Empirical Macroeconomics

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Structural Dynamic Systems

Dynamic econometric models

If we follow the approach initiated by Haavelmo we can think of

1. a time-dependent representation of the endogenous variables in terms of the available information. Basically the statistical model which is the likelihood.
2. the statistical model is the starting point and therefore must be well specified
3. once we have the statistical model we can impose a structure on the statistical system, intended to isolate or identify relationships interpretable by economic theory.

Structural Dynamic Systems

Structural semantics

Beware of the different definitions of what structural means:

- the specification of theoretical behavior econometricians want to uncover. We saw an equation describing consumption in the Haavelmo model from last class that can be thought as structural or not.
- the specification of the behavior of a stochastic process. Think of the data generating process of a random variable that time series statisticians usually define as structural.

Structural Dynamic Systems

Linear dynamic econometric models

In terms of our notation consider the joint data-density function $f_{X^T}(X^T;\psi)$ where $X^T = X_T, X_{T-1}, .., X_0$ is the all sample , ψ are the parameters and write it in terms of the sequential conditional density

$$f_{X^T}(X^T;\psi) = \prod_{t=1}^{T} f_{X_t|X_{t-1}}(X_t|X_{t-1};\psi)$$

where we omit the marginal density at t_0.
If we restrict to the analysis of linear, finite-lag systems, and that joint normality is a good approximation of $ln(f_{X^T}(X^T;\psi))$, the basic form of our sample density is a VAR

$$x_t = \sum_{j=1}^{q} A_j x_{t-j} + v_t \ where \ v_t \sim IN\left(0, \Omega\right)$$

Structural Dynamic Systems

From VAR to SVAR

Going back to Haavelmo in slide 1, we can have a well specified statistical model by carefully estimating the VAR. This requires a battery of tests that we will see below. Once the VAR is well estimated we have a well behaved $f_{X^T}(X^T;\psi)$ and we can apply our structural model suggested by economic theory. In the SVAR approach we are not so much interested in estimating the structural parameters as such but in imposing a structure that permit the estimation/identification of structural shocks, $\epsilon_t = A(0)^{-1}v_t$, where by structural shocks we define the exogenous stochastic shocks that have implications for the behavior of variable that is implied by the economic theory we choose to use.

Time Series

Review

- Hamilton, 1994, Time Series Analysis (difficult)
- Enders, Applied Econometric Time Series, 4th Edition (easier)
- here we review the theory used in Blanchard and Quah 1989 using for simplicity univariate time series but the results extend to multivariate
- the key point is to justify why the VAR is the basic form which permit to have a good statistical model upon which we apply the economic structure

Wold's Decomposition

Central piece of BQ approach

All covariance-stationary time-series can be written in the form

$$Y_t = \kappa_t + \sum_{j=0}^{\infty} \psi_j \epsilon_{t-j}$$

with $\sum_{j=0}^{\infty} \psi_j^2 < \infty$ and $\psi_0 = 1$. The term ϵ_t is white noise and represents the innovations

$$\epsilon_t \equiv Y_t - \hat{E}\left(Y_t \middle| Y_{t-1}, Y_{t-2}, ...\right)$$

and κ_t is uncorrelated with ϵ_{t-j} for all j.

Covariance-stationary

Introduction

Suppose we observe a sample of size T of some random variables, say i.i.d ϵ_t

$$\{\epsilon_1, \epsilon_2, ..., \epsilon_T\},$$

with

$$\epsilon_t \sim N(0, \sigma^2)$$

which is only one possible outcome of the underlying stochastic process. Same for $\{\epsilon\}_{t=-\infty}^{\infty}$: one single realization.

Covariance-stationary

Introduction

Now consider I realizations $\{y_t^1\}_{t=-\infty}^{\infty}, \{y_t^2\}_{t=-\infty}^{\infty}, ..., \left\{y_t^I\right\}_{t=-\infty}^{\infty}$ and select one observation associated with the I realizations:

$$\left\{y_t^1, y_t^2, ..., y_t^I\right\}$$

This is a sample of I realizations of the random variable Y_t. This random variable has some density, $f_{Y_t}(y_t)$ which is called unconditional density, for example

$$f_{Y_t}(y_t) = \frac{1}{\sqrt{2\pi}\sigma} e^{\left(\frac{-y_t^2}{2\sigma^2}\right)}$$

for the Gaussian white noise.

Covariance-stationary

Mean

The expectation of the tth observation of a time series refers to the mean of this probability distribution

$$E\left(Y_t\right) = \int_{-\infty}^{\infty} y_t f_{Y_t}(y_t) dy_t$$

which you might see as the probability limit of the ensemble average

$$E\left(Y_t\right) = plim_{I \to \infty} \left(1/I\right) \sum_{i=1}^{I} Y_t^{(i)}$$

which is sometimes called the unconditional mean of Y_t:

$$E(Y_t) = \mu_t$$

Covariance-stationary

Variance

The variance

$$\gamma_{0,t} = E\left(Y_t - \mu_t\right)^2 = \int_{-\infty}^{\infty} \left(y_t - \mu_t\right)^2 f_{Y_t}(y_t) dy_t$$

Covariance-stationary

Autocovariance

Given a particular realization such as $\{y_t^1\}_{t=-\infty}^{\infty}$ on a time series process consider x_t^1 consisting of the $[j+1]$ most recent observations on y as of date t for that realization:

$$x_t^1 = \begin{bmatrix} y_t^1 \\ y_{t-1}^1 \\ \dots \\ y_{t-j}^1 \end{bmatrix}$$

and think of each realization of y_t as generating one particular value of the vector x_t. The jth autocovariance of Y_t is

$$\gamma_{jt} = E\left(Y_t - \mu_t\right)\left(Y_{t-j} - \mu_{t-j}\right) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\dots\int_{-\infty}^{\infty}\left(y_t - \mu_t\right)\left(y_{t-j} - \mu_{t-j}\right)$$
$$\times f_{Y_t, Y_{t-1}, \dots, Y_{t-j}}(y_t, y_{t-1}, \dots, y_{t-j}) dy_t dy_{t-1} \dots dy_{t-j}$$

Covariance-stationary
Autocovariance

Notice it has the form of a covariance on lagged values. The Variance is the zero lag autocovariance. Again

$$\gamma_{jt} = plim_{I \to \infty} (1/I) \sum_{i=1}^{I} \left[Y_t^{(i)} - \mu_t \right] \left[Y_{t-j}^{(i)} - \mu_{t-j} \right]$$

Covariance Stationarity

Definition

If neither the mean μ_t nor the autocovariances γ_{jt} depend on the date t then the process for Y_t is said to be covariance-stationary or weakly stationary. It follows that for a covariance stationary process

$$\gamma_{-j} = \gamma_j$$

Strict stationarity is related to the joint distribution of $Y_t, Y_{t+j}, ...$ depending only on the intervals and not on t (higher moments)

Ergodicity

Definition

Consider the sample mean which in this case in not an ensemble average but rather a time average

$$\bar{y} = (1/T)\sum_{t=1}^{T} y_t^1$$

whether time averages eventually converge to ensemble averages has to do with ergodicity. A Gaussian covariance-stationary process is ergodic for the mean if the autocovariances γ_j goes to zero sufficiently quickly for as j becomes large: $\sum_{j=0}^{\infty} |\gamma_j| < \infty$.

Ergodicity

Ergodicity vs Stationarity

Usually stationarity and ergodicity coincide but not always

$$Y_t^i = \mu^i + \epsilon_t$$

with μ generated from a $N(0, \lambda^2)$ distribution. Which is covariance-stationary but not ergodic.

ARMA Processes

MA

consider a MA(1)

$$Y_t = \mu + \epsilon_t + \theta\epsilon_{t-1}$$

Mean and Variance, autocovariances -> covariance-stationary and ergodicity condition is satisfied.

$$E\left(Y_t\right) = \mu$$

$$E\left(Y_t - \mu\right)^2 = \left(1 + \theta^2\right)\sigma^2$$

ARMA Processes

MA

Autocovariance

$$E\left(Y_t-\mu\right)\left(Y_{t-1}-\mu\right)=\theta\sigma^2$$

Ergodicity

$$\sum_{j=0}^{\infty}\left|\gamma_j\right|=\left(1+\theta^2\right)\sigma^2+\left|\theta\sigma^2\right|$$

ARMA Processes

MA

The autocorrelation of a covariance-stationary process (denoted ρ_j) is defined as

$$\rho_j = \frac{\gamma_j}{\gamma_0}$$

first order autocorrelation

$$\frac{\theta}{1+\theta^2}$$

higher order autocorrelation are zero. max,min,same if θ is replaced by $1/\theta$

ARMA Processes

MA(∞)

$$Y_t = \mu + \sum_{j=0}^{\infty} \psi_j \epsilon_{t-j}$$

condition for covariance-stationary is

$$\sum_{j=0}^{\infty} \psi_j^2 < \infty$$

with $\sum_{j=0}^{\infty} |\psi_j| < \infty$ we also have ergodicity.

ARMA Processes

AR(1)

Consider

$$Y_t = c + \rho Y_{t-1} + \epsilon_t$$

$$Y_t = \sum_{j=0}^{\infty} \rho^j \left(c + \epsilon_{t-j} \right)$$

if $|\rho| \geq 1$ there is no covariance stationary process, if $|\rho| < 1$ there is a covariance stationary process that can be viewed as an $MA(\infty)$.

ARMA Processes

AR(1)

The unconditional mean is

$$\mu = \frac{c}{1-\rho}$$

and the variance is

$$\frac{\sigma^2}{1-\rho^2}$$

while the autocovariance

$$\gamma_j = \left[\rho^j/(1-\rho^2)\right]\sigma^2$$

you arrive at these formulas using the $MA(\infty)$ or assuming the $AR(1)$ is covariance stationary.

Likelihood

Estimation

Take a AR(1)

$$Y_t = c + \rho Y_{t-1} + \epsilon_t$$

the first observation Y_1 has a normal distribution (like ϵ) with mean $\mu = \frac{c}{1-\rho}$ and variance $\frac{\sigma^2}{1-\rho^2}$. The second observation Y_2 conditional on Y_1 is a normal with mean $c + \rho y_{t-1}$ and variance σ^2 and so for up to T the sample size, the likelihood is therefore

$$f_{\cdot Y_T, Y_{T-1}, \ldots, Y_1}(y_T, y_{T-1}, \ldots, y_1; \theta) = f_{Y_1}(y_{1;}\theta) \prod_{t=2}^{T} f_{Y_t|y_{t-1}}(y_t|y_{t-1}; \theta)$$

and the log-likelihood

$$L(\theta) = \log f_{Y_1}(y_{1;}\theta) + \sum_{t=2}^{T} \log f_{Y_t|Y_{t-1}}(y_t|y_{t-1}; \theta)$$

Summary

BQ approach (builds on Chris Sims 1980)

1. We can correctly model a multivariate time series by a VMA(∞) using Wold
2. We can correctly approximate the VMA(∞) using a finite VAR(q) and invert it
3. next class: we can impose a structure suggested by economic theory on the estimated VAR of VMA