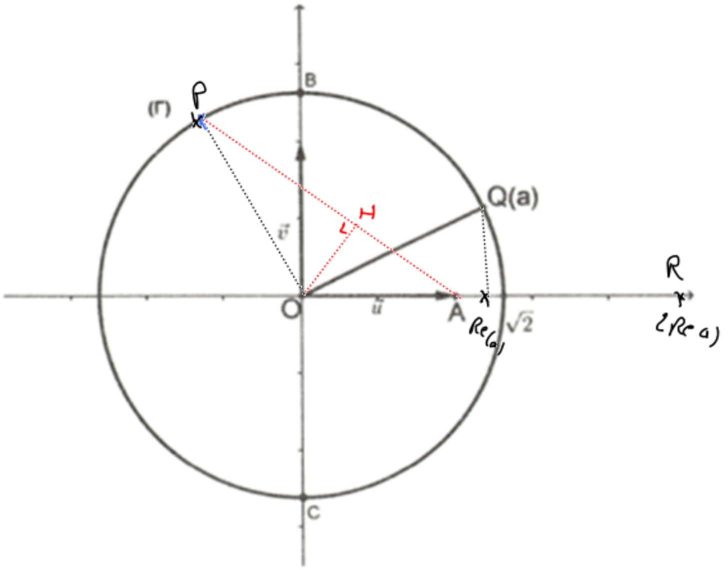
Bac complexes

Exo 2 (Juin 2020)



1) a) $z_R = a+\bar{a} = 2Re(a) \in \mathbb{R}$

donc $R \in (O,\vec{u})$

b) O, R et Q sont alignés et comme O et $R \in (O,\vec{u})$ alors $Q \in (O,\vec{u})$ et $Q \in (\Gamma)$

donc $a = \sqrt{2}$ ou $a = -\sqrt{2}$

2) a) $z_P = ia \iff z_P = i\, z_Q$

$\iff \dfrac{z_P}{z_Q} = i$

$\iff \begin{cases} \left|\dfrac{z_P}{z_Q}\right| = 1 \\ (\widehat{\vec{OQ},\vec{OP}}) \equiv \frac{\pi}{2}\,[2\pi] \end{cases}$

$\iff \begin{cases} |z_P| = |z_Q| \\ (\widehat{\vec{OQ},\vec{OP}}) \equiv \frac{\pi}{2}\,[2\pi] \end{cases} \iff \begin{cases} OQ = OP \\ (\widehat{\vec{OQ},\vec{OP}}) \equiv \frac{\pi}{2}\,[2\pi] \end{cases} \iff R_{(O,\frac{\pi}{2})}(Q) = P$

On a: $OQ = OP$ donc $P \in (\Gamma)$

et $(\widehat{\vec{OQ},\vec{OP}}) \equiv \frac{\pi}{2}\,[2\pi]$

b) * Pour $M = A$ donc $z = 1$ donc A, P et M sont alignés

$\iff (i\bar{a}+1)\times 1 + (i\bar{a}-1)\times 1 = i(a+\bar{a})$ (V)

* Pour $M \neq A$

A, P et M $\iff \dfrac{Aff(\vec{AP})}{Aff(\vec{AM})} = \dfrac{z_P - z_A}{z_M - z_A} = \dfrac{ia-1}{z-1} \in \mathbb{R}$

$\iff \dfrac{ia-1}{z-1} = \overline{\left(\dfrac{ia-1}{z-1}\right)} \iff \dfrac{ia-1}{z-1} = \dfrac{-i\bar{a}-1}{\bar{z}-1}$

$\iff (ia-1)(\bar{z}-1) = (-i\bar{a}-1)(z-1) \iff ia\bar{z} - ia - \bar{z} + \cancel{1} = -i\bar{a}z + i\bar{a} - z + \cancel{1}$

$\iff (ia-1)\bar{z} + (i\bar{a}+1)z = i(a+\bar{a})$

c) $(AP) \perp (OM) \iff \dfrac{Aff(\vec{AP})}{Aff(\vec{OM})} = \dfrac{ia-1}{z} \in i\mathbb{R} \iff$

$\dfrac{ia-1}{z} = -\dfrac{-i\bar{a}-1}{\bar{z}} \iff \dfrac{ia-1}{z} = \dfrac{i\bar{a}+1}{\bar{z}}$

$\iff (ia-1)\bar{z} = (i\bar{a}+1)z$

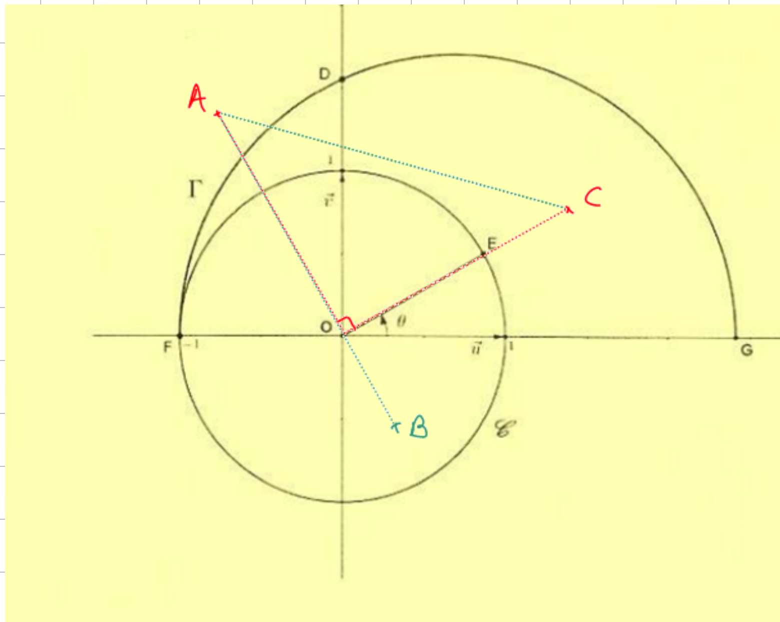
$\iff (i\bar{a}+1)z - (ia-1)\bar{z} = 0$

d) H le projeté orthogonal de O sur (AP) $\iff \begin{cases} \text{A, P et H sont alignés} \\ (AP) \perp (OH) \end{cases}$

$\iff \begin{cases} (i\bar{a}+1)z_H + (ia-1)\bar{z}_H = i(a+\bar{a}) \\ (i\bar{a}+1)z_H - (ia-1)\bar{z}_H = 0 \end{cases}$ (+)

alors $2(i\bar{a}+1)z_H = i(a+\bar{a}) \iff \boxed{z_H = \dfrac{i(a+\bar{a})}{2(i\bar{a}+1)}}$

Ex2 (Juin 2018)



1) a) Dans le triangle FDG rectangle en D
(car $D \in (\Gamma)$ de diamètre [FG] privé de F et G)
[OD] est la hauteur issue de D
(car $(OD) \perp (FG)$)

donc $OD^2 = OF \times OG = 1 \times (1+\sqrt{2})$

$$\boxed{OD^2 = 1+\sqrt{2}}$$

b) $\boxed{z_A} = i\sqrt{1+\sqrt{2}}\, e^{i\theta}$

$= \sqrt{1+\sqrt{2}}\, e^{i(\theta+\frac{\pi}{2})}$

$= \sqrt{OD^2}\, e^{i(\theta+\frac{\pi}{2})} = \boxed{OD\, e^{i(\theta+\frac{\pi}{2})}}$

On a : $z_A = OD\, e^{i(\theta+\frac{\pi}{2})}$

$$\begin{cases} |z_A| = OD \\ \operatorname{Arg}(z_A) \equiv (\theta+\frac{\pi}{2})\ [2\pi] \end{cases}$$

2) $(E) : z^2 + \dfrac{\sqrt{2}}{i\sqrt{1+\sqrt{2}}} e^{i\theta} z + e^{2i\theta} = 0$

a) $z_A^2 + \dfrac{\sqrt{2}}{i\sqrt{1+\sqrt{2}}} e^{i\theta} z_A + e^{2i\theta} = -(1+\sqrt{2})e^{2i\theta} + \dfrac{\sqrt{2}}{\cancel{i}\cancel{\sqrt{1+\sqrt{2}}}} e^{i\theta} \times \cancel{i}\cancel{\sqrt{1+\sqrt{2}}}\, e^{i\theta} + e^{2i\theta}$

$= -(1+\sqrt{2})e^{2i\theta} + \sqrt{2}\, e^{2i\theta} + e^{2i\theta} = e^{2i\theta}(-1-\sqrt{2}+\sqrt{2}+1) = 0$

donc z_A est une solution de (E)

b) $z_B \times z_A = \dfrac{e^{2i\theta}}{1} \iff z_B = \dfrac{e^{2i\theta}}{z_A} = \dfrac{1}{i\sqrt{1+\sqrt{2}}} e^{i\theta} = \boxed{\dfrac{-i}{\sqrt{1+\sqrt{2}}} e^{i\theta}}$

$= \boxed{\dfrac{1}{\sqrt{1+\sqrt{2}}} e^{i(\theta-\frac{\pi}{2})}}$

3) a) $\dfrac{\operatorname{Aff}(\vec{OA})}{\operatorname{Aff}(\vec{OB})} = \dfrac{i\sqrt{1+\sqrt{2}}\, \cancel{e^{i\theta}}}{\dfrac{1}{i\sqrt{1+\sqrt{2}}}\, \cancel{e^{i\theta}}} = \left(i\sqrt{1+\sqrt{2}}\right)^2 = -(1+\sqrt{2}) \in \mathbb{R}$

donc $\vec{OA}$ et $\vec{OB}$ sont colinéaires d'où O, A et B sont alignés

b) $z_C = OD\, e^{i\theta} \iff \begin{cases} OC = OD \\ (\widehat{\vec{u}, \vec{OC}}) \equiv \theta\ [2\pi] \end{cases}$

c) $\dfrac{\operatorname{Aff}(\vec{AC})}{\operatorname{Aff}(\vec{AB})} = \dfrac{\sqrt{1+\sqrt{2}}\, \cancel{e^{i\theta}} - i\sqrt{1+\sqrt{2}}\, \cancel{e^{i\theta}}}{\dfrac{-i}{\sqrt{1+\sqrt{2}}}\, \cancel{e^{i\theta}} - i\sqrt{1+\sqrt{2}}\, \cancel{e^{i\theta}}} = \dfrac{\sqrt{1+\sqrt{2}}\,(1-i)}{\dfrac{-i - i(1+\sqrt{2})}{\sqrt{1+\sqrt{2}}}}$

$= \dfrac{(1+\sqrt{2})(1-i)}{-i(2+\sqrt{2})} = \dfrac{(1+\sqrt{2})(1-i)\,i}{2+\sqrt{2}} = \dfrac{1+\sqrt{2}}{2+\sqrt{2}}(1+i)$

$= \dfrac{(1+\sqrt{2})(2-\sqrt{2})}{2^2-\sqrt{2}^2}(1+i) = \dfrac{\sqrt{2}}{2}(1+i)$

$\dfrac{\sqrt{2}}{2}(1+i) = \dfrac{\sqrt{2}}{2} \times \sqrt{2}\, e^{i\frac{\pi}{4}} = e^{i\frac{\pi}{4}}$

donc $\left|\dfrac{z_C - z_A}{z_B - z_A}\right| = 1$ $\quad (\vec{AB}, \vec{AC}) \equiv \frac{\pi}{4}\ [2\pi] \iff \begin{cases} AB = AC \\ (\vec{AB}, \vec{AC}) \equiv \frac{\pi}{4}\ [2\pi] \end{cases}$ équivaut ABC est un triangle isocèle en A

d) On a : O, A et B sont alignés ; $(\vec{AB}, \vec{AC}) \equiv \frac{\pi}{4}\ [2\pi]$
et $AB = AC$

Ex 2 (Juin 2016)

1) $(E) : z^2 - (1+2i)\,m\,z - (1-i)\,m^2 = 0$

a) $\Delta = m^2$ Soit S : la racine carrée de Δ

$S = m$

$z_1 = \dfrac{(1+2i)m - m}{2} = im$; $z_2 = \dfrac{(1+2i)\,m + m}{2} = m + im$

$S_{\mathbb{C}} = \{ im \,;\, m+im \}$

b) $z_1 \times z_2 \in \mathbb{R}_+ \Leftrightarrow \operatorname{Arg}(z_1 z_2) \equiv 0\,[2\pi] \Leftrightarrow \operatorname{Arg}\left[-(1-i)m^2\right] \equiv 0\,[2\pi]$

$\Leftrightarrow \operatorname{Arg}(-1+i) + \operatorname{Arg}(m^2) \equiv 0\,[2\pi]$

$\Leftrightarrow \dfrac{3\pi}{4} + 2\operatorname{Arg}(m) \equiv 0\,[2\pi] \Leftrightarrow 2\operatorname{Arg}(m) \equiv -\dfrac{3\pi}{4}\,[2\pi]$

$2\operatorname{Arg}(m) = -\dfrac{3\pi}{4} + 2k\pi$; $k \in \mathbb{Z}$

$\operatorname{Arg}(m) = -\dfrac{3\pi}{8} + k\pi$; $k \in \mathbb{Z}$

$\operatorname{Arg}(m) \equiv -\dfrac{3\pi}{8}\,[2\pi]$ ou $\operatorname{Arg}(m) \equiv \dfrac{5\pi}{8}\,[2\pi]$

comme $\operatorname{Arg}(m) \in \,]0, \pi[$

donc $\operatorname{Arg}(m) \equiv \dfrac{5\pi}{8}\,[2\pi]$

c) $z_1 \cdot z_2 = -(1-i)m^2 = (-1+i)\,m^2 = \sqrt{2}\,e^{i\frac{3\pi}{4}}\,|m|^2\,e^{i2\theta} = \sqrt{2}\,\underline{\underline{e^{i\frac{3\pi}{4}}}}\,|m|^2\,\underline{\underline{e^{i\frac{5\pi}{4}}}}$

$= \sqrt{2}\,|m|^2$ (car $e^{i\frac{3\pi}{4}} e^{i\frac{5\pi}{4}} = e^{i0} = 1$)

3) a) $E \in \mathcal{C}_{[BC]}$ privé de B et C donc

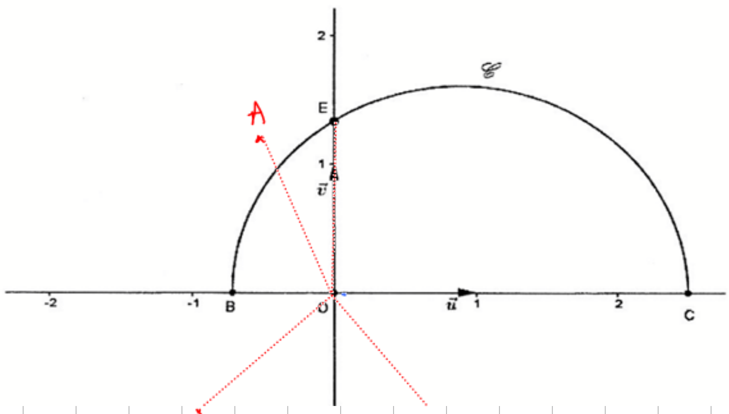
EBC est un triangle rectangle en E

et $[OE]$ l'hauteur issue de E (car $(OE) \perp (BC)$)

ainsi $OE^2 = OB \times OC$

b) on a : $OE^2 = OB \times OC = \dfrac{\sqrt{2}}{2} \times t = \dfrac{t}{\sqrt{2}}$

$\Leftrightarrow OE = \dfrac{\sqrt{t}}{\sqrt[4]{2}} = |m|$



$\left(\text{car } m = \dfrac{\sqrt{t}}{\sqrt[4]{2}}\,e^{i\frac{5\pi}{8}}\right)$

4) a) on a : $m = |m|\,e^{i\frac{5\pi}{8}} = OE\,e^{i\frac{5\pi}{8}}$

donc $\begin{cases} OA = OE \\ (\widehat{\vec{u}, \vec{OA}}) \equiv \dfrac{5\pi}{8}\,[2\pi] \end{cases}$

b) * $m_1 = i\,m$

$\Leftrightarrow \dfrac{m_1}{m} = i$

$\begin{cases} \left|\dfrac{z_1}{z_A}\right| = 1 \\ \operatorname{Arg}\left(\dfrac{z_1}{z_A}\right) \equiv \dfrac{\pi}{2}\,[2\pi] \end{cases} \Leftrightarrow \begin{cases} OA = OM_1 \\ (\widehat{\vec{OA}, \vec{OM_1}}) \equiv \dfrac{\pi}{2}\,[2\pi] \end{cases}$

$\Leftrightarrow R_{(O, \frac{\pi}{2})}(A) = M_1$

* $m_2 = m\,(1+i)$