Unknotting Problem

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What is the Unknotting Problem?

Given a knot presentation, determine if it is the unknot, and give a series of Reidemeister moves to reduce it to the unknot

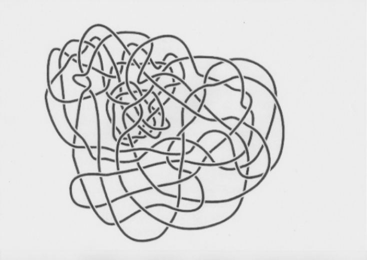

Figure: Haken's Gordian Knot, a complicated-looking unknot

Approaches

- What are some invariants which can distinguish the unknot?
    - Crossing number
    - Genus
    - Knot Group
    - Khovanov Homology
    - Jones Polynomial?

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  - P $\stackrel{?}{=}$ NP, NP $\stackrel{?}{=}$ co-NP

Approaches (cont.)

- Crossing number
  - (Lackenby 2015) The number of moves needed to reduce a knot to the unknot is at most $(286c)^{11}$
- Genus
  - Knot Floer Homology detects the genus of a knot. Manolescu, Ozsváth, Sarkar showed it can be computed combinatorially (2009)
  - Thurston norm
- Knot Group
  - Residual finiteness of the knot group is a key part of Kuperberg's proof that Knottedness is NP assuming GRH (2015).
- Khovanov Homology
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If K is a non-trivial knot, then there is a non-commutative representation

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- The verifier checks that the matrices satisfy the group relations and a pair do not commute

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Theorem

Let $h(x) \in \mathbb{Z}[x]$ be an irreducible polynomial of degree D and let R denote the max of the norm of the coefficients of $h(x)$. Assuming the GRH, there exists a p which is polynomial in D, R such that $h(x)$ has a root in $\mathbb{Z}[x]$.

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- Thurston norm of a homology class is NP $\rightarrow$ Classical knot genus is NP $\rightarrow$ Knottedness is NP

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Figure: from "Quantum money from knots"

References

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