Tropical Geometry

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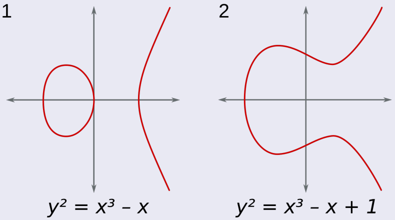
Ideals and Varieties

We work primarily in the polynomial ring $K[x_1, ..., x_n]$ or the Laurent polynomial ring $K[x_1^{\pm 1}, ..., x_n^{\pm 1}]$.

Given an ideal $I \subset K[x_1, ..., x_n]$ (resp. $K[x_1^{\pm 1}, ..., x_n^{\pm 1}]$), we define a variety $V(I) \subset K^n$ (resp. K^{*n}) as the zeros of the polynomials in I.

If $I = \langle f_1, ..., f_n \rangle$, then $V(I) = \{x \in K^n \mid f_i(x) = 0\}$

Example



The Tropical Semiring

Definition

The tropical semiring $\overline{\mathbb{R}}$ has underlying set $\mathbb{R} \cup \{\infty\}$ with tropical addition $\oplus$ and tropical multiplication $\otimes$

a$\oplus b := \min\{a, b\}$
a$\otimes b := a + b$

The additive identity is ∞ and the multiplicative identity is 0

Advantages:

1. Computations are easier
2. Tropical polynomials are piecewise linear
3. Some combinatorial optimization problems can be phrased in terms of tropical arithmetic

Valuation Fields

Given a field K, a valuation on K is a function

$$\mathrm{val} : K \to \overline{\mathbb{R}}$$

such that

1. $\mathrm{val}(a) = \infty$ iff $a = 0$ (so can consider val: $K^* \to \mathbb{R}$)
2. $\mathrm{val}(ab) = \mathrm{val}(a) \otimes \mathrm{val}(b)$
3. $\mathrm{val}(a + b) \geq \mathrm{val}(a) \oplus \mathrm{val}(b)$

Examples

-Trivial valuation: $\mathrm{val}(a) = 0$ for all $a \in K^*$.

-p-adic valuation: on $\mathbb{Q}$, take $\mathrm{val}(\frac{a}{b})$ to be the power of p appearing in $\frac{a}{b}$

-What valuations exist on finite fields?

Trop(f)

Given $f \in K[x_1^{\pm 1}, ..., x_n^{\pm 1}]$ we can write $f = \sum_{u \in \mathbb{Z}^n} c_u x^u$

Definition

trop(f) is attained by replacing classical operations with tropical operations

$$\begin{aligned} \operatorname{trop}(f)(w) &= \min_{u \in \mathbb{Z}^n} (\operatorname{val}(c_u) \otimes \bigotimes_i w_i^{\otimes u_i}) \\ &= \min_{u \in \mathbb{Z}^n} (\operatorname{val}(c_u) + u \cdot w) \end{aligned}$$

Example (elliptic curves)

If

$$f = Ax^3 + Bx^2y + Cxy^2 + Dy^3 + Ex^2 + Fxy + Gy^2 + Hx + Iy + J$$

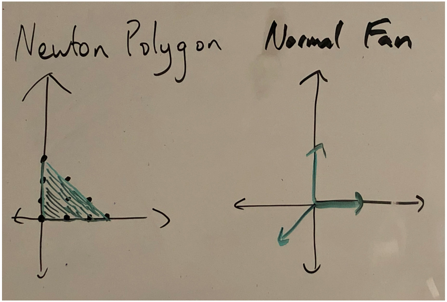
Then $\operatorname{trop}(f) = \min(3x, 2x + y, x + 2y, 3y, 2x, x + y, 2y, x, y, 0)$

Tropical Hypersurfaces

Before, we could associate hypersurfaces to classical polynomials, as the 0-set of the polynomial. For trop(f), the zeros are where the minimum of trop(f) is attained twice.

Example

For trivial valuation fields, the tropical hypersurface $V(\text{trop}(f))$ is the dual of the Newton polytope of f.



Tropical Varieties

Given an ideal $I \subset K[x_1^{\pm 1}, ..., x_n^{\pm 1}]$, and $X = V(I)$ we define

$$\text{trop}(X) = \bigcap_{f \in I} V(\text{trop}(f)) \subseteq \mathbb{R}^n$$

Cautionary Example

Given a generating set $I = \langle f_1, ..., f_n \rangle$ one can't take the intersection to be over $V(\text{trop}(f_i))$

If $I = \langle x_1 + x_2 + 3, x_1 + 5x_2 + 7 \rangle \subset \mathbb{C}[x_1^{\pm 1}, x_2^{\pm 1}]$, then $V(\text{trop}(I)) = \{(0,0)\}$, even though tropicalizing each line yields the same tropical line

Fundamental Theorem

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If K is a field with nontrivial valuation, I an ideal in its Laurent polynomial ring, $X = V(I)$ its variety in the torus, then the following coincide:

(1) The tropical variety $\mathrm{trop}(X)$
(2) $\{w \in \mathbb{R}^n \mid \mathrm{in}_w(I) \neq \langle 1 \rangle\}$
(3) The Euclidean closure in $\mathbb{R}^n$ of the image of X under pointwise valuation,

$$\mathrm{val}(X) = \{\mathrm{val}(y_1), \ldots, \mathrm{val}(y_n) \mid y \in X\}$$

Structure Theorem

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Let X be an irreducible $d-$dimensional subvariety of the torus $(K^*)^n$. Then trop(X) is the support of a balanced, weighted, Γ_{val}-rational polyhedral complex pure of dimension d. Moreover, the polyhedral complex is connected through codimension 1.