \alpha b) \in W_{T_p}(\alpha f)$ and $\alpha f \in T_p(G)$.

If $f_1, f_2 \in T_p(G)$ and $(a_i, b_i) \in W_{T_p}(f_i)$, for $i = 1,2$, then

$$f_1(st) + f_2(st) = (a_1 + a_2)(s,t) + (b_1 + b_2)(s,t)$$

and

$$\sup_s \|(a_1 + a_2)(s,\cdot)\|_p \leq \sup_s(\|a_1(s,\cdot)\|_p + \|a_2(s,\cdot)\|_p) \leq \sup_s \|a_1(s,\cdot)\|_p + \sup_s \|a_2(s,\cdot)\|_p < \infty \tag{10}$$

by the triangle inequality in $l^p(G)$. The same holds for $b_1 + b_2$, hence $(a_1 + a_2, b_1 + b_2) \in W_{T_p}(f_1 + f_2)$ and $f_1 + f_2 \in T_p(G)$. $\square$

Proposition 2.8. *The map $\|\cdot\|_{T_p(G)}$ is a norm on $T_p(G)$.*

Proof. The fact that $\|\cdot\|_{T_p(G)} \geq 0$ is obvious. If $f \equiv 0$, then we can choose $a, b \equiv 0$ and $\|f\| = 0$. If $f \not\equiv 0$, there exists some t with $|f(t)| > 0$, so that $\|f\|_\infty \geq |f(t)| > 0$ but using Equation (11) below, for any $(a,b) \in W_{T_p}(f)$,

$$|f(t)| \leq \sup_s \|a(s,\cdot)\|_p + \sup_t \|b(\cdot,t)\|_p,$$

hence $0 < |f(t)| \leq \|f\|_{T_p(G)}$. Fix $f \in B(G)$, and $\alpha \in \mathbb{C}$; if $\alpha = 0$, then $|\alpha|\|f\| = \|\alpha f\|$, and we can now assume without loss of generality that $\alpha \neq 0$. Then, noting that $(a,b) \in W_{T_p}(\alpha f)$ if and only if $(\alpha^{-1}a, \alpha^{-1}b) \in W_{T_p}(f)$, we have:

$$\begin{aligned}
\|\alpha f\| :=& \inf\{\sup_s \|a(s,\cdot)\|_p + \sup_t \|b(\cdot,t)\|_p : (a,b) \in W_{T_p}(\alpha f)\} \\
=& \inf\{\sup_s \|\alpha a(s,\cdot)\|_p + \sup_t \|\alpha b(\cdot,t)\|_p : (a,b) \in W_{T_p}(f)\} \\
=& \inf\{\sup_s |\alpha|\|a(s,\cdot)\|_p + \sup_t |\alpha|\|b(\cdot,t)\|_p : (a,b) \in W_{T_p}(f)\} \\
=& |\alpha| \inf\{\sup_s \|a(s,\cdot)\|_p + \sup_t \|b(\cdot,t)\|_p : (a,b) \in W_{T_p}(f)\} = |\alpha|\|f\|.
\end{aligned}$$

For the triangle inequality, fix $f_1, f_2 \in T_p(G)$ and $\varepsilon > 0$ of slack. Take $(a_i, b_i) \in W_{T_p}(f_i)$ with

$$\sup_s \|a_i(s,\cdot)\|_p + \sup_t \|b_i(\cdot,t)\|_p < \|f_i\| + \varepsilon/2, \qquad \text{for } i = 1, 2.$$

Then, we know $(a_1 + a_2, b_1 + b_2) \in W_{T_p}(f_1 + f_2)$ and

$$\begin{aligned}\|f_1 + f_2\| &\le \sup_s \|(a_1 + a_2)(s,\cdot)\|_p + \sup_t \|(b_1 + b_2)(\cdot,t)\|_p \\ &\le \sup_s \|a_1(s,\cdot)\|_p + \sup_s \|a_2(s,\cdot)\|_p + \sup_t \|b_1(\cdot,t)\|_p + \sup_t \|b_2(\cdot,t)\|_p \\ &\le \|f_1\| + \|f_2\| + \varepsilon\end{aligned}$$

and as this holds for any $\varepsilon > 0$, the triangle inequality follows. □

If $f \in T_p(G)$ with $f = a + b$, we have in particular, from the bounds on a, b, that

$$\sup_{s,t} |a(s,t)| \le \sup_s \|a(s,\cdot)\|_p < \infty, \qquad \sup_{s,t} |b(s,t)| \le \sup_t \|b(\cdot,t)\|_p < \infty$$

so that

$$\|f\|_\infty = \sup_{s,t} |f(st)| \le \sup_{s,t} |a(s,t)| + |b(s,t)| \le \sup_s \|a(s,\cdot)\|_p + \sup_t \|b(\cdot,t)\|_p \tag{11}$$

and as this holds for any a, b, we have $\|f\|_\infty \le \|f\|_{T_p(G)}$, so that $T_p(G) \subseteq l^\infty(G)$ continuously.

Also, if $f \in l^p(G)$, we can set $a(s,t) = f(st)$ and $b \equiv 0$, which implies

$$\|f\|_{T_p(G)} \le \sup_s \|a(s,\cdot)\|_p = \sup_s \|f(s\cdot)\|_p = \sup_s \|\delta_{s^{-1}} * f\|_p = \sup_s \|f\|_p = \|f\|_p,$$

and $l^p(G) \subseteq T_p(G)$ continuously.

The space $T_p(G)$ is actually Banach, and it remains to show completeness.

Proposition 2.9. *$T_p(G)$ is a complete space.*

Proof. Fix $\langle f_n \rangle_n$ a Cauchy sequence in $T_p(G)$ and assume without loss of generality that, for all $n \ge 1$,

$$\|f_n - f_{n-1}\|_{T_p(G)} < 2^{-n},$$

which implies in particular that for all $n \ge 1$, there exists a_n, b_n with

$$f_n(st) - f_{n-1}(st) = a_n(s,t) + b_n(s,t)$$

and

$$\sup_s \|a_n(s,\cdot)\|_p + \sup_t \|b_n(\cdot,t)\|_p < 2^{-n}. \tag{12}$$

Because $\|\cdot\|_\infty \le \|\cdot\|_{T_p(G)}$, $\langle f_n \rangle_n$ is still Cauchy in $l^\infty(G)$ and we can take the pointwise limit f of $\langle f_n \rangle$.

Let $f_{-1} :\equiv 0$ and take some $(a_0, b_0) \in W_{T_p}(f_0)$. Then, by definition of the pointwise limit, we have, for all s, t:

$$f(st) = \lim_n f_n(st) = \lim_N \sum_{n=0}^N f_n(st) - f_{n-1}(st) = \lim_N \sum_{n=0}^N a_n(s,t) + b_n(s,t).$$

Fix some s (resp. t) and define

$$a(s,\cdot) := \sum_{n=0}^\infty a_n(s,\cdot), \qquad b(\cdot,t) := \sum_{n=0}^\infty b_n(\cdot,t) \qquad \text{in } l^p(G),$$

i.e. as limits in $l^p(G)$. We first verify that those are well-defined, that is, that the sums converge in $l^p(G)$. We have, for any s,

$$\sum_n \|a_n(s,\cdot)\|_p = \|a_0(s,\cdot)\|_p + \sum_{n\ge1} \|a_n(s,\cdot)\|_p \le \|a_0(s,\cdot)\|_p + \sum_{n\ge1} 2^{-n} \le \|a_0(s,\cdot)\|_p + 1,$$

which shows that the sum defining $a(s,\cdot)$ converges in norm, so that $a(s,\cdot)$ is well-defined and

$$\|a(s,\cdot)\|_p \le \|a_0(s,\cdot)\|_p + 1,$$

for any s, and

$$\sup_s \|a(s,\cdot)\|_p \le \sup_s \|a_0(s,\cdot)\|_p + 1$$

and similarly for b.

Now, as $\sum_{n=0}^N a_n(s,\cdot) \xrightarrow{N} a(s,\cdot)$ in $l^p(G)$, the convergence also holds in $l^\infty(G)$ (because the ∞-norm is weaker than the p-norm), and we actually have pointwise convergence. That is, for any s,t:

$$\sum_{n=0}^N a_n(s,t) \to a(s,t).$$

and similarly for b. For any s,t, we then have

$$f(st) = \lim_n f_n(st) = \lim_N \sum_{n=0}^N a_n(s,t) + b_n(s,t) = \lim_N \sum_{n=0}^N a_n(s,t) + \lim_N \sum_{n=0}^N b_n(s,t) = a(s,t) + b(s,t)$$

which allows us to conclude that f is in $T_p(G)$. It remains to show that $f_n \to f$ in $T_p(G)$. But we have

$$f(st) - f_N(st) = \sum_{n=N+1}^\infty a_n(s,t) + b_n(s,t) = \sum_{n=N+1}^\infty a_n(s,t) + \sum_{n=N+1}^\infty b_n(s,t),$$

and writing $a^N(s,\cdot) = \sum_{n=N+1}^\infty a_n(s,\cdot)$ and $b^N(\cdot,t) = \sum_{n=N+1}^\infty b_n(\cdot,t)$ for the sake of readability, it suffices to show that

$$\sup_s \|a^N(s,\cdot)\|_p \to 0,$$

and similarly for b. Indeed, given $\varepsilon > 0$, choose N_0 large enough for $\sup_s \|a^N(s,\cdot)\|_p < \varepsilon/2$ to hold whenever $N \ge N_0$. Then $f(st) - f_N(st) = a^N(s,t) + b^N(s,t)$, and $\|f - f_N\|_{T_p(G)} < \varepsilon/2 + \varepsilon/2 = \varepsilon$.

Now,

$$\|\sum_{n=N+1}^\infty a_n(s,\cdot)\|_p \le \sum_{n=N+1}^\infty \|a_n(s,\cdot)\|_p < \sum_{n=N+1}^\infty 2^{-n} \le 2^{-N},$$

for all s, so that

$$\sup_s \|a^N(s,\cdot)\|_p \le 2^{-N} \to 0,$$

and we are done. $\square$

2.2.1 A Stronger Condition on $T_p(G)$

Given a group G, the space $T_p(G)$ can be defined with a seemingly stronger definition, which is that $f \in T_p(G)$ if there exist $a, b : G \times G \to \mathbb{C}$ such that:

- $f(st) = a(s,t) + b(s,t)$ for all s,t;
- $\sup_s \|a(s,\cdot)\|_p$ and $\sup_t \|a(\cdot,t)\|_p$ are finite; and
- a and b have disjoint support: i.e. $a(s,t)b(s,t) = 0$ for all s,t.

Proposition 2.10. *The "stronger" definition for $T_p(G)$ is equivalent to the original one.*

Proof. The only thing to verify is that if there exist $(a,b) \in W_{T_p}(f)$, then there exist $(a',b') \in W_{T_p}(f)$ *and* of disjoint support.

Fix $f \in T_p(G)$ and $(a,b) \in W_{T_p}(f)$. Define $A \subseteq G \times G$ by

$$A := \{(s,t) \in G \times G : |a(s,t)| \ge |b(s,t)|\}$$

and $B := G \times G \setminus A$, that is:

$$B = \{(s,t) \in G \times G : |a(s,t)| < |b(s,t)|\}.$$

Define $a', b' : G \times G \to \mathbb{C}$ by:

$$a'(s,t) := \begin{cases} a(s,t) + b(s,t) & \text{if } (s,t) \in A, \\ 0 & \text{if } (s,t) \in B, \end{cases}$$

and

$$b'(s,t) := \begin{cases} a(s,t)+b(s,t) & \text{if } (s,t)\in B, \\ 0 & \text{if } (s,t)\in A. \end{cases}$$

We see that a' and b' have disjoint support and that $f = a' + b'$ by construction, and

$$\sup_s \|a'(s,\cdot)\|_p^p = \sup_s \sum_t |a'(s,t)|^p \overset{(*)}{\leq} \sup_s \sum_t |2a(s,t)|^p = 2^p \sup_s \sum_t |a(s,t)|^p = 2^p \sup_s \|a(s,\cdot)\|_p^p,$$

where $(*)$ is due to the fact that at any (s,t), either $a'(s,t)$ is zero (when $(s,t)\in B$), or $a'(s,t) = a(s,t)+b(s,t)$, in which case $|b(s,t)| \leq |a(s,t)|$ and $|a'(s,t)| = |a(s,t)+b(s,t)| \leq |a(s,t)| + |b(s,t)| \leq 2|a(s,t)|$. Taking the p-th root, we get

$$\sup_s \|a'(s,\cdot)\|_p \leq 2\sup_s \|a(s,\cdot)\|_p,$$

and by a similar argument:

$$\sup_t \|b'(\cdot,t)\|_p \leq 2\sup_t \|b(\cdot,t)\|_p.$$

We conclude that a' and b' sum to f, are of disjoint support and are bounded as needed, and we are done. □

The above proof actually shows that for any $(a,b)\in W_{T_p}(f)$, we can get $(a',b')\in W_{T_p}(f)$ with disjoint support so that $\sup_s \|a'(s,\cdot)\|_p + \sup_t \|b'(\cdot,t)\|_p \leq 2(\sup_s \|a(s,\cdot)\|_p + \sup_t \|b(\cdot,t)\|_p)$. Define, then

$$\|f\|_{T_p^d} := \inf\{\sup_s \|a(s,\cdot)\|_p + \sup_t \|b(\cdot,t)\|_p : (a,b)\in W_{T_p}(f) \text{ and } a,b \text{ of disjoint support}\}.$$

Then the maps $\|\cdot\|_{T_p}$ and $\|\cdot\|_{T_p^d}$ are within a factor 2 of each other: The inequality $\|f\|_{T_p} \leq \|f\|_{T_p^d}$ is obvious, since the infimum for the first is taken over a larger set. Also, $\|f\|_{T_p^d} \leq 2\|f\|_{T_p}$, as for any pair $(a,b)\in W_{T_p}(f)$, we can construct $(a',b')\in W_{T_p}(f)$ with disjoint support and

$$\sup_s \|a'(s,\cdot)\|_p + \sup_t \|b'(\cdot,t)\|_p \leq 2(\sup_s \|a(s,\cdot)\|_p + \sup_t \|b(\cdot,t)\|_p),$$

so that taking the infimum over the RHS yields $\|f\|_{T_p^d} \leq 2\|f\|_{T_p}$.

Note that $\|\cdot\|_{T_p^d}$ doesn't define a norm , as far as I know.

2.2.2 A Different Norm on $T_p(G)$

Define the norm $\|\cdot\|_s$ on $T_p(G)$ by

$$\|f\|_s := \sup_{A,B\subseteq_f G} \left(\frac{1}{\max(|A|,|B|)} \sum_{x\in A, y\in B} |f(xy)|^p\right)^{1/p}.$$

First, note that in this definition, the supremum can be taken over $A,B\subseteq_f G$ of same finite cardinality: Indeed, given any A,B finite with $|A|\leq|B|$ (without loss of generality), we can extend A to some $A\subseteq A'$ with $|A'|=|B|$, and then:

$$\left(\frac{1}{\max(|A|,|B|)} \sum_{x\in A, y\in B} |f(xy)|^p\right)^{1/p} \leq \left(\frac{1}{\max(|A'|,|B|)} \sum_{x\in A', y\in B} |f(xy)|^p\right)^{1/p}$$

because $\max(|A|,|B|) = |B| = \max(|A'|,|B|)$.

Proposition 2.11. $\|\cdot\|_s$ *defines a norm on* $T_p(G)$.

Proof. The only non-obvious property to check is the triangle inequality, but by Minkowski's inequality:

$$\left(\sum_{x\in A, y\in B} |f_1(xy)+f_2(xy)|^p\right)^{1/p} \leq \left(\sum_{x\in A, y\in B} |f_1(xy)|^p\right)^{1/p} + \left(\sum_{x\in A, y\in B} |f_2(xy)|^p\right)^{1/p}$$

for any $f_1, f_2 \in T_p(G)$ and $A, B \subseteq_f G$. Therefore,

$$\left(\frac{1}{n}\sum_{x\in A, y\in B} |f_1(xy)+f_2(xy)|^p\right)^{1/p} \leq \left(\frac{1}{n}\sum_{x\in A, y\in B} |f_1(xy)|^p\right)^{1/p} + \left(\frac{1}{n}\sum_{x\in A, y\in B} |f_2(xy)|^p\right)^{1/p} \leq \|f_1\|_s + \|f_2\|_s$$

for any A, B of same finite cardinality n, and taking the supremum over those:

$$\|f_1+f_2\|_s \leq \|f_1\|_s + \|f_2\|_s, \qquad \forall f_1, f_2.$$

□

Proposition 2.12. *On $T_p(G)$, the norms $\|\cdot\|_{T_p}$ and $\|\cdot\|_s$ are compatible. More precisely,*

$$\frac{1}{2}\|f\|_s \leq \|f\|_{T_p} \leq 2\|f\|_s.$$

Proof. Fix $f \in T_p(G)$ and $(a, b) \in W_{T_p}(f)$, of disjoint support, and $A, B \subseteq_f G$ of same cardinality n. Then $\max(|A|, |B|) = n$ and

$$\begin{aligned}
\frac{1}{n}\sum_{x\in A, y\in B} |f(xy)|^p &= \frac{1}{n}\sum_{x\in A, y\in B} |a(x,y) + b(x,y)|^p \\
&= \frac{1}{n}\sum_{x\in A, y\in B} |a(x,y)|^p + |b(x,y)|^p \\
&= \frac{1}{n}\sum_{x\in A, y\in B} |a(x,y)|^p + \frac{1}{n}\sum_{x\in A, y\in B} |b(x,y)|^p \\
&= \frac{1}{n}\sum_{x\in A}\sum_{y\in B} |a(x,y)|^p + \frac{1}{n}\sum_{y\in B}\sum_{x\in A} |b(x,y)|^p \\
&\leq \frac{1}{n}\sum_{x\in A}\sum_{y\in G} |a(x,y)|^p + \frac{1}{n}\sum_{y\in B}\sum_{x\in G} |b(x,y)|^p \\
&= \frac{1}{n}\sum_{x\in A} \|a(x,\cdot)\|_p^p + \frac{1}{n}\sum_{y\in B} \|b(\cdot,y)\|_p^p \\
&\leq \sup_x \|a(x,\cdot)\|_p^p + \sup_y \|b(\cdot,y)\|_p^p,
\end{aligned}$$

using the fact that a, b have disjoint support in the second equality, and the fact that $\frac{1}{n}\sum$ is just an average, at the last inequality. Then, taking the p-th root:

$$\left(\frac{1}{n}\sum_{x\in A, y\in B} |f(xy)|^p\right)^{1/p} \leq \left(\sup_x \|a(x,\cdot)\|_p^p + \sup_y \|b(\cdot,y)\|_p^p\right)^{1/p} \leq \sup_x \|a(x,\cdot)\|_p + \sup_y \|b(\cdot,y)\|_p,$$

using the fact that for $\alpha, \beta \geq 0$ and $p \geq 1$, we have $\alpha^p + \beta^p \leq (\alpha+\beta)^p$, so that, considering $\alpha^{1/p}$ and $\beta^{1/p}$ we have $(\alpha^{1/p}+\beta^{1/p})^p \geq \alpha+\beta$, and $\alpha^{1/p}+\beta^{1/p} \geq (\alpha+\beta)^{1/p}$.

This inequality holding for any A, B and any $(a, b) \in W_{T_p}(f)$ of disjoint support, we can take the supremum over the LHS and the infimum over the RHS, and conclude

$$\|f\|_s \leq \|f\|_{T_p^d} \leq 2\|f\|_{T_p}.$$

Conversely, assume $\|f\|_s = k$. Then, for any $A, B \subseteq_f G$ of cardinality n, we have

$$\left(\frac{1}{n}\sum_{x\in A, y\in B} |f(xy)|^p\right)^{1/p} \leq k,$$

that is,

$$\sum_{x\in A, y\in B} |f(xy)|^p \leq nk^p.$$

Applying Corollary 3.10 to the function $g(s,t) = f(st)$, we get $a, b : G\times G \to \mathbb{C}$ with $g = a + b$, of disjoint support and such that:

$$\sup_s \sum_t |a(s,t)|^p \leq k^p, \qquad \sup_t \sum_s |b(s,t)|^p \leq k^p$$

so that $\sup_s \|a(s,\cdot)\|_p + \sup_t \|b(\cdot,t)\|_p \leq 2k$, and we conclude

$$\|f\|_{T_p} \leq \|f\|_{T_p^d} \leq 2\|f\|_s.$$

□

2.3 Some Verifications

Proposition 2.13. *Let G, G' be groups, $\varphi : G' \to G$ an isomorphism and $f : G \to \mathbb{C}$ a map. Then*

- *$f \in B(G)$ if and only if $f\circ\varphi \in B(G')$, in which case $\|f\circ\varphi\|_{B(G')} = \|f\|_{B(G)}$;*
- *$f \in T_p(G)$ if and only if $f\circ\varphi \in T_p(G')$, in which case $\|f\circ\varphi\|_{T_p(G')} = \|f\|_{T_p(G)}$.*

This proposition officializes the fact that $B(G)$ and $T_p(G)$ only depend on G up to isomorphism, as we expect them to.

Before the proof, note that for any $1 \leq p \leq \infty$, and bijection $\varphi : Y \to X$, if $f : X \to \mathbb{C}$ is in $l^p(X)$, then $f\circ\varphi$ is in $l^p(Y)$, and of same norm:

$$\|f\circ\varphi\|_p^p = \sum_y |f(\varphi(y))|^p = \sup_{F \subseteq_f Y} \sum_{y \in F} |f(\varphi(y))|^p = \sup_{E \subseteq_f X} \sum_{y \in \varphi^{-1}[E]} |f(\varphi(y))|^p = \sup_{E \subseteq_f X} \sum_{x \in E} |f(x)|^p = \|f\|_p^p,$$

which was for $p < \infty$, and

$$\|f\circ\varphi\|_\infty = \sup_{y \in Y} |f(\varphi(y))| = \sup_{x \in X} |f(x)| = \|f\|_\infty,$$

for $p = \infty$.

Proof. First, note that it suffices to show the $\Rightarrow$ direction in both cases, because the other follows from looking at $\varphi^{-1} : G \to G'$. Similarly, it suffices to show $\leq$ for the norms.

- Let $f \in B(G)$, and $(H, \pi, x, y) \in W_B(f)$, that is, $f(t) = \langle \pi(t)x, y\rangle\ \forall t \in G$. Then,

  $$f\circ\varphi(t') = \langle \pi\circ\varphi(t')x, y\rangle, \quad \forall t' \in G',$$

  and $\pi\circ\varphi$ is a unitary representation of G' (using the fact that φ is a homomorphism), but of G', so that $(H, \pi\circ\varphi, x, y) \in W_B(f\circ\varphi)$. Furthermore,

  $$\|f\circ\varphi\|_{B(G')} \leq \|x\|\|y\|$$

  and ranging over the witnesses (H, π, x, y) of f, we conclude:

  $$\|f\circ\varphi\|_{B(G')} \leq \|f\|_{B(G)}.$$

- Let $f \in T_p(G)$ and $(a, b) \in W_{T_p}(f)$, that is $f(st) = a(s,t) + b(s,t)$ for all $s, t \in G$, and

  $$\sup_s \|a(s,\cdot)\|_{l^p(G)} < \infty, \qquad \sup_t \|b(\cdot,t)\|_{l^p(G)} < \infty.$$

  Then $f(\varphi(s't')) = f(\varphi(s')\varphi(t')) = a(\varphi(s'), \varphi(t')) + b(\varphi(s'), \varphi(t'))$ for all $s', t' \in G'$, and writing $c'(s,t) = c(\varphi(s), \varphi(t))$ for $c = a, b$, we have $f\circ\varphi = a' + b'$ and

  $$\begin{aligned} \sup_{s' \in G'} \|a'(s',\cdot)\|_{l^p(G')} &= \sup_{s' \in G'} \|a(\varphi(s'), \varphi(\cdot))\|_{l^p(G')} \\ &= \sup_{s \in G} \|a(s, \varphi(\cdot))\|_{l^p(G')} \\ &= \sup_{s \in G} \|a(s,\cdot)\|_{l^p(G)} < \infty, \end{aligned}$$

  where the second equality holds because φ is a bijection between $s \in G$ and $s' \in G'$, and the third by the note just above the proof; and similarly for b'. This shows that $(a', b') \in W_{T_p}(f\circ\varphi)$, and

  $$\|f\circ\varphi\|_{T_p(G')} \leq \sup_{s' \in G'} \|a'(s',\cdot)\|_{l^p(G')} + \sup_{t' \in G'} \|b'(\cdot,t')\|_{l^p(G')} = \sup_{s \in G} \|a(s,\cdot)\|_{l^p(G)} + \sup_{t \in G} \|b(\cdot,t)\|_{l^P(G)}$$

  and ranging over the witnesses (a, b) of f, we conclude that

  $$\|f\circ\varphi\|_{T_p(G')} \leq \|f\|_{T_p(G)}.$$

$\square$

Proposition 2.14. *Let G be a group and H a subgroup of G. Then*

1. *If $f \in T_p(G)$, then $f|_H \in T_p(H)$;*
2. *If $f \in B(G)$, then $f|_H \in B(H)$.*

Proof. The proof essentially consists of restricting the witnesses for f to ones for $f|_H$.

1. If $f \in T_p(G)$, there exists $a, b : G \times G \to \mathbb{C}$ with $f(st) = a(s,t) + b(s,t)$ for all $s, t \in G$, and

$$\sup_{s\in G} \sum_{t\in G} |a(s,t)|^p < \infty, \qquad \sup_{t\in G} \sum_{s\in G} |b(s,t)|^p < \infty.$$

   Let $f' := f|_H$ and $a' := a|_{H\times H}, b' := b|_{H\times H}$. Then

$$f'(st) = f(st) = a(s,t) + b(s,t) = a'(s,t) + b'(s,t)$$

   for all $s, t \in H$. Also,

$$\begin{aligned}\sup_{s\in H} \sum_{t\in H} |a'(s,t)|^p = \sup_{s\in H} \sum_{t\in H} |a(s,t)|^p &= \sup_{s\in H} \sup_{E\subseteq_f H} \sum_{t\in E} |a(s,t)|^p \\ &\le \sup_{s\in G} \sup_{E\subseteq_f G} \sum_{t\in E} |a(s,t)|^p \\ &= \sup_{s\in G} \sum_{t\in G} |a(s,t)|^p < \infty\end{aligned}$$

   and similarly for b', so that $(a', b') \in W_{T_p}(f')$ and $f' \in T_p(H)$.

2. If $f \in B(G)$, there exists X, $\pi : G \to B^I(X)$ a unitary representation, and $x, y \in X$ with $f(t) = \langle \pi(t)x, y\rangle$ for all t. Then, if $\iota : H \to G$ is the inclusion morphism, $\pi\circ\iota$ is still a unitary representation. Also, for all $t \in H$

$$f|_H(t) = f(t) = \langle \pi(t)x, y\rangle = \langle \pi(\iota(t))x, y\rangle$$

   and $(X, \pi\circ\iota, x, y) \in W_B(f|_H)$, so that $f|_H \in B(X)$.

□

3 Non-Unitarizable Uniformly Bounded Group Representations

This section, starting with a first condition on group unitarizability (Theorem 3.1), then shows that the free group of countably infinite rank F_∞ doesn't satisfy it, hence isn't unitarizable (Lemma 3.2). Two results (Theorem 3.7, Lemma 3.8) relating amenability with the function spaces developed above follow; the second yielding the characterisation of amenability of a group G as $T_p(G)$ being equal to $l^p(G)$ for all $p \in [1, \infty)$. Finally, Lemma 3.11 and Corollary 3.12 extend the non-unitarizability of F_∞ to any group merely containing F_∞ as a subgroup.

The section follows the beginning of chapter 2 of [Pis01] closely, until Lemma 2.7 of [Pis01].

3.1 Unitarizability, $T_1(G)$ & $B(G)$

Theorem 3.1 ([Pis01, Theorem 2.1])**.** *If G is a unitarizable group, then $T_1(G) \subseteq B(G)$.*

Proof. Fix $f \in T_1(G)$. Then, by definition (see Remark 2.6), there exists $a, b : G \times G \to \mathbb{C}$ with

$$f(s^{-1}t) = a(s,t) + b(s,t), \qquad \forall s,t \in G,$$

and some C with

$$\sup_s \sum_t |a(s,t)| \le C, \qquad \sup_t \sum_s |b(s,t)| \le C. \tag{13}$$

Constructing a uniformly bounded family of operators. For any $t \in G$, note that

$$a(t^{-1}i, j) - a(i, tj) = b(i, tj) - b(t^{-1}i, j), \qquad \forall i, j.$$

Indeed, subtracting the RHS to the LHS yields:

$$\begin{aligned} a(t^{-1}i,j) - a(i,tj) - (b(i,tj) - b(t^{-1}i,j)) &= a(t^{-1}i,j) + b(t^{-1}i,j) - (a(i,tj) + b(i,tj)) \\ &= f((t^{-1}i)^{-1}j) - f(i^{-1}tj) \\ &= f(i^{-1}tj) - f(i^{-1}tj) = 0. \end{aligned}$$

Define then, for all $t \in G$:

$$\begin{aligned} d(t)(i,j) &:= a(t^{-1}i,j) - a(i,tj) \\ &= b(i,tj) - b(t^{-1}i,j), \qquad \forall i,j. \end{aligned}$$

We know that a and b define bounded matrices (of bounded operators on $l^1(G)$ and $l^\infty(G)$ respectively; see Appendix A.3):

$$\sup_i \sum_j |a(i,j)| \le C, \qquad \sup_j \sum_i |b(i,j)| \le C,$$

which allows us to bound the matrix $d(t)$:

$$\begin{aligned} \sup_j \sum_i |d(t)(i,j)| = \sup_j \sum_i |b(i,tj) - b(t^{-1}i,j)| &\le \sup_j \sum_i |b(i,tj)| + |b(t^{-1}i,j)| \\ &= \sup_j \sum_i |b(i,tj)| + \sum_i |b(t^{-1}i,j)| \\ &\le \sup_j \sum_i |b(i,tj)| + \sup_j \sum_i |b(t^{-1}i,j)| \\ &= \sup_j \sum_i |b(i,j)| + \sup_j \sum_i |b(i,j)| \le 2C, \end{aligned}$$

for all t, and similarly, using the formulation of $d(t)$ in terms of a:

$$\sup_i \sum_j |d(t)(i,j)| = \sup_i \sum_j |a(t^{-1}i,t) - a(i,tj)| \le 2C,$$

for all t. Summarising, we have:

$$\sup_i \sum_j |d(t)(i,j)| \le 2C, \qquad \sup_j \sum_i |d(t)(i,j)| \le 2C, \tag{14}$$

for all t.

Define $D(t) : \mathbb{C}[G] \to \mathbb{C}^G$ as the operator of matrix $d(t)$, i.e. $D(t) : \langle x_i \rangle_i \mapsto \langle \sum_j d(t)(i,j)x_j \rangle_i$. Applying Proposition A.7, we can conclude, from Equation (14), that, for all t, $D(t)$ defines an operator $l^2(G) \to l^2(G)$ of norm at most $2C$.

Finally, note that $D(st) = \lambda(s)D(t) + D(s)\lambda(t)\ \forall s, t$. Indeed, we have

$$\big(D(st)(\delta_j)\big)_i = d(st)(i,j) = a(t^{-1}s^{-1}i, j) - a(i, stj), \qquad \forall i, j,$$

and

$$\begin{aligned}\big((\lambda(s)D(t) + D(s)\lambda(t))(\delta_j)\big)_i &= d(t)(s^{-1}i, j) + d(s)(i, tj) \\ &= a(t^{-1}s^{-1}i, j) - a(s^{-1}i, tj) + a(s^{-1}i, tj) - a(i, stj) \\ &= a(t^{-1}s^{-1}i, j) - a(i, stj), \qquad \forall i, j,\end{aligned}$$

which shows that $D(st)$ and $\lambda(s)D(t) + D(s)\lambda(t)$ agree on an orthornormal basis, hence are equal. This implies in particular that $D(e) = \lambda(e)D(e) + D(e)\lambda(e) = D(e) + D(e)$, so that $D(e) = 0$.

Constructing a uniformly bounded representation. Let $H := l^2(G) \oplus l^2(G)$, and define, for all t,

$$\begin{aligned}\pi(t) &: H \to H \\ &x \oplus y \mapsto (\lambda(t)x + D(t)y) \oplus (\lambda(t)y),\end{aligned}$$

or, in matrix notation:

$$\pi(t) = \begin{pmatrix} \lambda(t) & D(t) \\ 0 & \lambda(t) \end{pmatrix} : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \lambda(t)x + D(t)y \\ \lambda(t)y \end{pmatrix}.$$

Then, π is a uniformly bounded representation. First, for all t, linearity and boundedness of $\pi(t)$ follows from its definition as a finite a matrix with linear bounded coefficients $D(t)$ and $\lambda(t)$. We then verify that π is a homomorphism:

- $\pi(e_G)(x \oplus y) = (\lambda(e)x + D(e)y) \oplus \lambda(e)y = (x + 0y) \oplus y = x \oplus y$, for any $x \oplus y \in H$, and $\pi(e_G)$ really is the identity.
- For any $s, t \in G$:

$$\begin{aligned}\pi(st)(x \oplus y) &= (\lambda(st)x + D(st)y) \oplus (\lambda(st)y) \\ &= (\lambda(s)\lambda(t)x + D(st)y) \oplus (\lambda(s)\lambda(t)y) \\ &= (\lambda(s)\lambda(t)x + D(s)\lambda(t)y + \lambda(s)D(t)y) \oplus (\lambda(s)\lambda(t)y) \\ &= \pi(s)\ (\lambda(t)x + D(t)y) \oplus (\lambda(t)y) = \pi(s)\pi(t)(x \oplus y),\end{aligned}$$

  for any $x \oplus y \in H$, using the fact that λ is itself a representation.

Finally, we show that π is uniformly bounded. Fix any t and $x \oplus y \in H$ of norm 1, i.e. $\|x\|_2^2 + \|y\|_2^2 = 1$, so that in particular $\|x\|_2, \|x\|_2^2, \|y\|_2, \|y\|_2^2 \le 1$. Then:

$$\begin{aligned}\|\pi(t)(x \oplus y)\|^2 =&\|\lambda(t)x + D(t)y \oplus \lambda(t)y\|^2 \\ =&\langle \lambda(t)x + D(t)y, \lambda(t)x + D(t)y\rangle + \langle \lambda(t)y, \lambda(t)y\rangle \\ =&\langle \lambda(t)x, \lambda(t)x\rangle + \langle \lambda(t)y, \lambda(t)y\rangle + \langle D(t)y, D(t)y\rangle + 2Re(\langle \lambda(t)x, D(t)y\rangle) \\ \le&1 + \langle D(t)y, D(t)y\rangle + 2|\langle \lambda(t)x, D(t)y\rangle| \\ \le&1 + \|D(t)\|^2\|y\|^2 + 2\|D(t)\|\|\lambda(t)\|\|x\|\|y\| \\ \le&1 + \|D(t)\|(\|D(t)\| + 2) \\ \le&1 + 2C(2C + 2) = (1 + 2C)^2\end{aligned}$$

using the fact that $\lambda(t)$ is unitary and the bound on the norm of $D(t)$. Thus, $\|\pi(t)\| \le 1 + 2C$ for all t, and $|\pi| \le 1 + 2C$, and π is uniformly bounded.

Writing f as an element of $B(G)$. The representation π being uniformly bounded, we can use the hypothesis that uniformly bounded representations are unitarizable and get a unitary representation $\overline{\pi}$ and some $S \in B^I(H)$ with $S\pi(\cdot)S^{-1} = \overline{\pi}(\cdot)$, i.e.

$$S^{-1}\overline{\pi}(\cdot)S = \pi(\cdot),$$

which means that for any $x, y \in H$ and $t \in G$, we have

$$\langle \pi(t)x, y\rangle = \langle \overline{\pi}(t)Sx, S^{-1^*}y\rangle. \tag{15}$$

Consider now the elements $x := 0 \oplus \delta_e$ and $y := \delta_e \oplus 0$ of H:

$$\begin{aligned} \langle \pi(t)x, y\rangle &= \langle D(t)\delta_e \oplus \lambda(t)\delta_e, \delta_e \oplus 0\rangle = \langle D(t)\delta_e, \delta_e\rangle \\ &= D(t)(\delta_e)_e = d(t)(e,e) \\ &= b(e,t) - b(t^{-1}, e), \qquad \forall t. \end{aligned} \tag{16}$$

Let $\psi : G \to \mathbb{C}$ be defined by $\psi : t \mapsto b(e,t) - b(t^{-1}, e)$. Combining Equation (15) and Equation (16), we have that

$$\psi(t) = \langle \overline{\pi}(t)Sx, S^{-1^*}y\rangle, \qquad \forall t$$

so that $(H, \overline{\pi}, Sx, S^{-1^*}y) \in W_B(\psi)$ and ψ is in $B(G)$.

Consider also the maps $t \mapsto a(e,t)$ and $t \mapsto b(t^{-1}, e)$. Both are in $l^1(G)$, since

$$\sum_t |a(e,t)| \leq \sup_s \sum_t |a(s,t)| \leq C, \quad \text{and} \quad \sum_t |b(t^{-1}, e)| = \sum_t |b(t,e)| \leq \sup_s \sum_t |b(t,s)| \leq C.$$

Then, as $l^1(G) \subseteq l^2(G) \subseteq B(G)$, both maps are in $B(G)$. Finally, f can be written:

$$f(t) = a(e,t) + b(e,t) = a(e,t) + b(t^{-1}, e) + b(e,t) - b(t^{-1}, e) = a(e,t) + b(t^{-1}, e) + \psi(t), \quad \forall t,$$

that is, as a sum of three elements of $B(G)$, and we conclude that it itself is in $B(G)$. $\square$

Lemma 3.2 ([Pis01, Lemma 2.2])**.** *Let F be the free group on $\mathbb{N}$ generators, and $f : F \to \mathbb{C}$ the indicator function of words of length 1. Then f is in $T_1(F)$ but not in $B(F)$.*

This result implies the existence of uniformly bounded, non-unitarizable representations of F, using Theorem 3.1. Write g_n, g_n^{-1} (for $n \in \mathbb{N}$) for the letters generating F, and their inverses, respectively; with this notation, the set of words of length 1 in F is

$$\{g_n : n \in \mathbb{N}\} \cup \{g_n^{-1} : n \in \mathbb{N}\}.$$

Proof. We first show that $f \in T_1(F)$, and then that $f \in B(F)$ would yield a contradiction.

f is in $T_1(F)$. Define $a, b : F \times F \to \mathbb{C}$ by

$$\begin{aligned} a(s,t) &:= \mathbb{1}_{|st|=1 \wedge |s|>|t|}(s,t), \\ b(s,t) &:= \mathbb{1}_{|st|=1 \wedge |t|>|s|}(s,t). \end{aligned}$$

Let us verify that $f = a + b$: Fix $s, t \in F$, if $f(st) \neq 1$, then $|st| \neq 1$ and $f(st) = a(s,t) = b(s,t) = 0$. If $f(st) = 1$, either $|s| > |t|$ or $|s| < |t|$ by Proposition C.12. Therefore, exactly one of $a(s,t), b(s,t)$ is 1, and so is $f(s,t)$.

We know that for any $s \in F$, there exists at most one $t \in F$ with $|t| < |s|$ and $|st| = 1$, by Proposition C.13. This means that

$$\sup_s \sum_t |a(s,t)| \leq \sup_s 1 = 1,$$

and symmetrically

$$\sup_t \sum_s |b(s,t)| \leq \sup_t 1 = 1,$$

so that $(a,b) \in W_{T_1}(f)$ and $f \in T_1(G)$.

f is not in $B(F)$. Suppose, towards a contradiction, that $f \in B(F)$. There exists $\pi : G \to B^I(H)$ unitary and $x, y \in H$ with, by definition of f:

$$\begin{aligned} \langle \pi(g_j^{\pm 1})x, y\rangle &= 1 \qquad\qquad \forall j, \\ \langle \pi(t)x, y\rangle &= 0 \quad \text{ if } |t| \neq 1. \end{aligned}$$

As π is unitary, we have in particular that $\pi(t)$ is unitary of inverse $\pi(t^{-1})$ for all t, which implies that $\pi(t^{-1}) = \pi(t)^*$ (Proposition B.10).

Define $a_j := \frac{1}{2}(\pi(g_j) + \pi(g_j^{-1})) = \frac{1}{2}(\pi(g_j) + \pi(g_j)^*)$, for all j, and note that $\|a_j\| \leq \frac{1}{2}(\|\pi(g_j)\| + \|\pi(g_j^{-1})\|) = 1$, as $\pi(t)$ is unitary for all t. Note also that $a_j^* = a_j$. Define, for all n (with i the imaginary unit)

$$R_n := \prod_{j=1}^{n} (I + \frac{i}{\sqrt{n}} a_j).$$

To bound the norm of R_n, we first have:

$$\begin{aligned}
\|(I + \frac{i}{\sqrt{n}} a_j)\|^2 &= \|(I + \frac{i}{\sqrt{n}} a_j)^*(I + \frac{i}{\sqrt{n}} a_j)\| \\
&= \|(I - \frac{i}{\sqrt{n}} a_j^*)(I + \frac{i}{\sqrt{n}} a_j)\| \\
&= \|(I + \frac{1}{n} a_j^* a_j)\| \\
&\leq \|I\| + \frac{1}{n}\|a_j^* a_j\| = 1 + \frac{1}{n}\|a_j\|^2 \leq 1 + \frac{1}{n},
\end{aligned}$$

using the fact that $\|TT^*\| = \|T\|^2$ for an operator T, so that (with $e \approx 2.7$)

$$\|R_n\|^2 \leq \prod_{j=1}^{n} (1 + \frac{1}{n}) = (1 + \frac{1}{n})^n \leq e, \qquad \forall n.$$

(Recall that e is the limit of the increasing sequence $(1 + \frac{1}{n})^n$). But we also have, by expanding the product in R_n (c.f. Proposition C.15):

$$\begin{aligned}
R_n = \prod_{j=1}^{n} (I + \frac{i}{\sqrt{n}} a_j) &= I + \sum_{j=1}^{n} \frac{i}{\sqrt{n}} a_j + \sum_k \psi_k \\
&= I + \frac{i}{2\sqrt{n}} \sum_{j=1}^{n} (\pi(g_j) + \pi(g_j^{-1})) + \sum_k \psi_k
\end{aligned} \tag{17}$$

for some ψ of finite index set, and such that for each index k, ψ_k is a product of at least two $\frac{i}{\sqrt{n}} a_j$s, and such that no $\frac{i}{\sqrt{n}} a_j$ appears twice; that is, ψ_k can be written

$$\psi_k = (\frac{i}{\sqrt{n}})^{l_k} a_{j_{1,k}} \cdot \ldots \cdot a_{j_{l_k,k}} = (\frac{i}{\sqrt{n}})^{l_k} (\pi(g_{j_{1,k}}) + \pi(g_{j_{1,k}}^{-1})) \cdot \ldots \cdot (\pi(g_{j_{l_k,k}}) + \pi(g_{j_{l_k,k}}^{-1}))$$

for some $l_k \geq 2$ and some sequence of indices $j_{\lambda,k}$, with $j_{\lambda,k} \neq j_{\lambda',k}$ whenever $\lambda \neq \lambda'$. Expanding ψ_k (Proposition C.16), we get:

$$\psi_k = (\frac{i}{\sqrt{n}})^{l_k} \sum_{\substack{\varepsilon_\lambda = \pm 1 \\ 1 \leq \lambda \leq l_k}} \pi(g_{j_{1,k}}^{\varepsilon_1}) \cdot \ldots \cdot \pi(g_{j_{l_k,k}}^{\varepsilon_{l_k}}) = (\frac{i}{\sqrt{n}})^{l_k} \sum_{\substack{\varepsilon_\lambda = \pm 1 \\ 1 \leq \lambda \leq l_k}} \pi(g_{j_{1,k}}^{\varepsilon_1} \cdot \ldots \cdot g_{j_{l_k,k}}^{\varepsilon_{l_k}}).$$

As we know that $l_k \geq 2$, and the $g_{j_{\lambda,k}}$ are all distinct, the word $g_{j_{1,k}}^{\varepsilon_1} \cdot \ldots \cdot g_{j_{l_k,k}}^{\varepsilon_{l_k}}$ is reduced and of length ≥ 2.

From this, we conclude that ψ_k is a polynomial in the variables $\pi(t)$, for a finite number of words t of length ≥ 2. This holds for any k, and we can therefore rewrite

$$\sum_k \psi_k = \sum_{|t|>1} \Psi_t \pi(t)$$

for $\Psi : F \to \mathbb{C}$ of finite support, by aggregating the sums in each ψ_k.

As ε (the neutral element of F) has length 0, and $I = \pi(\varepsilon)$ is one of the summands in Equation (17), we can let $\Psi_\varepsilon = 1$, and rewrite

$$R_n = \frac{i}{2\sqrt{n}} \sum_{j=1}^{n} (\pi(g_j) + \pi(g_j^{-1})) \;+\; \sum_{|t| \neq 1} \Psi_t \pi(t), \quad \forall n.$$

Then, expanding $\langle R_n x, y \rangle$ yields:

$$\begin{aligned}
\langle R_n x, y \rangle &= \frac{i}{2\sqrt{n}} \sum_{j=1}^{n} (\langle \pi(g_j) x, y \rangle + \langle \pi(g_j^{-1}) x, y \rangle) + \sum_{|t| \neq 1} \Psi_t \langle \pi(t) x, y \rangle \\
&= \frac{i}{2\sqrt{n}} \sum_{j=1}^{n} (1 + 1) \qquad\qquad + 0 \\
&= \frac{i}{\sqrt{n}} n = i\sqrt{n}, \qquad \forall n,
\end{aligned}$$

so that $\sqrt{n} = |i\sqrt{n}| = |\langle R_n x, y\rangle| \leq \|R_n\|\|x\|\|y\| \leq \sqrt{e}\|x\|\|y\|$. This is a contradiction, when $n \to \infty$, and we conclude that $f \notin B(G)$. ☐

Corollary 3.3 ([Pis01, Corollary 2.3]). *The free group F_∞ is not unitarizable.*

Proof. Indeed, Lemma 3.2 shows that $T_1(F_\infty) \not\subseteq B(F_\infty)$, which implies, by Theorem 3.1, that F_∞ is not unitarizable. ☐

From this, we can already conclude that

Corollary 3.4. *Any group containing F_∞ as a subgroup is non-unitarizable.*

Proof. Unitarizability passes to subgroups, and F_∞ is not unitarizable. ☐

Remark 3.5. The construction of Theorem 3.1 shows the existence of non unitarizable, uniformly bounded representations, of bound arbitrarily close to 1. Indeed, consider again $G = F_\infty$, f the indicator function of words of length 1, and fix $\varepsilon > 0$ arbitrary. Then, εf can be decomposed into εf_1 and εf_2, satisfying:

$$\sup_s \|\varepsilon f_1(s, \cdot)\|_1 \leq \varepsilon, \quad \sup_t \|\varepsilon f_2(\cdot, t)\|_1 \leq \varepsilon$$

as shown in Lemma 3.2. Then, following the construction in Theorem 3.1, the associated family of operators D has uniform bound at most 2ε, and the associated representation π satisfies $|\pi| \leq 1 + 4(\varepsilon^2 + \varepsilon)$, which can get arbitrarily close to 1. If π was unitarizable, εf would be an element of $B(G)$, which we know not to be true.

3.2 A Characterisation of Amenability

Theorem 3.6 ([Pis01, Theorem 2.4]). *Let G be a (discrete) group. The following are equivalent:*

1. *G is amenable;*
2. *There is a constant C such that for any $f \in \mathbb{C}^{(G)}$ with $f \geq 0$:*
$$\sum_{t \in G} f(t) \leq C \|\sum_{t \in G} f(t)\lambda(t)\|_{B(l^2(G))};$$
3. *For any $f \in \mathbb{C}^{(G)}$ with $f \geq 0$:*
$$\sum_{t \in G} f(t) \leq \|\sum_{t \in G} f(t)\lambda(t)\|_{B(l^2(G))}.$$

Proof. The fact that 3.⇒2. is obvious.

1. ⇒ 3. Assume that G is amenable. Equivalently, it satisfies the so-called "Reiter" property (Proposition C.5); that is:
There exists a net $\langle h_\alpha \rangle_{\alpha \in J}$ of positive elements of $l^1(G)$ of norm 1 such that:

$$\forall t \in G \ : \ \|\delta_t * h_\alpha - h_\alpha\|_1 \to 0.$$

Define $g_\alpha(x) := h_\alpha(x)^{\frac{1}{2}}$ for all $\alpha \in J$. We show that $\|g_\alpha\|_2 = 1$ and $\|\delta_t * g_\alpha - g_\alpha\|_2 \to 0$ for all t. First,

$$\|g_\alpha\|_2^2 = \sum_x |g_\alpha(x)|^2 = \sum_x h_\alpha(x) = \|h_\alpha\|_1 = 1,$$

and

$$\begin{aligned}
\|\delta_t * g_\alpha - g_\alpha\|_2^2 &= \sum_x |(\delta_t * g_\alpha)(x) - g_\alpha(x)|^2 \\
&= \sum_x |h_\alpha(t^{-1}x)^{\frac{1}{2}} - h_\alpha(x)^{\frac{1}{2}}|^2 \\
&\leq \sum_x |h_\alpha(t^{-1}x) - h_\alpha(x)| = \|\delta_t * h_\alpha - h_\alpha\| \to 0,
\end{aligned}$$

using the fact that for $a, b \geq 0$, $|a^2 - b^2| \geq |a - b|^2$. Thus, we have defined a net $\langle g_\alpha \rangle_{\alpha \in J}$ of elements of $l^2(G)$ of norm 1 and such that $\|\delta_t * g_\alpha - g_\alpha\|_2 \to 0$, for all t.

We wish to show that for any $f : G \to \mathbb{C}$ with $f \geq 0$ and of finite support:

$$\sum_t f(t) \leq \|\sum_t f(t)\lambda(t)\|_{B(l^2(G))}.$$

First, note that we can assume $\|\sum_t f(t)\lambda(t)\|_{B(l^2(G))} = 1$. Indeed, if this norm is C, let $f'(t) := f(t)/C$, then assuming the result holds $(*)$ when the norm is 1:

$$\sum_t f(t) = C\sum_t f'(t) \overset{(*)}{\leq} C\|\sum_t f'(t)\lambda(t)\|_{B(l^2(G))} = \|\sum_t Cf'(t)\lambda(t)\|_{B(l^2(G))} = \|\sum_t f(t)\lambda(t)\|_{B(l^2(G))},$$

and the result holds for any C.

Now, fix $f \geq 0$ with finite support and with $\|\sum_t f(t)\lambda(t)\|_{B(l^2(G))} = 1$, and let:

$$\begin{aligned} A(\alpha) &:= \langle \sum_t f(t)\lambda(t)g_\alpha, g_\alpha\rangle, \qquad \forall \alpha \in J \\ B(\alpha) &:= \langle \sum_t f(t)g_\alpha, g_\alpha\rangle, \qquad \forall \alpha \in J, \end{aligned}$$

which are well-defined because $\sum_t f(t)\lambda(t)$ is a finite sum of operators of $l^2(G)$, and g_α an element of $l^2(G)$. We have that

$$\begin{aligned} B(\alpha) = \sum_x \left(\sum_t f(t)g_\alpha(x)\right)\overline{g_\alpha(x)} &= \sum_x \left(\sum_t f(t)g_\alpha(x)\overline{g_\alpha(x)}\right) \\ &= \sum_t f(t)\sum_x |g_\alpha(x)|^2 = \sum_t f(t)\|g_\alpha\|_2^2 = \sum_t f(t), \end{aligned}$$

and

$$|A(\alpha)| \leq \|\sum_t f(t)\lambda(t)\|_{B(l^2(G))}\|g_\alpha\|_2^2 = \|\sum_t f(t)\lambda(t)\|_{B(l^2(G))} = 1.$$

Furthermore,

$$\begin{aligned} |A(\alpha) - B(\alpha)| &= |\langle \sum_t f(t)\lambda(t)g_\alpha, g_\alpha\rangle - \langle \sum_t f(t)g_\alpha, g_\alpha\rangle| \\ &= |\langle \sum_t f(t)\lambda(t)g_\alpha - \sum_t f(t)g_\alpha, g_\alpha\rangle| \\ &= |\langle \sum_t f(t)(\lambda(t) - 1)g_\alpha, g_\alpha\rangle| \\ &\leq \|\sum_t f(t)(\lambda(t)-1)g_\alpha\|_2\|g_\alpha\|_2 \\ &\leq \sum_t f(t)\|(\lambda(t)-1)g_\alpha\|_2 \cdot 1 \\ &= \sum_t f(t)\|\delta_t * g_\alpha - g_\alpha\|_2. \end{aligned}$$

As $\|\delta_t * g_\alpha - g_\alpha\|_2 \to 0$ for all t, and this last sum is finite, we conclude

$$|A(\alpha) - B(\alpha)| \to 0.$$

Given that $B(\alpha) = \sum_t f(t)$, our goal is to show that $B(\alpha) \leq 1$ for some (all) α. As $|A(\alpha)| \leq 1$ for all α, the net $A(\alpha)$ lies in the (compact) closed unit ball, and there exists a subnet $\langle A(\alpha_i)\rangle_i$ that converges to some point of norm ≤ 1. Then, for any index i_0

$$|B(\alpha_{i_0})| = \lim_i |B(\alpha_i)| \leq \lim_i |A(\alpha_i)| + |B(\alpha_i) - A(\alpha_i)| = \lim_i |A(\alpha_i)| + \lim_i |B(\alpha_i) - A(\alpha_i)| \leq 1 + 0 = 1.$$

We have shown that

$$\sum_t f(t) = B(\alpha_{i_0}) \leq 1 = \|\sum_t f(t)\lambda_t\|_{B(l^2(G))},$$

as needed, and we are done.

2. $\Rightarrow$ **3.** First, recall that the convolution operator, "$*$", is defined as:

$$\begin{aligned} * : \mathbb{C}[G] \times \mathbb{C}[G] &\to \mathbb{C}[G] \\ f, g &\mapsto \langle \sum_s f(s)g(s^{-1}t)\rangle_t, \end{aligned}$$

and note that, for any $f, g \in \mathbb{C}[G]$:

$$\sum_x (f * g)(x) = \sum_x\sum_s f(s)g(s^{-1}x) = \sum_s f(s)\sum_x g(s^{-1}x) = \sum_s f(s)\sum_t g(t) = (\sum_t f(t))(\sum_t g(t)),$$

so that, by induction:

$$\left(\sum_t f(t)\right)^n = \sum_t f^{*n}(t), \quad \forall f \in \mathbb{C}[G],\ \forall n.$$

Also, for any $f, g \in \mathbb{C}[G]$:

$$\begin{aligned}
\sum_x (f*g)(x)\lambda(x) &= \sum_x(\sum_s f(s)g(s^{-1}x))\lambda(x) = \sum_x(\sum_s f(s)g(s^{-1}x))\lambda(ss^{-1}x) \\
&= \sum_x(\sum_s f(s)g(s^{-1}x))\lambda(s)\lambda(s^{-1}x) = \sum_s f(s)\lambda(s)\sum_x g(s^{-1}x)\lambda(s^{-1}x) \\
&= (\sum_s f(s)\lambda(s))(\sum_t g(t)\lambda(t)),
\end{aligned}$$

using the fact that λ is a representation. Then, again, by induction:

$$\left(\sum_t f(t)\lambda(t)\right)^n = \sum_t f^{*n}(t)\lambda(t) \quad \forall f \in \mathbb{C}[G],\ \forall n.$$

Now, assume 2. holds for the constant C, and fix some positive $f \in \mathbb{C}[G]$. We have that

$$\left(\sum_t f(t)\right)^n = \sum_t f^{*n}(t) \le C\|\sum_t f^{*n}(t)\lambda(t)\| = C\|\left(\sum_t f(t)\lambda(t)\right)^n\| \le C\|\sum_t f(t)\lambda(t)\|^n, \quad \forall n,$$

and taking the n-th root:

$$\sum_t f(t) \le C^{1/n}\|\sum_t f(t)\lambda(t)\|, \quad \forall n.$$

Since $C^{1/n} \to 1$ when $n \to \infty$, taking the limit allows us to conclude 3..

3. $\Rightarrow$ **1.** Assuming 3., we know in particular that for any finite subset E of G, its indicator function $\mathbb{1}_E$ satisfies the inequality (being positive and of finite support), so that

$$|E| = \sum_t \mathbb{1}_E(t) \le \|\sum_t \mathbb{1}_E(t)\lambda(t)\|_{B(l^2(G))} = \|\sum_{t\in E}\lambda(t)\|_{B(l^2(G))}.$$

Because $\|\sum\lambda(t)\| \le \sum\|\lambda(t)\|$ (triangle inequality) and $\lambda(t)$ is unitary, it holds that

$$\|\sum_{t\in E}\lambda(t)\|_{B(l^2(G))} \le |E|$$

too, and we have an equality. By definition of the operator norm, this implies that there exists some sequence $\langle g_n^E\rangle_n \in l^2(G)$ with $\|g_n^E\| = 1$ and

$$\|\sum_{t\in E}\lambda(t)g_n^E\| \to |E|,$$

that is

$$\|\sum_{t\in E}\delta_t * g_n^E\| \to |E|.$$

By considering the sequence of $|E|$-tuples $\langle\delta_t * g_n^E\rangle_{t\in E}$, we can apply Proposition A.11 (because l^2 is Hilbert, hence uniformly convex), and get that, for any $s, t \in E$:

$$\|\delta_s * g_n^E - \delta_t * g_n^E\| \to 0,$$

and in particular, if $e \in E$ (e being the neutral element of G), for any $t \in E$:

$$\|\delta_t * g_n^E - g_n^E\| \to 0.$$

In summary, we have constructed, for all $E \subseteq_f G$ containing e, a sequence $\langle g_n^E\rangle_{n\in\mathbb{N}} \in l^2(G)$ with $\|g_n^E\|_2 = 1$ and

$$\|\delta_t * g_n^E - g_n^E\|_2 \to 0 \qquad \forall t \in E,$$

that is,

$$\forall t \in E\ \forall\varepsilon > 0\ \exists n_0(t,\varepsilon)\ \forall n \ge n_0(t,\varepsilon)\ :\ \|\delta_t * g_n^E - g_n^E\|_2 < \varepsilon.$$

This implies in particular that:

$$\forall E \subseteq_f G\ \forall\varepsilon > 0\ \exists g \in l^2(G)\ :\ \|g\|_2 = 1 \text{ and } \forall t \in E\ \|\delta_t * g - g\|_2 < \varepsilon. \tag{18}$$

Indeed, fix some $E \subseteq_f G$ and $\varepsilon > 0$, let $E' := E \cup \{e\}$ and

$$N := \max\{n_0(t, \varepsilon) : t \in E'\}.$$

Then

$$\forall t \in E' \ : \ \|\delta_t * g_N^{E'} - g_N^{E'}\|_2 < \varepsilon,$$

which shows Equation (18).

If $g \in l^2(G)$ is such that $\|g\|_2 = 1$ and $\|\delta_t * g - g\|_2 < \varepsilon$, let $h(t) := |g(t)|^2$, then

$$\sum_t |h(t)| = \sum_t |g(t)|^2 = \|g\|_2^2 = 1$$

so that $\|h\|_1 = 1$, and

$$\begin{aligned}
\|\delta_t * h - h\|_1 &= \sum_x |(\delta_t * h)(x) - h(x)| \\
&= \sum_x \big||g(t^{-1}x)|^2 - |g(x)|^2\big| \\
&= \sum_x \big||g(t^{-1}x)| - |g(x)|\big| \cdot \big||g(t^{-1}x)| + |g(x)|\big| \\
&\leq \left(\sum_x \big||g(t^{-1}x)| - |g(x)|\big|^2\right)^{\frac{1}{2}} \left(\sum_x \big||g(t^{-1}x)| + |g(x)|\big|^2\right)^{\frac{1}{2}} \\
&\leq \left(\sum_x \big|g(t^{-1}x) - g(x)\big|^2\right)^{\frac{1}{2}} \left(\sum_x \big|g(t^{-1}x) + g(x)\big|^2\right)^{\frac{1}{2}} \\
&= \|\delta_t * g - g\|_2 \|\delta_t * g + g\|_2 \\
&\leq \|\delta_t * g - g\|_2 (\|\delta_t * g\|_2 + \|g\|_2) \leq \|\delta_t * g - g\|_2 2 \leq 2\varepsilon
\end{aligned}$$

using Cauchy-Schwarz. It follows that

$$\forall E \subseteq_f G \ \forall \varepsilon > 0 \ \exists h \in l^1(G) \ : \ \|h\|_1 = 1, \ h \geq 0 \text{ and } \forall t \in E \ \|\delta_t * h - h\|_2 < \varepsilon, \tag{19}$$

by taking $h = |g|^2$ with g given by $\varepsilon/2$ in Equation (18), and noting that h is positive by construction.

Now, consider the directed set $\Lambda = \mathcal{P}_f(G) \times \mathbb{R}^{>0}$ with order $(E, \varepsilon) \geq (F, \delta)$ if and only if $F \subseteq E$ and $\varepsilon \leq \delta$, and define the net $\langle h_{(E,\varepsilon)} \rangle_{(E,\varepsilon) \in \Lambda}$ where $h_{(E,\varepsilon)}$ is given by Equation (19) for E and ε. This net is such that

$$\forall t \in G \ : \ \|\delta_t * h_\lambda - h_\lambda\|_1 \to 0.$$

Indeed, given any $t \in G$ and $\varepsilon > 0$, let $\lambda_0 := (\{t\}, \varepsilon)$; then, any h_λ with $\lambda \geq \lambda_0$ will satisfy $\|\delta_t * h_\lambda - h_\lambda\|_1 < \varepsilon$ by definition. We conclude that the Reiter property is satisfied by the net $\langle h_\lambda \rangle_\lambda$, and G is amenable.

□

Theorem 3.7 ([Pis01, Theorem 2.5])**.** *A discrete group G is amenable if and only if $T_p(G) \subseteq l^p(G)$, for all $1 \leq p < \infty$.*

In particular, as $l^p(G) \subseteq T_p(G)$ always holds, we have an equality whenever G is amenable.

Proof. G **amenable** $\Rightarrow$ $T_p \subseteq l^p$**.** Assume that G is amenable and let $\varphi : l^\infty(G) \to \mathbb{C}$ be an invariant mean, as given by amenability. Take any $f \in T_p(G)$, and any $f_1, f_2 \in W_{T_p}(f)$; that is, f_1, f_2 satisfy:

$$C_1 := \sup_s \sum_t |f_1(s,t)|^p < \infty, \quad C_2 := \sup_t \sum_s |f_2(s,t)|^p < \infty \tag{20}$$

and

$$f(st) = f_1(s,t) + f_2(s,t) \quad \forall s, t.$$

First, observe that for any s, t and x in G:

$$f(st) = f(sxx^{-1}t) = f_1(sx, x^{-1}t) + f_2(sx, x^{-1}t). \tag{21}$$

Define the family $F_i(s,t) : G \to \mathbb{C}$ by

$$F_i(s,t)(x) = f_i(sx, x^{-1}t), \qquad i = 1, 2, \ s, t \in G$$

Then, for any s,t, we have $F_i(s,t) \in l^\infty(G)$. Indeed, the inequalities in Equation (20) imply in particular that

$$\sup_{s,t} |f_i(s,t)| < \infty,$$

so that

$$\|F_i(s,t)\|_\infty = \sup_x |f_i(sx, x^{-1}t)| < \infty.$$

Now, fix s,t, and apply φ to Equation (21), as functions of the variable x:

$$\varphi(x \mapsto f(st)) = \varphi(x \mapsto f_1(sx, x^{-1}t) + f_2(sx, x^{-1}t))$$

which by linearity, the fact that φ maps constants to themselves, and by definition of the maps F_i, yields

$$f(st) = \varphi(F_1(s,t)) + \varphi(F_2(s,t)).$$

Define $\tilde{f}_i(s,t) := \varphi(F_i(s,t))$, we can then rewrite the previous equation:

$$f(st) = \tilde{f}_1(s,t) + \tilde{f}_2(s,t).$$

Now, note that

$$\tilde{f}_i(sy, y^{-1}t) = \varphi\left(x \mapsto f_i(syx, x^{-1}y^{-1}t)\right) = \varphi\left(\delta_{y^{-1}} * F_i(s,t)\right) = \varphi(F_i(s,t)) = \tilde{f}_i(s,t)$$

by invariance of φ, so that in particular $\tilde{f}_i(e, st) = \tilde{f}_i(s,t) = \tilde{f}_i(st, e)$ (for $i = 1,2$), and

$$f(x) = \tilde{f}_1(e,x) + \tilde{f}_2(x,e), \quad \forall x.$$

We will show that:

$$\sum_t |\tilde{f}_1(e,t)|^p \leq \sup_s \sum_t |f_1(s,t)|^p < \infty, \qquad \text{and} \qquad \sum_s |\tilde{f}_2(s,e)|^p \leq \sup_t \sum_s |f_2(s,t)|^p < \infty$$

which will imply that $\tilde{f}_1(e,\cdot)$ and $\tilde{f}_2(\cdot,e)$ are both in l^p, and therefore so is f.

The goal is now to show $\|\tilde{f}_1(e,\cdot)\|_p^p \leq \sup_s \|f_1(s,\cdot)\|_p^p$, and we actually show

$$\sup_{s'} \|\tilde{f}_1(s',\cdot)\|_p^p \leq \sup_s \|f_1(s,\cdot)\|_p^p,$$

and similarly for $\tilde{f}_2$ and f_2.

Fix any $E \subseteq_f G$, and $s \in G$, and recall that by Proposition C.8, for any $F \in l^\infty(G)$ and $p \geq 1$, we have $|\varphi(F)|^p \leq \varphi(|F|^p)$; then:

$$\begin{aligned}
\sum_{t\in E} |\tilde{f}_1(s,t)|^p &= \sum_{t\in E} |\varphi(F_1(s,t))|^p \\
&\leq \sum_{t\in E} \varphi(|F_1(s,t)|^p) && \text{(Proposition C.8)} \\
&= \varphi(\sum_{t\in E} |F_1(s,t)|^p) && \text{(linearity of } \varphi) \\
&\leq \|\sum_{t\in E} |F_1(s,t)|^p\|_\infty && (\|\varphi\| = 1) \\
&= \sup_x \left|\sum_{t\in E} |f_1(sx, x^{-1}t)|^p\right| \\
&= \sup_x \sum_{t\in E} |f_1(sx, x^{-1}t)|^p \\
&\leq \sup_x \sum_{t\in G} |f_1(sx, x^{-1}t)|^p \\
&= \sup_x \sum_{t'\in G} |f_1(sx, t')|^p && (\text{letting } t' = x^{-1}t) \\
&= \sup_{x'} \sum_{t'\in G} |f_1(x', t')|^p && (\text{letting } x' = sx).
\end{aligned}$$

Then, taking the supremum over the finite subsets E of G and over $s \in G$:

$$\sup_s \sum_t |\tilde{f}_1(s,t)|^p \le \sup_s \sum_t |f_1(s,t)|^p$$

and in particular

$$\sum_t |\tilde{f}_1(e,t)|^p \le \sup_s \sum_t |f_1(s,t)|^p = C_1. \tag{22}$$

A similar argument shows that

$$\sum_s |\tilde{f}_2(s,e)|^p \le \sup_t \sum_s |f_2(s,t)|^p = C_2. \tag{23}$$

Therefore, as $f = \tilde{f}_1(e,\cdot) + \tilde{f}_2(\cdot,e)$, is a sum of elements of $l^p(G)$, we have $f \in l^p(G)$.

We can actually show a bit more: $\|f\|_p \le \|f\|_{T_p}$. Taking the p-th root in Equation (22) and Equation (23), and using the triangle inequality for $f = \tilde{f}_1(e,\cdot) + \tilde{f}_2(\cdot,e)$, we get:

$$\|f\|_p \le \|\tilde{f}_1(e,\cdot)\|_p + \|\tilde{f}_2(\cdot,e)\|_p \le \sup_s \|f_1(s,\cdot)\|_p + \sup_t \|f_2(\cdot,t)\|_p$$

but as the T_p-norm is defined as the infimum of the RHS over the possible decompositions $f = f_1 + f_2$, we reach the desired inequality.

$T_p \subseteq l^p \Rightarrow G$ **amenable.** Assume $T_p(G) \subseteq l^p(G)$ for all p, and in particular for $p = 1$.

We show that G is amenable using the characterisation given by Theorem 3.6. Thus, we are looking for some C such that for any $f \in \mathbb{C}^{(G)}$ with $f \ge 0$, we have:

$$\sum_t f(t) \le C \|\sum_t f(t)\lambda(t)\|_{B(l^2(G))}.$$

By Proposition A.4, we know that $T_1(G) \subseteq l^1(G)$ implies the existence of some C_0 with

$$\|f\|_1 \le C_0 \|f\|_{T_1}$$

for all $f \in T_1(G)$, and a fortiori for all $f \in \mathbb{C}^{(G)}$. In particular, if $f \ge 0$, we have:

$$\sum_t f(t) \le C_0 \|f\|_{T_1}.$$

So, we are done if we can find some C_1 such that for all positive $f \in \mathbb{C}^{(G)}$:

$$\|f\|_{T_1} \le C_1 \|\sum_t f(t)\lambda(t)\|_{B(l^2(G))}.$$

Fix E and F finite subsets of G, and let $v_f := \sum_t f(t)\lambda(t)$. We know that

$$\begin{aligned} v_f(\mathbb{1}_F)(s) &= (\sum_t f(t)\lambda(t)\mathbb{1}_F)(s) = \sum_t f(t)(\lambda(t)\mathbb{1}_F)(s) \\ &= \sum_t f(t)\mathbb{1}_F(t^{-1}s) = \sum_{t'} f(st'^{-1})\mathbb{1}_F(t') \\ &= \sum_{t' \in F} f(st'^{-1}), \end{aligned}$$

so that

$$\langle \mathbb{1}_E, v_f(\mathbb{1}_F)\rangle = \sum_{s \in E} v_f(\mathbb{1}_F)(s) = \sum_{s \in E, x \in F} f(sx^{-1}).$$

We also have, by Cauchy-Schwarz:

$$|\langle \mathbb{1}_E, v_f(\mathbb{1}_F)\rangle| \le \|v_f\|_{B(l^2(G))} \|\mathbb{1}_E\|_2 \|\mathbb{1}_F\|_2 = \|v_f\|_{B(l^2(G))} \sqrt{|E||F|}.$$

Combining the preceding results, we get

$$\sum_{s \in E, x \in F} f(sx^{-1}) = \left| \sum_{s \in E, x \in F} f(sx^{-1}) \right| \le \|v_f\|_{B(l^2(G))} \sqrt{|E||F|},$$

which implies

$$\sum_{s\in E, t\in F} f(st) = \sum_{s\in E, t\in F^{-1}} f(st^{-1}) \le \|v_f\|_{B(l^2(G))}\sqrt{|E||F^{-1}|} = \|v_f\|_{B(l^2(G))}\sqrt{|E||F|}$$

and in particular, if both E, F have cardinality n, we have

$$\sum_{s\in E, t\in F} f(st) \le \|v_f\|_{B(l^2(G))} n.$$

We can now apply Corollary 3.9 below on $a(s,t) := f(st)$ (recall that $f \ge 0$) and get a decomposition

$$f(st) = b_1(s,t) + b_2(s,t)$$

with

$$\sup_s \sum_t |b_1(s,t)| \le \|v_f\|, \qquad \sup_t \sum_s |b_2(s,t)| \le \|v_f\|,$$

so that

$$\|f\|_{T_p} \le 2\|v_f\|_{B(l^2(G))} = 2\|\sum_t f(t)\lambda(t)\|_{B(l^2(G))}$$

and $C_1 = 2$ satisfies what it needs to. That is, for any $f \in \mathbb{C}^{(G)}$ with $f \ge 0$, we have

$$\sum_t f(t) \le C_0\|f\|_{T_1} \le C_0 C_1 \|\sum_t f(t)\lambda(t)\|_{B(l^2(G))}$$

and by the second characterisation of amenability of Theorem 3.7, G is amenable.

□

Lemma 3.8 ([Pis01, Lemma 2.6]). *Let $a : S \times T \to \mathbb{C}$ be an "infinite matrix". If*

$$\forall E \subseteq S,\ F \subseteq T \textit{ with } |E| = |F| = n \in \mathbb{N} : \quad \sum_{(s,t)\in E\times F} |a(s,t)| \le n, \tag{24}$$

then there exists a decomposition $a = b + c$ (that is, $a(s,t) = b(s,t) + c(s,t),\ \forall s \in S\ \forall t \in T$) such that:

$$\sup_s \sum_t |b(s,t)| \le 1, \quad \textit{and} \quad \sup_t \sum_s |c(s,t)| \le 1, \tag{25}$$

and b and c have disjoint support, i.e. $b(s,t)c(s,t) = 0\ \forall (s,t) \in S \times T$.

Proof. We first show that the lemma holds for finite S, T of same cardinality, and then for arbitrary S, T, using the finite case. For clarity only, let us define:

$$\Phi(S,T,a) :\equiv \text{ For all } E \subseteq S,\ F \subseteq T \text{ with } |E| = |F| = n \in \mathbb{N}\text{:} \quad \sum_{(s,t)\in E\times F} |a(s,t)| \le n,$$

and

$$\begin{aligned}\Psi(S,T,a,b,c) :\equiv \forall s \in S\ \forall t \in T : \quad & a(s,t) = b(s,t) + c(s,t),\ b(s,t)c(s,t) = 0,\\ \text{and} \quad & \sup_{s\in S}\sum_{t\in T}|b(s,t)| \le 1, \quad \sup_{t\in T}\sum_{s\in S}|c(s,t)| \le 1,\end{aligned}$$

so that the lemma can be written:

$$\forall S\ \forall T\ \forall a \in \mathbb{C}^{S\times T} : \quad \left(\Phi(S,T,a) \quad \Rightarrow \quad \exists b, c \in \mathbb{C}^{S\times T} :\ \Psi(S,T,a,b,c)\right). \tag{26}$$

We first note that if $\Phi(S,T,a)$ holds and $S' \subseteq S, T' \subseteq T$ and a' is the restriction of a to $S' \times T'$, then $\Phi(S',T',a')$ holds.

$|S| = |T|$ **finite.** We first show that Equation (26) holds when S, T are finite of same cardinality $\mathbb{N}$. For the base case, if $N = 1$, and given $a \in \mathbb{C}^{S\times T}$ satisfying $\Phi(S,T,a)$, simply choose $b = a$ and $c = 0$. For the induction step, fix N, assume Equation (26) holds when S, T are of size $N - 1$, and let S, T of size N and $a \in \mathbb{C}^{S\times T}$ satisfying $\Phi(S,T,a)$. Then, letting $E := S, F := T$, we get (dividing by N):

$$\frac{1}{N}\sum_{s\in S}\left(\sum_{t\in T}|a(s,t)|\right) = \frac{1}{N}\sum_{t\in T}\left(\sum_{s\in S}|a(s,t)|\right) \le 1$$

which implies the existence of s_0 and t_0 such that:

$$\sum_{t\in T}|a(s_0,t)|\leq 1 \quad \text{and} \quad \sum_{s\in S}|a(s,t_0)|\leq 1.$$

(If an average is $\leq$ some k, at least one of the averaged values is $\leq k$). Let $S' := S-\{s_0\}$,$T' := T-\{t_0\}$ and a' the restriction of a to $S'\times T'$. Then $\Phi(S',T',a')$ holds, and $|S'|=|T'|=N-1$, and by the induction hypothesis, there exists $b',c' : S'\times T'\to\mathbb{C}$ such that $\Psi(S',T',a',b',c')$ holds. Define $b,c : S\times T\to\mathbb{C}$ as extensions of b',c' respectively by:

$$\begin{aligned} b(s_0,t) =& \quad a(s_0,t), & c(s_0,t) =& \quad 0, & \forall t\in T',\\ b(s,t_0) =& \quad 0, & c(s,t_0) =& a(s,t_0), & \forall s\in S',\\ b(s_0,t_0) =& a(s_0,t_0), & c(s_0,t_0) =& \quad 0. \end{aligned}$$

(we could have chosen $b(s_0,t_0)=0, c(s_0,t_0)=a(s_0,t_0)$ as well) and we verify that $\Psi(S,T,a,b,c)$ holds:

$$\begin{aligned} s\in S': \quad & \sum_{t\in T}|b(s,t)| = |b(s,t_0)| + \sum_{t\in T'}|b(s,t)| = 0 + \sum_{t\in T'}|b'(s,t)|\leq 1,\\ s_0: \quad & \sum_{t\in T}|b(s_0,t)|\leq 1, \end{aligned}$$

where the first line holds by the choice of b', and the second by the choice of s_0. It works out similarly for c. The fact that $a=b+c$ is direct, and $b(s,t)c(s,t)=0$ holds on $S'\times T'$ by the induction hypothesis, and on $\{s_0\}\times T\cup S\times\{t_0\}$ by our choice of b,c on those values.

S,T **arbitrary.** Fix S,T and $a\in\mathbb{C}^{S\times T}$ satisfying $\Phi(S,T,a)$. We know that for any $\sigma\subseteq_f S$ and $\tau\subseteq_f T$ with $|\sigma|=|\tau|$, $\Phi(\sigma,\tau,a|_{\sigma\times\tau})$ holds, and by the finite case, there exists $b'_{\sigma,\tau},c'_{\sigma,\tau} : \sigma\times\tau\to\mathbb{C}$ such that $\Psi(\sigma,\tau,a|_{\sigma\times\tau},b'_{\sigma,\tau},c'_{\sigma,\tau})$ holds. Choose one such pair for each σ,τ, and let $b_{\sigma,\tau},c_{\sigma,\tau}$ be the extensions (to $S\times T$) with zeros of $b'_{\sigma,\tau},c'_{\sigma,\tau}$ respectively. Note that $b_{\sigma,\tau}(s,t)\in\overline{B}(0,1)$ at all s,t, because the condition in Ψ implies in particular that

$$\sup_{s,t}|b(s,t)|\leq 1.$$

Consider the directed set $\mathcal{P}_f(S)\times\mathcal{P}_f(T)$, whose elements are the pairs σ,τ of *same* (finite) cardinality, and the net

$$(b_{\sigma,\tau},c_{\sigma,\tau})_{\sigma,\tau}\in\overline{B}(0,1)^{S\times T}\times\overline{B}(0,1)^{S\times T}.$$

As this set is compact, the net has an accumulation point $(b,c)\in\overline{B}(0,1)^{S\times T}\times\overline{B}(0,1)^{S\times T}$, and we show that this point satisfies $\Psi(S,T,a,b,c)$.

We first show that $a=b+c$. Assume there is some point s,t with $a(s,t)\neq b(s,t)+c(s,t)$, so that there exists $\delta>0$ with $|a(s,t)-(b(s,t)+c(s,t))|>\delta>0$. Fix $\sigma_0=\{s\},\tau_0=\{t\}$, and the open neighbourhood U of b,c:

$$U := \{f : |f(s,t)-b(s,t)|<\delta/2\}\times\{g : |g(s,t)-c(s,t)|<\delta/2\}.$$

We know that there exists $(\sigma,\tau)\geq(\sigma_0,\tau_0)$ (that is, $\sigma_0\subseteq\sigma$ and $\tau_0\subseteq\tau$) with $(b_{\sigma,\tau},c_{\sigma,\tau})\in U$. Then, $s\in\sigma_0\subseteq\sigma$ and $t\in\tau_0\subseteq\tau_0$, and as $b'_{\sigma,\tau}+c'_{\sigma,\tau}=a|_{\sigma\times\tau}$ (by our choice of $b'_{\sigma,\tau},c'_{\sigma,\tau}$), and $s\in\sigma,t\in\tau$:

$$b_{\sigma,\tau}(s,t)+c_{\sigma,\tau}(s,t) = b'_{\sigma,\tau}(s,t)+c'_{\sigma,\tau}(s,t) = a|_{\sigma\times\tau}(s,t) = a(s,t).$$

We also have:

$$|b_{\sigma,\tau}(s,t)-b(s,t)|<\delta/2, \quad |c_{\sigma,\tau}(s,t)-c(s,t)|<\delta/2$$

because the pair $(b_{\sigma,\tau},c_{\sigma,\tau})$ is in U, so that:

$$\begin{aligned} |a(s,t)-(b(s,t)+c(s,t))| &= |a(s,t)-(b_{\sigma,\tau}(s,t)+c_{\sigma,\tau}(s,t))+(b_{\sigma,\tau}(s,t)+c_{\sigma,\tau}(s,t))-(b(s,t)+c(s,t))|\\ &\leq |a(s,t)-(b_{\sigma,\tau}(s,t)+c_{\sigma,\tau}(s,t))|+|b_{\sigma,\tau}(s,t)-b(s,t)|+|c_{\sigma,\tau}(s,t)-c(s,t)|\\ &\leq 0+\delta/2+\delta/2=\delta \end{aligned}$$

which is a contradiction, and we conclude that $a=b+c$.

We now show that b and c have disjoint support. Assume there exists s,t with $b(s,t)\neq 0$ and $c(s,t)\neq 0$, and choose some δ with $|b(s,t)|,|c(s,t)|>\delta>0$. Let $\sigma_0 := \{s\}$ and $\tau_0 := \{t\}$, and the open neighbourhood U of (b,c):

$$U := \{f : |f(s,t)-b(s,t)|<\delta\}\times\{f : |f(s,t)-c(s,t)|<\delta\}.$$

Then, there exists some σ, τ with $\sigma_0 \subseteq \sigma$, $\tau_0 \subseteq \tau$ and $(b_{\sigma,\tau}, c_{\sigma,\tau}) \in U$. As $s \in \sigma_0 \subseteq \sigma$ and $t \in \tau_0 \subseteq \tau$, $b_{\sigma,\tau}(s,t)c_{\sigma,\tau}(s,t) = b'_{\sigma,\tau}(s,t)c'_{\sigma,\tau}(s,t) = 0$; assume without loss of generality that $b_{\sigma,\tau}(s,t) = 0$. Then $\delta > |b_{\sigma,\tau}(s,t) - b(s,t)| = |b(s,t)|$ (since $(b_{\sigma,\tau}, c_{\sigma,\tau}) \in U$), which contradicts the fact that $|b(s,t)| > \delta$, and we conclude that $b(s,t)c(s,t) = 0$.

Finally, we verify the bounds on b and c. We only consider b, the argument being similar for c. Assume

$$\sup_s \sum_t |b(s,t)| > 1.$$

There must exist some s and a finite $F \subseteq_f T$ with some $\delta > 0$ and:

$$\sum_{t \in F} |b(s,t)| > \delta + 1 > 1.$$

Similarly as before, take $\sigma_0 := \{s\}$ and $\tau_0 := F$, and let (with $n := |F|$)

$$U := \{f : |f(s,t) - b(s,t)| < \delta/n, \ \forall t \in F\} \times \overline{B}(0,1)^{S \times T},$$

(here we don't care about the c component, we could just argue that b is an accumulation point of the net $\langle b_{\sigma,\tau} \rangle_{\sigma,\tau}$ instead) and there exists some σ, τ with $\sigma_0 \subseteq \sigma$ and $\tau_0 \subseteq \tau$ and $(b_{\sigma,\tau}, c_{\sigma,\tau}) \in U$. Then, we know that

$$\sum_{t' \in F} |b_{\sigma,\tau}(s,t')| = \sum_{t' \in F} |b'_{\sigma,\tau}(s,t')| \le \sup_{s' \in \sigma} \sum_{t' \in \tau} |b'_{\sigma,\tau}(s',t')| \le 1$$

by the choice of $b'_{\sigma,\tau}$ (satisfying Ψ), and as $F \subseteq \tau$ and $s \in \sigma$:

$$\sum_{t \in F} |b(s,t)| \le \sum_{t \in F} |b_{\sigma,\tau}(s,t)| + \sum_{t \in F} |b(s,t) - b_{\sigma,\tau}(s,t)| \le 1 + n\delta/n = 1 + \delta$$

because $(b_{\sigma,\tau}, c_{\sigma,\tau}) \in U$ which is again a contradiction.

We conclude that b, c satisfy $a = b + c$, having disjoint support and being bounded as needed, and $\Psi(S,T,a,b,c)$ holds, and we are done. $\square$

This fact still holds if we "scale" the bounds.

Corollary 3.9. *Let $a : S \times T \to \mathbb{C}$, and k such that, for all n and $\sigma \subseteq_f S$, $\tau \subseteq_f T$ of same finite cardinality n:*

$$\sum_{s \in \sigma} \sum_{t \in \tau} |a(s,t)| \le nk.$$

Then, there exists $b, c : S \times T \to \mathbb{C}$ such that $a = b + c$, b and c have disjoint support, and

$$\sup_s \sum_t |b(s,t)| \le k, \qquad \sup_t \sum_s |c(s,t)| \le k.$$

Proof. Define $a' : S \times T \to \mathbb{C}$ by $a'(s,t) = a(s,t)/k$. Then a' satisfies the premise of Lemma 3.8, and there exists $b', c' : S \times T \to \mathbb{C}$ of disjoint support, such that $a' = b' + c'$ and

$$\sup_s \sum_t |b'(s,t)| \le 1, \qquad \sup_t \sum_s |c'(s,t)| \le 1.$$

Define $b(s,t) = kb'(s,t)$ and $c(s,t) = kc'(s,t)$. Then $a(s,t) = ka'(s,t) = k(b'(s,t) + c'(s,t)) = b(s,t) + c(s,t)$, b and c have disjoint support and:

$$\sup_s \sum_t |b(s,t)| = \sup_s \sum_t k|b'(s,t)| = k \sup_s \sum_t |b'(s,t)| \le k$$

and similarly for c. $\square$

Corollary 3.10. *Fix $p \ge 1$ and let $a : S \times T \to \mathbb{C}$, and k such that, for all n and $\sigma \subseteq_f S$, $\tau \subseteq_f T$ of same finite cardinality n:*

$$\sum_{s \in \sigma} \sum_{t \in \tau} |a(s,t)|^p \le nk.$$

Then, there exists $b, c : S \times T \to \mathbb{C}$ such that $a = b + c$, b and c have disjoint support, and

$$\sup_s \sum_t |b(s,t)|^p \le k, \qquad \sup_t \sum_s |c(s,t)|^p \le k.$$

Proof. Define $a'(s,t) = |a(s,t)|^p$. Then a' satisfies the premise of Corollary 3.9, and there exists b', c' of disjoint support, such that $a' = b' + c'$ and

$$\sup_s \sum_t |b'(s,t)| \le k, \qquad \sup_t \sum_s |c'(s,t)| \le k.$$

Note that both b' and c' must be ≥ 0, because so is a', and having disjoint support, they can't compensate each other. Let $b(s,t) := \varphi(s,t)b'(s,t)^{1/p}$ and $c(s,t) := \varphi(s,t)c'(s,t)^{1/p}$, where $\varphi(s,t)$ is the argument of $a(s,t)$ (so that $a(s,t) = \varphi(s,t)|a(s,t)|$). Then

$$\begin{aligned}
b(s,t) + c(s,t) &= \varphi(s,t)b'(s,t)^{1/p} + \varphi(s,t)c'(s,t)^{1/p} \\
&= \varphi(s,t)(b'(s,t)^{1/p} + c'(s,t)^{1/p}) \\
&= \varphi(s,t)(b'(s,t) + c'(s,t))^{1/p} \\
&= \varphi(s,t)(a'(s,t)^{1/p}) \\
&= \varphi(s,t)(|a(s,t)|^{p\,1/p}) = a(s,t),
\end{aligned}$$

where the third equality comes from the fact that either $b'(s,t)$ or $c'(s,t)$ is zero at any (s,t). Also, b and c have disjoint support because so do b' and c', and

$$\sup_s \sum_t |b(s,t)|^p = \sup_s \sum_t |\varphi(s,t)|^p |b'(s,t)^{1/p}|^p = \sup_s \sum_t |b'(s,t)| \le k,$$

because $|\varphi(s,t)| = 1\ \forall s,t$, and similarly for c. □

3.3 Free Subgroup & Non-Unitarizability

If G is a group, a *free set* in G is a subset E of G such that the subgroup generated by E is free of base E; said differently, no non-trivial reduced formal word on the elements of E (and their formal inverses) evaluates to the identity. Write $\mathbb{C}^{D\subseteq G}$ for the subset of $\mathbb{C}^G$ of maps with support in D, for any $D \subseteq G$.

Lemma 3.11 ([Pis01, Lemma 2.7])**.** *Let E be a free set in some group G, and let $E^{-1} := \{t^{-1} : t \in E\}$.*

1. *For any $f \in B(G)$ supported on $E \cup E^{-1}$:*

$$\|f\|_2 \le 4\sqrt{e}\|f\|_{B(G)}.$$

2. *For any subgroup F of G, $T_1(F)$ can be isometrically identified with $T_1(G) \cap \mathbb{C}^{F\subseteq G}$.*
3. *If E is a countable infinite free set, the indicator function of $E \cup E^{-1}$ is in $T_1(G)$ but not in $B(G)$.*
4. *For all $N \ge 2$: $T_1(F_N) \not\subseteq B(F_N)$.*

Proof. 1. Fix some $f \in B(G) \cap \mathbb{C}^{E\cup E^{-1}\subseteq G}$. Assume we have shown that for all $\alpha : G \to \mathbb{C}$ finitely supported with $\|\alpha\|_2 = \sum_t |\alpha(t)|^2 \le 1$, and for all $(H, \pi, x, y) \in W_B(f)$, we have

$$|\sum_t \alpha(t)\overline{f(t)}| \le 4\sqrt{e}\|x\|\|y\|. \tag{27}$$

This is enough to conclude that $\|f\|_2 \le 4\sqrt{e}\|f\|_{B(G)}$. Indeed, by definition of the $B(G)$ norm, Equation (27) implies that, for all α finitely supported and of l^2-norm at most 1:

$$|\sum_t \alpha(t)\overline{f(t)}| \le 4\sqrt{e}\|f\|_{B(G)}. \tag{28}$$

Then, consider the map:

$$\begin{aligned}
\varphi : \mathbb{C}[G] &\to \mathbb{C} \\
\alpha &\mapsto \sum_t \alpha(t)\overline{f(t)}.
\end{aligned}$$

The supposed bound in Equation (28) shows that φ is bounded when $\mathbb{C}[G]$ is viewed as a dense subspace of $l^2(G)$. It is also linear:

$$c\varphi(\alpha) + \varphi(\beta) = c\sum_t \alpha(t)\overline{f(t)} + \sum_t \beta(t)\overline{f(t)} = \sum_t (c\alpha(t) + \beta(t))\overline{f(t)} = \varphi(c\alpha + \beta),$$

(all sums are finite), and therefore φ is continuous linear. As $\varphi : \mathbb{C}[G] \to \mathbb{C}$ is defined on a dense subspace of $l^2(G)$ and $\mathbb{C}$ is complete, φ extends uniquely to some $\tilde{\varphi} : l^2(G) \to \mathbb{C}$, of same norm, $\|\tilde{\varphi}\| = \|\varphi\| \leq 4\sqrt{e}\|f\|_{B(G)}$. But then, $\tilde{\varphi}$ is an element of $l^2(G)^*$, and as l^2 is Hilbert (hence reflexive), to $\tilde{\varphi}$ corresponds a unique $\tilde{f} \in l^2(G)$ with

$$\tilde{\varphi} = \langle -, \tilde{f} \rangle$$

and $\|\tilde{f}\|_2 = \|\tilde{\varphi}\| \leq 4\sqrt{e}\|f\|_{B(G)}$. But then, for any s, $\delta_s \in \mathbb{C}[G]$, and

$$\overline{f(s)} = \sum_t \delta_s(t)\overline{f(t)} = \varphi(\delta_s) = \tilde{\varphi}(\delta_s) = \langle \delta_s, \tilde{f} \rangle = \sum_t \delta_s(t)\overline{\tilde{f}(t)} = \overline{\tilde{f}(s)}$$

and we conclude that $f = \tilde{f}$, hence

$$\|f\|_2 \leq 4\sqrt{e}\|f\|_{B(G)},$$

and we are done.

We will actually show that for all $\alpha \in \mathbb{C}[G]$ with $\|\alpha\|_2 \leq 1$, we have:

$$|\sum_t \alpha(t)f(t)| \leq 4\sqrt{e}\|x\|\|y\|.$$

This is enough, as pointwise complex-conjugation is a bijection on those $\alpha \in \mathbb{C}[G]$ with $\|\alpha\|_2 \leq 1$, and

$$|\sum_t \overline{\alpha(t)}f(t)| = |\sum_t \alpha(t)\overline{f(t)}|.$$

Now, fix some $\alpha \in \mathbb{C}[G]$ of l^2-norm ≤ 1. Consider the set $(E \cup E^{-1}) \cap \operatorname{supp}(\alpha)$: fix some order and write its elements as

$$\{t_1, ..., t_n\} = E \cap \operatorname{supp}(\alpha), \qquad \{t'_1, ..., t'_m\} = E^{-1} \cap \operatorname{supp}(\alpha),$$

and define

$$A := (-i)\prod_{j=1}^{n}\left(I + i\Re(\alpha(t_j)\pi(t_j))\right), \quad B := \prod_{j=1}^{n}\left(I + i\Im(\alpha(t_j)\pi(t_j))\right),$$
$$A' := (-i)\prod_{j=1}^{m}\left(I + i\Re(\alpha(t'_j)\pi(t'_j))\right), \quad B' := \prod_{j=1}^{m}\left(I + i\Im(\alpha(t'_j)\pi(t'_j))\right),$$

where $\Re(T) = \frac{1}{2}(T + T^*)$ and $\Im(T) = \frac{1}{2}i(T^* - T)$, and finally, define

$$R := A + A' + B + B'.$$

We first bound the norms of the factors of A (and A'):

$$\begin{aligned}
\|I + i\Re(\alpha(t)\pi(t))\|^2 &= \|\left(I + i\Re(\alpha(t)\pi(t))\right)^*\left(I + i\Re(\alpha(t)\pi(t))\right)\| \\
&= \|\left(I - i\Re(\alpha(t)\pi(t))\right)\left(I + i\Re(\alpha(t)\pi(t))\right)\| \\
&= \|\left(I + \Re(\alpha(t)\pi(t))^2\right)\| \\
&\leq 1 + \|\Re(\alpha(t)\pi(t))\|^2 \\
&= 1 + \|\frac{1}{2}\alpha(t)\pi(t) + \frac{1}{2}\overline{\alpha(t)}\pi(t^{-1})\|^2 \leq 1 + |\alpha(t)|^2,
\end{aligned}$$

and similarly for B (and B'):

$$\|I + i\Im(\alpha(t)\pi(t))\|^2 \leq 1 + |\alpha(t)|^2,$$

for all t (using the facts that both $\Re(T)$ and $\Im(T)$ are self-adjoint, $\pi(t)^* = \pi(t^{-1})$, and $\pi(t)$ has norm 1). This implies that

$$\|A\| \leq \prod_{j=1}^{n}\|I + i\Re(\alpha(t_j)\pi(t_j))\| \leq \prod_{j=1}^{n}(1 + |\alpha(t_j)|^2)^{\frac{1}{2}},$$

and similarly for $\|A'\|$, $\|B\|$ and $\|B'\|$, and we can bound the norm of R:

$$\begin{aligned}
\|R\| &\leq \|A\| + \|A'\| + \|B\| + \|B'\| \\
&\leq \prod_{j=1}^{n}(1+|\alpha(t_j)|^2)^{\frac{1}{2}} + \prod_{j=1}^{m}(1+|\alpha(t'_j)|^2)^{\frac{1}{2}} + \prod_{j=1}^{n}(1+|\alpha(t_j)|^2)^{\frac{1}{2}} + \prod_{j=1}^{m}(1+|\alpha(t'_j)|^2)^{\frac{1}{2}} \\
&= 2\prod_{j=1}^{n}(1+|\alpha(t_j)|^2)^{\frac{1}{2}} + 2\prod_{j=1}^{m}(1+|\alpha(t'_j)|^2)^{\frac{1}{2}} \\
&= 2\prod_{t\in E\cap \mathrm{supp}(\alpha)}(1+|\alpha(t)|^2)^{\frac{1}{2}} + 2\prod_{t\in E^{-1}\cap \mathrm{supp}(\alpha)}(1+|\alpha(t)|^2)^{\frac{1}{2}} \\
&\leq 4\prod_{t\in \mathrm{supp}(\alpha)}(1+|\alpha(t)|^2)^{\frac{1}{2}} = 4\exp\left(\ln\left(\prod_{t\in \mathrm{supp}(\alpha)}(1+|\alpha(t)|^2)^{\frac{1}{2}}\right)\right) \\
&= 4\exp\left(\sum_{t\in \mathrm{supp}(\alpha)}\frac{1}{2}\ln((1+|\alpha(t)|^2))\right) \leq 4\exp\left(\frac{1}{2}\sum_{t\in \mathrm{supp}(\alpha)}|\alpha(t)|^2\right) \\
&\leq 4\exp(\frac{1}{2}) = 4\sqrt{e},
\end{aligned}$$

using the fact that $e^x \geq x+1$, and therefore $x \geq \ln(x+1)$, for all $x \geq 0$.

Expanding the product A (Proposition C.15), we get

$$A = -iI \ + \sum_{j=1}^{n}\Re(\alpha(t_j)\pi(t_j)) + \sum_k \psi_k \tag{29}$$

for some ψ of finite index set, and such that for each index k, ψ_k is a product of at least two $i\Re(\alpha(t_j)\pi(t_j))$s, and such that no $i\Re(\alpha(t_j)\pi(t_j))$ appears twice; that is, ψ_k can be written

$$\psi_k = i\Re(\alpha(t_{j_{1,k}})\pi(t_{j_{1,k}}))\cdot ... \cdot i\Re(\alpha(t_{j_{l_k,k}})\pi(t_{j_{l_k,k}}))$$

for some $l_k \geq 2$ and some sequence of indices $j_{\lambda,k}$, with $j_{\lambda,k} \neq j_{\lambda',k}$ whenever $\lambda \neq \lambda'$. Expanding ψ_k (Proposition C.16), we get:

$$\begin{aligned}
\psi_k &= i\Re(\alpha(t_{j_{1,k}})\pi(t_{j_{1,k}}))...i\Re(\alpha(t_{j_{l_k,k}})\pi(t_{j_{l_k,k}})) \\
&= \left(\frac{i}{2}\right)^{l_k}\left(\alpha(t_{j_{1,k}})\pi(t_{j_{1,k}}) + \overline{\alpha(t_{j_{1,k}})}\pi(t^{-1}_{j_{1,k}})\right)...\left(\alpha(t_{j_{l_k,k}})\pi(t_{j_{l_k,k}}) + \overline{\alpha(t_{j_{l_k,k}})}\pi(t^{-1}_{j_{l_k,k}})\right) \\
&= \left(\frac{i}{2}\right)^{l_k}\sum_{\substack{\varepsilon_\lambda=\pm1,\\ 1\leq\lambda\leq l_k}}\left(\prod_{\lambda=1}^{l_k}\alpha(t_{j_{\lambda,k}})^{\varepsilon_\lambda}\right)\pi(t^{\varepsilon_\lambda}_{j_{\lambda,k}}) \\
&= \left(\frac{i}{2}\right)^{l_k}\sum_{\substack{\varepsilon_\lambda=\pm1,\\ 1\leq\lambda\leq l_k}}\left(\prod_{\lambda=1}^{l_k}\alpha(t_{j_{\lambda,k}})^{\varepsilon_\lambda}\right)\pi(t^{\varepsilon_1}_{j_{\lambda,1}}\cdot ... \cdot t^{\varepsilon_{l_k}}_{j_{l_k,k}}),
\end{aligned}$$

with the convention that $\alpha(t)^{+1} = \alpha(t)$ and $\alpha(t)^{-1} = \overline{\alpha(t)}$. As $l_k \geq 2$ and the t_js are all distinct, the word $w = t^{\varepsilon_1}_{j_{\lambda,1}}\cdot ... \cdot t^{\varepsilon_{l_k}}_{j_{l_k,k}}$ is reduced, when viewed as an element of F_E, the free group generated by E, and of length ≥ 2, so that $w \notin E\cup E^{-1}$. From this, we conclude that ψ_k is a polynomial in the variables $\pi(t)$, for a finite number of elements t in G but not in $E\cup E^{-1}$. This holds for any k, and we can therefore rewrite

$$\sum_k \psi_k = \sum_{t\notin E\cup E^{-1}}\Psi_t\pi(t)$$

for $\Psi : G \to \mathbb{C}$ of finite support, by aggregating the sums in each ψ_k. Adding $-i$, to the coefficient of Ψ_e, we can rewrite A (from Equation (29)) as

$$A = \sum_{j=1}^{n}\Re(\alpha(t_j)\pi(t_j)) + \sum_{t\notin E\cup E^{-1}}\Psi_t\pi(t).$$

The same argument shows that we can expand the products of A', B and B' similarly, and

$$\begin{aligned}
R = &\quad A + A' + B + B' \\
= &\sum_{t \in E \cap \mathrm{supp}(\alpha)} \Re(\alpha(t)\pi(t)) + i\Im(\alpha(t)\pi(t)) \quad + \sum_{t \in E^{-1} \cap \mathrm{supp}(\alpha)} \Re(\alpha(t)\pi(t)) + i\Im(\alpha(t)\pi(t)) \quad + \sum_{t \notin E \cup E^{-1}} \rho_t \pi(t) \\
= &\sum_{t \in (E \cup E^{-1}) \cap \mathrm{supp}(\alpha)} \Re(\alpha(t)\pi(t)) + i\Im(\alpha(t)\pi(t)) \quad + \sum_{t \notin E \cup E^{-1}} \rho_t \pi(t) \\
= &\sum_{t \in (E \cup E^{-1}) \cap \mathrm{supp}(\alpha)} \alpha(t)\pi(t) \quad + \sum_{t \notin E \cup E^{-1}} \rho_t \pi(t)
\end{aligned}$$

with $\rho_t := \Psi_t^A + \Psi_t^{A'} + \Psi_t^B + \Psi_t^{B'}$ (and Ψ^X constructed similarly for $X = A, A', B, B'$ as Ψ was constructed for A). Recalling that $f(t) = \langle \pi(t)x, y\rangle$ $\forall t$, and f is zero outside of $E \cup E^{-1}$, the above implies:

$$\begin{aligned}
\langle Rx, y\rangle \quad &= \sum_{t \notin E \cup E^{-1}} \rho_t \langle \pi(t)x, y\rangle \sum_{t \in (E \cup E^{-1}) \cap \mathrm{supp}(\alpha)} \alpha(t)\langle \pi(t)x, y\rangle \quad + \\
&= \sum_{t \in (E \cup E^{-1}) \cap \mathrm{supp}(\alpha)} \alpha(t) f(t) \quad + \sum_{t \notin E \cup E^{-1}} \rho_t f(t) \\
&= \sum_{t \in (E \cup E^{-1}) \cap \mathrm{supp}(\alpha)} \alpha(t) f(t) \quad + \quad 0,
\end{aligned}$$

so that

$$\left| \sum_t \alpha(t) f(t) \right| = \left| \sum_{t \in (E \cup E^{-1}) \cap \mathrm{supp}(\alpha)} \alpha(t) f(t) \right| = |\langle Rx, y\rangle| \le \|R\| \|x\| \|y\| \le 4\sqrt{e} \|x\| y\|.$$

This is what we aimed to prove, and we are done.

2. Let F be a subgroup of G, we show that we can isometrically embed $T_1(F)$ in $T_1(G)$ via

$$\begin{aligned}
\phi : T_1(F) &\to T_1(G) \\
f &\mapsto f \sqcup 0_{G \setminus F},
\end{aligned}$$

that is, by setting

$$\phi(f)(t) := \begin{cases} f(t) & \text{if } t \in F; \\ 0 & \text{otherwise,} \end{cases}$$

and that this embedding is surjective on $T_1(G) \cap \mathbb{C}^{F \subseteq G}$.

ϕ is well-defined. It is obvious that ϕ sends a map in $\mathbb{C}^F$ to a map in $\mathbb{C}^{F \subseteq G}$, but we still have to show that if $f \in T_1(F)$, then $\phi(f) \in T_1(G)$. So, let $f \in T_1(F)$ and extend it with zeros to get an element $f' := \phi(f)$ of $\mathbb{C}^G$. Fix any $(f_1, f_2) \in W_{T_1}(f)$. The goal now is to extend f_i $(i = 1, 2)$ to $G \times G$ in such a way that $f_1' + f_2' = f'$.

Choose $R = \{t_n : n \in J\}$ a class of representatives for the right cosets of F in G. Note first that given $s, t \in G$,

$$(st \in F) \quad \Leftrightarrow \quad \left(\forall m, n \ : \ s \in Ft_n \ \wedge \ t^{-1} \in Ft_m \ \Rightarrow \ m = n\right).$$

Indeed, if $s \in Ft_n$ and $t^{-1} \in Ft_n$, we can write $s = ht_n$ and $t^{-1} = gt_n$ with $h, g \in F$; then $st = ht_n t_n^{-1} g^{-1} = hg^{-1} \in F$. Conversely, if $st \in F$ and $s = ht_n$ and $t^{-1} = gt_m$, then $ht_n t_m^{-1} g \in F$, so that $t_n \in h^{-1} F g^{-1} t_m = Ft_m$, so that $t_n = t_m$ and $n = m$.

Now, define, for $s, t \in G$:

$$\begin{aligned}
&f_1'(s,t) = f_1(st_m^{-1}, t_m t), \quad f_2'(s,t) = f_2(st_m^{-1}, t_m t) \qquad \text{if } st \in F, \text{ and } s \in Ft_m, t^{-1} \in Ft_m, \\
&f_1'(s,t) = 0, \qquad\qquad\qquad f_2'(s,t) = 0 \qquad\qquad\qquad\qquad\qquad\qquad \text{if } st \notin F.
\end{aligned}$$

This is well-defined because if $s, t^{-1} \in Ft_m$, then $st_m^{-1} \in F$ and $t_m t \in F$ too. Finally, we verify that $f' = f_1' + f_2'$ and that $\|f'\|_{T_1(G)} \le \|f\|_{T_1(F)}$ (not quite the isometry, yet). Fix any s, t: If $st \in F$, then let t_m be as above for s, t:

$$f'(st) = f(st) = f(st_m^{-1} t_m t) = f_1(st_m^{-1}, t_m t) + f_2(st_m^{-1}, t_m t) = f_1'(s,t) + f_2'(s,t)$$

and if $st \notin F$, then $f'(st) = 0$, and so are both $f'_1(s,t)$and $f'_2(s,t)$. Furthermore,

$$\begin{aligned}
\sup_{s\in G} \sum_{t\in G} |f'_1(s,t)| &= \sup_{s\in F, t_m\in R} \sum_{t\in F, t_n\in R} |f'_1(st_m, t_n^{-1}t)| \\
&= \sup_{s\in F, t_m\in R} \sum_{t\in F} |f'_1(st_m, t_m^{-1}t)| \\
&= \sup_{s\in F, t_m\in R} \sum_{t\in F} |f_1(s,t)| \\
&= \sup_{s\in F} \sum_{t\in F} |f_1(s,t)|
\end{aligned}$$

(using the fact that R^{-1} is a class of representatives for left cosets if R is one for right cosets, and the definition of f'_1) and similarly for f_2, so that

$$\|f'\|_{T_1(G)} \leq \sup_{s\in G} \sum_{t\in G} |f'_1(s,t)| + \sup_{t\in G} \sum_{s\in G} |f'_2(s,t)| = \sup_{s\in F} \sum_{t\in F} |f_1(s,t)| + \sup_{t\in G} \sum_{s\in G} |f_2(s,t)|$$

for all $(f_1, f_2) \in W_{T_1}(f)$, which shows that ϕ is well-defined (that is, it really takes value in $T_p(G)$), and, ranging over the choices of $(f_1, f_2) \in W_{T_1}(f)$, we get:

$$\|f'\|_{T_1(G)} \leq \|f\|_{T_1(F)}, \tag{30}$$

which is half of what is needed to show that ϕ is an isometry.

ϕ is surjective on $T_1(G) \cup \mathbb{C}^{F\subseteq G}$. If $f \in T_1(G) \cap \mathbb{C}^{F\subseteq G}$ and $(f_1, f_2) \in W_{T_1}(f)$, i.e. $f(st) = f_1(s,t) + f_2(s,t)$ and

$$\begin{aligned}
C_1 &:= \sup_{s\in G} \sum_{t\in G} |f_1(s,t)| < \infty \\
C_2 &:= \sup_{t\in G} \sum_{s\in G} |f_2(s,t)| < \infty
\end{aligned}$$

then we also have $f_F(st) = f_1|_{F\times F}(s,t) + f_2|_{F\times F}(s,t)$, for all $s, t \in F$, and

$$\begin{aligned}
\sup_{s\in F} \sum_{t\in F} |f_1|_{F\times F}(s,t)| &\leq C_1 \\
\sup_{t\in F} \sum_{s\in F} |f_2|_{F\times F}(s,t)| &\leq C_2
\end{aligned}$$

so that $(f_1|_{F\times F}, f_2|_{F\times F}) \in W_{T_1}(f|_F)$, and $f|_F \in T_1(F)$. As, obviously, $\phi(f|_F) = f$, ϕ is indeed surjective. Furthermore, $\|f|_F\|_{T_1(F)} \leq C_1 + C_2$, and ranging over $(f_1, f_2) \in W_{T_1}(f)$, we get

$$\|f|_F\|_{T_1(F)} \leq \|f\|_{T_1(G)}, \tag{31}$$

which is the other half of showing that

ϕ is an isometry. Putting together Equation (31) and Equation (30), we see that for all $f \in T_1(F)$:

$$\|\phi(f)\|_{T_1(G)} = \|f\|_{T_1(F)},$$

and we are done.

3. Let $f := \mathbb{1}_{E\cup E^{-1}}$ be the indicator function of $E \cup E^{-1}$; let f' be the restriction of f to $\langle E\rangle$, the free group of countable infinite rank generated by E. Let $\varphi : F_\infty \to \langle E\rangle$ be the isomorphism between the canonical free group of countable infinite rank and $\langle E\rangle$, as given by the universal mapping property of the free group. Then, $f' \circ \varphi$ is exactly the indicator function of words of length one in F_∞, and by Lemma 3.2, $f' \circ \varphi \in T_1(F_\infty)$, which implies that $f' \in T_1(\langle E\rangle)$, and by the point above, this in turn implies $f = \phi(f') \in T_1(G)$. Similarly, assuming $f \in B(G)$, we would have $f' \in B(\langle E\rangle)$, and $f' \circ \varphi \in B(F_\infty)$, which is false by Lemma 3.2; we conclude $f \notin B(G)$.
4. We know that for any $N \geq 2$, there exists a free set E of infinite countable cardinality in F_N. By the point above, the indicator function of $E \cup E^{-1}$ is in $T_1(F_N) \setminus B(F_N)$.

□

Note that for the third point of Lemma 3.11, instead of showing that $f \notin B(G)$ by passing to f', we could have used the first point: Indeed, given that f is supported on $E \cup E^{-1}$, and assuming $f \in B(G)$, we would have

$$\|f\|_2 \leq 4\sqrt{e}\|f\|_{B(G)} < \infty,$$

which is impossible, since f, being constant on an infinite set, has infinite l^2-norm.

Corollary 3.12 ([Pis01, Theorem 2.7*])**.** *Any group containing a free group of rank ≥ 2 as a subgroup is not unitarizable.*

Note that this fact has already been proved in Corollary 3.4, since a group contains the free group of countably infinite rank if and only if it contains some free subgroup of rank ≥ 2. The following proof differs from the one of Corollary 3.4 in that it doesn't use Lemma 1.9.

Proof. If G is such a group, it contains F_∞, and a countably infinite free set. Applying item 3 of Lemma 3.11, we get $T_1(G) \not\subseteq B(G)$, and by Theorem 3.1, G is not unitarizable. $\square$

4 Derivations

This section expands on a specific type of group representation, as constructed in Theorem 3.1. The central result (Theorem 4.5) is the condition “If G is unitarizable, then $T_1(G) \subseteq l^2(G)$”, which can be seen as a strenghtening of Theorem 3.1, since $l^2(G)$ is a subspace of $B(G)$.

The section is mainly based on [Oza11], and the proof of Lemma 4.5 (page 80) in [Pis01].

Fix a group G, and a family of operators $D : G \to B(l^2(G)))$. We say that D is a *derivation* if it satisfies the “Leibniz law”:

$$D(st) = D(s)\lambda(t) + \lambda(s)D(t), \quad \forall s, t \in G.$$

Note that if D is a derivation, then $D(e_G) = D(e_G e_G) = \lambda(e_G)D(e_G) + D(e_G)\lambda(e_G) = D(e_G) + D(e_G)$, so that $D(e_G) = 0$.

Recall that $B(l^2(G))$ can be seen as an algebra on $\mathbb{C}$. Thus, for any $S, T \in B(l^2(G))$, we can define their commutator $[S, T]$, which we recall to be:

$$[S, T] = ST - TS.$$

For any $T \in B(l^2(G))$, we can then consider the derivation D_T defined as

$$D_T(t) = [T, \lambda(t)] = T\lambda(t) - \lambda(t)T \in B(l^2(G))).$$

If a derivation D is of the form D_T for some T, we say that it is *inner*.

Lemma 4.1 ([Oza11, p. 15])**.** *Fix $D : G \to B(l^2(G)))$ and*

$$\pi_D(t) := \begin{pmatrix} \lambda(t) & D(t) \\ 0 & \lambda(t) \end{pmatrix} \in B(l^2(G) \oplus l^2(G))).$$

Then, π_D is a representation of G if and only if D is a derivation.

Note that $\pi_D(t)$ is necessarily linear and bounded, as a matrix of linear bounded operators ($D(t)$ and $\lambda(t)$).

Proof. We know that

$$\begin{aligned}
\pi_D(s)\pi_D(t) &= \begin{pmatrix} \lambda(s) & D(s) \\ 0 & \lambda(s) \end{pmatrix} \begin{pmatrix} \lambda(t) & D(t) \\ 0 & \lambda(t) \end{pmatrix} \\
&= \begin{pmatrix} \lambda(s)\lambda(t) & D(s)\lambda(t) + \lambda(s)D(t) \\ 0 & \lambda(s)\lambda(t) \end{pmatrix} = \begin{pmatrix} \lambda(st) & D(s)\lambda(t) + \lambda(s)D(t) \\ 0 & \lambda(st) \end{pmatrix},
\end{aligned}$$

because λ is itself a representation, and

$$\pi_D(st) = \begin{pmatrix} \lambda(st) & D(st) \\ 0 & \lambda(st) \end{pmatrix}$$

so that $\pi_D(s)\pi_D(t) = \pi_D(st)$ if and only if $D(st) = D(s)\lambda(t) + \lambda(s)D(t)$.

If D is a derivation, then for all s, t, $\pi_D(st) = \pi_D(s)\pi_D(t)$, and $\pi_D(e_G) = I$ (because $\lambda(e_G) = I$ and $D(e_G) = 0$), so that π_D is a representation. Conversely, if π is a representation, then D satisfies the Leibniz law and is a derivation. □

Lemma 4.2 ([Oza11, p. 15])**.** *Fix $D : G \to B(l^2(G)))$ a derivation, and π_D the associated group representation, as in Lemma 4.1. Then D is uniformly bounded if and only if π_D is uniformly bounded.*

Proof. If D is uniformly bounded (i.e. $|D| := \sup_{t \in G} \|D(t)\| < \infty$), then, for any t and $x \oplus y$ of norm at most 1:

$$\begin{aligned}
\|\pi(t)(x \oplus y)\|^2 = \|(\lambda(t)x + D(t)y) \oplus (\lambda(t)y)\|^2 &= \|(\lambda(t)x + D(t)y)\|^2 + \|(\lambda(t)y)\|^2 \\
&= \|x\|^2 + \|D(t)y\|^2 + \langle \lambda(t)x, D(t)y \rangle + \langle D(t)y, \lambda(t)x \rangle + \|y\|^2 \\
&\leq \|x\|^2 + \|y\|^2 + \|D(t)\|^2\|y\|^2 + 2\|x\|\|D(t)\|\|y\| \\
&\leq 1 + \|D(t)\|^2 + 2\|D(t)\| \leq 1 + |D|^2 + 2|D|,
\end{aligned}$$

so that $\|\pi(t)\| \leq 1 + |D|$, for all t, and $|\pi| \leq 1 + |D|$, and π is uniformly bounded.

Conversely, if π is uniformly bounded, then, in particular, for any y of norm at most 1:

$$\|D(t)y\|^2 \leq \|D(t)y\|^2 + \|\lambda(t)y\|^2 = \|D(t)y \oplus \lambda(t)y\|^2 = \|\pi(t)(0 \oplus y)\|^2 \leq \|\pi(t)\|^2 \leq |\pi|^2$$

so that $|D| \leq |\pi|$. □

Lemma 4.3 ([Oza11, p. 15])**.** *Fix $D : G \to B(l^2(G))$ a derivation, and π_D the associated group representation. Then π_D is unitarizable if and only if D is inner.*

The proof of this fact follows the proof of [Pis01, Lemma 4.5].

Proof. If D is inner, say $D = D_T$ for some $T \in B(l^2(G))$, we can write

$$\pi(t) := \begin{pmatrix} \lambda(t) & D(t) \\ 0 & \lambda(t) \end{pmatrix} = \begin{pmatrix} I & T \\ 0 & I \end{pmatrix} \begin{pmatrix} \lambda(t) & 0 \\ 0 & \lambda(t) \end{pmatrix} \begin{pmatrix} I & -T \\ 0 & I \end{pmatrix}.$$

Then,

$$\begin{pmatrix} I & T \\ 0 & I \end{pmatrix}^{-1} = \begin{pmatrix} I & -T \\ 0 & I \end{pmatrix},$$

and

$$\begin{pmatrix} \lambda(t) & 0 \\ 0 & \lambda(t) \end{pmatrix}$$

is easily verified to be a unitary representation. We conclude that π is unitarizable.

Conversely, assume that π is unitarizable. Let $S \in B^I(l^2(G) \oplus l^2(G))$ and μ unitary with

$$\mu(t) = S\pi(t)S^{-1} \quad \forall t.$$

Define $a := SS^*$, which is then self-adjoint, and actually positive, and note that

$$\pi(t)a = S\mu(t)S^{-1}SS^* = S\mu(t)S^*,$$

and

$$\begin{aligned} \pi(t)a &= S\mu(t)S^* = (S\mu(t)^*S^*)^* = (S\mu(t^{-1})S^*)^* = (\pi(t^{-1})a)^* = a^*\pi(t^{-1})^* \\ &= a\pi(t^{-1})^* \end{aligned} \tag{32}$$

for all t, because a is self-adjoint and μ is a unitary representation (so that $\mu(t^{-1}) = \mu(t)^*$ for all t).

The operator a being self-adjoint, its matrix is of the form

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{12}^* & a_{22} \end{pmatrix}, \quad \text{with } a_{ij} \in B(l^2(G)),$$

and we can write Equation (32) as:

$$\begin{pmatrix} \lambda(t) & D(t) \\ 0 & \lambda(t) \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{12}^* & a_{22} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{12}^* & a_{22} \end{pmatrix} \begin{pmatrix} \lambda(t^{-1})^* & 0 \\ D(t^{-1})^* & \lambda(t^{-1})^* \end{pmatrix},$$

which expands to (using the fact that $\lambda(t^{-1})^* = \lambda(t)$):

$$\begin{pmatrix} \lambda(t)a_{11} + D(t)a_{12}^* & \lambda(t)a_{12} + D(t)a_{22} \\ \lambda(t)a_{12}^* & \lambda(t)a_{22} \end{pmatrix} = \begin{pmatrix} a_{11}\lambda(t) + a_{12}D(t^{-1})^* & a_{12}\lambda(t) \\ a_{12}^*\lambda(t) + a_{22}D(t^{-1})^* & a_{22}\lambda(t) \end{pmatrix},$$

and in particular, for all t:

$$\begin{aligned} \lambda(t)a_{12} + D(t)a_{22} &= a_{12}\lambda(t), \\ \lambda(t)a_{22} &= a_{22}\lambda(t). \end{aligned}$$

Assuming that a_{22} has an inverse a_{22}^{-1}, we are done, because we can then write:

$$\begin{aligned} & \lambda(t)a_{12} + D(t)a_{22} = a_{12}\lambda(t) \\ \Rightarrow \quad & \lambda(t)a_{12} - a_{12}\lambda(t) = D(t)a_{22} \\ \Rightarrow \quad & (\lambda(t)a_{12} - a_{12}\lambda(t))\, a_{22}^{-1} = D(t) \\ \Rightarrow \quad & \lambda(t)a_{12}a_{22}^{-1} - a_{12}a_{22}^{-1}\lambda(t) = D(t) \\ \Rightarrow \quad & [-a_{12}a_{22}^{-1}, \lambda(t)] = D(t), \end{aligned}$$

using the fact that a_{22}^{-1} and $\lambda(t)$ commute (because so do a_{22} and $\lambda(t)$). This shows that D is inner, and it therefore remains to verify that a_{22} does have an inverse.

We know that S is invertible, then so is S^*, thus a, as their composition. As a is positive, there exists (Proposition B.14) $\delta > 0$ with

$$\langle a(x \oplus y), x \oplus y \rangle \geq \delta \langle x \oplus y, x \oplus y \rangle$$

for all $x \oplus y \in l^2(G) \oplus l^2(G)$. Expanding the above yields:

$$\langle a_{11}x + a_{12}y, x\rangle + \langle a_{12}^* x + a_{22}y, y\rangle \geq \delta\langle x, x\rangle + \delta\langle y, y\rangle,$$

so that letting $x = 0$, we get

$$\langle a_{22}y, y\rangle \geq \delta\langle y, y\rangle,$$

for all y. In particular, a_{22} is positive and $a_{22} \geq \delta I$, so that it is invertible (Proposition B.14), and we are done. □

Lemma 4.4 ([Oza11, p. 17]). *Fix $f \in T_1(G)$ and $(a, b) \in W_{T_1^{-1}}(f)$. Let, for all t, $D(t) \in B(l^2(G))$ be the operator defined by the matrix*

$$\begin{aligned} d(t)(i, j) &:= a(t^{-1}i, j) - a(i, tj) \\ &= b(i, tj) - b(t^{-1}i, j), \end{aligned}$$

as in Theorem 3.1. Then, if D is inner, $f \in l^2(G)$.

Proof. Assume that D is inner, and can be written:

$$D(t) = D_T(t) = [T, \lambda(t)], \qquad \forall t,$$

for some $T \in B(l^2(G))$. Consider the map

$$\begin{aligned} \varphi : G &\to \mathbb{C} \\ t &\mapsto \langle D(t)\delta_e, \delta_e\rangle = \langle (T\lambda(t) - \lambda(t)T)\delta_e, \delta_e\rangle. \end{aligned}$$

This map can be written $\varphi = \varphi_1 - \varphi_2$, with

$$\begin{aligned} \varphi_1 &: t \mapsto \langle T\lambda(t)\delta_e, \delta_e\rangle, \\ \varphi_2 &: t \mapsto \langle \lambda(t)T\delta_e, \delta_e\rangle, \end{aligned}$$

which are both in $l^2(G)$. Indeed,

$$\varphi_1(t) = \langle T\lambda(t)\delta_e, \delta_e\rangle = \langle T\delta_t, \delta_e\rangle = \langle \delta_t, T^*\delta_e\rangle = \overline{(T^*\delta_e)(t)},$$

so that

$$\sum_t |\varphi_1(t)|^2 = \sum_t |\overline{T^*\delta_e(t)}|^2 = \sum_t |T^*(\delta_e)(t)|^2 < \infty,$$

because $T \in B(l^2(G))$, thus $T^* \in B(l^2(G))$, which implies by definition that $T^*(\delta_e)$ is square-summable. Showing $\varphi_2 \in l^2(G)$ works by a similar argument (passing to the adjoint of $\lambda(t)$ rather than T), and we conclude that φ, being a sum of elements of $l^2(G)$, it is itself an element of $l^2(G)$. Now, first observing that:

$$\varphi : t \mapsto \langle D(t)\delta_e, \delta_e\rangle = d(t)(e, e) = a(t^{-1}, e) - a(e, t),$$

and that both the maps $t \mapsto a(e, t)$ and $t \mapsto b(t^{-1}, e)$ are in $l^1(G)$, thus in $l^2(G)$, we can write

$$f(t) = f(te) = a(t^{-1}, e) + b(t^{-1}, e) = a(t^{-1}, e) - a(e, t) + a(e, t) + b(t^{-1}, e) = \varphi(t) + a(e, t) + b(t^{-1}, e), \quad \forall t,$$

and conclude that f is in $l^2(G)$, as a sum of elements of $l^2(G)$. □

Theorem 4.5 ([Oza11, p. 17]). *If G is unitarizable, then $T_1(G) \subseteq l^2(G)$.*

Proof. Let $f \in T_1(G)$, $(a, b) \in W_{T_1^{-1}}(f)$, and let D be the family of operators as in Lemma 4.4. We know that D is a uniformly bounded derivation (as shown in Theorem 3.1), which implies that π_D (as defined in Lemma 4.1) is a uniformly bounded representation, and is unitarizable, since G is a unitarizable group, by hypothesis. By Lemma 4.3, D is inner, and Lemma 4.4 implies that $f \in l^2(G)$. □

Theorem 4.6 ([Oza11, p. 17]; Bożejko-Fendler). *Let G be a group. If G has F_2 (the free group of rank 2) as a subgroup, then there exists some derivation*

$$D : G \to B(l^2(G)),$$

uniformly bounded but not inner.

Proof. As $F_\infty \hookrightarrow F_2 \hookrightarrow G$, let $f := \mathbb{1}_{E \cup E^{-1}}$, with E a countably infinite free set in G. We know that $f \in T_1(G)$, and D (as defined in Lemma 4.4) is a uniformly bounded derivation. If D was inner, we could apply Lemma 4.4 and would have $f \in l^2(G) \subseteq B(G)$, which we know not to be true, by Lemma 3.2. □

5 Unitarizability & Fixed points

This section describes two metric spaces of operators on Hilbert spaces, on which any group representation can be translated to an isometric action of said group. In both cases, unitarizability of the representation is then equivalent to a fixed point property, and uniform boundedness is related to boundedness of the orbits.

The first metric space described here ($B^I(H)/\mathbb{C}^*U(H)$) comes from discussions with Prof. N. Monod; the second ($P(H)$) from [Sch15]. The alternate definitions for the metric of $P(H)$ come from [Mol09] and [ACS00].

Fix some Hilbert space H. Let $U(H)$ be the subgroup of $B^I(H)$ of unitary operators, and $\mathbb{C}^*U(H) := \{cV \,:\, c \in \mathbb{C}^*, V \in U(H)\}$, which is still a subgroup of $B^I(H)$. Let $P(H)$ be the cone of positive invertible operators, that is:

$$P(H) := \{S \in B^I(H) \;:\; \langle Sx, x\rangle \geq 0 \;\forall x \in H\}.$$

Recall that $P(H)$ being a cone means that it is closed under addition and multiplication by a strictly positive scalar; this is easily verified for $P(H)$.

5.1 $B^I(H)/\mathbb{C}^*U(H)$

On $B^I(H)$, define first the map

$$\begin{aligned} d' : B^I(H) \times B^I(H) &\to \mathbb{R}^{\geq 0} \\ S, T &\mapsto \log\left(\|S^{-1}T\|\|T^{-1}S\|\right). \end{aligned}$$

This is well-defined, because for any $S, T \in B^I(H)$,

$$\|S^{-1}T\|\|T^{-1}S\| \geq \|S^{-1}TT^{-1}S\| = \|I\| = 1$$

so that $\log\left(\|S^{-1}T\|\|T^{-1}S\|\right) \geq 0$. The map d' satisfies all of the requirements for a metric, except that of separating points:

- For any $S \in B^I(H)$, $d'(S,S) = \log\left(\|I\|\|I\|\right) = 0$.
- For any $S, T \in B^I(H)$, $d'(S,T) = \log\left(\|S^{-1}T\|\|T^{-1}S\|\right) = \log\left(\|T^{-1}S\|\|S^{-1}T\|\right) = d'(T,S)$.
- For any $R, S, T \in B^I(H)$,

$$\begin{aligned} d'(R,S) + d'(S,T) &= \log\left(\|R^{-1}S\|\|S^{-1}R\|\right) + \log\left(\|S^{-1}T\|\|T^{-1}S\|\right) = \log\left(\|R^{-1}S\|\|S^{-1}R\|\|S^{-1}T\|\|T^{-1}S\|\right) \\ &\geq \log\left(\|R^{-1}SS^{-1}T\|\|T^{-1}SS^{-1}R\|\right) = \log\left(\|R^{-1}T\|\|T^{-1}R\|\right) = d'(R,T). \end{aligned}$$

If $d'(S,T) = 0$, then S is not necessarily equal to T, but $\|S^{-1}T\|\|T^{-1}S\| = 1$ implies that $S^{-1}T \in \mathbb{C}^*U(H)$ (Proposition B.16).

Finally, note that d' is invariant under left multiplication:

$$d'(RS, RT) = \log(\|(RS)^{-1}RT\|\|(RT)^{-1}RS\|) = \log(\|S^{-1}T\|\|T^{-1}S\|) = d'(S,T).$$

Letting $X := B^I(H)/\mathbb{C}^*U(H)$, we would like to get a metric out of d' by passing to the quotient. So, define

$$\begin{aligned} d : X \times X &\to \mathbb{R}^{\geq 0} \\ [S], [T] &\mapsto d'(S,T). \end{aligned}$$

The first thing to do is verify that d is well-defined. Assume $[S] = [S']$, so that there exists a scalar α and a unitary operator U with $S' = S\alpha U$; similarly, assume $[T'] = [T]$ with $T' = T\beta V$. Then

$$\begin{aligned} d'(S', T') &= \log\left(\|S'^{-1}T'\|\|T'^{-1}S'\|\right) \\ &= \log\left(\|U^{-1}\alpha^{-1}S^{-1}T\beta V\|\|V^{-1}\beta^{-1}T^{-1}S\alpha U\|\right) \\ &= \log\left(|\alpha^{-1}||\alpha||\beta||\beta^{-1}|\|S^{-1}T\|\|T^{-1}S\|\right) \\ &= \log\left(\|S^{-1}T\|\|T^{-1}S\|\right) = d'(S,T), \end{aligned}$$

using the fact that composition with unitary maps preserves the norm (that is, if U is unitary, then $\|UT\| = \|T\| = \|TU\|$ for any T).

It is straightforward to check that d inherits all the right properties (that is, almost being a metric, and being invariant under left-multiplication) from d', and

$$d([S],[T]) = 0 \quad \Leftrightarrow \quad S^{-1}T \in \mathbb{C}^*U(H) \quad \Leftrightarrow \quad [S] = [T],$$

so that d really is a metric, invariant under left-multiplication (in other words, left-multiplication is an isometry of X, under the metric d).

Given a representation $\pi : G \to B^I(H)$, we can define a G-action on $B^I(H)$ by

$$\begin{aligned} \alpha'_\pi : G \times B^I(H) &\to B^I(H) \\ g, S &\mapsto \pi(g)S \end{aligned}$$

which induces a G-action on X:

$$\begin{aligned} \alpha_\pi : G \times X &\to X \\ g, [S] &\mapsto [\pi(g)S] \end{aligned}$$

and as left-multiplication is an isometry, this action acts isometrically.

Definition 5.1. Given a representation $\pi : G \to B^I(H)$, we say that it is *projectively uniformly bounded* if:

$$\sup_g \|\pi(g)\|\|\pi(g^{-1})\| < \infty.$$

Recall that the diameter of a subset A of a metric spaces is defined as

$$\operatorname{diam} A = \sup_{x,y\in A} d(x,y),$$

and that A is said to be bounded whenever it has a finite diameter.

In our specific context (isometric group action), and when the set A is an orbit, the diameter can be written:

$$\begin{aligned} \operatorname{diam} \mathcal{O}_{\alpha_\pi}([S]) :&= \sup_{[T],[T']\in\mathcal{O}_{\alpha_\pi}([S])} d([T],[T']) \\ &= \sup_{g,h\in G} d(\alpha_\pi(g)[S], \alpha_\pi(h)[S]) \\ &= \sup_{g,h\in G} d(\alpha_\pi(h^{-1})\alpha_\pi(g)[S], \alpha_\pi(h^{-1})\alpha_\pi(h)[S]) \\ &= \sup_{g,h\in G} d(\alpha_\pi(h^{-1}g)[S],[S]) \\ &= \sup_{k\in G} d(\alpha_\pi(k)[S],[S]), \end{aligned}$$

by invariance of the metric under the group action.

Lemma 5.2. *Some orbit of the action α_π is bounded if and only if all are.*

Proof. Assume the orbit of $[S]$ to be bounded, and fix any $[T]$. Then,

$$\begin{aligned} \operatorname{diam} \mathcal{O}_{\alpha_\pi}([T]) &= \sup_{k\in G} d(\alpha_\pi(k)[T],[T]) \\ &\le \sup_{k\in G} d(\alpha_\pi(k)[T], \alpha_\pi(k)[S]) + d(\alpha_\pi(k)[S],[S]) + d([S],[T]) \\ &= \sup_{k\in G} d([T],[S]) + d(\alpha_\pi(k)[S],[S]) + d([S],[T]) = \operatorname{diam} \mathcal{O}_{\alpha_\pi}([S]) + 2d([S],[T]), \end{aligned}$$

and we conclude that the orbit of $[T]$ is also bounded. □

Note that this holds for any isometric group action.

Proposition 5.3. *The representation π is projectively uniformly bounded if and only if the orbit of $[I]$ is bounded:*

Proof. The orbit of $[I]$ is bounded if and only if:

$$\begin{aligned} &\sup_{k\in G} d([\pi(k)I],[I]) < \infty \\ \Leftrightarrow\quad &\sup_{k\in G} d'(\pi(k), I) < \infty \\ \Leftrightarrow\quad &\sup_{k\in G} \log(\|\pi(k)\|\|\pi(k^{-1})\|) < \infty \\ \Leftrightarrow\quad &\sup_{k\in G} \|\pi(k)\|\|\pi(k^{-1})\| < \infty \end{aligned}$$

if and only if π is projectively uniformly bounded. □

Proposition 5.4. *The representation π is unitarizable if and only if it is uniformly bounded and the induced action on X has a fixed point.*

Proof. Assume π unitarizable. We know that π being unitarizable implies uniformly bounded. Let $S \in B^I(H)$ be such that $S\pi(\cdot)S^{-1}$ is unitary. Then:

$$\begin{aligned}
& S\pi(t)S^{-1} \in U(H) && \forall t \\
\Rightarrow\quad & S\pi(t)S^{-1} \in \mathbb{C}^* U(H) && \forall t \\
\Leftrightarrow\quad & [S\pi(t)S^{-1}] = [I] && \forall t \\
\Leftrightarrow\quad & [\pi(t)S^{-1}] = [S^{-1}] && \forall t
\end{aligned}$$

and $[S^{-1}]$ is a fixed point of the action of G.

Conversely, assume that π is uniformly bounded and that the induced action has a fixed point $[S]$. Then:

$$\begin{aligned}
& [\pi(t)S] = [S] && \forall t \\
\Leftrightarrow\quad & [S^{-1}\pi(t)S] = [I] && \forall t \\
\Leftrightarrow\quad & S^{-1}\pi(t)S \in \mathbb{C}^* U(H) && \forall t
\end{aligned}$$

which means that for all t, there exists some scalar $\beta(t)$ such that $\beta(t)S^{-1}\pi(t)S$ is unitary. Without loss of generality, we can assume $\beta(t) \in \mathbb{R}^{>0}$, and $\beta : G \to \mathbb{R}^{>0}$ is then a morphism of groups. Indeed, fix some s and t in G, then:

$$(\beta(s)S^{-1}\pi(s)S)(\beta(t)S^{-1}\pi(t)S) = \beta(s)\beta(t)S^{-1}\pi(st)S$$

is a product of unitary maps, hence is unitary. This implies that $\beta(st) = \beta(s)\beta(t)$ (if T is an operator and β_1, β_2 are scalars with $\|\beta_1 T\| = \|\beta_2 T\|$, then $|\beta_1| = |\beta_2|$). But then, for all t, we have:

$$1 = \|\beta(t)S^{-1}\pi(t)S\| \leq \beta(t)\|S^{-1}\|\|S\|\|\pi(t)\| \leq \beta(t)\|S\|\|S^{-1}\||\pi|,$$

and thus

$$\beta(t^{-1}) \leq \|S\|\|S^{-1}\||\pi|.$$

This shows that β has a bounded range, and must therefore be constant of value 1 (any non-trivial morphism to $\mathbb{R}^{>0}$ has unbounded range). We conclude that $S^{-1}\pi(t)S$ is unitary for all t, and π is unitarizable. $\square$

5.2 $P(H)$

Recall that $P(H)$ is the set of positive invertible elements of $B(H)$. It is closed under multiplication by strictly positive scalars, addition (hence, it is a cone) and inversion (the inverse of a positive operator is still positive (Proposition B.14)).

For any pair of elements $S, T \in P(H)$, define

$$M(S,T) := \inf\{\gamma > 0 \ : \ T \leq \gamma S\}.$$

Note that this is well-defined, since $T \leq \|T\| I$ and S being invertible, there exists $\delta > 0$ with $\delta I \leq S$. We know that $M(S,T) = \|S^{-\frac{1}{2}} T S^{-\frac{1}{2}}\|$, by Proposition B.19.

Note here that:

$$\|S^{-\frac{1}{2}} T S^{-\frac{1}{2}}\| = \|S^{-\frac{1}{2}} T^{\frac{1}{2}} T^{\frac{1}{2}} S^{-\frac{1}{2}}\| = \|S^{-\frac{1}{2}} T^{\frac{1}{2}}\|^2 = \|T^{\frac{1}{2}} S^{-\frac{1}{2}}\|^2 = \|T^{\frac{1}{2}} S^{-1} T^{\frac{1}{2}}\|.$$

Let us define a metric d on $P(H)$ by:

$$\begin{aligned}
d : P(H) \times P(H) \to & \mathbb{R}^{\geq 0} \\
S, T \mapsto & \log \max(M(S,T), M(T,S)) \\
= & \log \max(\|S^{-\frac{1}{2}} T S^{-\frac{1}{2}}\|, \|T^{-\frac{1}{2}} S T^{-\frac{1}{2}}\|) \\
= & \log \max(\|S^{-\frac{1}{2}} T^{\frac{1}{2}}\|^2, \|T^{-\frac{1}{2}} S^{\frac{1}{2}}\|^2).
\end{aligned}$$

First, note that d is well-defined: letting $R := S^{-\frac{1}{2}} T^{\frac{1}{2}}$,

$$d(S,T) = \log \max(\|R\|^2, \|R^{-1}\|^2)$$

and if $\|R\|^2, \|R^{-1}\|^2 < 1$, we would have $1 = \|\mathbb{R}^{-1}\| \leq \|R\|\|R^{-1}\| < 1$, which is a contradiction; this shows that either of $\|R\|^2, \|R^{-1}\|^2$ is at least 1, and thus $d(S,T) \geq 0$.

Still with $R = S^{-\frac{1}{2}}T^{\frac{1}{2}}$, we see that

$$d(S,T) = \log\max(\|R\|^2, \|R^{-1}\|^2) = \log\max(\|R^{-1}\|^2, \|R\|^2) = d(T,S)$$

which shows symmetry. The fact that $d(S,S) = 0$ follows from noting that $R = S^{-\frac{1}{2}}S^{\frac{1}{2}} = I = R^{-1}$. Conversely, if $d(S,T) = 0$, it follows that

$$\max(\|R\|^2, \|R^{-1}\|^2) = 1,$$

and without loss of generality $\|R\| = 1$, which implies that:

$$\|R^{-1}\| = \|R^{-1}\|\|R\| \geq \|I\| = 1$$

so that $\|R^{-1}\| = 1$ too. This implies that R is unitary (Proposition B.16), so that:

$$\begin{aligned} & \langle Rx, Rx\rangle = \langle x, x\rangle && \forall x \\ \Rightarrow\quad & \langle S^{-\frac{1}{2}}T^{\frac{1}{2}}x, S^{-\frac{1}{2}}T^{\frac{1}{2}}x\rangle = \langle x, x\rangle && \forall x \\ \Rightarrow\quad & \langle T^{\frac{1}{2}}S^{-1}T^{\frac{1}{2}}x, x\rangle = \langle x, x\rangle && \forall x \end{aligned}$$

from which we conclude that $T^{\frac{1}{2}}S^{-1}T^{\frac{1}{2}} = I$ (Proposition B.20), and $S = T$.

The triangle inequality remains. Fix any S, T, U in $P(H)$:

$$\begin{aligned} d(S,T) + d(T,U) &= \log\max(\|S^{-\frac{1}{2}}T^{\frac{1}{2}}\|^2, \|T^{-\frac{1}{2}}S^{\frac{1}{2}}\|^2) + \log\max(\|T^{-\frac{1}{2}}U^{\frac{1}{2}}\|^2, \|U^{-\frac{1}{2}}T^{\frac{1}{2}}\|^2) \\ &= \log\left(\max(\|S^{-\frac{1}{2}}T^{\frac{1}{2}}\|^2, \|T^{-\frac{1}{2}}S^{\frac{1}{2}}\|^2) \cdot \max(\|T^{-\frac{1}{2}}U^{\frac{1}{2}}\|^2, \|U^{-\frac{1}{2}}T^{\frac{1}{2}}\|^2)\right) \\ &\geq \log\left(\max(\|S^{-\frac{1}{2}}T^{\frac{1}{2}}\|^2\|T^{-\frac{1}{2}}U^{\frac{1}{2}}\|^2, \|T^{-\frac{1}{2}}S^{\frac{1}{2}}\|^2\|U^{-\frac{1}{2}}T^{\frac{1}{2}}\|^2)\right) \\ &\geq \log\left(\max(\|S^{-\frac{1}{2}}T^{\frac{1}{2}}T^{-\frac{1}{2}}U^{\frac{1}{2}}\|^2, \|U^{-\frac{1}{2}}T^{\frac{1}{2}}T^{-\frac{1}{2}}S^{\frac{1}{2}}\|^2)\right) \\ &= \log\left(\max(\|S^{-\frac{1}{2}}U^{\frac{1}{2}}\|^2, \|U^{-\frac{1}{2}}S^{\frac{1}{2}}\|^2)\right) = d(S,U). \end{aligned}$$

Finally, note that this metric is invariant under "adjoint-conjugation" by an invertible operator. Indeed, fix $A \in B^I(H)$ and $S, T \in P(H)$; then

$$d(ASA^*, ATA^*) = \log\max(M(ASA^*, ATA^*), M(ATA^*, ASA^*)) = \log\max(M(S,T), M(T,S)) = d(S,T)$$

where we know that $M(ASA^*, ATA^*) = M(S,T)$ because (Proposition B.17):

$$ASA^* \leq \gamma ATA^* \quad \Leftrightarrow \quad S \leq \gamma T.$$

Now, given a group representation π of G in H, we can define an action of G on $P(H)$ by "adjoint-conjugation":

$$\begin{aligned} \alpha_\pi : G \times P(H) &\to P(H) \\ t, S &\mapsto \pi(t)S\pi(t)^*, \end{aligned}$$

and we know this action to be an isometry, because "adjoint-conjugation" is.

Proposition 5.5 ([Sch15, Lemma 2.1]). *Given a representation $\pi : G \to B^I(H)$, the action α_π has a fixed point if and only if π is unitarizable.*

Proof. Assume $S \in B^I(H)$ is such that $S\pi(\cdot)S^{-1}$ is unitary. In particular (the adjoint of a unitary operator is its inverse, Proposition B.10),

$$\begin{aligned} & S\pi(g)S^{-1}(S\pi(g)S^{-1})^* = I, && \forall g \\ \Leftrightarrow\quad & S\pi(g)S^{-1}S^{-1^*}\pi(g)^*S^* = I, && \forall g \\ \Leftrightarrow\quad & \pi(g)S^{-1}S^{-1^*}\pi(g)^* = S^{-1}S^{-1^*}, && \forall g \end{aligned}$$

which shows that $S^{-1}S^{-1^*}$ is a fixed point of the action (note that $S^{-1}S^{-1^*}$ is positive, and invertible because both S^{-1}, S^{-1^*} are).

Conversely, assume $T \in P(H)$ is a fixed point of the action, then

$$\begin{aligned}
& \pi(g)T\pi(g)^* && = T, && \forall g \\
\Leftrightarrow\quad & \pi(g)T^{\frac{1}{2}}T^{\frac{1}{2}}\pi(g)^* && = T^{\frac{1}{2}}T^{\frac{1}{2}}, && \forall g \\
\Leftrightarrow\quad & T^{-\frac{1}{2}}\pi(g)T^{\frac{1}{2}}T^{\frac{1}{2}}\pi(g)^*T^{-\frac{1}{2}} && = I, && \forall g \\
\Leftrightarrow\quad & T^{-\frac{1}{2}}\pi(g)T^{\frac{1}{2}}(T^{-\frac{1}{2}}\pi(g)T^{\frac{1}{2}})^* && = I, && \forall g
\end{aligned}$$

and as $T^{-\frac{1}{2}}\pi(g)T^{\frac{1}{2}}$ is bijective and has its adjoint as an inverse, it is unitary (Proposition B.10). □

Proposition 5.6 ([Sch15, Lemma 2.8])**.** *Given a representation $\pi : G \to B^I(H)$, the orbit of I under the action α_π is bounded if and only if π is uniformly bounded. More precisely, we have*

$$\operatorname{diam}(\mathcal{O}_{\alpha_\pi}(I)) = 2\log|\pi|.$$

Proof.

$$\begin{aligned}
\operatorname{diam}(\mathcal{O}_{\alpha_\pi}(I)) &= \sup_{g,h} d(\pi(g)I\pi(g)^*, \pi(h)I\pi(h)^*) \\
&= \sup_g d(\pi(g)\pi(g)^*, I) \\
&= \sup_g \log\max(\|(\pi(g)\pi(g)^*)^{-\frac{1}{2}}(\pi(g)\pi(g)^*)^{-\frac{1}{2}}\|, \|(\pi(g)\pi(g)^*)^{\frac{1}{2}}(\pi(g)\pi(g)^*)^{\frac{1}{2}}\|) \\
&= \sup_g \log\max(\|(\pi(g)\pi(g)^*)^{-1}\|, \|\pi(g)\pi(g)^*\|) \\
&= \sup_g \log\max(\|\pi(g)^{-1}\|^2, \|\pi(g)\|^2) \\
&= 2\log\sup_g \|\pi(g)\| \\
&= 2\log|\pi|
\end{aligned}$$

□

6 Cost

In this section, the notion of *cost* of a countable group is introduced. Some of the basic properties of the cost are exhibited, among which one of “multiplicativity” for finite index subgroups. After introducing the concept of *random graphs* on countable groups, a specific form of random graph satisfying “size” properties related to the cost is constructed. Finally, this construction is used to show that finitely generated, residually finite groups of cost greater than 1 are not unitarizable.

The first and second part (“Basics” and “Induction”) of this section follow [Gab00] mainly, with some bits from [KM04]. The third part (“Random Graphs”) is based on [KM04] for the tools, which are used following [EM09].

In the following, we will make heavy use of footnotes, as a mean to verify matters of measurability, in a hopefully non-obtrusive way.

6.1 Basics

In the following, a pair (X, μ) will always denote a standard measure space, that is, a standard Borel space X endowed with a non-zero, σ-finite measure μ. A relation on the standard Borel space X is said to be measurable if it is a measurable subset of $X \times X$ (in the product σ-algebra). A measurable isomorphism $f : A \to B$ ($A, B \subseteq X$ measurable) *preserves* μ if for all measurable subset C of B, $\mu(f^{-1}[C]) = \mu(C)$. If R is a relation on some set X, a subset Y of X is said to be (R-)saturated if it contains every R-orbit it intersects.

Definition 6.1 ((SP) relation)**.** Given (X, μ), an equivalence relation R on X:

(S) Is *standard* if: for all $x \in X$, the equivalence class (also called orbit) $R[x]$ of x is countable, and R is a measurable subset of $X \times X$ in the product σ-algebra.

(P) *Preserves* μ if: any measurable isomorphism between two measurable subsets of X, whose graph is a subset of R, preserves μ.

The relation R is called (SP) if it is both standard and preserves μ, and (SP1) if, in addition, $\mu(X) = 1$ (even though X having measure 1 is not a property of R).

If G is a group and (X, μ) a standard measure space, an action α of G on X is said to be measurable if each $\alpha(g)$ is a measurable map from X to X. Since X is a standard Borel space, and each $\alpha(g)$ is necessarily bijective, α acts then by isomorphisms (Proposition C.20 (6)).

Lemma 6.2 (Group action and (SP) relation)**.** *Let G be a countable group and α a measurable action of G on (X, μ), preserving μ. The equivalence relation $R_\alpha := \{(x, y) \in X \times X \;:\; \exists g \in G \; y = \alpha(g)x\}$ induced by the orbits of this action is (SP).*

Proof. First, as G is countable, so is

$$R_\alpha[x] = \{\alpha(g)x \mid g \in G\} = \mathcal{O}_\alpha(x),$$

for any $x \in X$. Also, note that R_α can be written

$$R_\alpha = \bigcup_{g \in G} \Gamma(\alpha(g)),$$

(where Γ denotes the graph of a function) which is a countable union of measurable subsets of $X \times X$ (Proposition C.20(3)), hence is measurable itself. Finally if $f : A \to B$ is an isomorphism of measurable subsets of X of graph contained in R_α, we can order the elements of G and inductively define

$$A_g := \{x \in A \;:\; f(x) = \alpha(g)x\} \setminus \bigcup_{h<g} A_h,$$ [1]

so that

$$A = \bigsqcup_g A_g \qquad \text{and} \qquad f|_{A_g} = \alpha(g)|_{A_g}.$$

Let then $B_g := f[A_g] = \alpha(g)[A_g]$, for all g. Since f is an isomorphism, it follows that

$$B = \bigsqcup_g B_g \qquad \text{and} \qquad f^{-1}|_{B_g} = \alpha(g)^{-1}|_{B_g}.$$

Then, for any $C \subseteq B$,

$$\mu(f^{-1}[C]) = \mu(f^{-1}[\bigsqcup_g C \cap B_g]) = \sum_g \mu(f^{-1}[C \cap B_g]) = \sum_g \mu(\alpha(g)^{-1}[C \cap B_g]) = \sum_g \mu(C \cap B_g) = \mu(C),$$

since $\alpha(g)$ preserves μ, for all g, and we conclude that f does preserve μ. ☐

In light of the above, let us call an action α (SP) if it acts by measurable isomorphisms and preserves the measure. Then, an action α is (SP) if and only if the induced equivalence relation is (SP). The above lemma gives one direction, and if α is a measurable action such that R_α is (SP), the graph of each $\alpha(g)$ is contained in R_α, and is an isomorphism, hence preserves the measure. Let us also call an action α (SP1) if it is (SP) and $\mu(X) = 1$.

The above lemma has a "converse", which we don't prove:

Theorem 6.3 (Feldman-Moore)**.** *If R is a (SP) relation on (X, μ), there exists a countable group G and a measurable action α of G on (X, μ) such that $R_\alpha = R$.*

Proof. [FM77, Theorem 1]. □

Note that the action α given by Feldman-Moore is then necessarily (SP), since R is.

Definition 6.4 (Isomorphic relations)**.** Given two standard measure spaces (X, μ) and (Y, ν) and (SP) equivalence relations R and S on X and Y respectively, we say that R and S are isomorphic (written $R \simeq S$), if there exists a measurable isomorphism $f : X \to Y$ such that $\nu(\cdot) = \mu(f^{-1}[\cdot])$, and for μ-a.e. $x \in X$, $f[R[x]] = S[f(x)]$; that is,

$$\{x \in X \ : \ f[R[x]] \neq S[f(x)]\} \text{ is a null set.}^2 \tag{33}$$

We write $R \simeq_f S$ to signify that $R \simeq S$, and that $f : X \to Y$ is a measurable bijection witnessing this isomorphism.

Let us verify that this defines an equivalence relation.

Proposition 6.5. *The relation $\simeq$ between (SP) equivalence relations on standard measure spaces is an equivalence relation.*

Proof. Reflexivity is straightforward, just choose $f = 1_X$. If R is a (SP) relation on (X, μ), and S a (SP) relation on (Y, ν), with $f : X \to Y$ is a measure-preserving isomorphism witnessing $R \simeq S$, then $f^{-1} : Y \to X$ is still measurable, and $\mu(A) = \mu(f^{-1}f[A]) = \nu(f[A])$, so that $\mu(\cdot) = \nu(f[\cdot])$. Furthermore, since

$$\{y \in Y \ : \ f^{-1}[S[y]] \neq R[f^{-1}(y)]\} = f[\{x \in X \ : \ S[f(x)] \neq f[R[x]]\}]$$

and $\{x \in X \ : \ S[f(x)] \neq f[R[x]]\}$ is a null set, so is $\{y \in Y \ : \ f^{-1}[S[y]] \neq R[f^{-1}(y)]\}$ (if $f : (X, \mu) \to (Y, \nu)$ is an isomorphism with $\nu(\cdot) = \mu(f^{-1}[\cdot])$, f sends μ-null sets to ν-null sets, and back). This shows symmetry.

Finally, given (X, μ), (Y, ν) and (Z, ρ), with $f : X \to Y$ and $g : Y \to Z$ witnesses for $R \simeq S$ and $S \simeq T$ respectively, then their composition is an isomorphism of measurable spaces, and $\rho(\cdot) = \nu(g^{-1}[\cdot]) = \mu((gf)^{-1}[\cdot])$. If $x \in X$ is such that

$$gf[R[x]] \neq T[gf(x)],$$

then in particular,

$$g[f[R[x]]] \neq g[S[f(x)]], \quad \text{or} \quad g[S[f(x)]] \neq T[g(f(x))],$$

which implies, as g is bijective, that:

$$f[R[x]] \neq S[f(x)], \qquad \text{or} \quad g[S[f(x)]] \neq T[g(f(x))].$$

Then,

$$\begin{aligned} \{x \in X \ : \ gf[R[x]] \neq T[gf(x)]\} &\subseteq \{x \in X \ : \ f[R[x]] \neq S[f(x)]\} \cup \{x \in X \ : \ g[S[f(x)]] \neq T[g(f(x))]\} \\ &= \{x \in X \ : \ f[R[x]] \neq S[f(x)]\} \cup f^{-1}[\{y \in Y \ : \ g[S[y]] \neq T[g(y)]\}] \end{aligned}$$

is a null set, as subset of a finite union of null sets. This shows transitivity, and we are done. □

Definition 6.6 (Graphing)**.** A *graphing* on (X, μ) is a countable family $\Phi = \langle \varphi_i : A_i \to B_i \rangle_{i \in I}$ of measurable, measure preserving isomorphisms between measurable subsets of X.

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[1] By induction, the set A_g is measurable, since $\{x \in A \ : \ f(x) = \alpha(g)x\} = \langle f, \alpha(g) \rangle^{-1}[\Delta_B]$ is.
(Given $f : A \to B$ and $g : C \to D$ with A, B, C, D measurable subsets of X, we can restrict f and g to $A \cap C$ and consider their codomain to be the entirety of X, and thus define the map $\langle f, g \rangle : A \cap C \to X$. From now on, and just above, we will implicitely always assume $\langle f, g \rangle$ to be defined like that.)

[2] The set $\{x \in X \ : \ f[R[x]] \neq S[f(x)]\}$ is measurable, by Theorem 6.3. Indeed, let α be the action of a countable group G on X with $R_\alpha = R$, and β the action of a countable group H on Y with $R_\beta = S$. Then

$$\{x \in X \ : \ f[R[x]] \not\subseteq S[f(x)]\} = \bigcup_{g \in G} \bigcap_{h \in H} \langle f \circ \alpha(g), \beta(h) \circ f \rangle^{-1}[Y^2 \setminus \Delta_Y],$$

is measurable. Similarly, $\{x \in X \ : \ S[f(x)] \not\subseteq f[R[x]]\}$ is measurable, thus so is $\{x \in X \ : \ f[R[x]] \neq S[f(x)]\}$.

Note that this definition allows for a same function $f : A \to B$ to appear multiple times; for instance $\Phi = \langle I_X \rangle_{i\in\mathbb{N}}$ is a valid graphing (with I_X the identity on X).

By a Φ-word, we mean a formal sequence $w = \varphi_{i_1}^{\varepsilon_1}...\varphi_{i_l}^{\varepsilon_l}$ ($\varepsilon_i = \pm 1$), of elements φ_{i_j} of Φ, and signs ε_j (more formally, a word is a finite sequence of pairs "index"-"sign"). Any Φ-word defines an evident measure preserving isomorphism of domain and codomain "as large as possible"; let us write $m(w)$ for this morphism, if w is a Φ-word. A Φ-letter will then denote a single element along with a sign, $\varphi_i^{\varepsilon_i}$, of Φ. In particular, any sequence of letters defines a word, with a potentially empty domain. We will only care about *reduced* words (as in, $...\varphi_i^{\varepsilon_i}\varphi_i^{-\varepsilon_i}...$ doesn't appear in the word); thus, from now on, Φ-word means *reduced* Φ-word. Let us write $\mathcal{W}(\Phi)$ for the (reduced) Φ-words; this defines a countable set of measurable, μ-preserving isomorphisms between measurable subsets of X.

Given a graphing Φ, consider the directed multigraph $\mathcal{G}_\Phi$ with vertex set X and a directed edge with label i from x to y if and only if $x \in \operatorname{dom}\varphi_i$ and $y = \varphi_i(x)$ (c.f. Definition C.26). The multigraph $\mathcal{G}_\Phi$ could be encoded as the tuple

$$\mathcal{G}_\Phi = (X,\ E_\Phi := \{(x, i, \varphi_i(x)) \mid i \in I, x \in \operatorname{dom}\varphi_i\},\ s : E_\Phi \to X,\ t : E_\Phi \to X)$$

where s assigns to each edge $(x, i, \varphi_i(x))$ the vertex x, and t the vertex $\varphi_i(x)$. Then, given any x, y in X, there is an evident bijection

$$\{\text{(reduced) }\Phi\text{-word } w \text{ with } x \in \operatorname{dom} m(w) \text{ and } y = m(w)(x)\} \longleftrightarrow \{\text{(reduced) undirected walk in } \mathcal{G}_\Phi \text{ from } x \text{ to } y\}.$$

By definition, the graph $\mathcal{G}_\Phi$ is then acyclic if and only if there exists no x and no non-trivial (reduced) Φ-word w with $x \in \operatorname{dom} m(w)$ and $m(w)(x) = x$.

In the following, given x and y, we generally won't distinguish between a word w with $x \in \operatorname{dom} m(w)$ and $m(w)(x) = y$, and the corresponding walk. Furthermore, a word will often be conflated with the induced map.

Let us write $v_\Phi(x) \in \mathbb{N} \sqcup \{\infty\}$ for the degree of x in $\mathcal{G}_\Phi$; that is, the number of incident edges, with multiplicity. Then

$$\begin{aligned} v_\Phi(x) &= |\{e \in E_\Phi \ :\ s(e) = x\}| + |\{e \in E_\Phi \ :\ t(e) = x\}| \\ &= |\{i \in I \ :\ x \in \operatorname{dom}\varphi_i\}| + |\{i \in I \ :\ x \in \operatorname{cod}\varphi_i\}| \\ &= \sum_i \mathbb{1}_{A_i}(x) + \sum_i \mathbb{1}_{B_i}(x) = \sum_i \mathbb{1}_{A_i}(x) + \mathbb{1}_{B_i}(x). \end{aligned}$$

Let R_Φ denote the equivalence relation defined by the connected component of $\mathcal{G}_\Phi$. We will write $\Phi[x] = R_\Phi[x] = \mathcal{G}_\Phi[x]$ for the orbit of the element x in the relation R_Φ. By definition, y is in the orbit of x if and only if there exists a (reduced) undirected walk from x to y in $\mathcal{G}_\Phi$, that is, if there exists a (reduced) Φ-word w with $x \in \operatorname{dom} m(w)$ and $y = m(w)(x)$. If $y \in R_\Phi[x]$, let $d_\Phi(x, y)$ be the length of a shortest undirected walk from x to y in $\mathcal{G}_\Phi$; this defines a metric on each orbit of R_Φ.

If $\Phi_1 = \langle \varphi_i : A_i \to B_i \rangle_{i\in I_1}$ and $\Phi_2 = \langle \varphi_i : A_i \to B_i \rangle_{i\in I_2}$ are graphings, define the graphing $\Phi_1 \vee \Phi_2$ as $\Phi_1 \vee \Phi_2 := \langle \varphi_i : A_i \to B_i \rangle_{i\in I_1 \sqcup I_2}$, where I_1 and I_2 have been made disjoint. More precisely, let $\Phi_1' := \langle \varphi_i \rangle_{(i,0)\in I_1\times\{0\}}$ and $\Phi_2' := \langle \varphi_i \rangle_{(i,1)\in I_2\times\{1\}}$ and define the graphing $\Phi_1 \vee \Phi_2$ as the union of Φ_1' and Φ_2'.

Remark 6.7. Note that if α is a (SP) action of a countable group G on (X, μ), then $\Phi := \langle \alpha(g) : X \to X \rangle_{g\in G}$ is a graphing and $R_\Phi = R_\alpha$. Indeed, if $y \in R_\alpha[x] = \mathcal{O}_\alpha(x)$, there exists some $g \in G$ with $y = \alpha(g)x$; but then, $\alpha(g)$ is a Φ-word w with $x \in \operatorname{dom} m(w)$ and $m(w)(x) = y$, so that $y \in R_\Phi[x]$. Conversely, if $y \in R_\Phi[x]$, there exists a Φ-word $w = \alpha(g_1)^{\varepsilon_1}, ..., \alpha(g_n)^{\varepsilon_n}$ such that $x \in \operatorname{dom} m(w)$ and $y = m(w)(x)$. But $m(w) = \alpha(g_1)^{\varepsilon_1} \circ ... \circ \alpha(g_n)^{\varepsilon_n} = \alpha(g_1^{\varepsilon_1}...g_n^{\varepsilon_n})$, and $y \in \mathcal{O}_\alpha(x)$.

Similarly to how the (SP) action of a group induces a (SP) equivalence relation, the same can be said for a graphing.

Proposition 6.8. *Let Φ be a graphing on (X, μ). Then R_Φ is (SP).*

Proof. We know that

$$R_\Phi[x] = \{w(x) \ \mid \ w \in \mathcal{W}(\Phi)\}$$

and the set of Φ-word is countable, so that $R_\Phi[x]$ is countable. If w is a Φ-word, its graph is measurable, and

$$R_\Phi = \bigcup_{w\in\mathcal{W}(\Phi)} \Gamma(w)$$

is a countable union of measurable sets, hence is measurable.

Finally, if $f : A \to B$ is a measurable isomorphism between measurable sets A, B, such that the graph $\Gamma(f)$ of f is contained in R_Φ, we can order the Φ-words, and inductively define

$$A_w := \{x \in \operatorname{dom} w \cap \operatorname{dom} f \ :\ f(x) = w(x)\} \setminus \bigcup_{w'<w} A_{w'}, \qquad \forall w \in \mathcal{W}(\Phi),$$

so that

$$A = \bigsqcup_w A_w \qquad \text{and} \qquad f|_{A_w} = w|_{A_w}.$$

Let then $B_w := f[A_w] = w[A_w]$, for all w. Since f is an isomorphism, it follows that

$$B = \bigsqcup_w B_w \qquad \text{and} \qquad f^{-1}|_{B_w} = w^{-1}|_{B_w}.$$

Then, for any $C \subseteq B$:

$$\mu(f^{-1}[C]) = \mu(f^{-1}[\bigsqcup_w C \cap B_w]) = \sum_w \mu(f^{-1}[C \cap B_w]) = \sum_w \mu(w^{-1}[C \cap B_w]) = \sum_w \mu(C \cap B_w) = \mu(C),$$

since each w preserves μ, and we conclude that f does preserve μ. ☐

Definition 6.9. If R is a (SP) relation and Φ a graphing with $R_\Phi \simeq R$, Φ is said to be a *graphing of R*. The graphing Φ is said to be a *treeing* of R if, for μ-a.e. x, the connected component of x in $\mathcal{G}_\Phi$ is a tree (i.e. has no reduced non-trivial undirected cycle).[3]

We know, by Theorem 6.3, that any (SP) relation has a graphing, but not necessarily a treeing.

Definition 6.10 (Cost)**. Graphing.** The cost of a graphing $\Phi = \langle \varphi_i : A_i \to B_i \rangle_{i \in \mathbb{N}}$ on a standard measure space (X, μ) is defined as:

$$\mathcal{C}_\mu(\Phi) := \sum_i \mu(A_i).$$

Equivalence relation. The cost of an (SP) equivalence relation R on a standard measure space (X, μ) is defined as:

$$\mathcal{C}_\mu(R) := \inf\{\mathcal{C}_\mu(\Phi) \mid \Phi \text{ a graphing of } R\}.$$

Group. The cost of a countable group G is defined as:

$$\mathcal{C}(G) := \inf\{\mathcal{C}_\mu(R_\alpha) \mid \alpha \text{ is a free (SP1) action of } G \text{ on the standard measure space } (X, \mu)\}.$$

Note that in these definitions, we allow for the costs of graphings, equivalence relations and groups to be infinite.

Remark 6.11. Given a graphing $\Phi = \langle \varphi_i : A_i \to B_i \rangle_{i \in I}$, we know that $\mu(A_i) = \mu(B_i)$ for all i (φ_i is a measure preserving isomorphism of measurable subsets), which implies that

$$\mathcal{C}_\mu(\Phi) = \sum_i \mu(B_i).$$

Also, recalling that $v_\Phi(x)$ denotes the degree of x in the multigraph $\mathcal{G}_\Phi$, we have

$$\int_X v_\Phi(x) d\mu(x) = \int_X \sum_{i \in I} (\mathbb{1}_{A_i}(x) + \mathbb{1}_{B_i}(x)) d\mu(x) = \sum_{i \in I} \int_X \mathbb{1}_{A_i}(x) + \mathbb{1}_{B_i}(x) d\mu(x) = \sum_{i \in I} \mu(A_i) + \mu(B_i) = 2\mathcal{C}_\mu(\Phi).$$[4]

Remark 6.12. Given two graphings $\Phi_1 = \langle \varphi_i \rangle_{i \in I_1}$ and $\Phi_1 = \langle \varphi_i \rangle_{i \in I_2}$, we have

$$\mathcal{C}_\mu(\Phi_1 \vee \Phi_2) = \sum_{i \in I_1 \sqcup I_2} \mu(\operatorname{dom} \varphi_i) = \sum_{i \in I_1} \mu(\operatorname{dom} \varphi_i) \ + \ \sum_{i \in I_2} \mu(\operatorname{dom} \varphi_i) = \mathcal{C}_\mu(\Phi_1) + \mathcal{C}_\mu(\Phi_2).$$

Remark 6.13. If α is a (SP) action of a countable group G on a standard measure space (X, μ) with $\mu(X) < \infty$, we can let $\mu_1(\cdot) := \frac{\mu(\cdot)}{\mu(X)}$, and this makes α into a (SP1) action on the standard measure space (X, μ_1). By the "linear" usage of the measure in our definitions of cost, it follows that

$$\mathcal{C}_{\mu_1}(\Phi) = \frac{\mathcal{C}_\mu(\Phi)}{\mu(X)} \qquad \text{if } \Phi \text{ is a graphing,}$$

$$\mathcal{C}_{\mu_1}(R) = \frac{\mathcal{C}_\mu(R)}{\mu(X)} \qquad \text{if } R \text{ is a (SP) relation.}$$

This shows that we can compute costs on finite measure spaces, and then scale accordingly.

[3]The set $\{x \in X \,:\, \mathcal{G}_\Phi[x] \text{ is a tree}\}$ is measurable: it can be written

$$\bigcap_{w \in \mathcal{W}(\Phi) \setminus \{\varepsilon\}} \langle w, I_{\operatorname{dom} w} \rangle^{-1}[X^2 \setminus \Delta_{\operatorname{dom} w}],$$

with ε the trivial word.

[4]Corollary C.24 guarantees that $v_\Phi : X \to [0, \infty]$ is measurable and allows us to commute integral and sum. Indeed, endowing the countable index set I with the ordering induced by an injection into $\mathbb{N}$, v_Φ is the pointwise limit of the non-decreasing measurable maps $\langle \sum_{i<j} \mathbb{1}_{A_i} + \mathbb{1}_{B_i} \rangle_{j \in I}$.

In the definition of the cost of a (SP) relation, the infimum is taken over a non-empty set, by Theorem 6.3. Similarly:

Proposition 6.14. *Given a countable group G, there exists a free (SP1) action α of G on a standard measure space (X, μ).*

Proof. Fix G a countable group. Let I be the unit interval $[0, 1]$ endowed with its Borel σ-algebra, and Lebesgue measure μ_I. Define $X := I^G$, endowed with the product measure μ. Recall that the σ-algebra on X is the σ-algebra generated by the measurable basis

$$\{\pi_g^{-1}[B] \mid g \in G,\ B \text{ Borel in } I\},$$

where $\pi_g : X \to I$ is the projection on the coordinate g. Recall also that the measure on X is the unique measure satisfying

$$\mu\left(\bigcap_{i=1}^{n} \pi_{g_i}^{-1}[B_i]\right) = \prod_{i=1}^{n} \mu_I(B_i),$$

for any $n \geq 1$, sequence $g_1, ..., g_n$ of distinct elements of G and sequence $B_1, ..., B_n$ of Borel sets in I. Define then the action α of G on (X, μ) by

$$\alpha(g)(\varphi(\cdot)) = \varphi(g^{-1}\cdot),$$

that is, the usual left shift.

We first need to verify that this action is measurable and preserves the measure. To show that $\alpha(h)$ is measurable, it is enough to verify that if C is an element of a measurable basis for X, then $\alpha(h)^{-1}[C]$ is measurable, but

$$\alpha(h)^{-1}[\pi_x^{-1}[B]] = \pi_{h^{-1}x}^{-1}[B], \quad \forall B \text{ Borel in } I,\ \forall x \in G,$$

so that $\alpha(h)$ actually takes an element of a measurable basis back to another one, and is indeed a measurable map.

Furthermore, for any integer n, sequence $g_1, ..., g_n$ of distinct elements of G, and Borel sets $B_i \subseteq I$ $(1 \leq i < n)$:

$$\alpha(h)^{-1}\left[\bigcap_{i=1}^{n} \pi_{g_i}^{-1}[B_i]\right] = \bigcap_{i=1}^{n} \alpha(h)^{-1}[\pi_{g_i}^{-1}[B_i]] = \bigcap_{i=1}^{n} \pi_{h^{-1}g_i}^{-1}[B_i],$$

so that, noting that the elements $h^{-1}g_1, ...h^{-1}g_n$ are still distinct:

$$\mu\left(\alpha(h)^{-1}\left[\bigcap_{i=1}^{n} \pi_{g_i}^{-1}[B_i]\right]\right) = \mu\left(\bigcap_{i=1}^{n} \pi_{h^{-1}g_i}^{-1}[B_i]\right) = \prod_{i=1}^{n} \mu(B_i).$$

Thus, the measure $\mu(\alpha(h)^{-1}[\cdot])$ on X is equal to μ on the finite rectangles, which implies that they are equal in general, since μ is uniquely determined by its value on the finite rectangles; we conclude that $\alpha(h)$ preserves the measure.

We now show that α is μ-a.e. free, that is, the set of elements of X on which G doesn't act freely is a null set. First, observe that

$$\{\varphi \in X \;:\; \exists g \neq e_G\ \alpha(g)\varphi = \varphi\} = \bigcup_{g \neq e_G} \{\varphi \in X \;:\; \alpha(g)\varphi = \varphi\}$$

and it suffices, by countability of G, to show that, for all $g \neq e_G$, the set

$$\{\varphi \in X \;:\; \alpha(g)\varphi = \varphi\},^{5}$$

is null. If $g \neq e_G$ is such that $\alpha(g)\varphi = \varphi$, then by definition $\varphi(g^{-1}x) = \varphi(x)$ for all $x \in G$, and in particular for $x = g$, that is, $\varphi(e_G) = \varphi(g)$. Now, letting $P_n^i := [i\frac{1}{2^{-n}}, (i+1)\frac{1}{2^{-n}}]$, for any integer $n \geq 1$ and $0 \leq i \leq 2^n - 1$, we have:

$$\{\varphi \in X \;:\; \varphi(e_G) = \varphi(g)\} = \bigcap_{n \geq 1} \bigcup_{i=0}^{2^n-1} \pi_{e_G}^{-1}[P_n^i] \cap \pi_g^{-1}[P_n^i].$$

Given that P_n^i has measure $\mu_I(P_n^i) = 2^{-n}$,

$$\mu(\pi_{e_G}^{-1}[P_n^i] \cap \pi_g^{-1}[P_n^i]) = 2^{-2n},$$

so that

$$\mu(\bigcup_{i=0}^{2^n-1} \pi_{e_G}^{-1}[P_n^i] \cap \pi_g^{-1}[P_n^i]) \leq 2^{-n}.$$

And we conclude that

$$\mu\{\varphi \in X \;:\; \varphi(e_G) = \varphi(g)\} \leq 2^{-n}, \quad \forall n,$$

which implies that it has zero measure. Finally, since

$$\{\varphi \in X \;:\; \alpha(g)\varphi = \varphi\} \subseteq \{\varphi \in X \;:\; \varphi(e_G) = \varphi(g)\},$$

we conclude that the LHS is a null set, and the action is μ-a.e. free.

Now that we know the action to be μ-a.e. free, we can get rid of the non-free part. First, observe that the free part is saturated. Indeed, fix some φ and $\psi \in \mathcal{O}_\alpha(\varphi)$: there exists g with $\psi = \alpha(g)\varphi$. If there exists some $h \neq e_G$ with $\alpha(h)\psi = \psi$, then $\alpha(h)\alpha(g)\varphi = \alpha(g)\varphi$ so that $\alpha(g^{-1}hg)\varphi = \varphi$, and as $h \neq e_G$, $g^{-1}hg \neq e_G$. This shows that if ψ is not acted on freely by α, then neither is φ. Let then X' be the free part, and let α' be the restriction, at each g, of $\alpha(g)$ to X'. We know that $\mu(X') = \mu(X) = 1$, and the action α' is free (by construction), and preserves the restriction $\mu|_{X'}$ of μ to X', since it preserves μ. We have thus constructed a free (SP1) action of G on the standard measure space X'. □

Recall that for any group G, the rank $\operatorname{rank} G$ of G is the cardinality of a least generating set of G.

Proposition 6.15 (Finitely Generated Group)**.** *A group G of finite rank n has cost $\mathcal{C}(G) \leq n$.*

Proof. Let G be a group of rank n, and $S \subseteq G$ a subset of G of finite cardinality n with $\langle S \rangle = G$. Fix any free (SP1) action α of G on a standard measure space (X, μ). Consider the graphing $\Phi := \langle \alpha(s) : X \to X \rangle_{s \in S}$; its cost is $\sum_{s \in S} \mu(X) = |S| \cdot 1 = n$, and we show that it is a graphing of R_α. First, if $y \in R_\alpha[x]$, there exists some $g \in G$ with $y = \alpha(g)x$. Since S generates G, there exists elements $s_1, ..., s_n$ of S and signs $\varepsilon_1, ..., \varepsilon_n$ with $g = s_1^{\varepsilon_1} ... s_n^{\varepsilon_n}$. Then, $w = \alpha(s_1)^{\varepsilon_1} ... \alpha(s_n)^{\varepsilon_n}$ is a Φ-word, with $x \in \operatorname{dom} w$, and

$$w(x) = \alpha(s_1)^{\varepsilon_1} ... \alpha(s_n)^{\varepsilon_n} x = \alpha(s_1^{\varepsilon_1} ... s_n^{\varepsilon_n}) x = \alpha(g)x = y$$

and y lies in $R_\Phi[x]$.

Conversely, if $y \in R_\Phi[x]$, there exists a Φ-word $w = \alpha(s_1)^{\varepsilon_1} ... \alpha(s_n)^{\varepsilon_n}$ with $y = w(x)$, but since $w(x) = \alpha(s_1^{\varepsilon_1} ... s_n^{\varepsilon_n})x \in \mathcal{O}_\alpha(x)$, it follows that $y \in R_\alpha[x]$.

Since Φ is a graphing of R_α of cost n, we conclude that:

$$\mathcal{C}(G) \leq \mathcal{C}_\mu(R_\alpha) \leq n.$$

□

Definition 6.16 (μ-a.e. smooth relation)**.** If R is a (SP) relation on (X, μ) and $\mu(X)$ is finite, then R is said to be μ-a.e. *smooth* if there exists some measurable subset S of X intersecting μ-a.e. orbit of R at exactly one point, and not intersecting the other orbits at all; that is, if:

$$\forall x \in X \ : \ |S \cap R[x]| \leq 1, \quad \text{and} \quad \{x \in X \ : \ S \cap R[x] = \emptyset\} \text{ is null.}^6$$

The set S is called a *μ-a.e. fundamental domain* for X.

Removing the "μ-a.e." part, a fundamental domain is a measurable set intersecting each orbit at exactly one point, and a smooth measurable relation is a measurable relation possessing a fundamental domain.

The following result is not proved here.

Proposition 6.17. *A (SP1) relation is μ-a.e. smooth if and only if μ-a.e. orbit is finite.*

Proof. See [KM04, p. 66] for an idea of the proof. Also, [Slu16, Proposition 1.4.4] and [Slu16, Proposition 1.6.4] show the *non-μ*-a.e. case, which can then easily be used for the "general" one.

□

The following proposition and its proof follow [Slu16, Proposition 1.4.2].

Proposition 6.18. *Let R be a (SP) relation on (X, μ). Then, X can be partitioned as:*

$$X = \bigsqcup_{n \in \mathbb{N}^{\geq 1}} X_n \sqcup X_\infty,$$

where X_n is the set of elements of X with orbit of cardinality n. The sets X_n are all measurable and saturated with respect to R.

---

[5] The set $\{\varphi \in X \ : \ \alpha(g)\varphi = \varphi\} = \langle \alpha(g), 1_X \rangle^{-1}[X^2 \setminus \Delta_X]$ is measurable, by its second formulation.

[6] The set $\{x \in X \ : \ S \cap R[x] = \emptyset\}$ is measurable, by Theorem 6.3: Fix some measurable action α of a countable group G on X with $R = R_\alpha$. Then

$$\{x \in X \ : \ S \cap R[x] = \emptyset\} = \{x \in X \ : \ S \cap \mathcal{O}_\alpha(x) = \emptyset\} = \bigcap_{g \in G} \alpha(g)^{-1}[X \setminus S]$$

is measurable.

Proof. The partition is evident, since any orbit has a unique cardinality.

To show that each X_n (for $n < \infty$) is measurable, we use Theorem 6.3. Fix a countable group G and an (SP) action α of G on (X, μ). The set X_n can be described informally as those elements x of X such that the distinct actions of n elements of G cover the orbit $R[x]$:

$$X_n = \bigcup_{\substack{g_1,\cdots g_n \in G \\ \text{distinct}}} \left(\left(\bigcap_{1 \leq i \neq j \leq n} \langle \alpha(g_i), \alpha(g_j) \rangle^{-1}[X^2 \setminus \Delta_X] \right) \cap \left(\bigcap_{g \in G} \bigcup_{1 \leq i \leq n} \langle \alpha(g), \alpha(g_i) \rangle^{-1}[\Delta_X] \right) \right). \tag{34}$$

Given that all the intersections and unions are countable, and done over measurable sets, we conclude that X_n is measurable. Furthermore, X_∞ can be written as the complement of $\bigsqcup_n X_n$, which is measurable.

Finally, if the orbit of some x has cardinality n, then so does the orbit of any element in the orbit of x, which implies that X_n is R-saturated for all n. $\square$

Lemma 6.19. *Let R be a measurable equivalence relation on a standard Borel space X such that every R-orbit has cardinality n. There exists measurable subsets $S_1, ..., S_n$ of X, each a fundamental domain for R, satisfying*

$$X = \bigsqcup_{i=1}^{n} S_i,$$

and such that for each $1 \leq i < n$, the map $f_i : S_i \to S_{i+1}$ sending $x \in S_i$ to the unique element of $S_{i+1} \cap R[x]$ is a measurable isomorphism.

Note here that the graph of each f_i is contained in R: fix some $i < n$ and $x \in S_i$; $f_i(x)$ is the unique element in $S_{i+1} \cap R[x]$, so that $f_i(x) \in R[x]$ and $(x, f_i(x)) \in R$.

Proof. We first reduce the proof to the case where X is $[0,1]$ with its Borel σ-algebra. The space X being standard Borel, it is either discrete, or measurable isomorphic to $[0,1]$. If X is discrete, the lemma trivially holds, since any set is measurable, and the sets S_i can be picked as we please. Now, assume the lemma has been proven for $[0,1]$ and any (SP) relation on $[0,1]$. Fix X a standard Borel space isomorphic to $[0,1]$, R a measurable relation on X with all orbits of finite cardinality n, and $\varphi : X \to [0,1]$ a measurable isomorphism. Define the relation $R' := \{(\varphi(x), \varphi(y)) \mid (x,y) \in R\} = (\varphi \times \varphi)[R]$ on $[0,1]$; R' is a measurable relation, all orbits of which have cardinality n, since φ is a measurable bijection. We can therefore apply the lemma on R', and get Borel subsets $S'_1, ..., S'_n$ of $[0,1]$, each a fundamental domain for R', such that $[0,1] = \bigsqcup_{i=1}^n S_i$ and such that the maps $f'_i : S'_i \to S'_{i+1}$ sending an element $x \in S'_i$ to the unique element of $S'_{i+1} \cap R'$ are measurable, for each $1 \leq i < n$. Let then $S_i := \varphi^{-1}[S'_i]$, for each i; a straightforward verification shows that the sets S_i form a Borel partition of X, each being a fundamental domain for R, and such that the maps $f_i : S_i \to S_{i+1}$ described in the statement of the lemma are measurable.

Let us now prove the result for $X = [0,1]$.

Define the measurable subset A_n of $[0,1]^n$ by:

$$A_n := \{(x_1, ...x_n) \in [0,1]^n \ : \ x_1 < x_2 < ... < x_n\}. \tag{35}$$

This set is open (if the tuple $p = (x_1, ..., x_n)$ is in A_n and $\varepsilon > 0$ is such that each x_i is at a distance greater than 2ε from its successor/predecessor, then any tuple p' in the ε-box around p will still be in A_n), and therefore measurable. Define also Q_n^i, for $1 \leq i \leq n-1$ by

$$Q_n^i := \{(x_1, ...x_n) \in [0,1]^n \ : \ (x_i, x_{i+1}) \in R\} = \underbrace{[0,1] \times ... \times [0,1]}_{i-1 \text{ times}} \times R \times \underbrace{[0,1] \times ... \times [0,1]}_{n-i-1 \text{ times}},$$

which is measurable by its second characterisation. Finally, define Q_n by

$$Q_n := \bigcap_{i=1}^{n-1} Q_n^i = \{(x_1, ...x_n) \in [0,1]^n \ : \ \forall i \ (x_i, x_{i+1}) \in R\}, \tag{36}$$

which is measurable, as an intersection of measurable sets. The set Q_n consists of the n-tuples $(x_1, ..., x_n)$ of elements of X such that each x_i is in the same orbit as x_{i+1}; by transitivity, this implies that all the elements of the tuple are in the same orbit. Now, consider the set $S := A_n \cap Q_n$; it consists of the strictly increasing n-tuples of $[0,1]^n$ such that the underlying set is included in some orbit. But as the orbits are all of cardinality n, S is just the set of n-tuples corresponding to the orbits of R, ordered increasingly. Indeed, for any $p = (x_1, ..., x_n) \in S$, write $o(p) := \{x_1, ..., x_n\}$, the underlying set of p. We show that o is a bijection from S to the set of orbits of R. Consider any tuple $p = (x_1, ..., x_n)$ of S: since $x_i < x_{i+1}$ for all i, all x_is are distinct; since $p \in Q_n$, all x_is are in the same orbit. Then, $o(p) = \{x_1, ..., x_n\} = R[x_1]$ is the orbit of x_1. Furthermore, if $p = (x_1, ..., x_n)$ and $q = (y_1, ..., y_n)$ are two tuples of S and $\{x_1, ..., x_n\} = o(p) = o(q) = \{y_1, ..., y_n\}$, then $x_1 = y_1$, since they are both the least elements of the orbit, and repeating this on the remaining sets $n-1$ times, $x_i = y_i$ for all i, so that $p = q$. Finally, any orbit can be written $\{x_1, ..., x_n\}$ with $x_i < x_{i+1}$ for all i, and the tuple $(x_1, ..., x_n)$ in S is such that $o(p) = \{x_1, ..., x_n\}$. Thus, o really is a bijection.

For each $1 \leq i \leq n$, let $\pi_i : [0,1]^n \to [0,1]$ be the (continuous, hence measurable) projection on the coordinate i. We show that each π_i is injective on S. First, note that for any $p \in S$ and i, we have $\pi_i(p) \in o(p)$, by construction. Now, if p and q are two elements of S with $\pi_i(p) = \pi_i(q)$, then $\pi_i(p) \in o(q) \cap o(p)$, and the orbits $o(p)$ and $o(q)$ share an element, hence are equal; we conclude $p = q$.

Let then $S_i := \pi_i[S]$, for each i. Since $\pi_i|_S$ is injective, and measurable, it has a measurable image (Proposition C.20 (5)), and S_i is measurable. The set S_i consists in picking, for each orbit $\{x_1, ..., x_n\}$, the i-th element of the orbit.

We now show that the sets S_i form a partition of X, and each is a fundamental domain for R. Fix some $x \in X$; x is in some orbit, and thus, there exists some $p = (x_1, ..., x_n)$ and j with $x = x_j$. Then $x = \pi_j(p)$ and $x \in S_j$. Furthermore, if $x \in S_i \cap S_j$, there exists $p, q \in S$ with $\pi_j(p) = x = \pi_i(q)$, and $x \in o(p) \cap o(q)$, so that $p = q$; then, if $p = (x_1, ..., x_n)$, we have $x_i = \pi_i(p) = x = \pi_j(p) = x_j$, which implies $i = j$. This shows that the sets S_i form a partition of X. It remains to show that each S_j intersects each R-orbit at exactly one point. Fix some orbit $o(p)$, with $p = (x_1, ..., x_n)$, and some S_j and consider the intersection $o(p) \cap S_j = o(p) \cap \pi_j[S]$. The element $x_j = \pi_j(p)$ is in the intersection, and if y is any element of this intersection, then $y = x_k$ for some k, and $y = \pi_j(q)$, for some q; then $y \in o(q) \cap o(p)$ so that $p = q$, and $y = \pi_j(p) = x_j$, so that $y = x_j$. This shows the first part of the lemma.

For the second part, note that the restriction of $\pi_i|_S$ to its range is a measurable isomorphism, thus has a measurable inverse $\pi_i|_S^{-1} : S_i \to S$; then $\pi_{i+1}|_S \pi_i|_S^{-1} : S_i \to S_{i+1}$ is bijective measurable, but this map is exactly the one sending an element $x \in S_i$ to the unique element in $R[x] \cap S_{i+1}$, and we are done. ☐

Corollary 6.20. *If R is a (SP) equivalence relation on (X, μ) such that every R-orbit has cardinality n, then R has a fundamental domain S such that $\mu(S) = \frac{1}{n}\mu(X)$.*

Proof. By Lemma 6.19, we know that there exists a Borel partition $S_1, ..., S_n$, of X, with each S_i a fundamental domain for R. Furthermore, for all i, the map $f_i : S_i \to S_{i+1}$ is a measurable isomorphism; its graph is contained in R, since $f_i(x) \in R[x]$ for any $x \in S_i$, and it must therefore preserve the measure, because R is (SP). Then, $\mu(S_i) = \mu(S_{i+1})$, for all i, and:

$$\mu(X) = \mu(\bigsqcup_i S_i) = \sum_{i=1}^{n} \mu(S_i) = \sum_{i=1}^{n} \mu(S_1) = n\mu(S_1),$$

and $S := S_1$ is a fundamental domain for R of measure $\frac{1}{n}$ that of X. ☐

Lemma 6.21. *Let R be a measurable equivalence relation on a standard Borel space X such that every R-orbit has finite cardinality n, and S a fundamental domain for R. Letting $S_1 := S$, there exists measurable subsets $S_1, ..., S_n$ of X, each a fundamental domain for R, satisfying*

$$X = \bigsqcup_{i=1}^{n} S_i,$$

and such that for all $1 \leq i < n$, the map $f_i : S_i \to S_{i+1}$ sending $x \in S_i$ to the unique element of $S_{i+1} \cap R[x]$ is a measurable isomorphism.

Proof. It suffices to adapt the argument of the previous lemma, to take the "initial" set S into account. Define

$$A_n^S := \{(x_1, ...x_n) \in [0,1]^n \ : \ (x_1 \in S \ \wedge \ x_2 < ... < x_n) \ \wedge \ (x_1 \neq x_i \ \forall i \geq 2)\} = S \times (A_{n-1} \cap ([0,1] \setminus S)^n),$$

with A_{n-1} defined as in Equation (35). The set A_n^S is measurable since both S and A_{n-1} are. Let Q_n be as defined in Equation (36); $A^S \cap Q_n$ is measurable and in bijection with the orbits of R. The same argument as in Lemma 6.19, with $A^S \cap Q$ instead of $A \cap Q$, proves the lemma. ☐

Corollary 6.22. *If R is a (SP) equivalence relation on (X, μ) such that every orbit has cardinality n, and S is a fundamental domain for R, then*

$$\mu(S) = \frac{1}{n}\mu(X).$$

Proof. By Lemma 6.21, there exists a partition $S = S_1, ..., S_n$ of X, with each S_i a fundamental domain for R and measurable isomorphisms $f_i : S_i \to S_{i+1}$. As above, we know that the isomorphisms $f_i : S_i \to S_{i+1}$ preserve the measure, since their graphs are contained in R. Then, each S_i has the same measure as S, and since there are n of them and they form a partition of X, we conclude that

$$\mu(S) = \frac{1}{n}\mu(X).$$

□

Proposition 6.23 ([Gab00, Proposition I.9])**.** *Let R be a μ-a.e. smooth, (SP) relation on (X, μ), with $\mu(X)$ finite, and S any μ-a.e. fundamental domain for R. Then*

1. *$\mathcal{C}_\mu(R) = \mu(X) - \mu(S)$;*
2. *R admits a treeing;*
3. *Every treeing of R has cost $\mathcal{C}_\mu(R)$.*

Proof. By Proposition 6.17, μ-a.e. R-orbit is finite.

We first show that for any graphing Φ of R,

$$\mathcal{C}_\mu(\Phi) \geq \mu(X) - \mu(S)$$

with equality if and only if Φ is a treeing. Secondly, we show that R has some treeing. Item 2 is then direct, and from the inequality, we know

$$\mathcal{C}_\mu(R) \geq \mu(X) - \mu(S).$$

By the existence of a treeing, whose cost attains the bound, Item 1 and Item 3 follow.

Fix a graphing Φ of R. As μ-a.e. R-orbit is finite, so is μ-a.e. Φ-orbit. Let

$$Z := \{x \in X \;:\; (\Phi[x] \text{ is not finite}) \vee (S \cap \Phi[x] = \emptyset)\}$$

and $X' := X \setminus Z$. The set Z is null because μ-a.e. Φ-orbit is finite, and S intersects μ-a.e. orbit. Partition X' into

$$X' = \bigsqcup_{n \in \mathbb{N}^{\geq 1}} X_n, \qquad \text{where} \qquad X_n := \{x \in X' \;:\; |\Phi[x]| = n\},$$

and $S' := S \setminus Z$ into

$$S' = \bigsqcup_{n \in \mathbb{N}^{\geq 1}} S_n, \qquad \text{where} \qquad S_n := S \cap X_n.$$

Then, S' has the same measure as S, and each S_n is a fundamental domain for the (SP) relation $R_\Phi|_{X_n}$ on X_n, all orbits of which are of cardinality n. We then have, by Corollary 6.22:

$$\begin{aligned}
\mu(X) = \mu(X') &= \sum_{n \geq 1} \mu(X_n) \\
&= \sum_{n \geq 1} n\mu(S_n) \\
&= \sum_{n \geq 1} \int_{S_n} n d\mu(x) \\
&= \sum_{n \geq 1} \int_{S_n} |\Phi[x]| d\mu(x) \\
&= \int_{S'} |\Phi[x]| d\mu(x) = \int_{S} |\Phi[x]| d\mu(x).^7
\end{aligned}$$

We also have:

$$2\mathcal{C}(\Phi) = \int_X v_\Phi(x) d\mu(x) = \int_S \sum_{y \in \Phi[x]} v_\Phi(y) d\mu(x).^8$$

Indeed, by decomposing the integrals along the partitions $X' = \bigsqcup_n X_n$ and $S' = \bigsqcup_n S_n$ it is enough to show that

$$\int_{X_n} v_\Phi(x) d\mu(x) = \int_{S_n} \sum_{y \in \Phi[x]} v_\Phi(y) d\mu(x),$$

[7] The map $x \mapsto |\Phi[x]|$ is measurable: by Proposition 6.18, we know that for each n, the set $\{x \in X \;:\; |\Phi[x]| = n\}$ is measurable.

[8] The map $x \mapsto \sum_{y \in \Phi[x]} v_\Phi(y)$ is measurable. First, note that for any x,

$$\sum_{y \in \Phi[x]} v_\Phi(y) = \sum_{y \in \Phi[x]} \sum_i (\mathbb{1}_{A_i}(y) + \mathbb{1}_{B_i}(y)) = \sum_i \sum_{y \in \Phi[x]} (\mathbb{1}_{A_i}(y) + \mathbb{1}_{B_i}(y)) = \sum_i |A_i \cap \Phi[x]| + |B_i \cap \Phi[x]|,$$

and the maps $x \mapsto |A_i \cap \Phi[x]|$ and $x \mapsto |B_i \cap \Phi[x]|$ are measurable, by an argument similar to Proposition 6.18: replacing X^2 by A_i^2 and Δ_X by Δ_{A_i} in Equation (34) shows that $\{x \in X \;:\; |\Phi[x] \cap A_i| = n\}$ is measurable for any n. Then, Corollary C.24 allows us to conclude that the map $x \mapsto \sum_{y \in \Phi[x]} v_\Phi(y)$ is measurable, as a pointwise limits of non-decreasing measurable maps (the partial sums).

but as S_n is a fundamental domain for $R_\Phi|_{X_n}$, all orbits of which are of cardinality n, there exists a partition $S_n^1, ..., S_n^n$ of X_n, with $S_n^1 = S_n$, and measurable isomorphisms $f_i : S_n^i \to S_n^{i+1}$ as given by Lemma 6.21. Since the graph of each f_i is contained in R_Φ, f_i is measure preserving. Let then $f_i' := f_i ... f_1 : S_n \to S_n^i$ (with $f_0' = 1_{S_n}$); we have:

$$\begin{aligned}
\int_{X_n} v_\Phi(x) d\mu(x) &= \sum_{i=1}^{n} \int_{S_n^i} v_\Phi(x) d\mu(x) \\
&= \sum_{i=0}^{n-1} \int_{f_i'[S_n]} v_\Phi(x) d\mu(x) \\
&= \sum_{i=0}^{n-1} \int_{S_n} v_\Phi(f_i'(x)) d\mu(x) \\
&= \int_{S_n} \sum_{i=0}^{n-1} v_\Phi(f_i'(x)) d\mu(x) \\
&= \int_{S_n} \sum_{y \in \Phi[x]} v_\Phi(y) d\mu(x),
\end{aligned}$$

and we are done.

Fix now any $x \in X'$, it has a finite orbit, and the connected component of x in $\mathcal{G}_\Phi$ is finite and connected, so that we have (by Proposition C.30):

$$\sum_{y \in \Phi[x]} v_\Phi(y) \geq 2(|\Phi[x]| - 1),$$

with an equality whenever the connected component is a tree. Integrating over S on both sides yields:

$$2\mathcal{C}(\Phi) = \int_S \sum_{y \in \Phi[x]} v_\Phi(y) d\mu(x) \geq 2 \int_S |\Phi[x]| d\mu(x) - 2\mu(S) = 2\mu(X) - 2\mu(S)$$

with an equality whenever the connected component of x is a tree, for μ-a.e. $x \in S$. That is,

$$\mathcal{C}(\Phi) \geq \mu(X) - \mu(S)$$

with an equality whenever the connected component of x is a tree, for μ-a.e. $x \in X$, by Lemma 6.24, since "lying in an acyclic connected component" is a saturated property. We conclude that

$$\mathcal{C}(\Phi) \geq \mu(X) - \mu(S)$$

with an equality whenever Φ is a treeing.

We now show that R possesses a treeing. Consider the partition:

$$X = \left(\bigsqcup_{n \in \mathbb{N}^{\geq 1}} X_n \right) \sqcup X_\infty, \qquad \text{where} \quad X_n := \{x \in X \ : \ |R[x]| = n\}.$$

For each X_n ($1 \leq n < \infty$), there exists a partition into measurable sets $X_n = \bigsqcup_{i=1}^n S_n^i$, and measurable isomorphisms $f_n^i : S_n^i \to S_n^{i+1}$, sending $x \in S_n^i$ to the unique element of $R[x] \cap S_n^{i+1}$, by Lemma 6.19. Since the graph of each f_n^i is contained in R, a (SP) relation, each f_n^i is measure preserving. Then, $\Psi := \langle f_n^i : S_n^i \to S_n^{i+1} \rangle_{1 \leq n < \infty, 1 \leq i \leq n}$ is a treeing of R. We first verify that it is a graphing of R. Fix $x \notin X_\infty$, then $x \in S_n^i$ for some n and i. The inclusion $\Psi[x] \subseteq R[x]$ holds since the graph of each f_n^i is contained in R. Conversely, if $y \in R[x]$, then $y \in S_n^j$ for some j, and assuming without loss of generality that $i \leq j$, $f_n^{j-1} ... f_n^i$ is a Ψ-word from x to y. This shows that whenever $x \notin X_\infty$, $R[x] = \Psi[x]$.
Now, we show that Φ is actually a treeing. Fix any $x \notin X_\infty$, then $x \in X_n$ for some n and has orbit $R[x]$ of cardinality n, and we show that the connected component of x in $\mathcal{G}_\Psi$ is a tree, for which it is equivalent to show (Proposition C.30):

$$\sum_{y \in \Phi[x]} v_{\mathcal{G}_\Psi}(y) = 2(n-1), \qquad \text{or} \qquad \sum_{y \in \Phi[x]} v^+_{\mathcal{G}_\Psi}(y) = n - 1.$$

Given $y \in \Psi[x]$, there exists a unique S_n^i with $y \in S_n^i$. If $i = n$, y is not in the domain of any element of Ψ, and $v^+_{\mathcal{G}_\Psi}(y) = 0$. If $i \leq n-1$, y is only in the domain of f_n^i, so that $v^+_{\mathcal{G}_\Psi}(y) = 1$. Given that $\Psi[x] = \{y_1, ..., y_n\}$ with $y_i \in S_n^i$ $\forall i$, the equality follows.

□

Lemma 6.24. *Let R be a μ-a.e. smooth (SP) relation on (X,μ), of μ-a.e. fundamental domain S. Then, for any R-saturated measurable subset Y of X:*

$$\mu(Y)=0 \quad\Leftrightarrow\quad \mu(Y\cap S)=0.$$

Proof. If Y has zero measure, then so does $Y\cap S$. Conversely, assume that $Y\cap S$ has zero measure. Let $Z:=\{x\in X \ : \ |R[x]|=\infty \ \vee \ R[x]\cap S=\emptyset\}$; the set Z is null by assumption. We can then partition $X':=X\setminus Z$ into $X'=\bigsqcup_n X_n$, with X_n the set of elements of X with orbit of cardinality n intersecting S. Then, for all n, let $S_n:=X_n\cap S$; by Lemma 6.21, each X_n can be partitioned into $X_n=\bigsqcup_{i=1}^n S_n^i$, with $S_n^1=S_n$, and with measurable isomorphisms $f_n^i:S_n^i\to S_n^{i+1}$. Each f_n^i preserves the measure since its graph is contained in R. Let then, for all n and $i\le n-1$, ${f_n^i}':=f_n^i...f_n^1:S_n\to S_n^i$; we have:

$${f_n^i}'^{-1}[Y\cap S_n^i]\subseteq Y\cap S_n\subseteq Y\cap S.$$

Indeed, the graph of ${f_n^i}'$ is contained in R, so that ${f_n^i}'^{-1}(y)$ is in the orbit of y for any $y\in Y$, and since Y is saturated, ${f_n^i}'^{-1}(y)\in Y$; also, by construction, ${f_n^i}'^{-1}[S_n^i]=S_n$. Then, since ${f_n^i}'$ preserves the measure, as a composition of maps that do:

$$\begin{aligned}
\mu(Y)&=\mu(Y\setminus Z)\\
&=\mu\left(\bigsqcup_n Y\cap X_n\right)\\
&=\mu\left(\bigsqcup_n\bigsqcup_{i=1}^n Y\cap S_n^i\right)\\
&=\sum_n\sum_{i=1}^n \mu(Y\cap S_n^i)\\
&=\sum_n\sum_{i=1}^n \mu({f_n^i}'^{-1}[Y\cap S_n^i])\\
&\le\sum_n\sum_{i=1}^n \mu(Y\cap S)=\sum_n\sum_{i=1}^n 0=0.
\end{aligned}$$

□

Corollary 6.25 ([Gab00, Corollary I.10]; Cost of finite groups)**.** *A finite group G has cost $\mathcal{C}(G)=1-\frac{1}{|G|}$.*

Proof. Fix a free (SP1) action α of G on a standard measure space (X,μ). For each $x\in X$, $R_\alpha[x]=\mathcal{O}_\alpha(x)$ has cardinality exactly $|G|$. By Lemma 6.19, there exists a fundamental domain S of R_α, and it has measure $\mu(S)=\frac{1}{|G|}\mu(X)=\frac{1}{|G|}$. Then, by Proposition 6.23, $\mathcal{C}_\mu(R_\alpha)=1-\frac{1}{|G|}$, and as this holds for any free action, we conclude

$$\mathcal{C}(G)=1-\frac{1}{|G|}.$$

□

Remark 6.26. Let R be a (SP) relation on (X_1,μ_1) and $\Phi=\langle\varphi_i:A_i\to B_i\rangle_{i\in I}$ a graphing on (X_2,μ_2). Assume that Φ is a graphing of R, via some measurable measure preserving isomorphism $f:X_2\to X_1$; that is, $R_\Phi\simeq_f R$. Then, the graphing Φ' on (X_1,μ_1) defined as $\Phi':=\langle f\varphi_i f^{-1}:f[A_i]\to f[B_i]\rangle_{i\in I}$ is also a graphing of R, via the identity map 1_{X_1}, and $\mathcal{C}(\Phi')=\mathcal{C}(\Phi)$.

Indeed, $\mu(\operatorname{dom} f\varphi_i f^{-1})=\mu(\operatorname{dom}\varphi_i)$, so that the cost is preserved. To see that $R_{\Phi'}\simeq_{1_{X_1}} R$, it suffices to show that $R_\Phi\simeq_f R_{\Phi'}$, but Φ' is constructed precisely to have $\Phi'[f(x)]=f[\Phi[x]]\ \forall x\in X_2$. Indeed, if $y\in\Phi[x]$, there exists a Φ-word $w=\varphi_l^{\varepsilon_l}...\varphi_1^{\varepsilon_1}$ with $w(x)=y$, and $w':=(f\varphi_l^{\varepsilon_l}f^{-1})...(f\varphi_1^{\varepsilon_1}f^{-1})$ is a Φ'-word satisfying $w'(f(x))=f(y)$, and the converse is shown similarly.

This tells us that given any graphing for a (SP) relation, we can assume, if we care only about the cost, that the graphing is actually done on the same space, and the isomorphism is the identity.

Definition 6.27 (Graphing Reorientation, Refinement)**.** Consider a graphing $\Phi=\langle\varphi_i:A_i\to B_i\rangle_{i\in I}$ on (X,μ).

- Let $J\subseteq I$ and
$$\Phi^J:=\langle\varphi_i^{\varepsilon_i}\rangle_{i\in I}$$
where ε_i is -1 if $i\in J$, and 1 otherwise.

- Let $\langle P_i \rangle_{i\in I}$ be a family of countable measurable partitions P_i of A_i, for all i, and define, for all $p \in P_i$.

$$\varphi_{i,p} := \varphi_i|_p : A_i \cap p \to B_i \cap \varphi_i[p],$$

and

$$\Phi^P := \langle \varphi_{i,p} \rangle_{i\in I, p\in P_i}.$$

We will say that Φ^J is a *reorientation* on $J \subseteq I$, and Φ^P a *refinement* along $\langle P_i \rangle_{i\in I}$, of Φ. Given a countable measurable partition P of X, we can define $\langle \{p \cap A_i \mid p \in P\} \rangle_{i\in I}$, which defines a family of partitions of the domains A_i, and still call Φ^P a partition along P.

Proposition 6.28. *Both reorientations and refinements preserve the costs and the induced equivalence relations.*

Proof. Fix $\Phi = \langle \varphi_i : A_i \to B_i \rangle_{i\in I}$.

If $J \subseteq I$ and Φ^J is a reorientation on J, for any $i \in J$, $\mu(\operatorname{dom} \varphi_i^{-1}) = \mu(\operatorname{cod} \varphi_i) = \mu(\operatorname{dom} \varphi_i)$, so that

$$\mathcal{C}_\mu(\Phi^J) = \sum_i \mu(\operatorname{dom} \varphi_i^{\varepsilon_i}) = \sum_i \mu(\operatorname{dom} \varphi_i) = \mathcal{C}_\mu(\Phi),$$

and the cost is preserved. Also, the underlying simple graph of $\mathcal{G}_{\Phi^J}$ is the same as the one of $\mathcal{G}_\Phi$:

$$\begin{aligned}
(x,y) \in \mathsf{G}(\mathcal{G}_{\Phi^J}) \quad &\Leftrightarrow \quad \exists i : \ (x \in \operatorname{dom}(\varphi_i^{\varepsilon_i}) \wedge y = \varphi_i^{\varepsilon_i}(x)) \vee (y \in \operatorname{dom}(\varphi_i^{\varepsilon_i}) \wedge x = \varphi_i^{\varepsilon_i}(y)) \\
&\Leftrightarrow \quad \exists i : \ (x \in \operatorname{dom}(\varphi_i) \wedge y = \varphi_i(x)) \vee (y \in \operatorname{dom}(\varphi_i) \wedge x = \varphi_i(y)) \quad \Leftrightarrow \quad (x,y) \in \mathsf{G}(\mathcal{G}_\Phi),
\end{aligned}$$

so that the connected components of $\mathcal{G}_\Phi$ and $\mathcal{G}_{\Phi^J}$ are the same, and $R_\Phi = R_{\Phi^J}$.

If $\langle P_i \rangle_{i\in I}$ is a family of countable partitions of the domains of the maps φ_i, then

$$\mu(A_i) = \sum_{p\in P_i} \mu(A_i \cap p)$$

and

$$\mathcal{C}_\mu(\Phi^P) = \sum_{i,p\in P_i} \mu(A_i \cap p) = \sum_i \sum_{p\in P_i} \mu(A_i \cap p) = \sum_i \mu(A_i) = \mathcal{C}_\mu(\Phi).$$

The underlying simple graph of $\mathcal{G}_{\Phi^P}$ is also equal to the one of $\mathcal{G}_\Phi$:

$$\begin{aligned}
(x,y) \in \mathsf{G}(\mathcal{G}_{\Phi^P}) \quad &\Leftrightarrow \quad \exists i,p : \ (x \in \operatorname{dom}(\varphi_i^p) \wedge y = \varphi_i^p(x)) \vee (y \in \operatorname{dom}(\varphi_i^p) \wedge x = \varphi_i^p(y)) \\
&\Leftrightarrow \quad \exists i : \ (x \in \operatorname{dom}(\varphi_i) \wedge y = \varphi_i(x)) \vee (y \in \operatorname{dom}(\varphi_i) \wedge x = \varphi_i(y)) \quad \Leftrightarrow \quad (x,y) \in \mathsf{G}(\mathcal{G}_\Phi),
\end{aligned}$$

which shows that $R_\Phi = R_{\Phi^P}$. □

Proposition 6.29. *Let R_1 (resp. R_2) be a (SP) relation on (X_1, μ_1) (resp. (X_2, μ_2)) and $f : X_1 \to X_2$ a measurable, measure preserving isomorphism with $R_1 \simeq_f R_2$. If $Y_1 \subseteq X_1$ is measurable, then $R|_{Y_1} := R_1 \cap Y_1^2$ is a (SP) relation on $(Y_1, \mu|_{Y_1})$ and $R_1|_{Y_1} \simeq_{f|_{Y_1}} R_2|_{f[Y_1]}$.*

In short, (SP) relation isomorphisms pass to restrictions.

Proof. If $R_1 \subseteq X_1^2$ is measurable, then so is $R_1|_{Y_1} = R_1 \cap Y_1^2 \subseteq Y_1^2$. Also, $R_1|_{Y_1}[x] = R_1[x] \cap Y_1$ is countable, and whenever $g : A \to B$ ($A, B \subseteq Y_1$ are measurable) is an isomorphism, with $\Gamma(g) \subseteq R_1|_{Y_1}$, then in particular $\Gamma(g) \subseteq R_1$ and g preserves μ, hence $\mu|_{Y_1}$ too.

Finally,

$$\begin{aligned}
\{y \in Y_1 \ : \ R_2|_{f[Y_1]}[f(y)] \neq f[R_1|_{Y_1}[y]]\} &= \{y \in Y_1 \ : \ R_2[f(y)] \cap f[Y_1] \neq f[R_1[y] \cap Y_1]\} \\
&= \{y \in Y_1 \ : \ R_2[f(y)] \cap f[Y_1] \neq f[R_1[y]] \cap f[Y_1]\} \\
&\subseteq \{y \in Y_1 \ : \ R_2[f(y)] \neq f[R_1[y]]\},
\end{aligned}$$

which is a null set. □

6.2 Induction

Lemma 6.30 ([Gab00, Lemma II.8]). *Let R be a (SP) relation on (X, μ), Y a measurable subset of X intersecting μ-a.e. orbit of R, and Φ^* a graphing of R. Then, Φ^* can be made into a graphing Φ of R, of same cost, which can be written as a union of disjoint graphings*

$$\Phi = \Phi_v \vee \Phi_h$$

such that:

1. *Φ_v is a treeing (actually, no Φ_v-orbit has a cycle), and Y is a μ-a.e. fundamental domain for R_{Φ_v}.*
2. *Φ_h can be slid along Φ_v (in a sense to be made precise later) to get a graphing Ψ_h of $R|_Y$.*
3. *For any other graphing Ψ' of $R|_Y$, $\Phi_v \vee \Psi'$ is a graphing of R.*
4. *If $\mu(X)$ is finite, then Φ_v has finite orbits and*
   (a) *$\mathcal{C}_\mu(\Phi_v) = \mu(X) - \mu(Y)$.*
   (b) *$\mathcal{C}_\mu(\Phi_h) = \mathcal{C}_{\mu|_Y}(\Psi_h)$.*

Proof. Fix the graphing $\Phi^* = \langle \varphi_i^* : A_i^* \to B_i^* \rangle_{i \in I}$.

To simplify the proof, we first want to remove all edges not in a connected component of $\mathcal{G}_{\Phi^*}$ intersecting Y. As Y intersects μ-a.e. R-orbit, it also intersects μ-a.e. Φ^*-orbit. Let

$$Y_\infty(\Phi^*) := \{x \in X \ : \ \Phi^*[x] \cap Y = \emptyset\}, [9]$$

that is, the elements of X whose Φ^*-orbits don't intersect Y. By assumption, $Y_\infty(\Phi^*)$ is a null set. The goal is now to modify Φ^* so that no $\varphi_i^* : A_i^* \to B_i^*$ has domain or codomain intersecting $Y_\infty(\Phi^*)$, while still keeping a graphing of R, of same cost.

Given any $\varphi_i^* : A_i \to B_i$, we can first partition φ_i^* along $Y_\infty(\Phi^*)$ into $\varphi_i^\infty = \varphi_i^*|_{A_i^* \cap Y_\infty(\Phi^*)}$ and $\varphi_i^0 = \varphi_i^*|_{A_i^* \setminus Y_\infty(\Phi^*)}$. Note that both the domain and codomain of φ_i^∞ are included in $Y_\infty(\Phi^*)$, and neither of φ_i^0 intersect $Y_\infty(\Phi^*)$, because an element of the domain is in $Y_\infty(\Phi^*)$ if and only if its image is; they are in the same orbit. We have refined Φ^* along $Y_\infty(\Phi^*)$.

Then, let $\Phi^0 := \langle \varphi_i^0 : A_i^0 \to B_i^0 \rangle_{i \in I}$, with $A_i^0 = A_i^* \setminus Y_\infty(\Phi^*)$, and $B_i^0 = B_i^* \setminus Y_\infty(\Phi^*) = \varphi_i^*[A_i \setminus Y_\infty(\Phi^*)]$; we show that Φ^0 is a graphing of R_{Φ^*}, hence a graphing of R. For that, it suffices to show that the set

$$Z^* := \{x \in X \ : \ \Phi^0[x] \neq \Phi^*[x]\}$$

is null. The inclusion $\Phi^0[x] \subseteq \Phi^*[x]$ always holds, because to any Φ^0-word w^0 containing x in its domain corresponds such a Φ^*-word w^*. Conversely, if $\Phi^*[x] \nsubseteq \Phi^0[x]$, there exists a Φ^*-word $w^* = {\varphi_{i_l}^*}^{\varepsilon_l} \dots {\varphi_{i_1}^*}^{\varepsilon_1}$ with $x \in \operatorname{dom} w^*$ and such that $x \notin \operatorname{dom} w^0$ (with $w^0 = {\varphi_{i_l}^0}^{\varepsilon_l} \dots {\varphi_{i_1}^0}^{\varepsilon_1}$ the corresponding Φ^0-word). This means that there exists a first index k with ${\varphi_{i_{k-1}}^0}^{\varepsilon_{k-1}} \dots {\varphi_{i_1}^0}^{\varepsilon_1}(x) = {\varphi_{i_{k-1}}^*}^{\varepsilon_{k-1}} \dots {\varphi_{i_1}^*}^{\varepsilon_1}(x) \notin \operatorname{dom} {\varphi_{i_k}^0}^{\varepsilon_k}$, and as ${\varphi_{i_{k-1}}^*}^{\varepsilon_{k-1}} \dots {\varphi_{i_1}^*}^{\varepsilon_1}(x) \in \operatorname{dom} {\varphi_{i_k}^*}^{\varepsilon_k}$, we conclude

$${\varphi_{i_{k-1}}^*}^{\varepsilon_{k-1}} \dots {\varphi_{i_1}^*}^{\varepsilon_1}(x) \in Y_\infty(\Phi^*)$$

and $x \in ({\varphi_{i_{k-1}}^*}^{\varepsilon_{k-1}} \dots {\varphi_{i_1}^*}^{\varepsilon_1})^{-1}[Y_\infty(\Phi^*)]$ so that:

$$Z^* \subseteq \bigcup_{w^* \in \mathcal{W}(\Phi^*)} {w^*}^{-1}[Y_\infty(\Phi^*)],$$

which is a countable union of null sets, hence is a null set. This shows that Φ^0 is indeed a graphing of R. The graphing Φ^0 has the same cost as Φ^*, since $\mu(A_i^0) = \mu(A_i^* \setminus Y_\infty(\Phi^*)) = \mu(A_i^*)$ for all i, $Y_\infty(\Phi^*)$ being a null set.

It remains to show that Φ^0 satisfies the property that no edge lies in a connected component not intersecting Y. Thus, consider

$$Y_\infty(\Phi^0) := \{x \in X \ : \ \Phi^0[x] \cap Y = \emptyset\}.$$

We want to show that no $\varphi_i^0 : A_i^0 \to B_i^0$ in Φ^0 has domain or codomain intersecting $Y_\infty(\Phi^0)$. This follows, by construction of Φ^0, from the fact that $Y_\infty(\Phi^0) \subseteq Y_\infty(\Phi^*)$, which we now show. Assume $Y_\infty(\Phi^0) \subseteq Y_\infty(\Phi^*)$ doesn't hold; this means that there exists some x with:

$$\Phi^*[x] \cap Y \neq \emptyset, \quad \text{and} \quad \Phi^0[x] \cap Y = \emptyset.$$

[9] The set $Y_\infty(\Phi^*)$ is measurable; it can be written:

$$Y_\infty(\Phi^*) = \{x \in X \ : \ \forall w^* \in \mathcal{W}(\Phi^*) \ w^*(x) \notin Y\} = \bigcap_{w^* \in \mathcal{W}(\Phi^*)} {w^*}^{-1}[X \setminus Y].$$

In particular, there exists a Φ^*-word $w^* = {\varphi^*_{i_l}}^{\varepsilon_l}...{\varphi^*_{i_1}}^{\varepsilon_1}$ with $x \in \operatorname{dom} w^*$ and $w^*(x) \in Y$, and $x \notin \operatorname{dom} w^0$, where w^0 is the corresponding Φ^0-word. As above, this implies that there exists a first index $k \leq l$ with

$$ {\varphi^0_{i_{k-1}}}^{\varepsilon_{k-1}}...{\varphi^0_{i_1}}^{\varepsilon_1}(x) = {\varphi^*_{i_{k-1}}}^{\varepsilon_{k-1}}...{\varphi^*_{i_1}}^{\varepsilon_1}(x) \in Y_\infty(\Phi^*). $$

But then, as x is in the same Φ^*-orbit as ${\varphi^*_{i_{k-1}}}^{\varepsilon_{k-1}}...{\varphi^*_{i_1}}^{\varepsilon_1}(x)$, $x \in Y_\infty(\Phi^*)$, which is a contradiction, and we conclude that no φ^0_i of Φ^0 has domain or codomain intersecting $Y_\infty(\Phi^0)$.

Summarising, we have a graphing Φ^0 of R such that no φ^0_i in Φ^0 has domain or codomain intersecting $Y_\infty := Y_\infty(\Phi^0)$, and $\mathcal{C}_\mu(\Phi^0) = \mathcal{C}_\mu(\Phi^*)$. In particular, all elements x of Y_∞ are such that $\Phi^0[x] = \{x\}$. For the sake of readability, let us write $\Phi := \langle \varphi_i : A_i \to B_i \rangle$ instead of $\Phi^0 = \langle \varphi^0_i : A^0_i \to B^0_i \rangle$.

Recall the metric d_Φ on the connected components of the graph $\mathcal{G}_\Phi$ corresponding to the length of a shortest (undirected) walk. Let $X' := X \setminus Y_\infty$, that is, the elements of X lying in a connected component intersecting Y, and define the following subsets of X':

$$ Y_n := \{x \in X' \ : \ d_\Phi(x, Y) = n\}, \quad n \in \mathbb{N}, $$

where the distance to a set is the infimum of the distances to the elements of this set (in this case the minimum, since the metric is discrete). Then $Y_0 = Y$ and

$$ X = Y \sqcup \left(\bigsqcup_{n \geq 1} Y_n \right) \sqcup Y_\infty. $$

Given any φ_i, we can split it in a countable number of $\varphi_i^{m,n}$ whose domains (resp. codomains) are contained in some Y_m with $m < \infty$ (resp. Y_n with $n < \infty$) , and exchange a $\varphi_i^{m,n}$ with its inverse whenever necessary, so as to have, for any $\varphi_i^{m,n} : A \to B$, $A \subseteq Y_m$ and $B \subseteq Y_n$ with $m \geq n$. Formally, this can be done by first refining Φ along the partition $\{Y_n\}_{n \in \mathbb{N}}$, then reorienting on the whole index set, and refining along the partition $\{Y_n\}_{n \in \mathbb{N}}$ again, to finally reorient as needed. Let us still call this graphing $\Phi = \langle \varphi_i \rangle_{i \in I}$.

Order the index set I and define, for any $n \geq 1$ and $i \in I$:

$$ Z^i_n := \{x \in Y_n \ : \ \varphi_i \text{ is the } I\text{-first element of } \Phi \text{ such that } \varphi_i(x) \in Y_{n-1}\}. $$

For any $x \in Y_n$ $(1 \leq n < \infty)$, we know that $d_\Phi(x, Y) = n$, so that there exists a walk from x to Y of length n; the first edge φ_i on that walk must then be from x to some $x' \in Y_{n-1}$, and by the order on I, we know then that there exists a *I-first* such φ_i. Then, $Y_n = \bigsqcup_{i \in I} Z^i_n$, and

$$ X' \setminus Y = \bigsqcup_{n \geq 1, i \in I} Z^i_n. $$

For any element $\varphi_i : A_i \to B_i$ of Φ, assuming $A \subseteq Y_n$, for $n \geq 1$, define

$$ \overline{\delta_i} := \varphi_i|_{A_i \cap Z^i_n}, \qquad \delta_i := \varphi_i|_{A_i \setminus Z^i_n}, $$

and if $A \subseteq Y$

$$ \overline{\delta_i} := \emptyset, \qquad\qquad \delta_i := \varphi_i, $$

and define:

$$ \Phi_v := \langle \overline{\delta_i} \rangle_{i \in I}, \qquad \Phi_h := \langle \delta_i \rangle_{i \in I}. $$

In other words, Φ_v and Φ_h correspond to the refinement of Φ along the family of "partitions":

$$ P_i := \begin{cases} \{A_i \cap Z^i_n, A_i \setminus Z^i_n\} & \text{if } n \geq 1 \text{ is such that } A_i \subseteq Y_n, \\ \{\emptyset, A_i\} & \text{if } A_i \subseteq Y. \end{cases} $$

Since $\Phi_v \vee \Phi_h$ is a refinement of Φ, the cost and the orbits are preserved. We can now show that Φ_v and Φ_h satisfy all the needed properties.

1. Consider the graph $\mathcal{G}_{\Phi_v}$ of Φ_v. We show that it has no edge between elements of $Y = Y_0$ and that for each $x \in Y_n$ $(1 \leq n < \infty)$, there is exactly one edge with source x, and its destination lies in Y_{n-1}. For any $\varphi_i : A_i \to B_i$ with $A_i \subseteq Y$, $\overline{\delta_i} = \emptyset$, so that no element of Φ_v has domain in Y. Furthermore, for any given $x \in Y_n$ $(n \geq 1)$, let φ_i be the I-first map with $x \in \operatorname{dom} \varphi_i$ and $\varphi_i(x) \in Y_{n-1}$, then $x \in Z^i_n$, and $\overline{\delta_i}$ sends x

to Y_{n-1}; conversely, if φ_j is any other map sending x to Y_{n-1}, it must come I-after φ_i, so that $x \notin Z_n^j$, and $x \notin \operatorname{dom} \overline{\delta_j}$.

This is enough to show that $\mathcal{G}_{\Phi_v}$ is a forest, with each orbit (of an element) of X' intersecting Y at exactly one point (which, by co-nullity of X', proves Item 1):

Observe first that for any $x \in Y_n$ $(n \geq 1)$, there is exactly one edge with one endpoint x and the other in Y_{n-1}: assume e_1, e_2 are two such edges, if both have x as a source, we have a contradiction with the point above, but if either has its source in Y_{n-1}, we also have a contradiction, since its destination, x, then lies in Y_n. Similarly, observe that there is no edge with both endpoints in Y_n, for any n.

Now, first assume that there exists $y \neq y' \in Y$ in a same Φ_v-orbit; then there exists a *shortest* walk from y to y' in $\mathcal{G}_{\Phi_v}$, and this walk must pass through some $x \in Y_n$ with n maximised. Then, n can't be zero, since $\mathcal{G}_{\Phi_v}$ has no edge in Y_0. Consider the vertices just before and just after x, say x_0 and x_1, so that the walk is of the form $...x_0(e_0^{\varepsilon_0})x(e_1^{\varepsilon_1})x_1....$ We can't have $x_0 = x_1$ since this would imply that the walk can be shortened. Neither x_0 nor x_1 can be in Y_{n+1}, by hypothesis that x maximises n; neither can be in Y_n either, as there is no edge in Y_n; finally, as both must lie in Y_{n-1}, we have a contradiction, since then e_0 and e_1 would both be edges with one end x and the other in Y_{n-1}. The same argument shows that any *shortest* cycle in the orbit of some $x' \in X'$ must pass through some $x \in Y_n$ with n maximised, and a contradiction is reached. Finally, note that if $x \in X'$, then $x \in Y_n$ for some n, and an inductive argument on n shows that there exists a Φ_v-walk from x to Y, which implies that the orbit of x intersects Y. Thus, each orbit of an element of X' is a tree, intersecting Y, at exactly one point.

We conclude, since X' is conull, that Y is a μ-a.e. fundamental domain for Φ_v, which is a treeing.

Now that we have shown Φ_v to define a forest $\mathcal{G}_{\Phi_v}$ partitioning X' into trees each intersecting Y at a single point, we can refine Φ_h further. We know that for any $x \in X'$, there exists a unique Φ_v-word w such that $w(x) \in Y$. This implies that the domains of the Φ_v-words with codomain in Y form a countable, measurable partition of X'. Take the refinement of Φ_h (still called Φ_h) along this partition.
To any element $\delta_i : A_i \to B_i$ of Φ_h corresponds now a unique Φ_v-word w_{A_i} of domain containing A_i and codomain contained in Y (resp. a unique w_{B_i} of domain containing B_i and codomain contained in Y).

2. Define $\psi_i := w_{B_i} \delta_i w_{A_i}^{-1}$ and $\Psi_h := \langle \psi_i : w_{A_i}[A_i] \to w_{B_i}^{-1}[B_i] \rangle_{i \in I}$. The graphing Ψ_h is what is meant by the *sliding of* Φ_h *along* Φ_v; it "moves" the elements of Φ_h along Φ_v until they have domain and codomain in Y.

   To show that Ψ_h is a graphing of $R|_Y$, it is enough to show that it is a graphing of $R_\Phi|_Y$, because relation isomorphism passes to restrictions. But $R_\Phi|_Y = R_{\Phi_h \vee \Phi_v}|_Y$ has orbits those elements of Y for which there exists a $\Phi_h \vee \Phi_v$-walk linking them. Thus, we want to show that for any two elements y, y' of Y, there exists a $\Phi_h \vee \Phi_v$-walk linking them, whenever there exists a Ψ_h-walk linking them.

   Given that a Ψ_h-word is made up of letters from Φ_h and Φ_v, we already see that the orbits of R_{Ψ_h} are contained in those of $R_{\Phi_h \vee \Phi_v}|_Y$.

   Conversely, we want to show that for any non-empty $\Phi_h \vee \Phi_v$-word w and $y \in Y$ with $y \in \operatorname{dom} w$ and $y' := w(y) \in Y$, there exists a Ψ_h-word w_h with $y \in \operatorname{dom} w_h$ and $y' = w_h(y)$. First, we can assume without loss of generality that w contains at least one Φ_h letter; if it didn't, we would have $y' = w(y) \in \Phi_v[y]$, so that $y' = y$, and then $y' \in \Psi_h[y]$ trivially.

   We then show the statement by induction on the number of Φ_h-letters of w, and we will only look at such letters appearing with a positive sign, for brevity:
   First, assume $w = w_v' \delta_i w_v$, is a $\Phi_h \vee \Phi_v$-word with δ_i the only Φ_h-letter of w, and fix some $y \in Y \cap \operatorname{dom} w$ with $w(y) \in Y$. Then, w_v is a word with domain in Y and codomain intersecting the domain of δ_i, so that w_v must be $w_{A_i}^{-1}$; similarly, $w_v' = w_{B_i}$. We then have $w = w_{B_i} \delta_i w_{A_i}^{-1}$ and w is equal, as a map, to ψ_i, which is a Ψ_h-word, and $\psi_i(y) = w(y)$. This shows the base case.
   Now, assume the statement to be proven for words containing at most n Φ_h-letters, and let $w = w_v' \delta_j w_v \delta_i w_0$, a $\Phi_h \vee \Phi_v$-word with $n+1$ Φ_h-letters, and with δ_j the last, and δ_j the penultimate Φ_h-letter; fix also some $y \in Y \cap \operatorname{dom} w$ such that $w(y) \in Y$. The same argument as the case $n = 1$ shows that w_v' is equal to w_{B_j}. Let $x_0 := \delta_i w_0(y)$, and w_v'' the unique Φ_v-word from x_0 to Y, and $y_0 = w_v''(x_0)$. Then, consider both walks $w_v' \delta_j w_v''^{-1}$ and $w_v'' w_v \delta_i w_0$ from y to y_0 and y_0 to $w(y)$ respectively. The first has a single Φ_h-letter, and the second n Φ_h-letters. Applying the case $n = 1$ and the induction hypothesis, respectively, yields two Ψ_h walks, w_h and w_h', with $w_h(y) = y_0$ and $w_h(y_0) = w(y)$. Then, the walk $w_h' w_h$ is a Φ_h-walk satisfying $w_h' w_h(y) = w(y)$, and we are done.

Observe here that $\Psi_h \vee \Phi_v$ is a graphing of R, since $\Phi_h \vee \Phi_v$ is, and by construction of Ψ_h out of Φ_h.

3. We show that if Ψ' is a graphing of R_{Ψ_h}, then $\Phi' := \Psi' \vee \Phi_v$ is a graphing of $R_\Phi = R_{\Psi_h \vee \Phi_v}$. So, assume that

Ψ' is a graphing of R_{Ψ_h}; that is, the set

$$Z' := \{x \in Y : \Psi'[x] \neq \Psi_h[x]\}$$

is a null set. We aim to show that

$$Z'' := \{x \in X : (\Psi' \vee \Phi_v)[x] \neq (\Psi_h \vee \Phi_v)[x]\}$$

also is a null set. Note that we can actually restrict our attention to the elements of Z'' that are in $X' = X \setminus Y_\infty$, since Y_∞ is a null set. Fix some $x \in X'$ with $(\Psi' \vee \Phi_v)[x] \neq (\Psi_h \vee \Phi_v)[x]$, and without loss of generality $(\Psi' \vee \Phi_v)[x] \nsubseteq (\Psi_h \vee \Phi_v)[x]$, so that there exists some $\Psi' \vee \Phi_v$-word w' such that no $\Psi_h \vee \Phi_v$-word w satisfies $w'(x) = w(x)$; choose such a word w', and recall that it is reduced.

We claim that w' is of the form $w'_v w_p w_v$, with w_v, w'_v possibly empty Φ_v-words, and w_p a Ψ'-word. Indeed, if w' was a Φ_v-word, it would obviously also be a $\Psi_h \vee \Phi_v$-word, thus contradicting our assumption; this implies that w' has some Ψ' component. Let w_v be the longest prefix of w' consisting entirely of Φ_v-letters, and similarly w'_v the longest suffix of w' consisting entirely of Φ_v-letters. Then $w' = w'_v w_p w_v$, for some w_p and it remains to show that w_p is a Ψ'-word. Assume it isn't. By construction, we know that the first and last letters of w_p are elements of Ψ'. Then, let $\overline{\delta_i} : A_i \to B_i$ be the first element of Φ_v appearing with positive sign in w_p (such an element must exist: if w' contained only elements of Φ_v with negative sign, the last one to appear would have codomain some $A_i \subseteq Y_n$ for some $n \geq 1$, and the element of Ψ' following it would have a domain not intersecting A_i). We have $A_i \subseteq Y_n$ for some $n \geq 1$, and the letter preceding $\overline{\delta_i}$ must be some $\overline{\delta_j}^{-1} B_j \to A_j$, with $A_j \subseteq Y_n$; but then, as $A_j \cap A_i \neq \emptyset$ (if it wasn't the case the word w would have an empty domain), we must have $i = j$, contradicting reducedness of w_p, and of w'.

We know $w'_v w_p w_v(x) \notin (\Psi_h \vee \Phi_v)[x]$, which implies that $w_p w_v(x) \notin \Psi_h[w_v(x)]$, otherwise, there would exist some Ψ_h-word w_h with $w_h w_v(x) = w_p w_v(x)$, and thus $w'_v w_h w_v(x) = w'(x)$, so that $w'(x) \in (\Psi' \vee \Phi_v)[x]$, which is a contradiction. As w_p is a Ψ'-word, we conclude $\Psi'[w_v(x)] \nsubseteq \Psi_h[w_v(x)]$, so that $x \in w_v^{-1}[Z']$. This implies that

$$Z'' \subseteq \bigcup_{w_v \in \mathcal{W}(\Phi_v)} w_v^{-1}[Z']$$

is a null set.

4. Since Y intersects the Φ_v-orbits of all elements of X' at exactly one point, and doesn't intersect the Φ_v-orbit of any element of $X \setminus X'$, which has zero measure, Y is a μ-a.e. fundamental domain for R_{Φ_v}.
   (a) By Proposition 6.23, as Y is a μ-a.e. fundamental domain for R_{Φ_v}, and $\mu(X)$ is finite:

   $$\mathcal{C}(R_{\Phi_v}) = \mu(X) - \mu(Y)$$

   but as Φ_v is a treeing, it realises the cost of R_{Φ_v}, and (by Proposition 6.23)

   $$\mathcal{C}(\Phi_v) = \mu(X) - \mu(Y).$$

   (b) Recall that Ψ_h is defined as $\langle \psi_j \rangle_{j \in J}$, with $\psi_j = m_{B_j} \delta_j m_{A_j}^{-1}$, for all j. But $\operatorname{dom} \psi_j = m_{A_j}[\operatorname{dom} \delta_j]$, so that $\mu(\operatorname{dom} \psi_j) = \mu(\operatorname{dom} \delta_j)$, and

   $$\mathcal{C}_{\mu|_Y}(\Psi_h) = \sum_j \mu|_Y(\operatorname{dom} \psi_j) = \sum_j \mu(\operatorname{dom} \psi_j) = \sum_j \mu(\operatorname{dom} \delta_j) = \mathcal{C}_\mu(\Phi_h).$$

□

Proposition 6.31 ([Gab00, Proposition II.6]). *Let R be a (SP) relation on a standard measure space (X, μ) and Y a measurable subset of X intersecting μ-a.e. orbit of R. Assuming $\mu(X)$ is finite:*

$$\mathcal{C}_\mu(R) - \mu(X) = \mathcal{C}_{\mu|_Y}(R|_Y) - \mu|_Y(Y).$$

Proof. Fix some graphing Φ^* of R, and use the notation of Lemma 6.30:
First, note that as $\Phi = \Phi_h \vee \Phi_v$, $\mathcal{C}_\mu(\Phi) = \mathcal{C}_\mu(\Phi_v) + \mathcal{C}_\mu(\Phi_h)$. As $\mu(X)$ is finite, $\mathcal{C}_\mu(\Phi_v)$ is finite by Lemma 6.30(4a), and we can therefore write $\mathcal{C}_\mu(\Phi_h) = \mathcal{C}_\mu(\Phi) - \mathcal{C}_\mu(\Phi_v)$. Then, Ψ_h is a graphing of $R|_Y$, and:

$$\begin{aligned}
\mathcal{C}_{\mu|_Y}(R|_Y) - \mu(Y) &\leq \mathcal{C}_{\mu|_Y}(\Psi_h) - \mu(Y) \\
&= \mathcal{C}_\mu(\Phi_h) - \mu(Y) && \text{(Item 4b)} \\
&= \mathcal{C}_\mu(\Phi) - \mathcal{C}_\mu(\Phi_v) - \mu(Y) \\
&= \mathcal{C}_\mu(\Phi) - \mu(X) + \mu(Y) - \mu(Y) && \text{(Item 4a)} \\
&= \mathcal{C}_\mu(\Phi) - \mu(X) = \mathcal{C}_\mu(\Phi^*) - \mu(X),
\end{aligned}$$

and taking the infimum over the graphings Φ^* of R:

$$\mathcal{C}_{\mu|_Y}(R|_Y) - \mu(Y) \le \mathcal{C}_\mu(R) - \mu(X).$$

Conversely, fix some graphing Φ^* of R (recall that the existence of *some* graphing is guaranteed by Theorem 6.3) and keep the notation of Lemma 6.30. If Ψ' is a graphing of $R|_Y$, then $\Psi' \vee \Phi_v$ is a graphing of R (Item 3), and

$$\begin{aligned}
\mathcal{C}_\mu(R) - \mu(X) &\le \mathcal{C}_\mu(\Psi' \vee \Phi_v) - \mu(X) \\
&= \mathcal{C}_\mu(\Psi') + \mathcal{C}_\mu(\Phi_v) - \mu(X) \\
&= \mathcal{C}_\mu(\Psi') + \mu(X) - \mu(Y) - \mu(X) \qquad \text{(Item 4a)} \\
&= \mathcal{C}_{\mu|_Y}(\Psi') + \mu(Y) \qquad ,
\end{aligned}$$

and taking the infimum over the graphings Ψ' of $R|_Y$:

$$\mathcal{C}_\mu(R) - \mu(X) \le \mathcal{C}_\mu(R|_Y) + \mu(Y).$$

□

Theorem 6.32 ([Gab00, Theorem VI.19]; Finite index subgroup)**.** *Let Γ be a countable group and $\Lambda \le \Gamma$ a finite index subgroup of Γ. Then*

$$\mathcal{C}(\Lambda) - 1 = [\Gamma : \Lambda](\mathcal{C}(\Gamma) - 1).$$

Proof. We show both "$\le$" and "$\ge$", using the definition of the cost of a group as an infimum.

- Fix α a free (SP1) action of Λ on some standard measure space (X, μ). Define the set $\tilde{X} := X \times \Gamma$ and the actions:

  $$\begin{aligned}
  &\tilde{\alpha} : \Lambda \to \mathrm{Aut}(\tilde{X}), && \tilde{\beta} : \Gamma \to \mathrm{Aut}(\tilde{X}) \\
  &\tilde{\alpha}(\lambda)(x, \gamma) := (\alpha(\lambda)x, \gamma\lambda^{-1}), && \tilde{\beta}(\delta)(x, \gamma) := (x, \delta\gamma).
  \end{aligned}$$

  Then, $\tilde{\alpha}$ and $\tilde{\beta}$ commute, because they act on different sides of γ, and $\tilde{\beta}$ induces an (well-defined, because they commute) action $\overline{\beta}$ of Γ on $\overline{X} := \tilde{X}/\tilde{\alpha}$ (that is, on the orbits of $\tilde{\alpha}$):

  $$\begin{aligned}
  \overline{\beta} &: \Gamma \to \mathrm{Aut}(\tilde{X}/\tilde{\alpha}) \\
  \overline{\beta}(\delta)([(x, \gamma)]) &:= [\tilde{\beta}(\delta)(x, \gamma)].
  \end{aligned}$$

  We now want to endow $\overline{X}$ with a σ-algebra and a measure in such a way that $\overline{\beta}$ becomes a free (SP1) action.

  First, if $\gamma \neq 1$ is such that $[(x, \gamma)] = \overline{\beta}(\delta)([(x, \gamma)]) = [(x, \delta\gamma)]$, it follows that $(x, \delta\gamma)$ is in the $\tilde{\alpha}$-orbit of (x, γ), so that there exists some $\lambda \in \Lambda$ with

  $$(x, \delta\gamma) = \tilde{\alpha}(\lambda)(x, \gamma) = (\alpha(\lambda)x, \gamma\lambda^{-1}).$$

  By freeness of α, we must have $\lambda = 1$, and from $\delta\gamma = \gamma\lambda^{-1} = \gamma$, we conclude $\delta = 1$. This shows that $\overline{\beta}$ is indeed a free action.

  To endow $\overline{X}$ with a measure, first note that $\overline{X} = (X \times \Gamma)/\tilde{\alpha}$ is set-isomorphic to $X \times \Gamma/\Lambda$. Indeed, choose a set of representatives R for the left cosets of Λ in Γ, and consider first the map:

  $$\begin{aligned}
  \varphi' &: X \times \Gamma \to X \times \Gamma/\Lambda \\
  &(x, r\lambda) \mapsto (\alpha(\lambda)(x), [r]).
  \end{aligned}$$

  This map is surjective, because for any $(x, [r]) \in X \times \Gamma/\Lambda$, a preimage via φ' is (x, r). Now, the fibers of φ' are exactly the orbits of $\tilde{\alpha}$:

  $$\begin{aligned}
  \varphi'(\tilde{\alpha}(\delta)(x, r\lambda)) &= \varphi'((\alpha(\delta)(x), r\lambda\delta^{-1})) \\
  &= (\alpha(\lambda\delta^{-1})\alpha(\delta)(x), [r]) \\
  &= (\alpha(\lambda)(x), [r]) = \varphi'(x, r\lambda), \qquad \forall \delta \in \Lambda,\ \forall (x, r\lambda) \in X \times \Gamma,
  \end{aligned}$$

  which shows that φ' is equal on elements of the same orbit; and

  $$\begin{aligned}
  & \varphi'(x_1, r_1\lambda_1) = \varphi'(x_2, r_2\lambda_2) \\
  \Rightarrow\ & (\alpha(\lambda_1)(x_1), [r_1]) = (\alpha(\lambda_2)(x_2), [r_2]) \\
  \Rightarrow\ & \alpha(\lambda_2^{-1}\lambda_1)(x_1) = x_2 \ \text{ and } \ r_1 = r_2 \\
  \Rightarrow\ & r_2\lambda_2 = r_1\lambda_2 = r_1\lambda_1(\lambda_2^{-1}\lambda_1)^{-1}
  \end{aligned}$$

so that

$$(x_2, r_2\lambda_2) = \tilde{\alpha}(\lambda_2^{-1}\lambda_1)(x_1, r_1\lambda_1), \qquad \forall (x_1, r_1\lambda_1), (x_2, r_2\lambda_2) \in X{\times}\Gamma,$$

which shows that two elements with the same φ'-image must lie in the same orbit. This implies that φ' passes to the quotient $\tilde{X}/\tilde{\alpha}$ and induces a bijection $\varphi : \overline{X} \to X{\times}\Gamma/\Lambda$.

On Γ/Λ, consider the counting measure, and endow $X{\times}\Gamma/\Lambda$ with the product measure $\mu_\times$. We want to show that pushing this measure forward along φ makes $\overline{\beta}$ into a (SP) action; or, equivalently, that the action β^* on $X{\times}\Gamma/\Lambda$ induced by φ, defined by $\beta^*(\delta) := \varphi\overline{\beta}(\delta)\varphi^{-1}$, is (SP). Fix $\delta \in \Gamma$, and $(x, [r]) \in X{\times}\Gamma/\Lambda$ (with $r \in R$), then:

$$\beta^*(\delta)(x, [r]) = (\alpha(\rho(\delta r)^{-1}(\delta r))(x), [\rho(\delta r)]) = (\alpha(\rho(\delta r)^{-1}(\delta r))(x), [\delta r]),$$

where $\rho(\cdot) : \Gamma \to R$ maps an element to its representative in R. This shows that

$$\beta^*(\delta)|_{X\times\{[r]\}} = \alpha(\rho(\delta r)^{-1}(\delta r)){\times}m_{[\delta]},$$

where $m_{[\delta]} : \Gamma/\Lambda \to \Gamma/\Lambda$ is the left multiplication by $[\lambda]$ (and "$\times$" denotes the product map). Then, $\beta^*(\delta)|_{X\times\{[r]\}}$ is measurable and preserves the measure, for any r and δ, since both $\alpha(\rho(\delta r)^{-1}(\delta r))$ and $m_{[\delta]}$ do. Then, $\beta^*(\delta) = \bigsqcup_{r\in R} \beta^*(\delta)|_{X\times\{[r]\}}$ is measurable and preserves the measure, as a countable disjoint union of maps that do.[10] This shows that $\overline{\beta}$ preserves the induced measure $\overline{\mu} := \mu_\times(\varphi[\cdot])$ on $\overline{X}$, and the action is (SP).

Now, consider the quotient map $\pi : \tilde{X} \to \overline{X}$. It is injective on $X{\times}\{1\}$; indeed, if $[(x_1, 1)] = [(x_2, 1)]$, there exists some $\lambda \in \Lambda$ with

$$(x_2, 1) = \tilde{\alpha}(\lambda)(x_1, 1) = (\alpha(\lambda)(x_1), \lambda^{-1})$$

which implies $\lambda = 1$, and $(x_2, 1) = (x_1, 1)$.

Let $X_1 := \pi[X{\times}\{1\}]$. We show that X_1 intersects $\overline{\mu}$-a.e. orbit of $R_{\overline{\beta}}$ (actually, all orbits), and that it has measure $\overline{\mu}(X_1) = 1$. Given any $[(x, \gamma)] \in \overline{X}$,

$$\overline{\beta}(\gamma^{-1})([(x, \gamma)]) = [(x, 1)] \in X_1,$$

which shows that X_1 intersects all orbits. Also, we have:

$$\begin{aligned}\overline{\mu}(X_1) &= \mu_\times(\varphi[X_1]) = \mu_\times(\{\varphi([(x, 1)]) \mid x \in X\}) \\ &= \mu_\times(\{(x, [1]) \mid x \in X\}) = \mu_\times(X{\times}\{[1]\}) = 1\mu(X) = \mu(X) = 1.\end{aligned}$$

It remains to show that $\overline{\beta}$ induces, on X_1, a relation isomorphic to α on X; that is, that $R_{\overline{\beta}}|_{X_1} \simeq R_\alpha$. Consider the map $f : X \to X_1$ defined by $x \mapsto [(x, 1)]$; f is a bijection, and actually a measure-preserving measurable isomorphism. It is measurable, as it can be written $\varphi^{-1}\circ\iota_{[1]}$, where $\iota_{[1]} : X \to X \times \Gamma/\Lambda$ maps x to $(x, [1])$, and both those maps are measurable. Furthermore, we have:

$$\overline{\mu}(f[A]) = \mu_\times(\varphi f[A]) = \mu_\times(A{\times}\{[1]\}) = \mu(A) \cdot 1 = \mu(A), \quad \forall A \text{ measurable in } X,$$

which shows that it preserves the measure. We show that for any $x_1, x_2 \in X$, x_2 is in the α-orbit of x_1 if and only if $f(x_2)$ is in the $\overline{\beta}$-orbit of $f(x_1)$. Fix $x_1, x_2 \in X$: if $\alpha(\lambda)(x_1) = x_2$, then $f(x_2) = [(x_2, 1)] = [(\alpha(\lambda)(x_1), 1)] = [(x_1, \lambda^{-1})] = \overline{\beta}(\lambda^{-1})([(x_1, 1)]) = \overline{\beta}(\lambda^{-1})(f(x_1))$. Conversely, if $f(x_2) = \overline{\beta}(\delta)(f(x_1))$, then $[(x_2, 1)] = [(x_1, \delta)]$, so that there exists some $\lambda \in \Lambda$ with $(x_2, 1) = (\alpha(\lambda)(x_1), \delta\lambda^{-1})$, hence $\lambda = \delta^{-1}$, and $x_2 = \alpha(\delta^{-1})(x_1)$.

Finally, as $R_{\overline{\beta}}|_{X_1} \simeq R_\alpha$, their cost is equal. Note also that $\overline{\mu}(\overline{X}) = \mu_\times(\varphi[\overline{X}]) = \mu_\times(X \times \Gamma/\Lambda) = \mu(X)|\Gamma/\Lambda| = [\Gamma : \Lambda]$. Now, as X_1 intersects μ-a.e. $\overline{\beta}$-orbit, we have (Proposition 6.31):

$$\begin{aligned} & \mathcal{C}_{\overline{\mu}}(R_{\overline{\beta}}) - \overline{\mu}(\overline{X}) = \mathcal{C}_{\overline{\mu}|_{X_1}}(R_{\overline{\beta}}|_{X_1}) - \overline{\mu}|_{X_1}(X_1) \\ \Rightarrow \quad & \mathcal{C}_{\overline{\mu}}(R_{\overline{\beta}}) - [\Gamma : \Lambda] = \mathcal{C}_\mu(R_\alpha) - 1 \\ \Rightarrow \quad & [\Gamma : \Lambda](\mathcal{C}_{\overline{\mu}/\overline{\mu}(\overline{X})}(R_{\overline{\beta}}) - 1) = \mathcal{C}_\mu(R_\alpha) - 1. \end{aligned}$$

[10] More precisely, given that $\overline{\beta}(\delta)$ is a bijection for all δ and that $\beta^*(\delta) = \varphi\overline{\beta}(\delta)\varphi^{-1}$ is the composition of bijective maps, it also is. Then, consider $f : Z \to Y$ any bijection between measurable spaces and $Z = \bigsqcup_i Z_i$ a partition of Z into a countable number of measurable subsets Z_i, the restrictions $f_i := f|_{Z_i} : Z_i \to f[Z_i]$ have disjoint domain and codomains; assume also that $Y_i := f[Z_i]$ is measurable in Y for all i. Then, if each f_i is measurable (resp. preserves the measure), so is f, by the fact that $f^{-1}[A] = \bigsqcup_i f_i^{-1}[A \cap Y_i]$. Applying this to $f := \overline{\beta}(\delta)$ and $Z_i := X{\times}\{[r]\}$ shows that $\overline{\beta}(\delta)$ is measurable and preserves the measure.

In summary, for any free (SP1) action α of Λ on a space (X, μ), we have constructed a free (SP1) action $\overline{\beta}$ of Γ on the space $(\overline{X}, \frac{\overline{\mu}}{\overline{\mu}(\overline{X})})$, such that:

$$[\Gamma : \Lambda](\mathcal{C}(\Gamma) - 1) \leq [\Gamma : \Lambda](\mathcal{C}_{\overline{\mu}/\overline{\mu}(\overline{X})}(R_{\overline{\beta}}) - 1) = \mathcal{C}_{\mu}(R_{\alpha}) - 1.$$

Taking the infimum over free (SP1) actions α of Λ, we get:

$$[\Gamma : \Lambda](\mathcal{C}(\Gamma) - 1) \leq \mathcal{C}(\Lambda) - 1.$$

- Fix a free (SP1) action α of Γ on a space (X, μ), and fix $\Phi = \langle \varphi_i : A_i \to B_i \rangle_{i \in I}$ a graphing of R_α.

  Without loss of generality, we can assume that for each i, there exists $\gamma_i \in \Gamma$ with $\varphi_i = \alpha(\gamma_i)|_{A_i}$. Indeed, as Φ is a graphing of R_α, we know by definition that the set

  $$Z := \{x \in X \ : \ \Phi[x] \neq R_\alpha[x]\}$$

  has zero measure. For any $i \in I$, let

  $$Z_i := \{x \in A_i \ : \ \varphi_i(x) \neq \alpha(\gamma)(x) \ \forall \gamma \in \Gamma\}^{11}$$

  Then, Z_i is a null set, since $Z_i \subseteq Z$: if $\varphi(x) \neq \alpha(\gamma)(x)$ for all γ, then $\varphi(x) \notin \mathcal{O}_\alpha(x) = R_\alpha[x]$. Define

  $$\Phi' := \langle \varphi'_i := \varphi_i|_{A_i \setminus Z_i} \rangle_{i \in I}$$

  Because $\mu(Z_i) = 0$ for all i, we know that Φ' has the same cost as Φ, and we show that $R_{\Phi'} \simeq R_\Phi$. That is, we need to show that the set

  $$Z' := \{x \in X \ : \ \Phi'[x] \neq \Phi[x]\}$$

  also has zero measure. But $\Phi'[x] \subseteq \Phi[x]$ always holds, since the elements of Φ' are obtained by restricting those of Φ, and if $\Phi[x] \nsubseteq \Phi'[x]$, there must exist a Φ-word $w = \varphi_{i_l}^{\varepsilon_l} ... \varphi_{i_1}^{\varepsilon_1}$ with $x \in \operatorname{dom} w$, but $x \notin \operatorname{dom} w'$ (with $w' = {\varphi'_{i_l}}^{\varepsilon_l} ... {\varphi'_{i_1}}^{\varepsilon_1}$). In that case, there must exist some first index k with ${\varphi_{i_k}}^{\varepsilon_k} ... {\varphi_{i_1}}^{\varepsilon_1}(x) \notin \operatorname{dom} \varphi'_{i_{k+1}}$, that is, ${\varphi_{i_k}}^{\varepsilon_k} ... {\varphi_{i_1}}^{\varepsilon_1}(x) \in Z_{i_{k+1}}$. This implies that

  $$Z' \subseteq \bigcup_{w \in \mathcal{W}(\Phi)} \bigcup_{k \in I} w^{-1}[Z_k \cap \operatorname{cod} w]$$

  which is a countable union of null sets, and is null.

  We now have a graphing Φ' such that each $\varphi'_i : A_i \to B_i$ in Φ' agrees, at each point in its domain, with the action $\alpha(\gamma)$ of *some* $\gamma \in \Gamma$. Fixing some i, we can let

  $$A_i^\gamma := \{x \in A_i \ : \ \varphi'_i(x) = \alpha(\gamma)(x)\},^{12}$$

  and the sets $\langle A_i^\gamma \rangle_{\gamma \in \Gamma}$ form a partition of A_i (using freeness of the action). Thus we can take the refinement $\Phi'' = \langle \varphi_j \rangle_{j \in J}$ of Φ' over the family of partitions $\langle A_i^\gamma \rangle_{\gamma \in \Gamma}$ $(i \in I)$. This implies that, for all j, there exists some unique γ_j for which φ_j agrees with the action $\alpha(\gamma_j)$. Let us call the resulting graphing $\Phi = \langle \varphi_i \rangle_{i \in I}$. Note then that to any Φ-word $w = \varphi_{i_l}^{\varepsilon_{i_l}} ... \varphi_{i_1}^{\varepsilon_{i_1}}$ corresponds an element $\gamma_w := \gamma_{i_l}^{\varepsilon_{i_l}} \ \gamma_{i_1}^{\varepsilon_{i_1}}$ of Γ, such that $w(x) = \alpha(\gamma_w)(x)$ for all $x \in \operatorname{dom} w$.

  Define the space $\overline{X} := X \times \Gamma/\Lambda$, endowed with the product measure $\overline{\mu}$ (with the counting measure on Γ/Λ), and define the graphing:

  $$\overline{\Phi} := \langle \overline{\varphi_i} \rangle_{i \in I}, \qquad \text{with} \qquad \begin{aligned} \overline{\varphi_i} : A_i \times \Gamma/\Lambda &\to B_i \times \Gamma/\Lambda \\ (x, [\gamma]) &\mapsto (\varphi_i(x), [\gamma_i \gamma]) \end{aligned}$$

  where γ_i is the unique element of Γ such that $\alpha(\gamma_i)|_{A_i} = \varphi_i$. Note that $\overline{\varphi_i}$ is a measure-preserving measurable isomorphism because both φ_i and left-multiplication by γ_i are.

  Given that $\overline{\mu}(\operatorname{dom} \overline{\varphi_i}) = \overline{\mu}(A_i \times \Gamma/\Lambda) = [\Gamma : \Lambda]\mu(A_i)$, we have

  $$\mathcal{C}_{\overline{\mu}}(\overline{\Phi}) = [\Gamma : \Lambda]\mathcal{C}_\mu(\Phi).$$

  Consider the subset $X_1 := X \times \{[1]\}$ of $\overline{X}$ of measure 1. It is measurable and intersects $\overline{\mu}$-a.e. orbit of $R_{\overline{\Phi}}$. Indeed, let

  $$Z := \{x \in X \ : \ \Phi[x] \neq R_\alpha[x]\},$$

[11] For all i, the set Z_i is measurable, as it can be written $Z_i = \bigcap_\gamma \langle \varphi_i, \alpha(\gamma) \rangle^{-1}[B_i^2 \setminus \Delta_{B_i}]$.

[12] For each i and γ, A_i^γ is measurable, as it can be written $A_i^\gamma = \langle \varphi'_i, \alpha(\gamma) \rangle^{-1}[\Delta_{B_i}]$.

which has zero measure by hypothesis, and

$$\overline{W} := \{(x,[\gamma]) \in \overline{X} \ : \ X_1 \cap \overline{\Phi}[(x,[\gamma])] = \emptyset\}.^{13}$$

The goal is to show that $\overline{W}$ is a null set, but showing that for any $(x,[\gamma]) \in \overline{W}$, $x \in Z$ is enough, as that implies $\overline{W} \subseteq Z\times\Gamma/\Lambda$, and $Z\times\Gamma/\Lambda$ has measure zero ($\mu(Z\times\Gamma/\Lambda) = \mu(Z)[\Gamma:\Lambda] = 0$, as Z is a null set by assumption). So, fix some $(x,[\gamma]) \in \overline{W}$. By definition of $\overline{W}$, this means that given any Φ-word w with $x \in \operatorname{dom} w$, and letting $\overline{w}$ be the corresponding $\overline{\Phi}$-word, we have

$$(w(x),[\gamma_w\gamma]) = \overline{w}(x,[\gamma]) \notin X_1,$$

with γ_w the element of Γ corresponding to w. The above non-membership implies that $\gamma_w\gamma \notin \Lambda$, and in particular $\gamma_w\gamma \neq 1$. This shows that for any Φ-word w of domain containing x, $\gamma_w \neq \gamma^{-1}$, and hence $w(x) = \alpha(\gamma_w)(x) \neq \alpha(\gamma^{-1})(x)$, by freeness of the action α. We conclude that $\alpha(\gamma^{-1})(x) \notin \Phi[x]$, and $x \in Z$.

The restriction $R' := R_{\overline{\Phi}}|_{X_1}$ of $R_{\overline{\Phi}}$ to X_1 is isomorphic to the relation $R_{\alpha|_\Lambda}$ on X induced by the action $\alpha|_\Lambda$. Indeed, consider the map $f : X \to X_1$ sending x to $(x,[1])$; it is a measurable measure-preserving isomorphism. Then, let

$$\overline{Z} := \{x \in X \ : \ R'[(x,[1])] \neq f[R_{\alpha|_\Lambda}[x]]\} = \{x \in X \ : \ R'[(x,[1])] \neq \mathcal{O}_{\alpha|_\Lambda}(x)\times\{[1]\}\}.$$

The goal is to show that $\overline{Z}$ is null. Fix some $x \in X$; any element of $R'[(x,[1])]$ can be written as $(w(x),[\gamma_w])$ for some Φ-word w with $x \in \operatorname{dom} w$ and γ_w (the corresponding element of Γ) in Λ. But then $w(x) = \alpha(\gamma_w)(x)$, and $(w(x),[\gamma_w]) = (\alpha(\gamma_w)(x),[1]) \in \mathcal{O}_{\alpha|_\Lambda}(x)\times\{[1]\}$. This shows that $R'[(x,[1])] \subseteq f[R_{\alpha|_\Lambda}[x]]$ always holds.

Conversely, if $f[R_{\alpha|_\Lambda}[x]] \nsubseteq R'[(x,[1])]$, there must exist some $\lambda \in \Lambda$ such that there is no Φ-word w with $x \in \operatorname{dom} w$ and:

$$(\alpha(\lambda)(x),[1]) = (w(x),[\gamma_w]) = (\alpha(\gamma_w)(x),[\gamma_w]),$$

that is, there is no Φ-word w with $x \in \operatorname{dom} w$, $\gamma_w \in \Lambda$ and $\gamma_w = \lambda$ (by freeness of the action); thus, $x \in Z$. This shows $\overline{Z} \subseteq Z$, and nullity of $\overline{Z}$ follows.

Then, as $R_{\alpha|_\Lambda} \simeq R_\Phi|_{X_1}$, X_1 intersects $\overline{\mu}$-a.e. R_Φ-orbit, and $\alpha|_\Lambda$ is a free (SP1) action of Λ on (X,μ), we have:

$$\begin{aligned}\mathcal{C}(\Lambda) - 1 &\leq \mathcal{C}_\mu(R_{\alpha|_\Lambda}) - 1 \leq \mathcal{C}_{\overline{\mu}}(R_{\overline{\Phi}}|_{X_1}) - 1 = \mathcal{C}_{\overline{\mu}}(R_{\overline{\Phi}}|_{X_1}) - \overline{\mu}(X_1) = \mathcal{C}_{\overline{\mu}}(R_{\overline{\Phi}}) - \overline{\mu}(\overline{X}) \\ &\leq \mathcal{C}_{\overline{\mu}}(\overline{\Phi}) - [\Gamma:\Lambda] = [\Gamma:\Lambda](\mathcal{C}_\mu(\Phi) - 1)\end{aligned}$$

Taking the infimum over free (SP1) actions α of Γ and their graphings Φ, we conclude:

$$\mathcal{C}(\Lambda) - 1 \leq [\Gamma:\Lambda](\mathcal{C}(\Gamma) - 1).$$

Combining the two above items yields the desired equality:

$$\mathcal{C}(\Lambda) - 1 = [\Gamma:\Lambda](\mathcal{C}(\Gamma) - 1).$$

□

6.3 Random Graphs

6.3.1 Random Forests

Let H be a countable group. Endowing the set of relations on H, $\mathcal{P}(H\times H) \simeq 2^{H\times H}$, with the product topology (with $2 = \{0,1\}$ discrete), we get a compact topological space, by Tychonoff. In the following, we will somewhat abuse the notation and write $(g,h) \in P$ and $P(g,h) = 1$ (resp. $g,h \notin P$ and $P(g,h) = 0$) interchangeably, for $P \in \mathcal{P}(H\times H) \simeq 2^{H\times H}$.

Definition 6.33 (Graph, Tree)**.** Define the *graphs* on H to be the subset GRAPH_H of $\mathcal{P}(H\times H)$ of simple graphs, that is, antireflexive, symmetric relations. Define the *forests* on H to be the subset FOREST_H of GRAPH_H of graphs with no cycle, that is, no sequence $x_1, ...x_n$ ($n \geq 3$) of distinct elements such that (x_i, x_{i+1}) is in the graph for each i (with $\mod n$ arithmetic).

[13] The set $\overline{W}$ is measurable:

$$\overline{W} = \{p \in \overline{X} \ : \forall w \in \mathcal{W}(\overline{\Phi}) \ w(p) \notin X_1\} = \bigcap_{w\in\mathcal{W}(\overline{\Phi})} w^{-1}[\operatorname{cod} w \setminus X_1].$$

Given that $2 = \{0,1\}$ is a standard Borel space, so is $2^{H\times H}$, since a countable product of standard Borel spaces is standard.

Proposition 6.34. *Both* GRAPH_H *and* FOREST_H *are closed in* $\mathcal{P}(H\times H)$*.*

Proof. If P is not in GRAPH_H, it is either not antireflexive or not symmetric. If there exists some $x \in X$ with $(x,x) \in P$ (not antireflexive), then

$$\{P' \in \mathcal{P}(H\times H) \ : \ (x,x) \in P'\}$$

is an open set containing P and not intersecting GRAPH_H. If there exists $x, y \in X$ with $(x,y) \in P$ but $(y,x) \notin P$, then

$$\{P' \in \mathcal{P}(H\times H) \ : \ (x,y) \in P' \ \wedge \ (x,y) \notin P'\}$$

is an open set containing P and not intersecting GRAPH_H.

Now that GRAPH_H has been shown to be closed, it suffices to show that FOREST_H is closed in GRAPH_H. If P is in GRAPH_H but not FOREST_H, there exists $x_1, ...x_n$ forming a cycle in P, and

$$\{P' \in \mathcal{P}(H\times H) \ : \ \forall i \ (x_i, x_{i+1}) \in P'\} \cap \mathrm{GRAPH}_H \qquad \text{(with} \quad \bmod n \text{ arithmetic)}$$

is an open set of GRAPH_H containing P and not intersecting FOREST_H. ☐

As GRAPH_H and FOREST_H are closed (hence measurable) subsets of a standard Borel space, they are standard themselves.

The group H acts on $\mathcal{P}(H\times H)$ by:

$$g.S := \{(gh, gk) \mid (h,k) \in S\}, \qquad \forall S \in \mathcal{P}(H\times H), \ \forall g \in H.$$

and FOREST_H and GRAPH_H are both easily seen to be invariant under the action. A measure on GRAPH_H (resp. FOREST_H) is said to be H-invariant if the action of H on GRAPH_H (resp. FOREST_H) preserves the measure. Note that since the action is continuous, it is measurable.

Definition 6.35 (Random Graph, Random Forest)**.** A *random graph* on H is a H-invariant probability measure on GRAPH_H. A *random forest* on H is a H-invariant probability measure on FOREST_H.

We will only care about random forests.

Definition 6.36 (Expected Degree, f_ρ, Width)**.** Let ρ be a random forest on a countable group H.

The *expected degree* of ρ is defined as:

$$\deg \rho := \int_{\mathsf{F} \in \mathrm{FOREST}_H} v_{\mathsf{F}}(1) d\rho(\mathsf{F}),$$ [14]

with $v_{\mathsf{F}}(x) = |\mathsf{F}_x| = |\{y \in H \ : \ (x,y) \in \mathsf{F}\}|$ the degree of the vertex x in F.

Let f_ρ be the following map:

$$\begin{aligned} f_\rho : H &\to \mathbb{R}^{\geq 0} \\ g &\mapsto \rho\{\mathsf{F} \in \mathrm{FOREST}_H \ : \ (1,g) \in \mathsf{F}\}. \end{aligned}$$

The *width* of ρ is then defined as:

$$\mathrm{width}\, \rho := |\{g \in H \ : \ f_\rho(g) > 0\}|.$$

[14] The map $\mathsf{F} \mapsto v_{\mathsf{F}}(1)$ is measurable. Indeed, it can be written: $\mathsf{F} \mapsto v_{\mathsf{F}}(1) = \sum_{g\in H} \mathbb{1}_{\mathsf{F}}(1,g)$ and by Corollary C.24 it suffices to show that $\mathsf{F} \mapsto \mathbb{1}_{\mathsf{F}}(1,g)$ is measurable. But the set

$$\{\mathsf{F} \in \mathrm{FOREST}_H \ : \ \mathbb{1}_{\mathsf{F}}(1,g) = 1\} = \{\mathsf{F} \in \mathrm{FOREST}_H \ : \ (1,g) \in \mathsf{F}\}$$

is open by construction, hence measurable, and similarly for the set of those F with $\mathbb{1}_{\mathsf{F}}(1,g) = 0$; this shows that $\mathsf{F} \mapsto \mathbb{1}_{\mathsf{F}}(1,g)$ is a measurable map.

Note that the expected degree can be written:

$$
\begin{aligned}
\deg \rho &= \int\limits_{\mathsf{F}\in\mathrm{FOREST}_H} v_{\mathsf{F}}(1) d\rho(\mathsf{F}) \\
&= \int\limits_{\mathsf{F}\in\mathrm{FOREST}_H} \sum_{g\in H} \mathbb{1}_{\mathsf{F}}(1,g) d\rho(\mathsf{F}) \\
&= \sum_{g\in H} \int\limits_{\mathsf{F}\in\mathrm{FOREST}_H} \mathbb{1}_{\mathsf{F}}(1,g) d\rho(\mathsf{F}) \\
&= \sum_{g\in H} \rho\{\mathsf{F}\in\mathrm{FOREST}_H \ : \ (1,g)\in\mathsf{F}\} \\
&= \sum_{g\in H} f_\rho(g).^{15}
\end{aligned}
$$

By definition of the width, if $\operatorname{width}\rho$ is finite, then f_ρ has finite support. Also, given the above formula for the expected degree, we have:

$$\deg \rho = \sum_{g\in H} f_\rho(g) = \sum_{\substack{g\in H,\\ f_\rho(g)>0}} f_\rho(g) \leq \sum_{\substack{g\in H,\\ f_\rho(g)>0}} 1 = \operatorname{width}\rho,$$

because $f_\rho(g) \leq 1$ for all $g \in H$, ρ being a probability measure.

6.3.2 Simple Graphings & Cost

Definition 6.37 (Simple Graphing)**.** On a standard measure space (X,μ), a *simple graphing* is a measurable subset G of $X\times X$ such that:

- For all $x,y\in X$: $(x,x)\notin\mathsf{G}$ and $(x,y)\in\mathsf{G} \Leftrightarrow (y,x)\in\mathsf{G}$.
- For all $x\in X$: $v_G(x) = |\mathsf{G}_x| = |\{y\in X \ : \ (x,y)\in\mathsf{G}\}|$ is countable.

In other words, G is a symmetric, antireflexive measurable relation on X, with countable sections.

From now on, the term *Levitt* graphing will be used for what we originally defined as a graphing, in an effort to distinguish it better from *simple* graphings.

Definition 6.38 (Cost)**.** Given a simple graphing G on a standard measure space (X,μ), we define the cost of G as:

$$\mathcal{C}_\mu(\mathsf{G}) := \frac{1}{2}\int\limits_{x\in X} v_{\mathsf{G}}(x) d\mu(x),^{16}$$

where $v_{\mathsf{G}}(x)$ is, as defined above, the degree of x in G.

Recall that the reflexive, transitive closure R_{G} of a simple graphing G corresponds to the partition of X into its connected components.

Definition 6.39. If R is an (SP) relation, and the transitive, reflexive closure R_{G} of a simple graphing G is (SP) and satisfies $R\simeq R_{\mathsf{G}}$, we say that G is a simple graphing of R.

Lemma 6.40 ([KM04, Sections 17&18])**.** *Let R be a (SP) relation on (X,μ). Then:*

$$\mathcal{C}_\mu(R) = \inf\{\mathcal{C}_\mu(\mathsf{G}) \ \mid \ \mathsf{G} \textit{ a simple graphing of } R\}$$

Proof. Fix a (SP) relation R on (X,μ).

We show that to each Levitt graphing of R corresponds a simple graphing of R of no greater cost, and conversely.

[15] Once again, Corollary C.24 allows to commute sum and integral, at the third equality.

[16] The map $v_{\mathsf{G}} : X\to[0,\infty]$ is measurable. Indeed, by Corollary C.22, G can be written as the disjoint union of a countable number of graphs of measurable maps: $\mathsf{G} = \bigsqcup_{i\in\mathbb{N}}\Gamma(f_i)$. Then, for any $n\in\mathbb{N}$,

$$\{x\in X \ : \ v_{\mathsf{G}}(x) = n\} = \bigcup_{\substack{i_1,\dots,i_n\\ \text{distinct}}} \left(\left(\bigcap_{j=1}^{n} \operatorname{dom} f_{i_j} \right) \cap \left(\bigcap_{j\neq i_1,\dots,i_n} X\setminus \operatorname{dom} f_j \right) \right)$$

which is measurable.

Fix Φ a Levitt graphing of R, and define

$$\begin{aligned}\mathsf{G}_\Phi &:= \{(x,y)\in X^2\setminus\Delta_X :\ \exists i\ (x\in\operatorname{dom}\varphi_i\ \wedge\ y=\varphi_i(x)) \text{ or } (y\in\operatorname{dom}\varphi_i\ \wedge\ x=\varphi_i(y))\}\\ &= \left(\bigcup_{i\in I}\Gamma(\varphi_i)\cup\Gamma(\varphi_i^{-1})\right)\setminus\Delta_X,\end{aligned}$$

that is, G_Φ is the underlying simple graph of $\mathcal{G}_\Phi$ (Definition C.28). We claim that G_Φ is a simple graphing of R, of cost at most $\mathcal{C}_\mu(\Phi)$.

The second characterisation of G_Φ makes it obvious that it is measurable in $X\times X$. It is symmetric and antireflexive by construction. We know that the connected components of the underlying simple graph of a multigraph are the same as the original connected components (Proposition C.29), so that $\mathsf{G}_\Phi[x]=\mathcal{G}_\Phi[x]$ for all x (and this implies in particular that all vertices have a countable number of neighbours in G_Φ). Finally, still because G_Φ is the underlying simple graph of $\mathcal{G}_\Phi$ (Proposition C.29):

$$v_{\mathsf{G}_\Phi}(x)\le v_{\mathcal{G}_\Phi}(x)=v_\Phi(x),\qquad \forall x,$$

so that

$$\mathcal{C}_\mu(\mathsf{G}_\Phi)=\frac{1}{2}\int_X v_{\mathsf{G}_\Phi}(x)d\mu(x)\le\frac{1}{2}\int_X v_\Phi(x)d\mu(x)=\mathcal{C}_\mu(\Phi).$$

Conversely, for each simple graphing G of R, we construct a Levitt graphing Φ_G of R of cost at most $\mathcal{C}_\mu(\mathsf{G})$: Fix such a simple graphing G, and consider a measurable ordering $<$ of X as given by a measurable isomorphism of X with $[0,1]$ (or a subspace thereof). Define the subset G^+ of G by

$$\mathsf{G}^+ := \{(x,y)\in\mathsf{G}\ :\ x<y\}.$$

Then, G^+ is measurable, as it is equal to $\mathsf{G}\cap\{x<y\}$, and G^+_x is countable, for all x. Then, by Corollary C.22, there exists a countable set of measurable maps $\{f_i : A_i\to X\}_{i\in I}$, with each A_i measurable, and such that

$$\mathsf{G}^+=\bigsqcup_i\Gamma(f_i).$$

Now, for any $y\in X$, $f_i^{-1}[\{y\}]$ is countable, since $\Gamma(f_i)$ is contained in G, and we can apply Corollary C.22 again, to get, for each i, a countable set $\{g_{i,j} : B_{i,j}\to A_i\}_{i\in I,j\in J_i}$ of measurable maps with $B_{i,j}$ measurable for all i,j and

$$\Gamma(f_i)^{-1}=\bigsqcup_j\Gamma(g_{i,j})$$

(where $A^{-1}:=\{(y,x)\ \mid\ (x,y)\in A\}$, if $A\subseteq X^2$). The maps $g_{i,j}$ are injective, since if $g_{i,j}(x)=y=g_{i,j}(x')$, we have $(x,y),(x',y)\in\Gamma(g_{i,j})\subseteq\Gamma(f_i)^{-1}$, and $(y,x),(y,x')\in\Gamma(f_i)$, which implies $x=x'$. Since each $g_{i,j}$ is injective and measurable, its image is a measurable set, and $g_{i,j}$ can be restricted to its image, and hence assumed surjective.

Let $\Phi_\mathsf{G} := \langle g_{i,j}\rangle_{i\in I,j\in J_i}$. By construction Φ_G is a Levitt graphing, and it remains to show $R_{\Phi_\mathsf{G}}=R_\mathsf{G}$ and $\mathcal{C}_\mu(\Phi_\mathsf{G})\le\mathcal{C}_\mu(\mathsf{G})$. To that end, first note that

$$\mathsf{G}^+=\bigsqcup_i\bigsqcup_j\Gamma(g_{i,j})^{-1}.$$

This tells us that for any pair (x,y) of G^+, there exists a unique $g_{i,j}$ with $y\in\operatorname{dom}g_{i,j}$ and $x=g_{i,j}(y)$, and, conversely, if $x=g_{i,j}(y)$, then $(x,y)\in\mathsf{G}^+$. We conclude that G is then the underlying simple graph of $\mathcal{G}_{\Phi_\mathsf{G}}$, since:

$$\begin{aligned}(x,y)\in\mathsf{G}\quad &\Leftrightarrow\quad (x,y)\in\mathsf{G}^+\vee(y,x)\in\mathsf{G}^+\\ &\Leftrightarrow\quad (\exists i,j :\ x\in\operatorname{dom}g_{i,j}\wedge y=g_{i,j}(x))\vee(\exists i,j :\ y\in\operatorname{dom}g_{i,j}\wedge x=g_{i,j}(y))\\ &\Leftrightarrow\quad \big(\exists\text{ an edge } e \text{ in } \mathcal{G}_{\Phi_\mathsf{G}} \text{ with } (s(e),t(e))=(x,y)\big)\ \vee\ \big(\exists\text{ an edge } e \text{ in } \mathcal{G}_{\Phi_\mathsf{G}} \text{ with } (s(e),t(e))=(y,x)\big),\end{aligned}$$

and $R_{\mathcal{G}_{\Phi_\mathsf{G}}}=R_\mathsf{G}$. Furthermore:

$$\begin{aligned}v_\mathsf{G}(x)&=|\{y\in X\ :\ (x,y)\in\mathsf{G}\}|\\ &=|\{y\in X\ :\ (x,y)\in\mathsf{G}^+\}|+|\{y\in X\ :\ (y,x)\in\mathsf{G}^+\}|\\ &=|\{i,j\ :\ x\in\operatorname{cod}g_{i,j}\}|+|\{i,j\ :\ x\in\operatorname{dom}g_{i,j}\}|\\ &=v^-_{\mathcal{G}_{\Phi_\mathsf{G}}}(x)+v^+_{\mathcal{G}_{\Phi_\mathsf{G}}}(x)=v_{\mathcal{G}_{\Phi_\mathsf{G}}}(x)=v_{\Phi_\mathsf{G}}(x)\end{aligned}$$

for any x, so that $\mathcal{C}_\mu(\mathsf{G})=\mathcal{C}_\mu(\Phi_\mathsf{G})$.

As we have shown that to any Levitt graphing of R corresponds a simple graphing of R of no greater cost, and that to any simple graphing of R corresponds a Levitt graphing of equal cost, we conclude that the infimum of those agree.

□

The following fact will not be proved:

Lemma 6.41 ([KM04, Lemma 28.11]). *Let R a (SP) relation on (X, μ), G a simple graphing of R and T_0 an acyclic simple graphing with $\mathsf{T}_0 \subseteq \mathsf{G}$. Then, there exists an acyclic simple graphing T with $\mathsf{T}_0 \subseteq \mathsf{T} \subseteq \mathsf{G}$ such that, if $\mathcal{C}_\mu(\mathsf{G}) < \infty$,*

$$\mathcal{C}_\mu(R) \leq \mathcal{C}_\mu(\mathsf{T}).$$

Proof. [KM04, Lemma 28.11] □

6.3.3 Cost & Unitarizability

Lemma 6.42. *For any finitely generated group H, there exists a random forest ρ on H satisfying:*

$$\operatorname{width} \rho \leq 2 \operatorname{rank} H, \qquad \textit{and} \qquad \deg \rho \geq 2\mathcal{C}(H).$$

The construction used in this proof follows the paragraph after Proposition 4.1 in [EM09], and essentially consists in using [KM04, Lemma 28.11] and the first part of [KM04, Proposition 29.5], to construct a random forest on H out of (an appropriate acyclic subgraph of) a "Cayley" graphing of H on some space (X, μ).

Proof. Fix a group H, generated by some finite set $S \subseteq H$, of minimal cardinality $|S| = \operatorname{rank} H$. Fix some free (SP1) action α of H on a standard measure space (X, μ). Then, $\Phi_S := \langle \alpha(s) \rangle_{s \in S}$ is a Levitt graphing of R_α (see Proposition 6.15), and defines a simple graphing G_S of R_α, via the construction of Lemma 6.40. We know that $\mathcal{C}(H) \leq \mathcal{C}_\mu(R_\alpha)$. Also, $\mathcal{C}_\mu(\mathsf{G}_S)$ is finite, since $\mathcal{C}_\mu(\mathsf{G}_S) \leq \mathcal{C}_\mu(\Phi_S) = |S|$. We can apply Lemma 6.41 on G_S and $\mathsf{T}_0 := \emptyset$, so as to get an acyclic simple graphing T with $\mathsf{T} \subseteq \mathsf{G}_S$ and $\mathcal{C}_\mu(\mathsf{T}) \geq \mathcal{C}_\mu(R_\alpha)$, and in particular:

$$\mathcal{C}_\mu(\mathsf{T}) \geq \mathcal{C}(H).$$

Out of T, we now want to construct a random forest ρ on H, of degree $\deg \rho \geq 2\mathcal{C}(H)$, and width $\operatorname{width} \rho \leq 2 \operatorname{rank} H$.

Define the measurable map:

$$\begin{aligned} \varphi_{\mathsf{T}} : X &\to \mathrm{FOREST}_H \\ x &\mapsto \{(g, h) \in H \times H \ : \ (\alpha(g^{-1})x, \alpha(h^{-1})x) \in \mathsf{T}\}.^{17} \end{aligned}$$

By freeness of the action and the fact that T is acyclic (i.e. a forest), so is $\mathsf{F}_{\mathsf{T}}(x)$, for all $x \in X$. Indeed, assume $\mathsf{F}_{\mathsf{T}}(x)$ has a non-trivial shortest cycle $h_1, ..., h_n$; that is, $(h_i, h_{i+1}) \in \mathsf{F}_{\mathsf{T}}(x)$ for all i (with $\bmod n$ arithmetic), and all h_i are distinct. Then, by definition, $(\alpha(h_i^{-1})x, \alpha(h_{i+1}^{-1})x) \in \mathsf{T}$ for all i, and $\alpha(h_i^{-1})x \neq \alpha(h_j^{-1})x$ whenever $i \neq j$, by freeness of the action (that is, the cycle in $\mathsf{F}_{\mathsf{T}}(x)$ doesn't "collapse"). This implies that T has a (non-trivial) cycle, which is a contradiction.

Now, define the probability measure ρ on FOREST_H by:

$$\rho(P) := \mu(\varphi_{\mathsf{T}}^{-1}[P]).$$

To see that ρ is H-invariant, first observe that $\varphi_{\mathsf{T}}(\alpha(g).x) = g.\varphi_{\mathsf{T}}(x)$ for all g and x (it is equivariant):

$$\begin{aligned} g.\varphi_{\mathsf{T}}(x) &= g.\{(h, k) \ : \ (\alpha(h^{-1})x, \alpha(k^{-1})x) \in \mathsf{T}\} \\ &= \{(h, k) \ : \ (\alpha(h^{-1}g)x, \alpha(k^{-1}g)x) \in \mathsf{T}\} \\ &= \{(h, k) \ : \ (\alpha(h^{-1})\alpha(g)x, \alpha(k^{-1})\alpha(g)x) \in \mathsf{T}\}. = \varphi_{\mathsf{T}}(\alpha(g)x) \end{aligned}$$

This implies that for any measurable subset A of FOREST_H:

$$\rho(g.A) = \mu(\varphi_{\mathsf{T}}^{-1}[g.A]) = \mu(\alpha(g).\varphi_{\mathsf{T}}^{-1}[A]) = \mu(\varphi_{\mathsf{T}}^{-1}[A]) = \rho(A)$$

where the second equality is from the equivariance of φ_{T} and the third from the fact that α preserves μ, and we conclude that ρ really is H-invariant. We now show that $\deg \rho \geq 2\mathcal{C}(H)$.

$$\begin{aligned}
\deg \rho &= \int\limits_{\mathsf{F} \in \mathrm{FOREST}_H} v_{\mathsf{F}}(1) d\rho(\mathsf{F}) \\
&= \int\limits_{\mathsf{F} \in \mathrm{FOREST}_H} v_{\mathsf{F}}(1) d\mu(\varphi_{\mathsf{T}}^{-1}[\mathsf{F}]) \\
&= \int\limits_{x \in X} v_{\varphi_{\mathsf{T}}(x)}(1) d\mu(x) \\
&= \int\limits_{x \in X} |\{h \in H \ : \ (1,h) \in \varphi_{\mathsf{T}}(x)\}| d\mu(x) \\
&= \int\limits_{x \in X} |\{h \in H \ : \ (x, \alpha(h^{-1})x) \in \mathsf{T}\}| d\mu(x) \\
&= \int\limits_{x \in X} |\{y \in X : \ (x,y) \in \mathsf{T}\}| d\mu(x) \\
&= \int\limits_{x \in X} v_{\mathsf{T}}(x) d\mu(x) \\
&= 2\mathcal{C}_{\mu}(\mathsf{T}) \geq 2\mathcal{C}(H)
\end{aligned}$$

where the sixth equality is due to the fact that the map

$$\begin{aligned}
\{h \in H \ : \ (x, \alpha(h^{-1})x) \in \mathsf{T}\} &\to \{y \in X : \ (x,y) \in \mathsf{T}\} \\
h &\mapsto \alpha(h^{-1})x
\end{aligned}$$

is a bijection; indeed, if $\alpha(h^{-1})x = \alpha(k^{-1})x$, by freeness of α, $h = k$; conversely, for any $y \in X$ with $(x,y) \in \mathsf{T}$, $(x,y) \in \mathsf{G}_S$, and there must exist an edge $e \in \mathcal{G}_{\Phi_S}$ with $(x,y) = (s(e), t(e))$ or $(x,y) = (t(e), s(e))$, that is, an $s \in S$ with $(x,y) = (x, \alpha(s)x)$ or $(x,y) = (\alpha(s)y, y)$, and we conclude that y has preimage s or s^{-1}.

It remains to show that $\operatorname{width} \rho \leq 2|S|$. We have

$$\begin{aligned}
\operatorname{width} \rho &= |\{g \in H \ : \ f_\rho(g) > 0\}| \\
&= |\{g \in H \ : \ \rho\{\mathsf{F} \in \mathrm{FOREST}_H \ : \ (1,g) \in \mathsf{F}\} > 0\}| \\
&= |\{g \in H \ : \ \mu\{x \in X \ : \ (x, \alpha(g^{-1})x) \in \mathsf{T}\} > 0\}|
\end{aligned}$$

and if $g \in H$ is such that $f_\rho(g) > 0$, then, in particular, there must exist some x with $(x, \alpha(g^{-1})x) \in \mathsf{T}$, and thus $(x, \alpha(g^{-1})x) \in \mathsf{G}_S$. If $(x, \alpha(g^{-1})x) \in \mathsf{G}_S$, there must exist an edge e in $\mathcal{G}_{\Phi_S}$ with $(x, \alpha(g^{-1})x) = (s(e), t(e))$ or $(x, \alpha(g^{-1})x) = (t(e), s(e))$; that is, there exists some $s \in S$ and $y \in X$ with:

$$(x, \alpha(g^{-1})x) = (y, \alpha(s^{-1})y), \qquad \text{or} \qquad (x, \alpha(g^{-1})x) = (\alpha(s^{-1})y, y)$$

which implies, by freeness of the action, that $g = s^{\pm 1}$. We conclude that if $g \in H$ is such that $f_\rho(g) > 0$, then $g \in S \cup S^{-1}$, and $\operatorname{width} \rho \leq 2|S|$. ☐

Proposition 6.43 ([EM09, Proposition 2.3])**.** *Let ρ be a random forest of finite width on a countable group H. Then*

$$\|f_\rho\|_{T_1(H)} \leq 2 \quad \textit{and} \quad \|f_\rho\|_2 \geq \frac{\deg \rho}{\sqrt{\operatorname{width} \rho}}.$$

We note that f_ρ really is an element of $T_1(H)$ and $l^2(H)$, since ρ has finite width, and f_ρ is thus finitely supported.

Proof.

[17] φ_{T} is a measurable map: It suffices to show that the preimages of the elements of a countable topological subbasis of FOREST_H are measurable in X. A countable subbasis for FOREST_H consists of the sets of the form $B = \{\mathsf{F} \in \mathrm{FOREST}_H \ : \ (g,h) ? \mathsf{F}\}$, where "?" is either "$\in$" or "$\notin$", and $g, h \in H$. Then,

$$\varphi_{\mathsf{T}}^{-1}[B] = \{x \in X \ : \ (\alpha(g^{-1})x, \alpha(h^{-1})x) ? \mathsf{T}\} = \begin{cases} \langle \alpha(g^{-1}), \alpha(h^{-1}) \rangle^{-1}[\mathsf{T}] & \text{if "?" is "}\in\text{"}, \\ \langle \alpha(g^{-1}), \alpha(h^{-1}) \rangle^{-1}[X^2 \setminus \mathsf{T}] & \text{if "?" is "}\notin\text{"}, \end{cases}$$

which is measurable, since T is.

First inequality. We want to decompose f_ρ into some $f_\rho^+, f_\rho^- : H\times H \to \mathbb{C}$ such that $f_\rho(g^{-1}g') = f_\rho^+(g,g') + f_\rho^-(g,g')$ and

$$\sup_g \sum_{g'} |f_\rho^+(g,g')| \leq 1, \quad \text{and} \quad \sup_{g'} \sum_{g} |f_\rho^-(g,g')| \leq 1.$$

Let FOREST_H^+ be the set of orientations of forests on H, that is, the set of antisymmetric relations on H such that their symmetric closure is a forest. More informally, an element of FOREST_H^+ is constructed by choosing a forest $\mathsf{F} \in \mathrm{FOREST}_H$ and, for each edge of F, an orientation.

Order the (countable) group H as $H = \{g_i\}_{i\in\mathbb{N}}$, and define the map $O : \mathrm{FOREST}_H \to \mathrm{FOREST}_H^+$ as follows: For any forest $\mathsf{F} \in \mathrm{FOREST}_H$ and edge $(g,g') \in \mathsf{F}$, let n be the first integer such that g_n is in the same tree as g (or, equivalently, as g'; such an n must exist because g itself is in the same tree as g). Then, consider the (unique, because tree) path from g to g_n; two cases are possible: g' appears in that path, or it doesn't, in which case g appears in the path from g_n to g'. In the first case, define (g,g') to be the edge corresponding to (g,g') in $O(\mathsf{F})$, and in the second, let it be (g',g). Formally:

$$O(\mathsf{F}) = \{(g,g') \in \mathsf{F} \ : \ g' \text{ is on the unique path from } g \text{ to } g_n(g)\}.$$

where $g_n(g)$ is the "first" element of H to lie in the connected component of g.

Define then

$$\begin{aligned} f_\rho^+(g,g') &:= \rho\{\mathsf{F} \in \mathrm{FOREST}_H \ : \ (g,g') \in O(\mathsf{F})\}, \\ f_\rho^-(g,g') &:= \rho\{\mathsf{F} \in \mathrm{FOREST}_H \ : \ (g',g) \in O(\mathsf{F})\}. \end{aligned}$$

Given that for any $\mathsf{F} \in \mathrm{FOREST}_H$, and any edge of F, exactly one of its orientations is in $O(\mathsf{F})$, we have

$$\{\mathsf{F} \in \mathrm{FOREST}_H \ : \ (g,g') \in \mathsf{F}\} = \{\mathsf{F} \in \mathrm{FOREST}_H \ : \ (g,g') \in O(\mathsf{F})\} \sqcup \{\mathsf{F} \in \mathrm{FOREST}_H \ : \ (g',g) \in O(\mathsf{F})\},$$

so that:

$$\rho\{\mathsf{F} \in \mathrm{FOREST}_H \ : \ (g,g') \in \mathsf{F}\} = \rho\{\mathsf{F} \in \mathrm{FOREST}_H \ : \ (g,g') \in O(\mathsf{F})\} + \rho\{\mathsf{F} \in \mathrm{FOREST}_H \ : \ (g',g) \in O(\mathsf{F})\}.$$

By invariance of ρ under the H-action:

$$\rho\{\mathsf{F} \in \mathrm{FOREST}_H \ : \ (g,g') \in \mathsf{F}\} = \rho\{\mathsf{F} \in \mathrm{FOREST}_H \ : \ (1,g^{-1}g') \in \mathsf{F}\} = f_\rho(g^{-1}g')$$

and combining the above two equations:

$$f_\rho(g^{-1}g') = f_\rho^+(g,g') + f_\rho^-(g,g'), \qquad \forall g,g' \in H.$$

It remains to show that f_ρ^+ and f_ρ^- are bounded as needed, which we only do for f_ρ^+, given that f_ρ^- is defined symmetrically. Fix $g \in H$ and consider some $\mathsf{F} \in \mathrm{FOREST}_H$. Let n be the first integer with g_n in the same tree as g. For $(g,g') \in O(\mathsf{F})$ to hold, we must have g' in the path between g and g_n, which implies that a g' with $(g,g') \in O(\mathsf{F})$ is uniquely determined by g, as the first step on a path from g to g_n, and doesn't exist if $g_n = g$, since such a path has no first step in that case. This implies that given g and two distinct g_1', g_2',

$$\{\mathsf{F} \in \mathrm{FOREST}_H \ : \ (g,g_1') \in O(\mathsf{F})\} \quad \text{and} \quad \{\mathsf{F} \in \mathrm{FOREST}_H \ : \ (g,g_2') \in O(\mathsf{F})\}$$

are disjoint, since if both $(g,g_1'), (g,g_2')$ lie in $O(\mathsf{F})$, both g_1' and g_2' would be the first element on a path from g to g_n, in F, which is impossible. We conclude that

$$\sum_{g'} f_\rho^+(g,g') = \sum_{g'} \rho\{\mathsf{F} \in \mathrm{FOREST}_H \ : \ (g,g') \in O(\mathsf{F})\} = \rho\left(\bigsqcup_{g'} \{\mathsf{F} \in \mathrm{FOREST}_H \ : \ (g,g') \in O(\mathsf{F})\}\right) \leq 1$$

and we are done, since then:

$$\sup_g \sum_{g'} f_\rho^+(g,g') \leq 1.$$

Second inequality. As ρ has finite width, let $S = \{g \in G \ : \ f_\rho(g) > 0\}$. Then f_ρ is supported on S and $|S| = \operatorname{width}\rho$, so that (by Cauchy-Schwarz on $l^2(H)$):

$$\deg\rho = \sum_{g\in H} f_\rho(g) = \sum_{g\in S} f_\rho(g)1 = \langle f_\rho, \mathbb{1}_S\rangle \leq \|f_\rho\|_2 \|\mathbb{1}_S\|_2 = \|f_\rho\|_2 \sqrt{\operatorname{width}\rho},$$

and we conclude:

$$\|f_\rho\|_2 \geq \frac{\deg\rho}{\sqrt{\operatorname{width}\rho}}.$$

□

Proposition 6.44 ([EM09, Proposition 2.4]). *If G is a unitarizable group, there exists some K such that for any subgroup H of G, and for any $f \in T_1(H)$:*

$$\|f\|_{l^2(H)} \leq K\|f\|_{T_1(H)}.$$

From the previous sections, we already have all the necessary results to prove this.

Proof. By Theorem 4.5, we know that $T_1(G) \subseteq l^2(G)$, and as both spaces are continuously included in $\mathbb{C}^G$ (product topology), Proposition A.4 implies the existence of some K with

$$\|f\|_{l^2(G)} \leq K\|f\|_{T_1(G)}, \quad \forall f \in T_1(G).$$

Fix now some subgroup H of G. We know that $l^2(H)$ embeds isometrically in $l^2(G)$, by extending a map of $l^2(H)$ with zeros on $G \setminus H$. The same holds for $T_1(H)$ and $T_1(G)$, by Lemma 3.11, so that, writing f' for the extension of f:

$$\|f\|_{l^2(H)} = \|f'\|_{l^2(G)} \leq K\|f'\|_{T_1(G)} = K\|f\|_{T_1(H)}, \quad \forall f \in T_1(H),$$

since the embeddings of $l^2(H)$ and $T_1(H)$ both map f to f'. $\square$

Theorem 6.45 ([EM09, Theorem 1.2]). *A finitely generated, residually finite group G with $\mathcal{C}(G) > 1$ is not unitarizable.*

The proof follows the argument in the paragraph after proposition 4.1, and the proof of theorem 1.1, in [EM09].

We prove the contrapositive, that is: if G is finitely generated, residually finite, and unitarizable, then $\mathcal{C}(G) \leq 1$. Note here that G being finitely generated, it has finite cost.

Proof. Assume G to be finitely generated, residually finite, and unitarizable. If G is finite, $\mathcal{C}(G) = 1 - \frac{1}{|G|}$ and the proposition trivially holds. So, assume G to be infinite; then, by residual finiteness, it contains normal subgroups of arbitrarily large finite index. By Proposition C.17 (Schreier), we have, for any subgroup H of G of finite index:

$$\operatorname{rank} H \leq [G : H] \operatorname{rank} G. \tag{37}$$

This implies, in particular, that H has finite rank.

Combining Proposition 6.43 and Proposition 6.44 tells us that for any finite index subgroup H of G and any random forest ρ on H of finite width:

$$\frac{(\deg \rho)^2}{\operatorname{width} \rho} \leq \|f_\rho\|^2_{l^2(H)} \leq K^2\|f_\rho\|^2_{T_1(H)} \leq 4K^2 \tag{38}$$

for some constant K (depending on G only).

In particular, fix some finite index subgroup H of G, and consider the random forest ρ on H constructed in Lemma 6.42. This forest has finite width and satisfies:

$$\deg \rho \geq 2\mathcal{C}(H) \quad \text{and} \quad \operatorname{width} \rho \leq 2 \operatorname{rank} H. \tag{39}$$

This implies, by combining Equation (38) and Equation (39), that for any finite index subgroup H of G:

$$4\mathcal{C}(H)^2 \leq 8K^2(\operatorname{rank} H) \quad \text{and therefore} \quad \mathcal{C}(H) \leq 2^{\frac{1}{2}}K(\operatorname{rank} H)^{\frac{1}{2}}.$$

Finally, recall that $\mathcal{C}(H) - 1 = [G : H](\mathcal{C}(G) - 1)$ holds for any finite index subgroup H of G. This, along with Equation (37), yields, for any finite index subgroup H of G:

$$[G : H](\mathcal{C}(G) - 1) + 1 \leq 2^{\frac{1}{2}}K[G : H]^{\frac{1}{2}}(\operatorname{rank} G)^{\frac{1}{2}} \quad \text{and therefore} \quad \mathcal{C}(G) - 1 \leq 2^{\frac{1}{2}}K\frac{(\operatorname{rank} G)^{\frac{1}{2}}}{[G : H]^{\frac{1}{2}}} - \frac{1}{[G : H]}.$$

Now, since G has subgroup of arbitrarily large finite index, we can let $n = [G : H]$ tend to ∞:

$$(\mathcal{C}(G) - 1) \leq 2^{\frac{1}{2}}K\frac{(\operatorname{rank} G)^{\frac{1}{2}}}{n^{\frac{1}{2}}} - \frac{1}{n} \overset{n\to\infty}{\longrightarrow} 0$$

so that $\mathcal{C}(G) \leq 1$. $\square$

II Appendix

A Banach

A.1 l^p Spaces.

Fix some index set I. Recall that for $1 \leq p < \infty$, the space $l^p(I)$ is the $\mathbb{C}$-vector space of maps $f : I \to \mathbb{C}$ satisfying

$$\sum_i |f(i)|^p < \infty,$$

with the addition and $\mathbb{C}$-action defined pointwise. Defining a norm by

$$\|f\|_p := \left(\sum_i |f(i)|^p\right)^{1/p}$$

makes $l^p(I)$ into a Banach space. In the case $p = \infty$, the space $l^p(I)$ is defined as the $\mathbb{C}$-vector space of maps $f : I \to \mathbb{C}$ satisfying:

$$\sup_i |f(i)| < \infty,$$

that is, the bounded maps, with the addition and the $\mathbb{C}$-action pointwise. For $1 \leq p \leq \infty$, $l^p(I)$ is then a sub-vector space of the product space $\mathbb{C}^I$.

When $p = 2$, we can define an inner product

$$f, g \mapsto \langle f, g\rangle := \sum_i f(i)\overline{g(i)},$$

and this makes $l^2(I)$ into a Hilbert space.

If $I = G$ is a group, define the maps:

$$\begin{aligned} \lambda : G &\to B(l^p(G), l^p(G)) & \qquad \rho : G &\to B(l^p(G), l^p(G)) \\ \lambda(g)(f)(x) &:= f(g^{-1}x) & \rho(g)(f)(x) &:= f(xg), \end{aligned}$$

Then $\lambda(g)$ is a bijective isometry for each g, and any $1 \leq p \leq \infty$, since:

$$\sum_x |f(g^{-1}x)|^p = \sum_x |f(x)|^p,\ (p < \infty) \qquad \text{and} \quad \sup_x |f(g^{-1}x)| = \sup_x |f(x)|,\ (p = \infty),$$

because $g^{-1}\cdot$ just shifts the indices. The map λ is actually a unitary representation of G, as:

$$((\lambda(g)\lambda(h))f)(x) = (\lambda(h)f)(g^{-1}x) = f(h^{-1}g^{-1}x) = (\lambda(gh)f)(x).$$

It is called the *left regular representation*. By a similar argument, we see that ρ is also a unitary representation, called *right regular representation*.

Still with $I = G$ a group, the set $\mathbb{C}[G] = \mathbb{C}^{(G)}$ of maps $G \to \mathbb{C}$ with finite support is a sub-$\mathbb{C}$-vector space of $\mathbb{C}^G$, and is included in $l^p(G)$, for all p. The inclusion is dense if $p < \infty$. The elements $\delta_g \in \mathbb{C}[G]$ $(g \in G)$, defined by

$$\delta_g(x) = \begin{cases} 1 & \text{if } x = g, \\ 0 & \text{otherwise,} \end{cases}$$

form a basis of $\mathbb{C}[G]$, and are called the *Dirac*s (δ_g is the Dirac *at* g). The subspace $\mathbb{C}[G]$ is a group under the operation of involution $*$:

$$\begin{aligned} f, h &\mapsto f{*}h \\ f{*}h\ (x) &:= \sum_y f(y)h(y^{-1}x). \end{aligned}$$

Noting that $\mathbb{C}[G]$ is closed under the action of any left regular representation, we remark that the latter agrees with left convolution with a Dirac:

$$\lambda(g)(f) = \delta_g * f, \quad \forall g \in G, f \in \mathbb{C}[G].$$

Finally, with a slight abuse of notation, we will often write, for elements f of $l^p(G)$, but not necessarily of finite support, $\delta_g * f$ instead of $\lambda(g)(f)$.

A.2 Baire, Category & Consequences

Proposition A.1 (Open Mapping Theorem)**.** *Let $T : X \to Y$ be a continuous operator of Banach spaces. If T is bijective, then its inverse is also continuous.*

Proof. [Ped89, Corollary 2.2.5, page 53]. ☐

The structure and formulation of the two following results and their proofs are borrowed from [Tao09].

Proposition A.2 (Closed Graph Theorem)**.** *Let X, Y be Banach spaces and $T : X \to Y$ linear. Then, the following are equivalent:*

1. *T is continuous.*
2. *The graph $\Gamma(T) \subseteq X \times Y$ of T is closed in the product topology.*
3. *For any topology τ on Y, non-strictly weaker than the norm on Y, and Hausdorff, T is continuous with respect to τ.*
4. *There exists a topology τ on Y, non-strictly weaker than the norm on Y, and Hausdorff, such that T is continuous with respect to τ.*

Proof. 1. implies 3. because if T is continuous with respect to the norm topology on Y, it is still continuous for any weaker topology. 3. implies 4. by simply taking τ the norm topology.

Observe that the product topology on $X \times Y$ agrees with the topology induced by the norm $\|(x, y)\| := \max(\|x\|, \|y\|)$, which is complete.

4.⇒2. Recall that in a metric space, a set being closed is equivalent to it containing its limit points. So, let $\langle (x_n, T(x_n)) \rangle_n \in \Gamma(T)$ be a sequence of points of $\Gamma(T)$ with limit $(x, y) \in X \times Y$. Showing that $(x, y) \in \Gamma(T)$ is equivalent to showing that $y = T(x)$. We know that $x_n \to x$ and $Tx_n \to y$, because $X \times Y$ has the product topology. As T is continuous with respect to τ, $Tx_n \to Tx$ in τ, and as τ is weaker than the norm in Y, $Tx_n \to y$ in τ. As τ is Hausdorff, any sequence has at most one limit, and $y = Tx$.

2.⇒1. Consider the two norm-decreasing (hence continuous) projections $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$, and their restrictions π_1', π_2' to $\Gamma(T)$, and note that $\Gamma(T)$ is a vector subspace of $X \times Y$. The set $\Gamma(T)$ being closed in a complete space, it is itself complete. Since π_1' is a continuous bijection, it has a continuous inverse $\iota : X \to \Gamma(T)$, by the Open Mapping Theorem. But then, $T = \pi_2' \circ \iota$ is continuous.

☐

Proposition A.3. *Let X, Y be Banach spaces, Z a Hausdorff topological vector space with $Y \subseteq Z$ continuously and linearly, and $T : X \to Z$ continuous linear. Then, the following are equivalent:*

1. *$T[X] \subseteq Y$.*
2. *$T[X] \subseteq Y$, and there exists C such that for any $x \in X$: $\|Tx\|_Y \leq C\|x\|_X$.*
3. *For any D dense in X, $T[D] \subseteq Y$ and there exists C such that for any $x \in D$: $\|Tx\|_Y \leq C\|x\|_X$.*
4. *There exists D dense in X, $T[D] \subseteq Y$ and there exists C such that for any $x \in D$: $\|Tx\|_Y \leq C\|x\|_X$.*

Proof. That 2. implies 1., 3., and 4. is direct (for 4., simply take $D = X$). 3. also implies 2., and 4. by taking $D = X$ in 4..

1.⇒2. The map $T : X \to Z$ is continuous, and has range in Y by 1.; let $S : X \to Y$ be the restriction of T to the codomain Y. The map S is then continuous with respect to the topology τ on Y inherited from Z, as a subspace. By hypothesis, τ is Hausdorff and weaker than the norm topology of Y (i.e. $Y \subseteq Z$ continuously). By the Closed Graph Theorem, this implies that S is continuous with respect to the norm, hence bounded and there exist C with:

$$\forall x \in X \ : \ \|Sx\|_Y \leq C\|x\|_X$$

and as $Sx = Tx$ for all x, 2. follows.

4.⇒2. Write $\iota : Y \to Z$ for the (continuous) inclusion of Y in Z. Write $S : D \to Y$ for the restriction of T to the domain D and codomain Y (using the fact that $T[D] \subseteq Y$). We know S to be continuous of norm $\leq C$ by 4., when D has the induced subspace topology. As Y is complete and D dense in X, S uniquely extends to some $\tilde{S} : X \to Y$ still satisfying $\|\tilde{S}\| \leq C$. Then, both $\iota \circ \tilde{S} : X \to Z$ and $T : X \to Z$ are continuous, and being equal on D, a dense subspace of X, they must be equal. Then, $T[X] = \iota \circ \tilde{S}[X] \subseteq Y$, and

$$\forall x \in X \ : \ \|Tx\|_Y = \|\iota \circ \tilde{S}x\|_Y = \|\tilde{S}x\|_Y \leq C\|x\|_X.$$

$\square$

Proposition A.4. *Fix some set I and endow $\mathbb{C}^I$ with the product topology. Let V, W be Banach spaces, continuously and linearly included in $\mathbb{C}^I$. Then, the following are equivalent:*

1. *$V \subseteq W$.*
2. *$V \subseteq W$ and there exists some C such that for all $v \in V$, $\|v\|_W \leq C\|v\|_V$.*
3. *There exists a dense subspace D of V and some C such that for all $v \in D$, $\|v\|_W \leq C\|v\|_V$.*
4. *For all dense subspaces D of V, there exists some C such that for all $v \in D$, $\|v\|_W \leq C\|v\|_V$.*

Proof. Follows from Proposition A.3. $\square$

Note that in the direction $3 \Rightarrow 2$, the constant C can be kept.

A.3 Matrices

Given a bounded (linear) operator

$$T : l^p(I) \to l^q(J),$$

with $1 \leq p < \infty$ and $1 \leq q \leq \infty$, we can write, for any $f \in l^p(I)$,

$$Tf = T\left(\sum_i f_i \delta_i\right) = \sum_i f_i T\delta_i,$$

because any element of $l^p(I)$ can be written as an infinite sum of Diracs (this doesn't hold in l^∞), and by continuity of T. By continuity of the "evaluation at j", we therefore have

$$(Tf)_j = \sum_i f_i (T\delta_i)_j.$$

This tells us that T is entirely defined by the family $\langle (T\delta_i)_j \rangle_{i,j}$, and in particular, whenever two operators agree on this family, they must be equal. A more abstract explanation of this fact: For $1 \leq p < \infty$, the span of $\{\delta_i \mid i \in I\}$ is dense in $l^p(I)$ and a continuous map is entirely defined by its value on any dense subspace of its domain. By linearity, its value on the span is entirely defined by its value on the generators. Hence, any continuous linear operator T is entirely defined by its value on the elements of $\{\delta_i \mid i \in I\}$; if the codomain is $l^q(J)$, $T\delta_i$ is determined by its value when evaluated at any j, that is, $(T\delta_i)_j$.

Conversely, given a matrix $\langle a_{i,j} \rangle_{i,j}$ ($i \in I, j \in J$), we can define a linear operator:

$$\begin{aligned} A : \mathbb{C}^{(I)} &\to \mathbb{C}^J \\ \langle f_i \rangle_i &\mapsto \left\langle \sum_i a_{i,j} f_i \right\rangle_j . \end{aligned}$$

We then have:

Proposition A.5. *If $C := \sup_j \sum_i |a_{i,j}| < \infty$, then A defines an operator $l^\infty(I) \to l^\infty(J)$ of norm at most C.*

Proof. Assume $\langle f_i \rangle_i$ is an element of $l^\infty(I)$. Then, for any j,

$$|(Af)_j| = |\sum_i a_{i,j} f_i| \leq \sum_i |a_{i,j}||f_i| \leq \|f\|_\infty \sum_i |a_{i,j}| \leq \|f\|_\infty C,$$

so that $(Af)_j$ is well-defined, and it remains to verify that $Af \in l^\infty(J)$, but

$$\sup_j |(Af)_j| \leq \|f\|_\infty C,$$

which shows that A is well-defined as a map $l^\infty(I) \to l^\infty(J)$. The above inequality also shows boundedness ($\|A\| \leq C$) and linearity remains:

$$(A(f+g))_j = \sum_i a_{i,j}(f_i + g_i) = \sum_i a_{i,j} f_i + \sum_i a_{i,j} g_i = (Af)_j + (Ag)_j,$$

and

$$(A(cf))_j = \sum_i a_{i,j} c f_i = c \sum_i a_{i,j} f_i = c(Af)_j$$

by continuity of addition and multiplication by a scalar. $\square$

A "symmetrical" statement holds:

Proposition A.6. *If $C := \sup_i \sum_j |a_{i,j}| < \infty$, then A defines an operator $l^1(I) \to l^1(J)$, of norm at most C.*

Proof. First, if $f \in l^1(I)$, note that $|a_{i,j}| \leq C$ for all i, j, and

$$|(Af)_j| = |\sum_i a_{i,j} f_i| \leq \sum_i |a_{i,j}||f_i| \leq \sum_i C|f_i| = C\|f\|_1$$

which shows that $(Af)_j$ is well-defined, and

$$\|Af\|_1 = \sum_j |(Af)_j| = \sum_j |\sum_i a_{i,j} f_i| \leq \sum_j \sum_i |a_{i,j}||f_i| = \sum_i |f_i| \sum_j |a_{i,j}| \leq \sum_i |f_i| C = C\|f\|_1,$$

so that A is well-defined as a map $l^1(I) \to l^1(J)$. The above inequality also shows boundedness ($\|A\| \leq C$) and linearity holds by the same argument as in Proposition A.5. □

Converse facts to those above also hold:

If $A : l^1(I) \to l^1(J)$ is a continuous operator of norm C, defined by the matrix $a_{i,j}$ as

$$A : \langle f_i \rangle_i \mapsto \langle \sum_i a_{i,j} f_i \rangle_j,$$

then we have

$$\sup_j \sum_i |a_{i,j}| \leq C.$$

Indeed, we know

$$\sup_{\|f\|_1 = 1} \sum_j |\sum_i a_{i,j} f_i| = C$$

by definition of the operator norm and of A. But considering $f := \delta_k$ for any $k \in I$:

$$\sup_k \sum_j |a_{k,j}| \leq C.$$

Similarly, assume $A : l^\infty(I) \to l^\infty(J)$ to be a continuous linear operator of norm C defined by the matrix $a_{i,j}$; then, by definition:

$$\sup_{\|f\|_\infty = 1} \sup_j |\sum_i a_{i,j} f_i| \leq C.$$

Now, for any j, define $\langle g_i^j \rangle_i \in l^\infty(I)$ by $g_i^j := sgn(a_{i,j})$. Then

$$\sup_j \sum_i |a_{i,j}| = \sup_j \sum_i a_{i,j} g_i^j = \sup_j |\sum_i a_{i,j} g_i^j| \leq \sup_{\|f\|_\infty = 1} \sup_j |\sum_i a_{i,j} f_i| \leq C.$$

Proposition A.7 ([Pis01, Proposition 2.15])**.** *Let $a : I \times I \to \mathbb{C}$ be an "infinite matrix", satisfying*

$$S := \sup_i \sum_j |a_{i,j}| \leq 1 \tag{40}$$

$$T := \sup_j \sum_i |a_{i,j}| \leq 1. \tag{41}$$

Then, to the matrix $\langle a_{i,j} \rangle_{i,j}$ corresponds an operator $A : l^2(I) \to l^2(I)$ of norm $\|A\|_{B(l^2(I))}$ at most 1, defined as

$$A : \langle f_j \rangle_j \mapsto \langle \sum_j a_{i,j} f_j \rangle_i.$$

In light of the above facts, this proposition states that if a matrix defines a bounded operator both in $B(l^1(I))$ and $B(l^\infty(I))$, then this operator is actually bounded from $l^2(I)$ to $l^2(I)$.

Note that if $S \leq C$ and $T \leq C$ instead of 1, we can let $a'_{i,j} := a_{i,j}/C$; then, a' satisfies Equation (40) and Equation (41) so that it defines an operator $A' \in B(l^2(I))$ of norm at most 1. Letting $A = CA'$, we have $\|A\|_{B(l^2(I))} = C\|A'\|_{B(l^2(I))} \leq C$, and

$$A : \langle f_j \rangle_j \mapsto C\langle \sum_j a'_{i,j} f_j \rangle_i = \langle \sum_j C a'_{i,j} f_j \rangle_i = \langle \sum_j a_{i,j} f_j \rangle_i$$

so that A is the operator corresponding to the matrix $\langle a_{i,j} \rangle_{i,j}$ and has norm at most C. In short, the statement of Proposition A.7 stays true if we scale the bounds.

Proof. Define the map:

$$\varphi : l^2(I) \times l^2(I) \longrightarrow \mathbb{C}$$
$$f, g \longmapsto \langle Af, g\rangle = \sum_i (\sum_j a_{i,j} f_j)\overline{g_i}$$

(where the notation $\langle Af, g\rangle$ doesn't make sense yet: we don't know that $Af \in l^2(I)$, so that the sum defining $\langle\cdot,\cdot\rangle$ may not converge). We will show that φ is sesquilinear and bounded, and Proposition B.2 will imply the existence of some $B \in B(l^2(I))$ with $\varphi = \langle B\cdot,\cdot\rangle$.

First, note that if $f \in l^\infty(I)$ (hence also if $f \in l^p(I)$, for $p \geq 1$), then Af is well-defined:

$$|(Af)_i| = |\sum_j a_{i,j} f_j| \leq \sum_j |a_{i,j}||f_j| \leq \|f\|_\infty S.$$

This holds because the condition in Equation (40) tells us that A defines a bounded operator $l^\infty(I) \to l^\infty(I)$.

φ bounded (by 1). The argument will mainly consist of applying Cauchy-Schwarz. We want to show that:

$$\sup_{\|f\|_2 = \|g\|_2 = 1} |\varphi(f, g)| \leq 1,$$

so fix $f, g \in l^2(I)$ of norm 1. Some remarks:

- For any j, both $\langle |a_{i,j}|^{\frac{1}{2}}\rangle_i$ and $\langle |a_{i,j}|^{\frac{1}{2}}|g_i|\rangle_i$ are in $l^2(I)$:

  $$\sum_i |a_{i,j}|^{\frac{1}{2}\,2} = \sum_i |a_{i,j}| \leq T \leq 1,$$

  and:

  $$\sum_i |a_{i,j}|^{\frac{1}{2}\,2}|g_i|^2 \leq \sum_i |a_{i,j}|\|g\|_\infty^2 \leq \|g\|_\infty^2 T \leq \|g\|_\infty^2.$$

- Both $\langle |f_j|\rangle_j$ and $\langle \sqrt{\sum_i |a_{i,j}||g_i|^2}\rangle_j$ are in $l^2(I)$: The former is obvious, and for the latter:

  $$\begin{aligned}\sum_j \left(\sqrt{\sum_i |a_{i,j}||g_i|^2}\right)^2 &= \sum_j \sum_i |a_{i,j}||g_i|^2 \\ &= \sum_i |g_i|^2 \sum_j |a_{i,j}| \\ &\leq \sum_i |g_i|^2 S \\ &= \|g\|_2^2 S \leq \|g\|_2^2.\end{aligned}$$

Finally, recall that if $x, y \in l^2(I)$, then $|\sum_i x_i \overline{y_i}| = |\langle x, y\rangle| \leq \|x\|_2\|y\|_2$ by Cauchy-Schwarz.

Now,

$$|\langle Af, g\rangle| = \left|\sum_i (Af)_i \overline{g_i}\right| = \left|\sum_i \left(\sum_j a_{i,j} f_j\right)\overline{g_i}\right| g_i \leq \sum_i \sum_j |a_{i,j}||f_j||g_i|$$

which can be written

$$... = \sum_j |f_j| \left(\sum_i |a_{i,j}|^{\frac{1}{2}}(|a_{i,j}|^{\frac{1}{2}}|g_i|)\right)$$

and applying Cauchy-Schwarz to $\langle |a_{i,j}|^{\frac{1}{2}}\rangle_i$ and $\langle |g_i|\rangle_i$ (at all j)

$$\begin{aligned}... &\leq \sum_j |f_j| \left(\sum_i |a_{i,j}|\right)^{\frac{1}{2}} \left(\sum_i |a_{i,j}||g_i|^2\right)^{\frac{1}{2}} \\ &= \sum_j |f_j| T^{\frac{1}{2}} \left(\sum_i |a_{i,j}||g_i|^2\right)^{\frac{1}{2}} \\ &\leq \sum_j |f_j| \left(\sum_i |a_{i,j}||g_i|^2\right)^{\frac{1}{2}}\end{aligned}$$

and Cauchy-Schwarz again, but now to $\langle |f_j| \rangle_j$ and $\langle \sqrt{\sum_i |a_{i,j}||g_i|^2} \rangle_j$

$$
\begin{aligned}
... \leq \|f\|_2 \left(\sum_j \sum_i |a_{i,j}||g_i|^2 \right)^{\frac{1}{2}} &= \|f\|_2 \left(\sum_i |g_i|^2 \sum_j |a_{i,j}| \right)^{\frac{1}{2}} \\
\leq \|f\|_2 \left(\sum_i |g_i|^2 S \right)^{\frac{1}{2}} &= S^{\frac{1}{2}} \|f\|_2 \|g\|_2 \leq \|f\|_2 \|g\|_2 \leq 1
\end{aligned}
$$

Note that in addition to proving that φ is bounded, this shows it to be well-defined. Indeed, for any f, g in $l^2(I)$, the sum defining $\varphi(f, g)$ is absolutely bounded by $\|f\|_2 \|g\|_2$ and converges.

φ **sesquilinear.**

$$
\begin{aligned}
\varphi(\alpha f + f', g) &= \sum_i (A(\alpha f + f'))_i \overline{g_i} = \sum_i (\alpha A f + A f')_i \overline{g_i} \\
&= \sum_i \alpha (Af)_i \overline{g_i} + (Af')_i \overline{g_i} = \alpha \sum_i (Af)_i \overline{g_i} + \sum_i (Af')_i \overline{g_i} \\
&= \alpha \varphi(f, g) + \varphi(f', g)
\end{aligned}
$$

where we used the fact that A itself is linear, continuity of addition, and the fact that the sums are convergent (because φ is well-defined). (Concerning linearity of A, recall that A can be seen as an operator on $B(l^\infty(G))$; then, for any $f, g \in l^2(G)$, and $\alpha \in \mathbb{C}$, $f, g \in l^\infty(G)$, and $A(\alpha f + g) = \alpha Af + Ag$.) Similarly:

$$
\begin{aligned}
\varphi(f, \beta g + g') &= \sum_i (Af)_i \overline{\beta g_i + g'_i} = \sum_i (Af)_i (\overline{\beta} \overline{g_i} + \overline{g'_i}) \\
&= \sum_i \overline{\beta} (Af)_i \overline{g_i} + (Af)_i \overline{g_i} = \overline{\beta} \sum_i (Af)_i \overline{g_i} + \sum_i (Af)_i \overline{g'_i} \\
&= \overline{\beta} \varphi(f, g) + \varphi(f, g')
\end{aligned}
$$

so that φ is sesquilinear.

A **in** $B(l^2)$**.** Since φ is sesquilinear bounded, Proposition B.2 implies the existence of some $B \in B(l^2(I))$ with

$$
\varphi(f, g) = \langle Bf, g \rangle \quad \forall f, g \in l^2(I),
$$

and of norm $\|B\| = \|\varphi\| \leq 1$. But then, let $\langle b_{i,j} \rangle_{i,j}$ be the matrix of B, we have

$$
a_{i,j} = \langle A\delta_i, \delta_j \rangle = \varphi(\delta_i, \delta_j) = \langle B\delta_i, \delta_j \rangle = b_{i,j}
$$

and B has the same matrix as A, hence $A = B$ is in $B(l^2(I))$, of norm at most 1. □

A.4 Uniform Convexity

Definition A.8. A normed vector space X is said to be *uniformly convex* if, for all $\varepsilon > 0$, there exists $\delta > 0$ such that: for all x, y in X with $\|x\|, \|y\| \leq 1$:

$$
\|x + y\| > 2 - \delta \quad \Rightarrow \quad \|x - y\| < \varepsilon. \tag{42}
$$

Proposition A.9. *Hilbert spaces are uniformly convex.*

Proof. Let H a Hilbert space. Recall the parallelogram identity, stating that for all $x, y \in H$

$$
\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2,
$$

so that whenever $\|x\|, \|y\| \leq 1$ and $\|x - y\| \geq \varepsilon$ (and $\varepsilon \leq 2$):

$$
\begin{aligned}
\|x + y\| &= \sqrt{2(\|x\|^2 + \|y\|^2) - \|x - y\|^2} \\
&\leq \sqrt{2(1 + 1) - \|x - y\|^2} \\
&\leq \sqrt{4 - \varepsilon^2} = 2 - (2 - \sqrt{4 - \varepsilon^2}).
\end{aligned}
$$

Hence, assuming $\varepsilon < 2$, and showing the statement by the contrapositive of Equation (42), choose $\delta = 2 - \sqrt{4 - \varepsilon^2}$: the above inequality shows that whenever $\|x\|, \|y\| \leq 1$ and $\|x - y\| \geq \varepsilon$, then $\|x + y\| \leq 2 - \delta$.

If $\varepsilon \geq 2$, and $\|x - y\| \geq \varepsilon$, then $2 \leq \|x - y\| \leq \|x\| + \|-y\| \leq 1 + 1 = 2$, so that $\|x - y\| = 2$, and $\|x\| = \|y\| = 1$, which implies $x = -y$ (via the parallelogram identity), and therefore, whenever $\varepsilon \geq 2$ choosing for instance $\delta = 1$ works. □

Proposition A.10. *Let X be a uniformly convex space. For all $n \geq 2$ and $\varepsilon > 0$, there exists $\delta > 0$ such that for any n elements $x_1, ..., x_n$ of X, with $\|x_i\| \leq 1$ for all i:*

$$\|\sum_i x_i\| > n - \delta \quad \Rightarrow \quad \forall i, j : \|x_i - x_j\| < \varepsilon.$$

This follows from uniform convexity by induction:

Proof. The statement holds for $n = 2$, by assumption that X is uniformly convex.

Fix $n \geq 2$ and assume the statement holds for n; we show it does for $n+1$. Fix $\varepsilon > 0$ and let δ be as given by the induction hypothesis for n and ε. Fix any $x_0, ...x_n$, all of norm ≤ 1 and assume that

$$\|\sum_{i=0}^{n} x_i\| > (n+1) - \delta.$$

Fix any i, j, we show that $\|x_i - x_j\| < \varepsilon$. Choose some $k \neq i, j$ (which is possible since $n + 1 \geq 3$). The inequality above yields in particular (triangle inequality)

$$\|\sum_{i \neq k} x_i\| + \|x_k\| \geq \|\sum_{i=0}^{n} x_i\| > (n+1) - \delta$$

so that

$$\|\sum_{i \neq k} x_i\| > (n+1) - \|x_k\| - \delta \geq n - \delta$$

since $\|x_k\| \leq 1$. In particular, the subset $\{x_0, ..., x_n\} - \{x_k\}$ satisfies the hypothesis of the proposition for n, and by the choice of δ (as given by the induction hypothesis, for n and ε), for any $i', j' \neq k$, we have

$$\|x_{i'} - x_{j'}\| < \varepsilon$$

and obviously, in particular:

$$\|x_i - x_j\| < \varepsilon,$$

and we are done. $\square$

Proposition A.11. *Let X be a uniformly convex space and $n \geq 2$. If $\langle x_1^j, ..., x_n^j \rangle_j$ is a sequence of n-tuples of elements of X, all of norm ≤ 1, then:*

$$\|\sum_{i=1}^{n} x_i^j\| \to n$$

implies that for all $1 \leq i, i' \leq n$,

$$\|x_i^j - x_{i'}^j\| \to 0.$$

Proof. Fix $\varepsilon > 0$, and let $\delta > 0$ be as given by Proposition A.10. As

$$\|\sum_{i=1}^{n} x_i^j\| \to n,$$

there exists some N such that for all $j \geq N$

$$|\|\sum_{i=1}^{n} x_i^j\| - n| < \delta$$

and in particular

$$\|\sum_{i=1}^{n} x_i^j\| > n - \delta.$$

By Proposition A.10, this implies

$$\forall 1 \leq i, i' \leq n : \quad \|x_i^j - x_{i'}^j\| < \varepsilon.$$

In particular, this shows that for any fixed i, i', and $\varepsilon > 0$, taking δ and N as above, for all $j \geq N$,

$$\|x_i^j - x_{i'}^j\| < \varepsilon,$$

which shows the convergence of $\|x_i^j - x_{i'}^j\|$ to 0. $\square$

B Hilbert

B.1 Basics

Proposition B.1. *On a complex Hilbert space H, the inner product can be recovered from the norm by the formula*

$$\langle x, y \rangle = \frac{1}{4} \sum_{k=0}^{3} i^k \|x + i^k y\|^2,$$

where i stands for the imaginary unit.

Proof. [Ped89, Paragraph 3.1.2]. □

Proposition B.2. *Let H a Hilbert space and $\varphi : H \times H \to \mathbb{C}$ sesquilinear, and bounded in the following sense:*

$$M := \sup_{\|x\|=1=\|y\|} |\varphi(x, y)| < \infty.$$

Then, there exists a unique $S \in B(H, H)$ such that

$$\varphi(x, y) = \langle x, Sy \rangle \quad \forall x, y,$$

and $\|S\| = M$.

Proof. [Ped89, Lemma 3.2.2]. □

Proposition B.3. *Let $T \in B(H, H')$ be an operator of Hilbert spaces. Then*

$$\|T\| = \sup_{\|x\|=1=\|y\|} |\langle Tx, y \rangle|.$$

Proof. [Ped89, Lemma 3.2.2]. □

Proposition B.4. *Let $T \in B(H, H)$ be a self-adjoint operator on a Hilbert space H. Then*

$$\|T\| = \sup_{\|x\|=1} |\langle Tx, x \rangle|.$$

Proof. [Ped89, Proposition 3.2.25]. □

B.2 Bases

Recall that in a Hilbert space H, an orthonormal set E is a subset of H such that, for all $x, y \in E$:

$$\langle x, y \rangle = \begin{cases} 1 & \text{if } x = y, \\ 0 & \text{if } x \neq y. \end{cases}$$

In other words, the elements of E are pairwise orthogonal and of norm 1. An orthonormal basis is an orthonormal set E whose span is dense in H.

Recall that for any set E, we have

$$H = \overline{\operatorname{span} E} \oplus E^\perp, \quad \text{i.e.} \quad \overline{\operatorname{span} E} \cap E^\perp = \{0\} \ \wedge \ \overline{\operatorname{span} E} + E^\perp = H \tag{43}$$

with $E^\perp := \{x \in H \mid \langle x, e \rangle = 0 \ \forall e \in E\}$.

Let us say that a subset E of H is topologically independent if no $x \in E$ lies in the closed span of $E \setminus \{x\}$, that is:

$$\forall x \in E \ : \ x \notin \overline{\operatorname{span}(E \setminus \{x\})}.$$

Then, any orthonormal set E is topologically independent. Indeed, fixing some $x \in E$, and letting $E' := E \setminus \{x\}$, we have $x \in E'^\perp$, and since $x \neq 0$, we conclude by Equation (43) that $x \notin \overline{\operatorname{span} E'}$.

Proposition B.5. *Any Hilbert space H has an orthonormal basis.*

Proof. [Ped89, Proposition 3.1.12]. □

Proposition B.6. *If H_1, H_2 are two Hilbert spaces with respective orthonormal bases B_1, B_2 of same cardinality, there exists a unitary transformation $Q : H_1 \to H_2$.*

Proof. [Ped89, Proposition 3.1.14]. $\square$

Proposition B.7. *Let H be a Hilbert space, with inner product ρ, and let ρ' be another inner product on H, topologically compatible with ρ (the induced norms are compatible). Then, if B is an orthonormal basis for (H, ρ), there exists an orthonormal basis for (H, ρ') of cardinality $|B|$.*

Proof. We know that if C is a set of topologically independent vectors of a Hilbert space H with dense span in H, then there exists an orthonormal basis for H of cardinality $|C|$ (Proposition B.9). Fix B an orthonormal basis for (H, ρ); it is therefore enough to show that the span of B is dense in (H, ρ') and that no $b \in B$ lies in the ρ'-closure of $\operatorname{span} B \setminus \{b\}$. But as ρ and ρ' are topologically equivalent, $\overline{\operatorname{span} B}^{\rho'} = \overline{\operatorname{span} B}^{\rho} = H$ and $b \notin \overline{\operatorname{span} B \setminus \{b\}}^{\rho} = \overline{\operatorname{span} B \setminus \{b\}}^{\rho'}$, and we are done. $\square$

Let us write $[E]$ for $\overline{\operatorname{span} E}$.

Lemma B.8. *Let E be an orthonormal set and $x \notin [E]$. Up to a scalar of norm 1, there exists a unique y such that $[E \cup \{x\}] = [E \cup \{y\}]$ and $E \cup \{y\}$ is orthonormal.*

Proof. Write $x = x_{[E]} + x_{E^\perp} \in [E] \oplus E^\perp$ the decomposition of x, with $x_{[E]} \in [E]$ and $x_{E^\perp} \in E^\perp$. By assumption on x, we have $x_{E^\perp} \neq 0$. Define $y = x_{E^\perp}/\|x_{E^\perp}\|$; we show that y satisfies the requirements.

First, $\|y\| = 1$ by construction, and if $e \in E$, then $\langle e, y\rangle = \frac{1}{\|x_{E^\perp}\|}\langle e, x_{E^\perp}\rangle = 0$, because $x_{E^\perp} \in E^\perp$. This makes $E \cup \{y\}$ orthonormal.

Now, let z be any element of H, and U any subset of H; we have that

$$[U \cup \{z\}] = \overline{\operatorname{span}(U \cup \{z\})} = \overline{\operatorname{span}(U) + \langle z\rangle} = \overline{\overline{\operatorname{span}(U)} + \langle z\rangle} = \overline{[U] + \langle z\rangle}. \tag{44}$$

The second equality is purely algebraic. The "$\subseteq$" part of the third is obvious. Now, if $w \in \overline{\operatorname{span}(U)} + \langle z\rangle$, we can write $w = (\lim_n v_n) + \lambda z = \lim_n (v_n + \lambda z)$, with $v_n \in \operatorname{span}(U)$ for all n; but $\lim_n (v_n + \lambda z) \in \overline{\operatorname{span}(U) + \langle z\rangle}$, so that $w \in \overline{\operatorname{span}(U) + \langle z\rangle}$. This shows

$$\overline{\operatorname{span}(U)} + \langle z\rangle \subseteq \overline{\operatorname{span}(U) + \langle z\rangle},$$

and taking the closure of the LHS gets us the desired full equality.

We know that $[E] + \langle x\rangle = [E] + \langle y\rangle$, because $y = x_{E^\perp}/\|x_{E^\perp}\| = u + \alpha x$ for some $u \in [E]$ and $\alpha \in \mathbb{C}^*$. Using Equation (44), we can now conclude

$$[E \cup \{x\}] = [E \cup \{y\}].$$

Note also that $y \notin E$, because if it was, we would get $y \in [E] \cap E^\perp = \{0\}$, which contradicts the hypothesis of x not being in $[E]$; this is actually already implied by the proof that $E \cup \{y\}$ is orthonormal.

Concerning unicity: if y' satisfies the requirements, we have in particular $y' \in E^\perp$ (because $E \cup \{y'\}$ is orthonormal) and as $x \in [E \cup \{y'\}] = \overline{\operatorname{span}(E \cup \{y'\})}$, we can write

$$x_{[E]} + x_{E^\perp} = x = \lim_n \left(\sum_i \lambda_i u_i + \lambda'_n y'\right),$$

with each sum a finite linear combination of elements of E and y. Taking the projection $\pi_{E^\perp}$ on $E^\perp$ (which we know to be continuous) yields

$$x_{E^\perp} = \pi_{E^\perp}\left(\lim_n \sum_i \lambda_i u_i + \lambda'_i y'\right) = \lim_n \pi_{E^\perp}\left(\sum_{i=1}^{l_n} \lambda_{i,n} u_{i,n} + \lambda'_n y'\right) = \lim_n \lambda'_n y' = (\lim_n \lambda'_n) y'$$

(where the last equality holds because $y' \neq 0$). Recalling that $y := x_{E^\perp}/\|x_{E^\perp}\|$, and that y' has norm 1, we conclude that y and y' are equal up to a scalar of norm 1.

$\square$

Applying the above proposition, we get

Proposition B.9 (Gram-Schmidt)**.** *If B is a set of topologically independent elements of H, there exists an orthonormal set C such that $[C] = [B]$ and B and C have the same cardinality.*

Proof. First, using the axiom of choice, endow B with some well-ordering $B = \{b_\alpha : \alpha < \kappa\}$ with κ the cardinality of B. Let B_λ denote the initial segment $\{b_\alpha : \alpha < \lambda\}$ of B of length λ (so that $B_\kappa = B$).

By transfinite induction on $\lambda \leq \kappa$, we construct C_λ satisfying:

1. $C_\beta \subseteq C_\lambda$ for all $\beta \leq \lambda$;
2. $|C_\lambda| = |B_\lambda|$; more precisely, there is a bijection $f_\lambda : C_\lambda \to B_\lambda$ extending f_β for any $\beta < \lambda$;
3. C_λ is orthonormal;
4. $[C_\lambda] = [B_\lambda]$.

Fix λ and assume C_β to be defined for all $\beta < \lambda$, and satisfying the above conditions.

- If $\lambda = 0$, define $C_\lambda = \emptyset$.
- If λ is successor, say of λ', consider the element $b := b_{\lambda'} \notin B_{\lambda'}$ of B. We know by assumption on B that $b \notin [B_{\lambda'}]$, since $b \notin B_{\lambda'}$ and B is topologically independent. We can therefore apply the previous proposition and get some c such that $c \notin [B_{\lambda'}] = [C_{\lambda'}]$, $\{c\} \cup C_{\lambda'}$ is orthonormal, and with

  $$[\{c\} \cup C_{\lambda'}] = [\{b\} \cup C_{\lambda'}].$$

  But
  $$[\{b\} \cup C_{\lambda'}] = \overline{[C_{\lambda'}] + \langle b\rangle} = \overline{[B_{\lambda'}] + \langle b\rangle} = [\{b\} \cup B_{\lambda'}] = [B_\lambda]$$
  by Equation (44), and we conclude that $[C_{\lambda'} \cup \{c\}] = [B_\lambda]$. Finally, set $f_\lambda : C_{\lambda'} \cup \{c\} \to B_\lambda$ by extending $f_{\lambda'}$ with $c \mapsto b$. Setting $C_\lambda := C_{\lambda'} \cup \{c\}$, we are done.
- If λ is limit, let
  $$C_\lambda = \bigcup_{\beta<\lambda} C_\beta.$$
  Property 1. is obvious. Setting $f_\lambda = \bigcup_{\beta<\lambda} f_\beta$ shows property 2. If $c, c' \in C_\lambda$, there must exist β with $c, c' \in C_\beta$ (by nestedness), so that $\langle c, c'\rangle = 0$, and $\|c\| = 1$, by orthonormality of C_β; this shows property 3. Finally, if
  $$s = \sum_{i=1}^{n} \lambda_i c_i \in \operatorname{span} C_\lambda,$$
  there exists some C_β containing each c_i (by nestedness), so that $s \in \operatorname{span} C_\beta \subseteq [C_\beta] = [B_\beta] \subseteq [B_\lambda]$. This shows $\operatorname{span} C_\lambda \subseteq [B_\lambda]$, and taking the closure, $[C_\lambda] \subseteq [B_\lambda]$. The reverse inclusion is done similarly; this shows property 4, and we are done.

With $\lambda = \kappa$, the statement is proven (setting $C := C_\kappa$). □

B.3 Operators

Fix a Hilbert space X. Recall that a (bounded) operator T on X is said to be positive if it is self-adjoint and satisfies:

$$\langle Tx, x\rangle \in \mathbb{R}^{\geq 0}, \quad \forall x \in X.$$

Proposition B.10. *Let $T \in B(X)$ be an operator. Then, T is unitary if and only if T is invertible and $T^{-1} = T^*$.*

Proof. Assume T unitary, then
$$\langle Tx, y\rangle = \langle Tx, TT^{-1}y\rangle = \langle x, T^{-1}y\rangle \quad \forall x, y,$$
so that $T^* = T^{-1}$. Conversely, assume T invertible of inverse T^*, then:

$$\langle Tx, Ty\rangle = \langle x, T^*Ty\rangle = \langle x, T^{-1}Ty\rangle = \langle x, y\rangle,$$

and T is a bijective isometry, hence unitary. □

Proposition B.11. *If $S, T \in B(X)$ are two commuting positive operators, then ST is positive.*

Proof. [Ped89, Paragraph 3.2.8]. □

Proposition B.12. *Any unitary operator $T \in B(X)$ can be decomposed into $T = UP$ ($U, P \in B(X)$) with U unitary and P positive.*

Proof. [Ped89, Propositions 3.2.17 and 3.2.19]. □

Proposition B.13. *For any positive operator $T \in B(X)$, there exists a unique positive operator $T^{\frac{1}{2}} \in B(X)$, called the square-root of T, satisfying $T^{\frac{1}{2}}T^{\frac{1}{2}} = T$. Furthermore, $T^{\frac{1}{2}}$ commutes with any operator commuting with T.*

Proof. [Ped89, Proposition 3.2.11]. □

Proposition B.14. *For any positive operator $T \in B(X)$, T is invertible if and only if there exists some $\delta > 0$ with $T \geq \delta I$. In that case, the inverse of T is positive, $T^{\frac{1}{2}}$ is also invertible, and the inverse of $T^{\frac{1}{2}}$ is the square root of T^{-1}. If S is another positive invertible operator with $S \leq T$, then $T^{-1} \leq S^{-1}$.*

Proof. [Ped89, Proposition 3.2.12]. □

Proposition B.15. *If $T \in B(X)$ is a positive operator satisfying $\delta I \leq T \leq \varepsilon I$, then*

$$\delta^{\frac{1}{2}} I \leq T^{\frac{1}{2}} \leq \varepsilon^{\frac{1}{2}} I.$$

Proof. We have, using commutativity of $T^{\frac{1}{2}}$ and $\delta^{\frac{1}{2}} I$:

$$(T^{\frac{1}{2}} - \delta^{\frac{1}{2}} I)(T^{\frac{1}{2}} + \delta^{\frac{1}{2}} I) = T - \delta I.$$

As $T^{\frac{1}{2}} + \delta^{\frac{1}{2}} I \geq \delta^{\frac{1}{2}} I$ is invertible, we can write:

$$(T^{\frac{1}{2}} - \delta^{\frac{1}{2}} I) = (T - \delta I)(T^{\frac{1}{2}} + \delta^{\frac{1}{2}} I)^{-1}$$

so that $T^{\frac{1}{2}} - \delta^{\frac{1}{2}} I$ is a product of commuting positive operators, hence is itself positive, and

$$T^{\frac{1}{2}} \geq \delta^{\frac{1}{2}} I.$$

A similar argument applied to

$$(\varepsilon^{\frac{1}{2}} I - T^{\frac{1}{2}})(\varepsilon^{\frac{1}{2}} I + T^{\frac{1}{2}}) = \varepsilon I - T$$

shows that

$$T^{\frac{1}{2}} \leq \varepsilon^{\frac{1}{2}} I.$$

□

Proposition B.16. *Let $R \in B^I(X)$ be an invertible operator such that $\|R\|\|R^{-1}\| = 1$. Then, there exists a scalar α such that αR is unitary.*

Proof. Fix any $x \in X$. Then:

$$\|Rx\| \leq \|R\|\|x\|, \quad \text{and} \quad \|x\| = \|R^{-1}Rx\| \leq \|R^{-1}\|\|Rx\|,$$

so that:

$$\frac{1}{\|R^{-1}\|}\|x\| \leq \|Rx\| \leq \|R\|\|x\|.$$

Since $\|R\|\|R^{-1}\| = 1$, we have

$$\|Rx\| = \|R\|\|x\|$$

for all x, and $R/\|R\|$ is unitary. □

Proposition B.17. *Let $S, T \in B(X)$ be positive operators, and A invertible. Then*

$$S \leq T \quad \Leftrightarrow \quad ASA^* \leq ATA^*.$$

Proof. If $S \leq T$,

$$\forall x \ : \quad \langle Tx - Sx, x \rangle \geq 0.$$

Then, given any x, we have:

$$\langle T(A^*x) - S(A^*x), A^*x \rangle \geq 0$$

which can be rewritten as

$$\langle AT(A^*x) - AS(A^*x), x \rangle \geq 0$$

and $ASA^* \leq ATA^*$. A symmetric argument holds for the converse. □

Proposition B.18. *For any positive operator $T \in B(X)$, we have:*

$$\|T\| = \inf\{\gamma > 0 \ : \ T \leq \gamma I\}. \tag{45}$$

Proof. For any pair of positive operators S, T, if $S \leq T$, then $\|S\| \leq \|T\|$. This implies that for any γ in the infimum in Equation (45), $\|T\| \leq \|\gamma I\| = \gamma$, so that

$$\|T\| \leq \inf\{\gamma > 0 \ : \ T \leq \gamma I\}.$$

Also, for any x,

$$\langle Tx, x\rangle = |\langle Tx, x\rangle| \leq \|T\|\langle x, x\rangle$$

so that $T \leq \|T\| I$, which shows the other inequality. $\square$

Proposition B.19. *For any positive invertible operators $S, T \in B(X)$:*

$$\|S^{-\frac{1}{2}} T S^{-\frac{1}{2}}\| = \inf\{\gamma > 0 \ : \ T \leq \gamma S\}.$$

Proof. First, note that $S^{-\frac{1}{2}} T S^{-\frac{1}{2}}$ is positive because $S^{-\frac{1}{2}}$ is self-adjoint and T is positive. We know by Proposition B.18 that

$$\|S^{-\frac{1}{2}} T S^{-\frac{1}{2}}\| = \inf\{\gamma > 0 \ : \ S^{-\frac{1}{2}} T S^{-\frac{1}{2}} \leq \gamma I\}.$$

But for any $\gamma > 0$:

$$S^{-\frac{1}{2}} T S^{-\frac{1}{2}} \leq \gamma I \quad \Leftrightarrow \quad T \leq \gamma S,$$

which implies

$$\|S^{-\frac{1}{2}} T S^{-\frac{1}{2}}\| = \inf\{\gamma > 0 \ : \ T \leq \gamma S\}.$$

$\square$

Proposition B.20. *Let $S, T \in B(X)$ two self-adjoint operators. Then $S = T$ if and only if:*

$$\langle Sx, x\rangle = \langle Tx, x\rangle, \quad \forall x.$$

Proof. The $\Rightarrow$ direction is obvious. If $\langle Sx, x\rangle = \langle Tx, x\rangle$ for all x, then $\langle (S-T)x, x\rangle = 0$ for all x. As S, T are self-adjoint, $S - T$ also is, and:

$$\|S - T\| = \sup_{\|x\|=1} |\langle (S-T)x, x\rangle| = \sup_{\|x\|=1} 0 = 0$$

and we conclude $S - T = 0$, and $S = T$. $\square$

C Miscellaneous

C.1 Cauchy & Convergence

Proposition C.1. *If $\langle x_n \rangle_n$ is a Cauchy sequence in a metric space X, and $\varphi : \mathbb{N} \to \mathbb{R}^{>0}$, there exists a subsequence $\langle x_{n_i} \rangle_i$ of $\langle x_n \rangle_n$ satisfying:*

$$d(x_{n_i}, x_{n_{i+1}}) < \varphi(i), \quad \forall i.$$

Proof. We construct the subsequence by induction. For the base case ($i = 0$), we know by Cauchy-ness, that there exists $N(0)$ with

$$\forall m, n \geq N(0) \ : \ d(x_m, x_n) < \varphi(0)$$

and choose $n_0 := N(0)$. Assume that n_j has been defined up to $j = i$, and satisfies

1. $d(x_{n_j}, x_{n_{j+1}}) < \varphi(j)$ for all $j < i$; and
2. $d(x_{n_i}, x_m) < \varphi(i)$ for all $m \geq n_i$.

By Cauchy-ness, there exists $N(i + 1)$ such that

$$\forall m, n \geq N(i + 1) \ : \ d(x_m, x_n) < \varphi(i + 1)$$

and choose $n_{i+1} := \max(N(i + 1), n_i + 1)$. Then,

$$d(x_{n_i}, x_{n_{i+1}}) < \varphi(i)$$

by Item 2. and because $n_{i+1} \geq n_i$; and whenever $m \geq n_{i+1}$,

$$d(x_{n_{i+1}}, x_m) < \varphi(i + 1)$$

by the choice of $n_{i+1} \geq N$. Therefore the sequence up to $i + 1$ still satisfies Item 1 and Item 2. □

Proposition C.2. *If $\langle x_n \rangle_n$ is a Cauchy sequence in a metric space X, and $\langle x_{n_i} \rangle_i$ a subsequence converging to some x, then $\langle x_n \rangle_n$ converges to x too.*

Proof. Fix $\varepsilon > 0$, and J such that for all $i \geq J$, $d(x, x_{n_i}) < \varepsilon/2$. Let N such that for all $m, n \geq N$, $d(x_m, x_n) < \varepsilon/2$. We show that for all $n \geq N$, $d(x, x_n) < \varepsilon$.

Fix $n \geq N$, and take some $i \geq \max(J, N)$ so that $n_i \geq N$, because n_j is strictly increasing. Then $d(x, x_{n_i}) < \varepsilon/2$, because $i \geq J$, and $d(x_{n_i}, x_n) < \varepsilon/2$, because $n_i, n \geq N$, and we conclude

$$d(x, x_n) \leq d(x, x_{n_i}) + d(x_{n_i}, x_n) < 2\varepsilon/2 = \varepsilon.$$

□

C.2 Amenability, Jensen & Convexity

Fix G a discrete group.

Definition C.3. We say that G is amenable if there exists a map $\varphi : l^\infty(G) \to \mathbb{C}$ satisfying the following conditions:

1. φ is linear;
2. $\|\varphi\| := \sup_g |\varphi(g)| = 1$;
3. $\varphi(\mathbb{1}_G) = 1$;
4. $\varphi(f) \in \mathbb{R}^{\geq 0}$ whenever $f(x) \in \mathbb{R}^{\geq 0}$ $\forall x$;
5. $\varphi(\delta_g * f) = \varphi(f)$ for any $f \in l^\infty(G)$ and $g \in G$; that is, φ is left shift invariant;

We call φ a *mean* if it satisfies all conditions, except possibly for left shift invariance.

Definition C.4 (Reiter)**.** A group G satisfies the *Reiter* property if there exists a net $\langle f_\lambda \rangle_\lambda$, of positive elements of norm 1 of $l^1(G)$ with

$$\forall g \in G \ : \ \|\delta_g * f_\lambda - f_\lambda\|_1 \to 0.$$

Proposition C.5. *A group G is amenable if and only if it satisfies the Reiter property.*

Proof. [Pat88, Proposition 0.8]. $\square$

Note that Definition C.3(1) and Definition C.3(2) imply that a mean φ is actually an element of the topological dual $(l^\infty(G))^*$ of $l^\infty(G)$, which is itself the dual of $l^1(G)$. Recall that $l^1(G)$ embeds isometrically in $(l^\infty(G))^*$; if $g \in l^1(G)$, write $\hat{g} \in (l^\infty(G))^*$ for the image of g under the embedding.

Proposition C.6. *If φ is a mean on the group G, there exist a net $\langle \varphi_\lambda \rangle_\lambda$ of positive, finitely supported elements of $l^1(G)$ such that $\langle \hat{\varphi_\lambda} \rangle_\lambda$ converges to φ in the weak-$*$ topology on $(l^\infty(G))^*$.*

If X is a topological space and Y, Z are subsets, let us say that Y is dense *with respect to* Z if for any element z of Z and any open neighbourhood U of z, $U \cap Y \neq \emptyset$.

Proof. [Pat88, Proposition 0.1(iii)]. More precisely, [Pat88, Proposition 0.1(iii)] says that the embedding of the positive elements of norm 1 of $l^1(G)$ is weak-$*$ dense in the set of means. In particular, the embedding of the positive elements of $l^1(G)$ is weak-$*$ dense with respect to the set of means. We also know that the set of positive elements of $l^1(G)$ of finite support is dense in the set of positive elements of $l^1(G)$, so that the embedding of the former is norm-dense in the embedding of the latter, and also weak-$*$ dense, since the weak-$*$ topology is weaker than the norm topology. We conclude that the embedding of the set of positive elements of $l^1(G)$ of finite support is weak-$*$ dense with respect to the set of means, and a net can be constructed. $\square$

Proposition C.7 (Jensen's inequality)**.** *Fix a mean $\varphi : l^\infty(G) \to \mathbb{C}$. For any convex function $\rho : \mathbb{R} \to \mathbb{R}$, and any $f \in l^\infty(G)$ with values in $\mathbb{R}$ and such that $\rho \circ f \in l^\infty(G)$, we have:*

$$\rho(\varphi(f)) \leq \varphi(\rho \circ f).$$

The proof follows [Wika].

Note that for any $f \in l^\infty(G)$ with values in $\mathbb{R}$, $\varphi(f) \in \mathbb{R}$. Indeed, f can be written as $f = f^+ - f^-$, with $f^+ := f \cdot \mathbb{1}_{\{x : f(x) \geq 0\}}$ and $f^- := -f \cdot \mathbb{1}_{\{x : f(x) \leq 0\}}$), and since both are positive elements of $l^\infty(G)$, it follows by linearity and positivity of φ that $\varphi(f^+ - f^-) = \varphi(f^+) - \varphi(f^-) \in \mathbb{R}$.

Proof. Let $x_0 = \varphi(f)$. We know (Proposition C.9) that there exists a, b with

$$ax + b \leq \rho(x) \quad \forall x \quad \text{and} \quad ax_0 + b = \rho(x_0).$$

In particular,

$$af(x) + b \leq \rho(f(x)), \quad \forall x,$$

which, by monotonicity (Definition C.3(1) and Definition C.3(4)) of φ, yields:

$$\varphi(x \mapsto af(x) + b) \leq \varphi(x \mapsto \rho \circ f(x)).$$

But, by linearity and Definition C.3(3):

$$\varphi(x \mapsto af(x) + b) = a\varphi(f) + b\varphi(\mathbb{1}_X) = ax_0 + b = \rho(x_0) = \rho(\varphi(f)),$$

and we conclude

$$\rho(\varphi(f)) \leq \varphi(\rho \circ f).$$

$\square$

Proposition C.8. *Fix a mean $\varphi : l^\infty(G) \to \mathbb{C}$. Then, for any $f \in l^\infty(G)$ and $p \geq 1$:*

$$|\varphi(f)|^p \leq \varphi(|f|^p).$$

Proof. First, observe that it suffices to show that for any $f \in l^\infty(G)$, we have:

$$\varphi(|f|) \geq |\varphi(f)|.$$

Indeed, assume this to be true. Since, $|f| \in l^\infty(G)$ has values in $\mathbb{R}$, we have by Proposition C.7, and by convexity of the function $|\cdot|^p$:

$$\varphi(|f|^p) = \varphi(|(|f|)|^p) \geq |\varphi(|f|)|^p \geq |(|\varphi(f)|)|^p = |\varphi(f)|^p.$$

Fix any positive element ψ of $l^1(G)$ of finite support; the triangle inequality shows that:

$$\hat{\psi}(|f|) = \sum_i \psi_i |f_i| = \sum_i |\psi_i \bar{f}_i| \geq |\sum_i \psi_i \bar{f}_i| = |\hat{\psi}(f)|.$$

This shows that the result holds for the weak-$*$ dense subset of finitely supported elements of $l^1(G)$.

Let $\langle \varphi_\lambda \rangle_\lambda$ a net of positive elements of $l^1(G)$ of finite support such that their embedding converges to φ in the weak-$*$ topology, as given by proposition C.6; using weak-$*$ continuity of the evaluation at f allows us to conclude that:

$$\varphi(|f|) = (\lim_\lambda \hat{\varphi_\lambda})(|f|) = \lim_\lambda \hat{\varphi_\lambda}(|f|) \leq \lim_\lambda |\hat{\varphi_\lambda}(f)| = |\lim_\lambda \hat{\varphi_\lambda}(f)| = |\lim_\lambda \hat{\varphi_\lambda}(f)| = |\varphi(f)|,$$

and we are done. □

Proposition C.9 (Subgradient). *If $\rho : \mathbb{R} \to \mathbb{R}$ is a convex map, then, for any $x_0 \in \mathbb{R}$, there exists $a, b \in \mathbb{R}$ with:*

$$ax + b \leq \rho(x), \quad \forall x \in \mathbb{R}$$

with equality at x_0.

Proof. We use the Hyperplane Separation Theorem (Theorem C.10): Assume $\rho : \mathbb{R} \to \mathbb{R}$ is convex and $x_0 \in \mathbb{R}$. Let

$$\begin{aligned} \hat{\Gamma}(\rho) &:= \{(x, y) \in \mathbb{R} \times \mathbb{R} : y \geq \rho(x)\}, \\ \check{x}_0 &:= \{(x_0, y) \in \mathbb{R} \times \mathbb{R} : \ y < \rho(x_0)\}. \end{aligned}$$

Then both $\hat{\Gamma}(\rho)$ and $\check{x}_0$ are convex, non-empty, don't intersect, and there exists some $v \in \mathbb{R}^2 \setminus \{0\}$ and $c \in \mathbb{R}$ with:

$$\begin{aligned} \langle v, z \rangle &\geq c, \quad \forall z \in \hat{\Gamma}(\rho), \\ \langle v, z \rangle &\leq c, \quad \forall z \in \check{x}_0, \end{aligned}$$

by the Hyperplane Separation Theorem. This means that there exists $(a, b) \in \mathbb{R}^2 \setminus \{(0, 0)\}$ with

$$\begin{aligned} ax + b\rho(x) &\geq c, \quad \forall x \in \mathbb{R}, \\ ax_0 + by &\leq c, \quad \forall y < \rho(x_0). \end{aligned}$$

Then, $b \neq 0$, because that would imply $ax \geq c \ \forall x \in \mathbb{R}$, so that $a = 0$, and $v = (a, b) = 0$, which is a contradiction. We can rearrange those inequalities:

$$\begin{aligned} \rho(x) &\geq \frac{-a}{b}x + \frac{c}{b}, \quad \forall x \in \mathbb{R}, \\ y &\leq \frac{-a}{b} + \frac{c}{b}, \quad \forall y < \rho(x_0), \end{aligned}$$

and the second inequality then implies $\rho(x_0) = \frac{-a}{b}x_0 + \frac{c}{b}$. □

Theorem C.10 (Hyperplane Separation Theorem). *Let A, B be two non-empty convex subsets of $\mathbb{R}^n$ with $A \cap B = \emptyset$. There exist $v \in \mathbb{R}^n \setminus \{0\}$ and $c \in \mathbb{R}$ such that for all $a \in A$ and $b \in B$:*

$$\langle v, a \rangle \leq b \leq \langle v, b \rangle.$$

Proof. [BV04, Paragraph 2.5.1]. □

C.3 Free Group

Recall that for any set A, a free group on A is the data of a pair $(F_A, j : A \to F_A)$ where F_A is a group and $j : A \to F_A$ a map, such that for any group G and map $f : A \to G$, there exists a unique morphism of group $\varphi : F_A \to G$ satisfying $\varphi \circ j = f$. This condition is the *universal mapping property* (UMP) of the free group on A. A straightforward verification shows that the UMP of the free group guarantees that any two free groups on some set are isomorphic.

For any set A, let $F(A)$ denote the standard construction of a free group on A, as the subset of reduced words of $W(A)$, the set of formal words on the alphabet $A \sqcup A^{-1}$, along with $\iota : A \to F(A)$ the map assigning to any $a \in A$ the one-letter word a of $W(A)$. If w, w' are two elements of $F(A)$, their concatenation (as finite sequences) is written ww'. An elementary reduction is the cancelling of and adjacent pair of inverse letters of a word, and an arrow $w \to w'$ is used to show that w' is the result of an elementary reduction on w. The reduction $R(w)$ of a word w consists then on doing as many successive elementary reductions as possible (the order doesn't matter). Recall that the group operation on $F(A)$ is then $w \cdot w' := R(ww')$. Finally, the length of w as a finite sequence is written $|w|$.

Lemma C.11. *Let $w = w_1 ... w_n$ and $w' = w'_1 ... w'_{n'}$ be words of $F(A)$ of length n, n' respectively. Then*

$$w \cdot w' = w_1 ... w_i w'_{n'-i} ... w'_{n'}$$

for some $i \leq n, n'$.

Proof. We know that

$$w \cdot w' = R(ww') = R(w_1 ... w_n w'_1 ... w'_{n'})$$

and consider the sequence of elementary reductions:

$$ww' = r_0 \to r_1 \to ... \to r_i = w \cdot w'.$$

We show by induction that for any $k \leq i$,

$$r_k = w_1 ... w_{n-k} w'_{1+k} ... w' n'.$$

For $k = 0$, this obviously holds. Assume it does for $k - 1$, and $k \leq i$, then r_{k-1} is not reduced, and

$$r_{k-1} = w_1 ... w_{n-(k-1)} w'_{1+k-1} ... w'_{n'}.$$

but the only possible elementary reduction on r_{k-1} is at $w_{n-(k-1)} w'_{1+k-1}$, so that

$$r_k = w_1 ... w_{n-k} w'_{1+k} ... w'_{n'}.$$

and we are done ☐

Proposition C.12. *For any $w, w' \in F(A)$, $|w \cdot w'| \equiv |w| + |w'| \mod 2$.*

Proof. By definition, $w \cdot w' = R(ww')$, and any elementary reduction decreases the length by 2. ☐

Proposition C.13. *For any $w \in F(A)$, there exists at most one $z \in F(A)$ with $|w| > |z|$ and $|w \cdot z| = 1$.*

Proof. By Lemma C.11, we know that $w \cdot z$ can be written $w \cdot z = w'z'$ with w' a prefix of w and z' a suffix of z, but as $|w \cdot z| = 1$, either one of w', z' must be empty, and as $|z| < |w|$, it must be z'. This shows $w \cdot z = w_0$ the first letter of w, so that $z = w^{-1} \cdot w_0$, and z is indeed unique. ☐

Proposition C.14. *Let F_2 be the free group on two elements a, b. Then F_2 contains a free group on countably many elements, F_∞, as a subgroup.*

Proof. [LS01, Proposition 3.1]. ☐

C.4 Products in Rings

Proposition C.15. *Let $a_1, ..., a_n$ be elements of some ring R, then*

$$\prod_{i=1}^{n} (1 + a_i) = 1 + \sum_{i=1}^{n} a_i + \sum_{j=1}^{k} p_j$$

where k is some integer, and, for all j, p_j is a product $a_{i_{1,j}} a_{i_{2,j}} ... a_{i_{l_j,j}}$, with $l_j \geq 2$, and $i_{m,j} \neq i_{m',j}$ if $m \neq m'$.

Proof. This is easily seen by induction. It is true for $n = 1$ by setting $k = 0$; and for $n = 2$, as

$$(1 + a)(1 + b) = 1 + (a + b) + (ab).$$

Assume the statement holds for n, and fix $a_1, ..., a_n, a_{n+1}$; then

$$\begin{aligned}
\prod_{i=1}^{n+1} (1 + a_i) &= \left(\prod_{i=1}^{n} (1 + a_i) \right) (1 + a_{n+1}) = \left(1 + \sum_{i=1}^{n} a_i + \sum_{j=1}^{k} p_j \right) (1 + a_{n+1}) \\
&= \left(1 + \sum_{i=1}^{n} a_i + \sum_{j=1}^{k} p_j \right) + \left(a_{n+1} + \sum_{i=1}^{n} a_i a_{n+1} + \sum_{j=1}^{k} p_j a_{n+1} \right) \\
&= 1 + \left(\sum_{i=1}^{n+1} a_i \right) + \left(\sum_{j=1}^{k} p_j + \sum_{i=1}^{n} a_i a_{n+1} + \sum_{j=1}^{k} p_j a_{n+1} \right)
\end{aligned}$$

and we see that each term in

$$\sum_{j=1}^{k} p_j + \sum_{i=1}^{n} a_i a_{n+1} + \sum_{j=1}^{k} p_j a_{n+1}$$

is a product of at least two a_is, all of distinct index i. ☐

Proposition C.16. *Let $a_1^{\pm}, ...a_n^{\pm}$ be elements of some ring R, then:*

$$\prod_{i=1}^{n}(a_i^+ + a_i^-) = \sum_{\varepsilon_i \in \{+,-\}\ \forall i} a_1^{\varepsilon_1} a_2^{\varepsilon_2} ... a_n^{\varepsilon_n}.$$

Proof. By induction. The base case is obvious, and assume the statement holds for n:

$$\begin{aligned}\prod_{i=1}^{n+1}(a_i^+ + a_i^-) &= \left(\prod_{i=1}^{n}(a_i^+ + a_i^-)\right)(a_{n+1}^+ + a_{n+1}^-) \\ &= \left(\sum_{\varepsilon_i \in \{+,-\}\ \forall i \le n} a_1^{\varepsilon_1} a_2^{\varepsilon_2} ... a_n^{\varepsilon_n}\right)(a_{n+1}^+ + a_{n+1}^-) \\ &= \sum_{\varepsilon_i \in \{+,-\}\ \forall i \le n} a_1^{\varepsilon_1} a_2^{\varepsilon_2} ... a_n^{\varepsilon_n} a_{n+1}^+ + \sum_{\varepsilon_i \in \{+,-\}\ \forall i \le n} a_1^{\varepsilon_1} a_2^{\varepsilon_2} ... a_n^{\varepsilon_n} a_{n+1}^- \\ &= \sum_{\varepsilon_i \in \{+,-\}\ \forall i} a_1^{\varepsilon_1} a_2^{\varepsilon_2} ... a_n^{\varepsilon_n} a_{n+1}^{\varepsilon_{n+1}}.\end{aligned}$$

□

C.5 Rank & Subgroups

Proposition C.17 (Schreier)**.** *Let G be a finitely generated group and H a subgroup of finite index. Then*

$$\operatorname{rank} H \le [G : H] \operatorname{rank} G.$$

The construction is based on [Ser03, Lemma 4.2.1], [Gro], and [Wikb].

Proof. Assume G is generated by $S \subseteq G$, of cardinality $|S| = n$, and $H \le G$ is of finite index m, with R a system of representatives for the right cosets of H in G; assume also that the neutral element e of G is the representative of H in R. Consider the set

$$T := \{rs[rs]^{-1} \mid r \in R, s \in S\}.$$

where $[g]$ denotes the unique element r of R such that $g \in Hr$. Then, T has cardinality at most mn, and we show that it generates H. First, for any $r \in R, s \in S$, $rs \in H[rs]$ by definition, so that $rs[rs]^{-1} \in H$, and thus $T \subseteq H$. Note also that, for any $r \in R$ and $s \in S$, there exists $r' \in R$ with

$$(r's[r's]^{-1})^{-1} = rs^{-1}[rs^{-1}]^{-1}. \tag{46}$$

Indeed, let $r' := [rs^{-1}]$, then

$$\begin{aligned}r's[r's]^{-1} &= [rs^{-1}]s[[rs^{-1}]s]^{-1} \\ &= [rs^{-1}]s[rs^{-1}s]^{-1} \\ &= [rs^{-1}]s[r]^{-1} \\ &= [rs^{-1}]sr^{-1} = [rs^{-1}](rs^{-1})^{-1} = ((rs^{-1})[rs^{-1}]^{-1})^{-1},\end{aligned}$$

using the fact that for any $r \in R$ and $g, g' \in G$, $[r] = r$ and $[[g]g'] = [gg']$.

We now show by induction on $n \ge 1$ that $(S \cup S^{-1})^n \subseteq R^{-1}(T \cup T^{-1})^n$, for all n.

Let first $n = 1$. If $s \in S$, then there exists a unique $r \in R$ with $s \in r^{-1}H$, and $rs \in H$; then $[rs] = e$ and

$$R^{-1}T \ni r^{-1}(rs[rs]^{-1}) = r^{-1}(rse^{-1}) = s.$$

Similarly, given $s^{-1} \in S^{-1}$, there exists a unique r with $s^{-1} \in r^{-1}H$ and $rs^{-1} \in H$; then $[rs^{-1}] = e$ and

$$R^{-1}T^{-1} \ni r^{-1}(r's[r's]^{-1})^{-1} = r^{-1}(rs^{-1}[rs^{-1}]^{-1}) = r^{-1}(rs^{-1}e) = s^{-1},$$

with $r' := [rs^{-1}]$, as in Equation (46).

Now, for the induction step. Assume that $(S \cup S^{-1})^k \subseteq R^{-1}(T \cup T^{-1})^k$ holds for all $k \le n$ and let $x \in (S \cup S^{-1})^{n+1}$. Then x can be written as $x = s^{\varepsilon}w$ with $s \in S$, $\varepsilon = \pm 1$ and $w \in (S \cup S^{-1})^n$. Assume first that $\varepsilon = 1$. We know that there exists $r_x \in R$ and $w_x \in (T \cup T^{-1})^n$ such that $w = r_x^{-1}w_x$. Let $r := [r_x s^{-1}]$; then $Hr = Hr_x s^{-1}$ and $Hrs = Hr_x$ so that $r_x = [rs]$, and:

$$R^{-1}(T \cup T^{-1})^{n+1} \ni r^{-1}(rs)[rs]^{-1}w_x = sr_x^{-1}w_x = sw.$$

Now, assume that $\varepsilon = -1$. We know that there exists $r_x \in R$ and $w_x \in (T \cup T^{-1})^n$ such that $w = r_x^{-1} w_x$. Let $r := [r_x s]$; then $Hr = Hr_x s$ and $Hrs^{-1} = Hr_x$ so that $r_x = [rs^{-1}]$, and:

$$R^{-1}(T \cup T^{-1})^{n+1} \ni r^{-1}((r's)[r's]^{-1})^{-1} w_x = r^{-1}(rs^{-1})[rs^{-1}]^{-1} w_x = s^{-1} r_x^{-1} w_x = sw,$$

with $r' := [rs^{-1}]$, as in Equation (46), and we are done.

We now show that $H \subseteq \langle T \rangle$. We already know $\langle T \rangle \subseteq H$ since $T \subseteq H$, and show $H \subseteq \langle T \rangle$. Given any $h \in H$, $h \in (S \cup S^{-1})^n$ for some n. Since $(S \cup S^{-1})^n \subseteq R^{-1}(T \cup T^{-1})^n$, h can be written as $h = r^{-1} w_h$ with $w_h \in (T \cup T^{-1})^n$ and $r \in R$, which implies $r = w_h h^{-1}$, but since $w_h \in H$, we have $r \in H$, and thus $r = e$, and $w_h = h$; this shows that T does generate H. ☐

Definition C.18. A group G is said to be residually finite if, for any $1 \neq g \in G$, there exists a normal subgroup of G, of finite index, not containing g.

Proposition C.19. *An infinite residually finite group G contains normal subgroups of arbitrary large index.*

Proof. We show by induction that, for all $n \in \mathbb{N}$, there exists a normal subgroup H of G of finite index $[G : H] \geq n$.

First, fix some $g_0 \neq 1$ in G; by hypothesis, G has a finite index normal subgroup H of G not containing g, and $[G : H] \geq 2$, since $H \neq G$. Then, fix $n \geq 3$ and assume that for all $k < n$, G has a finite index normal subgroup H with $[G : H] \geq k$. Let then H_0 be a finite index normal subgroup with $[G : H_0] \geq \frac{1}{2}n$. Then, H_0 is not $\{1\}$, by the finite index condition, and there exists some $1 \neq g_0 \in H_0$. Let H be a finite index normal subgroup of G, not containing g_0. Then

$$[G : H \cap H_0] = [G : H_0][H_0 : H \cap H_0] \geq 2[G : H_0] \geq 2\frac{1}{2}n = n$$

because $H \cap H_0$ is a strict subgroup of H_0. Also,

$$[G : H \cap H_0] = [G : H_0][H_0 : H \cap H_0] = [G : H_0][H_0 H : H] \leq [G : H_0][G : H] < \infty$$

because both H and H_0 have finite index and by the second isomorphism theorem for groups. Then $H \cap H_0$ has finite index at least n, and we are done. ☐

C.6 Borel, Measure & Integration

Fix some set X. Recall that a σ-algebra on X is a subset of $\mathcal{P}(X)$, closed under complement, countable unions and intersections, and containing X and the empty set. A pair $(X, \mathcal{B})$, with $\mathcal{B}$ a σ-algebra on X will be called a *measurable space*.

If $\mathcal{B}$ is the smallest σ-algebra containing some $\mathcal{S} \subseteq \mathcal{P}(X)$, we call $\mathcal{S}$ a (measurable) *basis* for $\mathcal{B}$. This is written $\mathcal{B} = \sigma(\mathcal{S})$. If (X, τ) is a topological space (τ being the topology), the Borel σ-algebra on X is defined as $\sigma(\tau)$.

A subset of X which lies in $\mathcal{B}$ is called measurable. A morphism of measurable spaces $f : (X_1, \mathcal{B}_1) \to (X_2, \mathcal{B}_2)$ is a map $f : X_1 \to X_2$ such that for any $B \in \mathcal{B}_2$, $f^{-1}[B] \in \mathcal{B}_1$. If such a map is bijective and its inverse is also a morphism of measurable space, we call f an *isomorphism* of measurable spaces.

If $(X_1, \mathcal{B}_1)$ and $(X_2, \mathcal{B}_2)$ are two measurable spaces, we can take their product, defined as $(X_1 \times X_2, \mathcal{B}_1 \otimes \mathcal{B}_2)$, where $\mathcal{B}_1 \otimes \mathcal{B}_2$ is the smallest σ-algebra containing the sets of the form $B_1 \times B_2$, for $B_i \in \mathcal{B}_i$ $(i = 1, 2)$; that is, $\mathcal{B}_1 \otimes \mathcal{B}_2$ is the σ-algebra of basis the rectangles. Similarly, if $(X_i, \mathcal{B}_i)$ $(i \in I)$ is an indexed family of measurable spaces, their product is the set $\prod_i X_i$ endowed with the smallest σ-algebra containing all the finite rectangles, that is, the sets of the form

$$\{x \in \prod_i X_i \ : \ x_{i_j} \in B_{i_j}, \forall j = 1, ..., n\},$$

where n is some integer, $i_1, ..., i_n$ a sequence of distinct indices, and $B_{i_1}, ..., B_{i_n}$ each a measurable set in X_{i_j}.

If $(X, \mathcal{B})$ is a measurable space and $B \in \mathcal{B}$, we can take the restriction of $\mathcal{B}$ to B,

$$\mathcal{B}|_B := \{C \in \mathcal{B} \ : \ C \subseteq B\} = \{C \cap B \mid C \in \mathcal{B}\},$$

which defines a σ-algebra on B; the measurable space $(B, \mathcal{B}|_B)$ is then a *subspace* of $(X, \mathcal{B})$.

For any map $f : (X, \mathcal{B}) \to (Y, \sigma(\mathcal{S}))$, the measurability of f is equivalent to the condition that $f^{-1}[S] \in \mathcal{B}$, for any $S \in \mathcal{S}$, since any element of $\sigma(\mathcal{S})$ is constructed by successive unions and complementations out of elements of $\mathcal{S}$, and $f^{-1}[\cdot]$ respects those operations.

If $f : X \to Y$ and $g : X \to Z$ are two measurable maps on measurable spaces (the σ-algebras are omitted for brevity), then so is $\langle f, g\rangle : X \to Y \times Z$; indeed, fixing any measurable set A in Y and B in Z, $\langle f, g\rangle^{-1}[A \times B] = f^{-1}[A] \cap g^{-1}[B]$, which is a measurable set, by measurability of f and g; since the rectangles form a basis for the σ-algebra on $Y \times Z$, we conclude that $\langle f, g\rangle$ is measurable. Similarly, if $f : X \to Y$ and $g : W \to Z$ are both measurable, so is $f \times g : X \times W \to Y \times Z$, since $(f \times g)^{-1}[A \times B] = f^{-1}[A] \times g^{-1}[B]$.

C.6.1 Standard Borel Space

A Polish space is defined as a measurable space $(X, \mathcal{B})$, where $\mathcal{B} = \sigma(\tau)$ is the Borel σ-algebra for τ, a separable, completely metrizable topology on X. A measurable space is called *standard Borel* if it is measurable isomorphic to a measurable subspace of a Polish space.

Standard Borel spaces behave well:

Proposition C.20 (Properties of standard Borel spaces)**.**

1. *Any finite or countable product of standard Borel spaces is standard Borel.*
2. *Any measurable subset of a standard Borel space, seen as a subspace, is standard Borel.*
3. *The graph of any measurable function between standard Borel spaces is measurable (seen as a subset of the product domain$\times$codomain).*

If X, Y are standard Borel spaces and $\Gamma \subseteq X \times Y$ is measurable, and is such that $\forall x \in X\ \exists!\, y \in Y : (x, y) \in \Gamma$, then Γ is called a measurable graph.

4. *Any measurable graph in a product of standard Borel spaces defines a measurable map.*
5. *Any injective measurable map between standard Borel spaces has image a measurable set.*
6. *Any bijective measurable map between standard Borel spaces is a measurable isomorphism.*
7. *The Isomorphism Theorem:*
   - *A finite or countable standard Borel space is discrete (i.e. the σ-algebra is the whole powerset).*
   - *An uncountable standard Borel space is isomorphic to the interval $[0, 1]$, with the Borel σ-algebra.*

Proof.

1. [Kec95, p. 75].
2. [Kec95, Corollary (13.4)].
3. [Kec95, Theorem (14.12)].
4. [Kec95, Theorem (14.12)].
5. [Kec95, Corollary (15.2)].
6. [Kec95, Corollary (15.2)].
7. [Kec95, Theorem (15.6)]. More precisely, [Kec95, Theorem (15.6)] states that any two standard Borel spaces of same cardinality are isomorphic, and that a standard Borel space has cardinality at most $|\mathbb{R}|$. Our formulation follows, since $[0, 1]$, and any at most countable discrete measurable space, are standard.

□

We observe also that the diagonal $\Delta_X \subseteq X^2$ of any standard Borel space is measurable; this is easily concluded from Item 7, since the diagonal in $[0, 1]$ is closed.

Proposition C.21 ([Gao09, Theorem 7.1.2]; Luzin–Novikov)**.** *Let X, Y be standard Borel spaces and A a measurable subset of $X \times Y$. If $A_x = \{y \in X : (x, y) \in A\}$ is at most countable for all x, there exists a countable number of measurable functions $\{f_n : A_n \to Y\}_{n \in I}$, with A_n measurable in X for all n and*

$$A = \bigcup_n \Gamma(f_n).$$

Furthermore, the projection of A on its first coordinate, $\pi_1[A] = \{x \in X : \exists y \in Y\, (x, y) \in A\}$, is measurable in X.

Proof. [Gao09, Theorem 7.1.2]. □

Corollary C.22. *Let X be a standard Borel space and A a measurable subset of $X \times X$. If A_x is at most countable for all x, there exists a countable number of measurable functions $\{f_n : A_n \to X\}_{n \in I}$, with A_n measurable in X for all n and*

$$A = \bigsqcup_n \Gamma(f_n),$$

that is, the graphs are disjoint.

Proof. Fix $\{f_n : A_n \to X\}_{n \in I}$, as given by Proposition C.21, and let $\Gamma_n := \Gamma(f_n)$, for each n. Each Γ_n is a measurable subset of $A_n \times X$, and thus of $X \times X$. Order the index set I and define, by induction:

$$\Gamma'_n := \Gamma_n \setminus \bigcup_{i<n} \Gamma_i.$$

Then, $A = \bigsqcup_n \Gamma'_n$ holds and each Γ'_n is measurable. It remains to show that each Γ'_n still defines a measurable function on some measurable subset of X. But Γ'_n has countable sections, and thus its projection $A'_n := \pi_1[\Gamma'_n]$ is measurable, by Proposition C.21; Γ'_n is then a measurable graph on $A'_n \times X$, and defines a measurable function $f'_n : A'_n \to X$, and we are done. □

C.6.2 Measure & Integration

Consider the space $[0, \infty] = \mathbb{R}^{\geq 0} \sqcup \{\infty\}$ endowed of the obvious order. The Alexandroff one-point compactification of $\mathbb{R}^{\geq 0}$ agrees with the order topology on $[0, \infty]$. The addition on $\mathbb{R}^{\geq 0}$ can be continuously extended to $[0, \infty]$ by setting $a + \infty = \infty + a = \infty$ for any $a \in [0, \infty]$. Any non-decreasing sequence in $[0, \infty]$ converges; if it is bounded, to its supremum, and if it isn't, to ∞. Finally, endow $[0, \infty]$ with its Borel σ-algebra.

A measure μ on a measurable space $(X, \mathcal{B})$ is a map $\mu : \mathcal{B} \to [0, \infty]$ such that $\mu(\emptyset) = 0$ and μ is countably additive, that is, for any countable family of pairwise disjoint measurable sets $\{B_i\}_{i \in \mathbb{N}}$:

$$\mu(\bigsqcup_i B_i) = \sum_i \mu(B_i).$$

A *measure space* is a triple $(X, \mathcal{B}, \mu)$ where $(X, \mathcal{B})$ is a measurable space, and μ a measure on it; this will be abbreviated (X, μ), since the σ-algebra $\mathcal{B}$ can be recovered as the domain of μ. If μ is such that X can be written as the union of a countable number of measurable sets of finite measure, μ is called a σ-finite measure. If $(X, \mathcal{B})$ is a standard Borel space, and μ is non-zero and σ-finite, the triple $(X, \mathcal{B}, \mu)$ will be called a *standard measure space.*

A subset A of X is said to be *null* if it is contained in a measurable set of zero measure; a subset B of X is said to be *co-null* if $X \setminus B$ is null. A property $P(x)$ on the elements x of X is said to hold μ-a.e. (μ-almost everywhere), if $\{x \in X \ : \ P(x)\}$ is a null set.

Theorem C.23 ([Dud02, p. 4.3.2]; Monotone Convergence Theorem)**.** *Let (X, μ) be a measure space and $\langle f_n : X \to [0, \infty]\rangle_{n \in \mathbb{N}}$ a sequence of non-decreasing measurable functions ($f_n(x) \leq f_{n+1}(x)$ for all n and $x \in X$). Let f be the pointwise limit of $\langle f_n \rangle_n$. Then f is measurable and $\int_X f d\mu = \lim_n \int_X f_n d\mu$.*

Proof. [Dud02, p. 4.3.2]. □

Corollary C.24. *Let (X, μ) be a measure space and $\langle f_n : X \to [0, \infty]\rangle_{n \in \mathbb{N}}$ a sequence of measurable functions. Then, the function f defined as the pointwise limit $f(x) := \sum_n f_n(x)$ $\forall x$ is measurable and:*

$$\int_X f d\mu = \sum_n \int_X f_n d\mu.$$

Proof. Consider, for all n, the function $F_n = \sum_{i<n} f_i$, which is measurable, as a finite sum of measurable maps. Then, $\langle F_n \rangle_{n \in \mathbb{N}}$ satisfies the premise of Theorem C.23, so that f is measurable and

$$\int_X f d\mu = \lim_n \int_X F_n d\mu = \lim_n \int_X \sum_{i<n} f_i d\mu = \lim_n \sum_{i<n} \int_X f_n d\mu = \sum_n \int_X f_n d\mu.$$

□

C.7 Graphs

Fix a vertex set X.

Definition C.25 (Simple Graph)**.** A simple graph G on X is a tuple:

$$\mathsf{G} := (X, E)$$

where E, the *edges* of G, is a symmetric, antireflexive subset of $X \times X$. The *degree* of a vertex $x \in X$ is defined as

$$v_{\mathsf{G}}(x) = |\{y \in X \ : \ (x,y) \in \mathsf{G}\}|.$$

A walk in G is a sequence of vertices $x_1 ... x_n$ such that for each i, (x_i, x_{i+1}) is in E. Two vertices are connected if there exists a walk from one to the other. The connected component of some vertex x, $\mathsf{G}[x]$, is the set of all vertices connected to x. The graph G is connected if it has one connected component, that is, if all vertices are connected to each other.

Definition C.26 (Multigraph)**.** A multigraph $\mathcal{G}$ on X is a tuple:

$$\mathcal{G} := (X, E, s : E \to X, t : E \to X),$$

where E is to be interpreted as the set of edges of $\mathcal{G}$, and s and t the maps assigning to any edge e, respectively, its source and destination. By a *self edge*, we mean an edge e with $s(e) = t(e)$ (this is sometimes referred to as a loop). Let, for any $x \in X$,

$$v^+_{\mathcal{G}}(x) := |\{e \in E \ : \ s(e) = x\}|, \quad \text{and} \quad v^-_{\mathcal{G}}(x) := |\{e \in E \ : \ t(e) = x\}|$$

and define the *degree* of x to be:

$$v_{\mathcal{G}}(x) = v^+_{\mathcal{G}}(x) + v^-_{\mathcal{G}}(x).$$

A directed walk in $\mathcal{G}$ is a finite sequence of edges $e_1, ... e_n$ such that $t(e_i) = s(e_{i+1})$ for all i; the source of the walk is $s(e_1)$, and the destination $t(e_n)$. An undirected walk in $\mathcal{G}$ is a finite sequence of pairs $(e_1, \varepsilon_1), ...(e_n, \varepsilon_n)$, with $e_i \in E$, and $\varepsilon_i = \pm 1$, such that $r^{\varepsilon_i}(e_i) = r^{-\varepsilon_{i+1}}(e_{i+1})$ for all i, where $r^1 := t, r^{-1} := s$; the source is then $r^{-\varepsilon_1}(e_1)$, and the destination $r^{\varepsilon_n}(e_n)$. An undirected walk will be written $e_1^{\varepsilon_1} e_2^{\varepsilon_2} ... e_n^{\varepsilon_n}$, or $x_0(e_1^{\varepsilon_1})x_1(e_2^{\varepsilon_2})x_2 ... x_{n-1}(e_n^{\varepsilon_n})x_n$ to make the intermediary vertices explicit. A reduced undirected walk in $\mathcal{G}$ is an undirected walk that doesn't contain a subsequence of the form $... e_i^{\varepsilon_i} e_i^{-\varepsilon_i} ...$.

Two vertices $x, y \in X$ are said to be connected if there exists a (reduced) undirected walk from x to y. We write $\mathcal{G}[x]$ for the connected component of x, that is, the set of vertices that are connected to x. The graph is connected if any two of its vertices are connected. It is a forest, or acyclic, if for any $x \in X$, there exists no reduced undirected walk from x to x, other than the trivial one. It is a tree if it is connected and acyclic.

Remark C.27. We can write

$$v^+_{\mathcal{G}}(x) = |\{e \in E \ : \ s(e) = x\}| = |s^{-1}[\{x\}]|$$

and as $E = \bigsqcup_{x \in X} s^{-1}[\{x\}]$, we conclude that

$$|E| = \sum_{x \in X} v^+_{\mathcal{G}}(x)$$

where infinite cardinalities are not distinguished. The same argument, with t instead of s yields that

$$|E| = \sum_{x \in X} v^-_{\mathcal{G}}(x).$$

Definition C.28 (Underlying Simple Graph)**.** Given a multigraph $\mathcal{G}$, we define the *underlying simple graph* G of $\mathcal{G}$ as the simple graph on the vertices X and with edge set

$$E := \{(x,y) \in X^2 \setminus \Delta_X \ : \ \exists e \in E \ (x,y) = (s(e), t(e)) \vee (x,y) = (t(e), s(e))\}.$$

Proposition C.29. *Let $\mathcal{G}$ be a multigraph and G its underlying simple graph. Then, the connected components of $\mathcal{G}$ are exactly those of G, and the degree of any vertex in G is no greater than that of the vertex in $\mathcal{G}$.*

Proof. We first show that there exists an undirected walk from x to y in $\mathcal{G}$ if and only if there exists one in G.

Assume there exists a walk from x to y in $\mathcal{G}$. In particular, there exists a *shortest* one:

$$x = x_0(e_1^{\varepsilon_1})x_1(e_2^{\varepsilon_2})x_2 ... x_{n-1}(e_n^{\varepsilon_n})x_n = y.$$

Being a shortest walk, no vertex appears twice. Then, by definition, for all i, $(x_i, x_{i+1}) \in \mathsf{G}$, and the sequence $x, x_1, ..., x_{n-1}, y$ is a walk from x to y in G.

Conversely, assume $x = x_0, x_1, ..., x_{n-1}, x_n = y$ to be a walk in G. Then, for all i, there exists an edge e_i and some $\varepsilon_i = \pm 1$ with $(x_{i-1}, x_i) = (r^{-\varepsilon_i}(e_i), r^{\varepsilon_i}(e_i))$, and

$$x(e_i^{\varepsilon_i})x_1...x_{n-1}(e_n^{\varepsilon_n})y$$

is a walk from x to y in $\mathcal{G}$.

For the degree:

$$\begin{aligned}
v_{\mathsf{G}}(x) &= |\{y \in X \ : \ (x,y) \in \mathsf{G}\}| \\
&= |\{y \in X \setminus \{x\} : \ \exists e \in E \ (x,y) = (s(e), t(e)) \vee (x,y) = (t(e), s(e))\}| \\
&\le |\{y \in X \ : \ \exists e \in E \ (x,y) = (s(e), t(e)) \vee (x,y) = (t(e), s(e))\}| \\
&\le |\{y \in X \ : \ \exists e \in E \ (x,y) = (s(e), t(e))\}| + |\{y \in X \ : \ \exists e \in E \ (x,y) = (t(e), s(e))\}| \\
&\le |\{e \in E \ : \ x = s(e)\}| + |\{e \in E \ : \ x = t(e)\}| \\
&\le v^+_{\mathcal{G}}(x) + v^-_{\mathcal{G}}(x) = v_{\mathcal{G}}(x).
\end{aligned}$$

□

Proposition C.30. *If $\mathcal{G}$ is a connected multigraph on a finite set of vertices X, then*

$$\sum_{x \in X} v_{\mathcal{G}}(x) \ge 2(|X| - 1),$$

and this is an equality exactly when $\mathcal{G}$ is a tree.

Given that

$$\sum_{x \in X} v_{\mathcal{G}}(x) = \sum_{x \in X} v^+_{\mathcal{G}}(x) + \sum_{x \in X} v^-_{\mathcal{G}}(x) = 2|E|,$$

the proposition states that for a connected multigraph $\mathcal{G} = (X, E, s, t)$ with a finite vertex set, $|E| \ge |X| - 1$, with an equality whenever $\mathcal{G}$ is a tree; this is the form in which we prove it.

Proof. We first prove that if $\mathcal{G}$ is connected, then $|E| \ge |X| - 1$. This is done by induction on the number of vertices $|X|$. If $\mathcal{G}$ has one vertex, the statement is obvious. If $\mathcal{G}$ has two vertices x_1, x_2, and is connected, some edge must connect x_1 to x_2, and $|E| \ge 1 = |X| - 1$. Now, for the induction step: assume that for any connected graph with at most $n \ge 2$ vertices, the statement holds, and let $\mathcal{G}$ be a graph with $n + 1$ vertices and such that $|E| < |X| - 1$. Let us prove that $\mathcal{G}$ is not connected.

First, if there exists some vertex x_0 with no non-self edge, then the graph is not connected, since for any other vertex $x \ne x_0$, there is no walk between x and x_0, and we are done. So, assume without loss of generality that each vertex has at least one non-self edge. There must exist at least one vertex with exactly one non-self edge. Indeed, if all vertices had two or more non-self edges, we would have

$$2|E| = \sum_{x \in X} v_{\mathcal{G}}(x) \ge \sum_{x \in X} 2 = 2|X|$$

and thus $|E| \ge |X|$, which is a contradiction. Fix one such vertex x (that has exactly one non-self edge), and let e_x be this unique non-self edge of x, and x_e the other end of x_e. Remove x from the graph, along with all its edges, to get a graph:

$$\mathcal{G}' = (X \setminus \{x\}, E', s|_{E'}, t|_{E'}), \quad \text{where} \quad E' := E \setminus \{e \in E \ : \ s(e) = x \vee t(e) = x\}.$$

Then, the graph $\mathcal{G}'$ satisfies:

$$|E'| \le |E| - 1 < |X| - 2 = |X \setminus \{x\}| - 1$$

and is therefore not connected, by the induction hypothesis. Now, towards a contradiction, assume $\mathcal{G}$ to be connected, and choose vertices y, y' in $X \setminus \{x\}$ that are not connected in $\mathcal{G}'$. A walk between y and y' in $\mathcal{G}$ must pass through x, otherwise it would still be a walk in $\mathcal{G}'$; thus, there exists a first time it passes through x and a last time, and it must be of the form $y...x_e(e_x^{\pm 1})x...x(e_x^{\mp 1})x_e...y'$. Indeed, before reaching x for the first time, the walk must pass through x_e, and similarly after the last time it passes through x. But then, x could be skipped altogether, yielding a walk in $\mathcal{G}'$. Hence, $\mathcal{G}$ is not connected.

Now that we know that a connected graph must satisfy $|E| \ge |X| - 1$, we show that we have an equality whenever the graph is a tree. If $\mathcal{G}$ is not a tree, there exists a *shortest*, non-trivial walk $w = x(e_1^{\varepsilon_1})x_1...x_{n-1}(e_n^{\varepsilon_n})x$ connecting some x to itself; since w is shortest, it doesn't pass twice through the same vertex. Remove the first edge e_1 of the walk from $\mathcal{G}$. The resulting graph $\mathcal{G}' := (X, E \setminus \{e\}, s', t')$ is still connected. Indeed, for any pair y, y' of vertices, there exists a walk in $\mathcal{G}$ from one to the other. If this walk uses the edge e_1, replace e_1 by the inverse of the rest of w, which is a walk in $\mathcal{G}'$. Given that $\mathcal{G}'$ is connected:

$$|E| > |E \setminus \{e\}| \geq |X| - 1$$

and we conclude that $|E| \neq |X| - 1$.

Finally, the fact that a tree satisfies $|E| = |X| - 1$ is proven by induction on $|X|$. Note that in a tree, no vertex has a self edge, since such an edge would define a cycle. The base case $|X| = 1$ is trivial.

First, let us show that if $\mathcal{G}$ is a tree with at least 2 vertices, then it must possess at least one vertex x with $v_{\mathcal{G}}(x) = 1$. Indeed, if some vertex had degree 0, it would not be connected to the rest of the tree, which would contradict connectedness. If all vertices had degree ≥ 2, construct the following sequence of walks. Fix some vertex x_0; x_0 has degree ≥ 2, and there exists an edge e_1 and some ε_1 with $x_0 = r^{-\varepsilon_1}(e_1)$; let x_1 be the other end of e_1. Let $w_1 = x_0(e_1^{\varepsilon_1})x_1$. Now, by induction, assume w_n to be defined, and ending with $x_{n-1}(e_n^{\varepsilon_n})x_n$. Given that x_n has degree ≥ 2, there exists some $e_{n+1} \neq e_n$ and ε_{n+1} with $x_n = r^{-\varepsilon_{n+1}}(e_{n+1})$; let $w_{n+1} = w_n x(e_{n+1})x_{n+1}$, with x_{n+1} the other end of e_{n+1}. This walk is reduced, by construction, and eventually cycles, since the number of vertices is finite. This contradicts the fact that $\mathcal{G}$ is a tree, and we conclude that at least one vertex with degree 1 exists.

Now, assuming $\mathcal{G}$ is a tree with $|X| = n+1$ vertices, there exist some $x_0 \in X$ of degree 1. Let $\mathcal{G}' = (X \setminus \{x_0\}, E', s', t')$ be the graph obtained by removing x_0 and its corresponding edges, from $\mathcal{G}$. The graph $\mathcal{G}'$ is still acyclic, since any walk in $\mathcal{G}'$ is still one in $\mathcal{G}$, and it is connected, by the same argument as given in the first part of the proof (assume it isn't, and choose two non-connected vertices; since they are connected in $\mathcal{G}$, a $\mathcal{G}$ path from one to the other must pass though x_0, but then x_0 could be skipped, since the path uses the same edge to reach x_0 as the one taken to leave it). Then, by the induction hypothesis, $\mathcal{G}'$ satisfies $|E'| = |X \setminus \{x_0\}| - 1$, and since $|E| = |E'| + 1$ and $|X| = |X \setminus \{x_0\}| + 1$, we are done. $\square$

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