Business Cycle Accounting
Theory and Applications

Day 1 – Session 3
BCA in practice - Log-Linearization

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Solving the model with Log-Linearization

I. Solving for the Steady State.

II. Approximate the solution by log-linear decision rules.

III. Simulation and calibration.

Model

Suppose households own the capital stock and rent it out at rate[1] r_t. They also work for wages at rate w_t per unit of labor input. The problem is to solve

$$\max_{\{c_t, x_t, l_t\}} E_0 \sum_{t=0}^{\infty} \beta^t U\left(c_t, 1 - l_t\right) N_t \tag{1}$$

subject to:

$$c_t + \left(1 + \tau_{xt}\right) x_t = r_t k_t + \left(1 - \tau_{lt}\right) w_t l_t + T_t \tag{2}$$

$$N_{t+1} k_{t+1} = \left[\left(1 - \delta\right) k_t + x_t\right] N_t \tag{3}$$

$$c_t, x_t \geq 0 \text{ in all states} \tag{4}$$

[1] I use the following notation: italics for scalars, lowercase bold for vectors and uppercase bold for matrices.

Transfers are residually determined and made lump-sum after government expenditures have been incurred. Lowercase variables define per-capita quantities, and N_t is the population level at time t.

Firms are competitive and solve

$$\max_{\{k_t.l_t\}} F\left(k_t, Z_t l_t\right) - r_t k_t - w_t l_t \tag{5}$$

Finally, the resource constraint for the economy is given by:

$$c_t + x_t + g_t = y_t \tag{6}$$

Functional forms and detrending

Suppose that the following functional forms are used:

$$U\left\{c_t, l_t\right\}_{t=0}^{t=+\infty} = E_0 \sum_{t=0}^{\infty} \beta^t \left[\log c_t + \psi \log\left(1 - l_t\right)\right] N_t \tag{7}$$

$$F\left(k_t, Z_t l_t\right) = k_t^{\theta}\left(Z_t l_t\right)^{1-\theta} \tag{8}$$

and $Z_t = z_t\left(1+g_z\right)^t$ with $\ln z_t \sim N(0,1)$.

The utility function in detrended terms, is then given by:

$$U\left\{\hat{c}_t, l_t\right\}_{t=0}^{t=+\infty} = E_0 \sum_{t=0}^{\infty} \beta^t \left[\log \hat{c}_t + \psi \log\left(1 - l_t\right) + \log\left(1+\gamma\right)^t\right] N_t \tag{9}$$

where the notation for detrended variables follows $\hat{c}_t = \frac{c_t}{(1+\gamma)^t}$ and we assume that $N_{t+1} = (1+g_n)N_t$.

The household's budget constraint becomes:

$$\hat{c}_t + (1 + \tau_{xt})\,\hat{x}_t = r_t\hat{k}_t + (1 - \tau_{lt})\,\hat{w}_t l_t + \hat{\varphi}_t \tag{10}$$

where $\hat{\varphi}$ represents per-capita detrended transfers.

The production function is also detrended and expressed in per-capita terms:

$$\hat{y}_t = \hat{k}_t^{\theta}\,(z_t l_t)^{1-\theta} \tag{11}$$

and consequently wages and the rental rate for capital are the solution to the same problem:

$$\max_{\{\hat{k}_t . l_t\}} \hat{k}_t^{\theta}\,(z_t l_t)^{1-\theta} - r_t\hat{k}_t - \hat{w}_t l_t \tag{12}$$

Note also the capital accumulation equation in per-capita, detrended terms:

$$(1 + n)\,(1 + \gamma)\,\hat{k}_{t+1} = \hat{x}_t + (1 - \delta)\,\hat{k}_t \tag{13}$$

Optimization

The firms problem stated previously leads to the following first order conditions:

$$r_t = \theta \hat{k}_t^{\theta-1} (z_t l_t)^{1-\theta} \tag{14}$$

$$\hat{w}_t = (1-\theta) z_t \hat{k}_t^{\theta} (z_t l_t)^{-\theta} \tag{15}$$

In order to set up the Lagrangian function for the representative household problem, I will first solve the capital accumulation equation for $\hat{x}_t$:

$$\hat{x}_t = (1+n)(1+\gamma)\hat{k}_{t+1} - (1-\delta)\hat{k}_t \tag{16}$$

and substitute it in the household budget constraint to get:

$$\hat{c}_t + (1+\tau_{xt})\left[(1+n)(1+\gamma)\hat{k}_{t+1} - (1-\delta)\hat{k}_t\right] = r_t\hat{k}_t + (1-\tau_{lt})\hat{w}_t l_t + \hat{\varphi}_t \tag{17}$$

It is now time to set up the Lagrangian for the household problem. I now drop the $\log (1+\gamma)^t$ from preferences since by doing so, the preference ordering is not altered[2]:

$$L^{HH} = E_0 \sum_{t=0}^{\infty} \beta^t \Bigg\{ \left[\log \hat{c}_t + \psi \log (1-l_t)\right] N_t +$$

$$+\lambda_t \Big[r_t \hat{k}_t + (1-\tau_{lt}) \hat{w}_t l_t + \hat{\varphi}_t$$

$$-\hat{c}_t - (1+\tau_{xt}) \left[(1+n)(1+\gamma) \hat{k}_{t+1} - (1-\delta) \hat{k}_t \right] \Big] \Bigg\}$$

The Inada conditions are fulfilled for the above functional form assumptions and, together with the appropriate no-Maddoff and transversality conditions, the solution is defined by taking the necessary first order conditions w.r.t. consumption, capital and labor.

[2]See King, Plosser and Rebelo (2001), JME

First-Order Conditions

$$\hat{c}_t \quad : \quad \frac{1}{\hat{c}_t} N_t = \lambda_t \tag{18}$$

$$l_t \quad : \quad \frac{\psi}{1 - l_t} N_t = (1 - \tau_{lt})\, \hat{w}_t \lambda_t \tag{19}$$

$$k_{t+1} \quad : \quad \lambda_t (1 + \tau_{xt})\,(1 + n)\,(1 + \gamma) = \beta E_t \lambda_{t+1} \left[r_{t+1} - (1 + \tau_{xt+1})\,(1 - \delta) \right] \tag{20}$$

First Order Conditions (contd.)

By making use of (15), (18) and (19), we find the intratemporal condition, or labor/leisure choice:

$$\frac{\psi \hat{c}_t}{1 - l_t} = (1 - \tau_{lt})(1 - \theta) z_t \hat{k}_t^{\theta} (z_t l_t)^{-\theta} \tag{21}$$

The intertemporal condition or Euler equation reads

$$\frac{1}{\hat{c}_t}(1 + \tau_{xt}) = \hat{\beta} E_t \left\{ \frac{1}{\hat{c}_{t+1}} \left[r_{t+1} - (1 + \tau_{xt+1})(1 - \delta) \right] \right\} \tag{22}$$

where I assume that $N_t(1+n) = N_{t+1}$ and use $\hat{\beta} = \frac{\beta}{1+\gamma}$.

The model is closed and the solution implicitly defined by adding the resource constraint to the set of equations defining the optimum:

$$\hat{c}_t + (1 + \tau_{xt}) \left[(1+n)(1+\gamma) \hat{k}_{t+1} - (1 - \delta) \hat{k}_t \right] + \hat{g}_t = \hat{y}_t \tag{23}$$

Steady State Computation

In the steady state, the Euler equation is given by:

$$(1+\tau_x) = \hat{\beta}\left[r - (1+\tau_x)(1-\delta)\right] \tag{24}$$

Solving (24) w.r.t. r yields:

$$r = \frac{(1+\tau_x)\left[1-\hat{\beta}(1-\delta)\right]}{\hat{\beta}} \tag{25}$$

Remember from (14) that $r = \theta \hat{k}^{\theta-1}(zl)^{1-\theta}$ so that:

$$\hat{k} = \left[\frac{(1+\tau_x)\left[1-\hat{\beta}(1-\delta)\right]}{\theta\hat{\beta}}\right]^{\frac{1}{\theta-1}} zl \tag{26}$$

Steady State Solution – Exercise

Let A and B be

$$A = \left(\frac{z}{\frac{\hat{k}}{l}}\right)^{1-\theta} - (1+\gamma)(1+n) + (1-\delta)$$

$$B = \frac{(1-\tau_l)(1-\theta)\left(\frac{\hat{k}}{l}\right)^{\theta} z^{1-\theta}}{\psi}$$

Then

$$\hat{k} = \frac{B+g}{A+\frac{B}{\frac{\hat{k}}{l}}}$$

$$\hat{c} = A\hat{k} - g$$

$$l = \frac{\hat{k}}{\frac{\hat{k}}{l}}$$

$$\hat{y} = \hat{k}^{\theta}(zl)^{1-\theta}$$

$$\hat{x} = \hat{y} - \hat{c} - \hat{g}$$

Log-linear approximation to the policy functions

The aim now is to find a log-linear approximation to the solution, of the form[3]:

$$\log \hat{k}_{t+1} \quad = \quad \gamma_k \log \hat{k}_t + \boldsymbol{\gamma}' \mathbf{s}_t + \gamma_0, \tag{27}$$

where

$$\mathbf{s}_t = \begin{bmatrix} \log z_t & \tau_{lt} & \tau_{xt} & \hat{g}_t \end{bmatrix}'$$

We assume

$$\mathbf{s}_{t+1} = \mathbf{P}\mathbf{s}_t + P_0 + \boldsymbol{\epsilon}_{t+1} \tag{28}$$

with $\boldsymbol{\epsilon}_{\mathbf{t}} \sim N\left(\mathbf{0}, \mathbf{Q}'\mathbf{Q}\right)$.

[3] note that $\boldsymbol{\gamma}$ is a 4x1 vector

The stochastic nonlinear second order difference equation in capital

- First we can rearrange the resource constraint to get consumption explicitly as a function of capital and labor i.e. $c_t\left(k_t, k_{t+1}, l_t, \mathbf{s}_t\right)$.
- Then, using this function with the intratemporal condition (21) we can define labor implicitly as a function of capital i.e. $l_t\left(k_t, k_{t+1}, \mathbf{s}_t\right)$.
- Finally, making use of these two functions, we can write the intertemporal condition (22) as

$$0 = E_t\left[F\left(\hat{k}_{t+2}, \hat{k}_{t+1}, \hat{k}_t, \mathbf{s}_{t+1}, \mathbf{s}_t\right)\right] \tag{29}$$

- This is the implicit formulation of the equilibrium solution for this economy.

- We are going to approximate this nonlinear stochastic second-order difference equation with a log-linear specification.
- Assume that the residual from the dynamic first-order condition is given by:

$$0 = E_t \left[a_0 \log \hat{k}_{t+2} + a_1 \log \hat{k}_{t+1} + a_2 \log \hat{k}_t + \mathbf{b}_0' \mathbf{s}_{t+1} + \mathbf{b}_1' \mathbf{s}_t \right] \quad (30)$$

- Also, given our guesses (27) and (28)

$$\begin{aligned} 0 =& E_t \left\{ a_0 \left(\gamma_k \log \hat{k}_{t+1} + \boldsymbol{\gamma}' \mathbf{s}_{t+1} \right) + a_1 \left(\gamma_k \log \hat{k}_t + \boldsymbol{\gamma}' \mathbf{s}_t \right) + a_2 \log \hat{k}_t + \mathbf{b}_0' \mathbf{s}_{t+1} + \mathbf{b}_1' \mathbf{s}_t \right\} \\ =& \left(a_0 \gamma_k^2 + a_1 \gamma_k + a_2 \right) \log \hat{k}_t + \left(a_0 \gamma_k \boldsymbol{\gamma} + a_0 \boldsymbol{\gamma} \mathbf{P} + a_1 \boldsymbol{\gamma} + \mathbf{b}_0' \mathbf{P} + \mathbf{b}_1' \right) \mathbf{s}_t \end{aligned}$$

- The left-hand side can only be zero (in general) if γ_k and $\boldsymbol{\gamma}$ satisfy the following system equations:

$$\left\{ \begin{array}{c} a_0 \gamma_k^2 + a_1 \gamma_k + a_2 = 0 \\ a_0 \gamma_k \boldsymbol{\gamma} + a_0 \boldsymbol{\gamma} \mathbf{P} + a_1 \boldsymbol{\gamma} + \mathbf{b}_0' \mathbf{P} + \mathbf{b}_1' = \mathbf{0} \end{array} \right. \quad (31)$$

- Because of the quadratic equation in γ_k there will be two solutions for this system that are $1/\sqrt{\beta}$ reciprocals.
- The transversality condition imposes an upper bound for capital and therefore the solution to be chosen is the one associated with the root that is lower than $1/\sqrt{\beta}$.
- Then:

$$\boldsymbol{\gamma} = -\left[\left(a_0\gamma_k + a_1\right)\mathbf{I}_{4\times 4} + a_0\mathbf{P}'\right]^{-1}\left(\mathbf{b}_0'\mathbf{P} + \mathbf{b}_1'\mathbf{I}_{4\times 4}\right)' \tag{32}$$

- Once know the values of $\boldsymbol{\gamma}$ and γ_k, γ_0 is given by using the steady state values:

$$\gamma_0 = \left(1 - \gamma_k\right)\log\hat{k} - \boldsymbol{\gamma}'\mathbf{s} \tag{33}$$

Exercise

Parameters

```
gn      = (1.015)^(1/4)-1;
gz      = (1.016)^(1/4)-1;
beta    = .9722^(1/4);
delta   = 1-(1-.0464)^(1/4);
psi     = 2.24;
sigma   = 1.000001;
theta   = .35;

P       = [0.98,   -0.0138, -0.0117, 0.192; ...
          -0.033,   0.956,  -0.0451, 0.0569; ...
          -0.0702, -0.0460,  0.896,  0.104;...
           0.00481 -0.00811, 0.0488, 0.971];

sc = 1.05; % scale of the P matrix to make the AR(1) stationary
P = P./sc;

Sbar    = [-0.0239; 0.328; 0.483;-1.53];
P0      = (eye(4)-P)*Sbar;
Q       = [0.0116,0,0,0;0.00141,0.00644,0,0;-0.0105,0.00103,0.0158,0; ...
          -0.000575,0.00611,0.0142,0.00458];

param   = [gn;gz;beta;delta;psi;sigma;theta];
```

Steady State

```
zs          = exp(Sbar(1));
tauls       = Sbar(2);
tauxs       = Sbar(3);
gs          = exp(Sbar(4));
beth        = beta*(1+gz)^(-sigma);
kls         = ((1+tauxs)*(1-beth*(1-delta))/(beth*theta))^(1/(theta-1))*zs;
A           = (zs/kls)^(1-theta)-(1+gz)*(1+gn)+1-delta; %original
B           = (1-tauls)*(1-theta)*kls^theta*zs^(1-theta)/psi; %original
ks          = (B+gs)/(A+B/kls); %original
cs          = A*ks-gs; %original
ls          = ks/kls; %original
ys          = ks^theta*(zs*ls)^(1-theta);
xs          = ys-cs-gs;

display(['Steady state k/y is ',num2str(ks/ys)]);
display(['Steady state c/y is ',num2str(cs/ys)]);
display(['Steady state hours is ',num2str(ls)]);
```

res_wedge.m

Define a function that given parameters, values for $k_{t+2}, k_{t+1}, k_t, \mathbf{s}_{t+1}$ and $\mathbf{s}_t$, yields the residual of the Euler equation.

res_wedge.m (part I)

```
function R = res_wedge(Z,param);
%RES_WEDGE     residual for the growth model in ‘‘Business Cycle Accounting.’’
%

%              Ellen R. McGrattan, 11-1-02
%              Revised, ERM, 11-1-02

%_____________________________________________________________________________
%
% PARAMETERS

gn          = param(1);
gz          = param(2);
beta        = param(3);
delta       = param(4);
psi         = param(5);
sigma       = param(6);
theta       = param(7);
beth        = beta*(1+gz)^(-sigma);

%_____________________________________________________________________________
```

res_wedge.m (part II)

```
% VARIABLES APPEARING IN RESIDUALS

k2          = exp(Z(1));
k1          = exp(Z(2));
k           = exp(Z(3));
z1          = exp(Z(4));
z           = exp(Z(5));
taul1       = Z(6);
taul        = Z(7);
taux1       = Z(8);
taux        = Z(9);
g1          = exp(Z(10));
g           = exp(Z(11));
```

res_wedge.m (part III)

```
l           = 0.25; %guess for l[t]
l1          = 0.25; %guess for l[t+1]

% find the implied l[t] and l[t+1] by newton, assuming it solves in 5 iterations

for i=1:5;
y          = k^theta*(z*l)^(1-theta);
y1         = k1^theta*(z1*l1)^(1-theta);
c          = y-(1+gz)*(1+gn)*k1+(1-delta)*k-g;
c1         = y1-(1+gz)*(1+gn)*k2+(1-delta)*k1-g1;
res        = psi*c*l/y-(1-taul)*(1-theta)*(1-l);
res1       = psi*c1*l1/y1-(1-taul1)*(1-theta)*(1-l1);

lp         = l+.0001;
l1p        = l1+.0001;
y          = k^theta*(z*lp)^(1-theta);
y1         = k1^theta*(z1*l1p)^(1-theta);
c          = y-(1+gz)*(1+gn)*k1+(1-delta)*k-g;
c1         = y1-(1+gz)*(1+gn)*k2+(1-delta)*k1-g1;
dres       = (psi*c*lp/y-(1-taul)*(1-theta)*(1-lp)-res)/.0001;
dres1      = (psi*c1*l1p/y1-(1-taul1)*(1-theta)*(1-l1p)-res1)/.0001;

l          = l-res/dres;
l1         = l1-res1/dres1;
end;
```

res_wedge.m (part IV)

```
% Given the solution for l[t] and l[t+1] compute residual

y          = k^theta*(z*l)^(1-theta);
y1         = k1^theta*(z1*l1)^(1-theta);
c          = y-(1+gz)*(1+gn)*k1+(1-delta)*k-g;
c1         = y1-(1+gz)*(1+gn)*k2+(1-delta)*k1-g1;

R          = (1+taux)*c^(-sigma)*(1-l)^(psi*(1-sigma))- ...
beth*c1^(-sigma)*(1-l1)^(psi*(1-sigma))* ...
(theta*y1/k1+(1-delta)*(1+taux1));
```

Remember that:

$$0 = E_t \left[a_0 \log \hat{k}_{t+2} + a_1 \log \hat{k}_{t+1} + a_2 \log \hat{k}_t + \mathbf{b}_0' \mathbf{s}_{t+1} + \mathbf{b}_1' \mathbf{s}_t \right]$$

Find coefficients of the approximation to the Euler equation

```
% 3. Call subroutine with residuals:

Z          = [log(ks);log(ks);log(ks);log(zs);log(zs);tauls;tauls;
tauxs;tauxs;log(gs);log(gs)];
del        = max(abs(Z)*1e-5,1e-8);
for i=1:11;
Zp       = Z;
Zm       = Z;
Zp(i)    = Z(i)+del(i);
Zm(i)    = Z(i)-del(i);
dR(i,1)  = (res_wedge(Zp,param)-res_wedge(Zm,param))/(2*del(i));
end
```

Find coefficients of the approximation to the Euler equation

```
% 4. Solution:  log k[t+1] = gamma0 + gammak* log k[t] + gamma* S[t]
%

a0          = dR(1);
a1          = dR(2);
a2          = dR(3);
b0          = dR(4:2:11)';
b1          = dR(5:2:11)';
tem         = roots([a0,a1,a2]);
gammak      = tem(find(abs(tem)<1));
gamma       = -((a0*gammak+a1)*eye(4)+a0*P')\(b0*P+b1)';
gamma0      = (1-gammak)*log(ks)-gamma'*[log(zs);tauls;tauxs;log(gs)];
Gamma       = [gammak;gamma;gamma0];
```

Simulation

```
% 5. Generate a time series for the 4-dimensional vector of shocks
%    according to the specified process, the time series for capital
%    according to the above derived policy function, for 100 and 1000
%    observations


T = 1000;
t = 100;


st  = zeros(T,4);
lkt = zeros(T,1);

lkt(1)  = log(ks);
st(1,:) = [Sbar(1) Sbar(2) Sbar(3) Sbar(4)];

sq = 100; %scale the Q matrix

et = randn(T,4);

for i=2:T
st(i,:) = P0+P*st(i-1,:)'+sq*Q*Q'*et(i-1,:)';
lkt(i)    = gamma0 + gammak*lkt(i-1)+gamma'*st(i,:)';
end
```

Plotting

```
subplot(2,2,1)
plot((1:t)./4,exp(lkt(1:t))./ks)
title(['log(kt) for n=',num2str(t)])
xlim([0 t(end)/4]);
subplot(2,2,2)
plot((1:t)./4,st(1:t,:))
xlim([0 t(end)/4]);
title(['log(st) for n=',num2str(t)])

subplot(2,2,3)
plot((1:T)./4,lkt(1:T)./log(ks))
title(['log(kt) for n=',num2str(T)])
xlim([0 T(end)/4]);
subplot(2,2,4)
plot((1:T)./4,st)
xlim([0 t(end)/4]);
title(['log(st) for n=',num2str(T)])
```