In [15]:
```julia
using Interact
using Gadfly
using Distributions
```

In [2]:
```julia
set_default_plot_size(25cm,10cm)
```

# Chapter 1 - Numerical Differentiation and Integration

## 1. Numerical derivatives

### Exercise 1.

Let f be a continuous and twice differentiable function of x in the interval [0.05, 2], such that:

$$f(x) = 2 - 0.5x^{-0.5} - 0.5x^{-0.2}$$

Compute the analytical derivative and numerical derivatives with both forward and two-sided approximation. Evaluate the three functions in the [0.05, 2] interval and plot the absolute values of the difference between the analytical derivative and each numerical approximation. What do you conclude?

In [3]:
```julia
f(x) = 2-0.5x^(-0.5)-0.5x^(-.02);
```

Analytical derivative:

$$f'(x) = 0.25x^{-1.5} + 0.1x^{-1.2}$$

In [4]:
```julia
df(x) = 0.25x^(-1.5)+0.1x^(-1.2);
```

Numerical derivative - forward approximation:

$$f'(x_0) \approx \frac{f(x_0+h) - f(x_0-h)}{h}, \quad h = x - x_0$$

```
In [5]: df_forward(x) = (f(x+h)-f(x))/h; h = 10.0^-10;
```

Numerical derivative - two-sided approximation:

$$f'(x_0) \approx \frac{f(x_0+h)-f(x_0-h)}{2h}, \qquad h = x - x_0$$

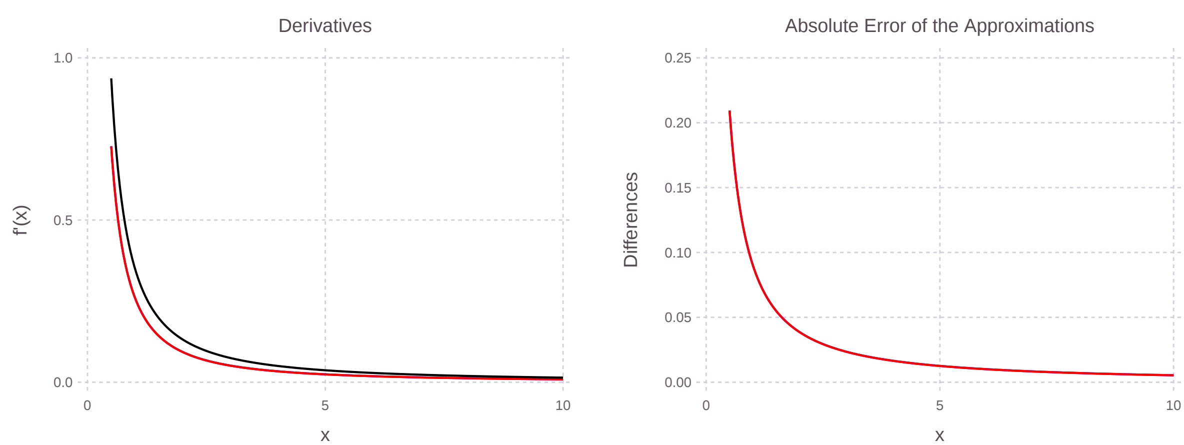
```
In [6]: df_2sided(x) = (f(x+h)-f(x-h))/(2h); h = 10.0^-10;
```

Let us now plot the results for the interval [0.5, 2]:

```
In [7]: function ex1_Differenciation()

            x = 0.5:0.01:10

            results = plot(
                layer(x=x, y=df.(x), Geom.line, Theme(default_color=colorant"black")),
                layer(x=x, y=df_forward.(x), Geom.line, Theme(default_color=colorant"
                layer(x=x, y=df_2sided.(x), Geom.line, Theme(default_color=colorant"b
                Guide.Title("Derivatives"),
                Guide.xlabel("x"),
                Guide.ylabel("f'(x)")
            )

            difference = plot(
                layer(x=x, y=abs.(df.(x)-df_forward.(x)), Geom.line, Theme(default_co
                layer(x=x, y=abs.(df.(x)-df_2sided.(x)), Geom.line, Theme(default_col
                Guide.Title("Absolute Error of the Approximations"),
                Guide.xlabel("x"),
                Guide.ylabel("Differences")
            )
            return hstack(results, difference)
        end;
```

```
In [8]: ex1_Differenciation()
```

Out[8]:



In [9]:
```julia
println("Sum of the absolute error of the forward approximation:","\n", sum(a
println("Sum of the absolute error of the two-sided approximation:","\n", sum
```

```
Sum of the absolute error of the forward approximation:
23.046544815621964

Sum of the absolute error of the two-sided approximation:
23.046518170269373
```

## 2. Numerical Integrals

### Exercise 2.

Let $f(x) = x^2$. Compute $\int_1^4 x^2 dx$ by making use of the left an right Riemann sums and the average of both. Compare the numerical approximations with the analytical solution for $n = 50,\ 500,\ 5000$.

Analytical solution:

$$\int_1^4 x^2 dx = \left[\frac{x^3}{3}\right]_1^4 = \frac{64}{3} - \frac{1}{3} = 21$$

Numerical approximation:

In [10]:
```julia
function ex2_RiemannSums()

    a = 1; b = 4; f(x) = x.^2
    results = ["n" "l_node" "r_node" "mean"; zeros(3, 4)]

    for (i, n) in enumerate([50, 500, 5000])
        dx = (b-a)/n
        l_int = dx*f.(a:dx:4-dx)
        r_int = dx*f.(a+dx:dx:4)
        results[i+1,:] = [n sum(l_int) sum(r_int) sum(.5*(l_int+r_int))]
    end

    return results
end;
```

In [11]:
```
ex2_RiemannSums()
```

Out[11]:
```
4×4 Array{Any,2}:
     "n"     "l_node"     "r_node"     "mean"
   50.0   20.5518      21.4518      21.0018
  500.0   20.955       21.045       21.0
 5000.0   20.9955      21.0045      21.0
```

---

## Exercise 3.

The manager of a supermarket has to decide each day how much bread to buy. For every bread that it is sold, there is a profit of 60 cents. However, for every one not sold, there is a loss of 40 cents. The demand is uniformily distributed in the interval [80, 140]. How much bread should the manager order, in order to maximize the profit? What if the demand follows the Gamma distribution, with parameters k = 2 and θ = 55?

---

Formalizing the problem. Let $P$ denote the profit. Then:

$$P = \begin{cases} 0.6Q, & \text{if } D \geq Q \\ 0.6D - 0.4(Q - D), & \text{if } D < Q \end{cases}$$

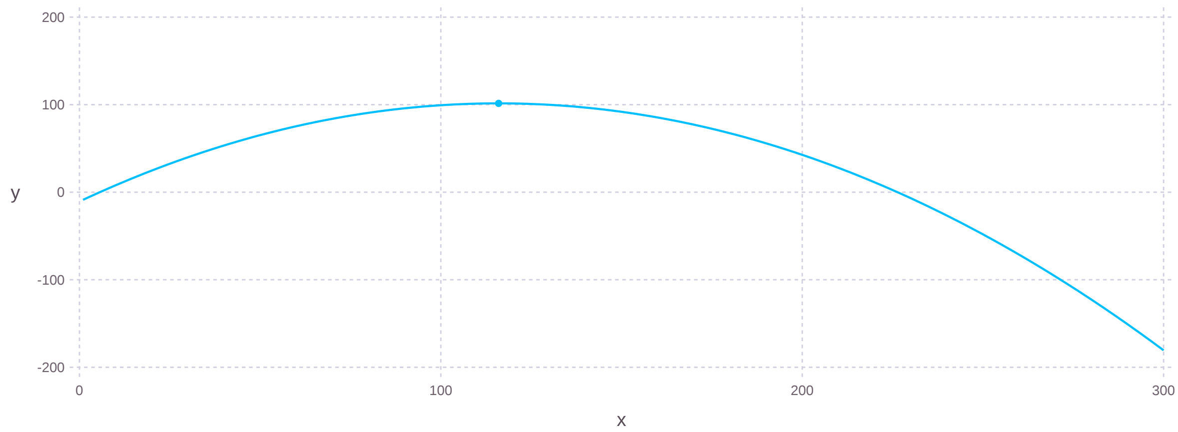
Uniform distribution - analytical solution:

$$\max_{\mathbf{Q}} E[P] = \max_{\mathbf{Q}} \left[ \int_{80}^{Q} [0.6x - 0.4(Q - x)] \frac{1}{60} dx + \int_{Q}^{140} 0.6Q \frac{1}{60} dx \right]$$

$$Q^* = 116 \quad E[P] = 101.47$$

In [12]:
```
function ex3_Uniform(Q)
    return (-1/120)Q^2+(29/15)Q-32/3
end

plot(layer(x=1:300, y=ex3_Uniform.(1:300), Geom.line),
    layer(x=[116], y=[101.47], Geom.point)
)
```

Out[12]:




Gamma distribution - analytical formulation:

$$\max_{\mathbf{Q}} E[P] = \max_{\mathbf{Q}} \left[ \int_{80}^{Q} [0.6x - 0.4(Q-x)] \frac{1}{\Gamma(k)\theta^k} x^{k-1} dx + \int_{Q}^{140} 0.6Q \frac{1}{\Gamma(k)\theta^k} x^{k-1} dx \right]$$

Gamma distribution - Numerical approximation:

In [13]:
```
function ex3_MonteCarlo()

    k = 2; θ = 55

    results = ["n" "Optimal Q" "E[P]"]

    for n in [5 500 5000]
        Q = quantile.(Gamma(k,θ), rand(n))

        gQ = pdf.(Gamma(k,θ),Q)
        P = zeros(n)
        D = rand(Gamma(k,θ),n)
        mP = zeros(n)

        for i in 1:n
            for j in 1:n
                if D[j] >= Q[i]
                    P[j] = 0.6*Q[i]
                else
                    P[j] = 0.6*D[j]-0.5(Q[i]-D[j])
                end
            end
            mP[i] = mean(P)
        end

        Pfunc = [Q mP]
        Q_EP = []
        max_pair = Pfunc[findfirst(Pfunc[:,2], maximum(Pfunc[:,2])),:]'

        results = [results; n max_pair[1] max_pair[2]]

    end

    return results
end;
```

In [14]:
```
ex3_MonteCarlo()
```

Out[14]:
```
4×3 Array{Any,2}:
     "n"      "Optimal Q"    "E[P]"
    5.0    69.3069        17.9061
  500.0    98.7182        32.4981
 5000.0   101.292         33.3719
```