TP 3 - Boubacar Sow

Exercice 1 : Hasting - Metropolis with Gibbs - Stochastic Approximation EM

We observe N individuals, for the i-th individual, we have k_i measurements denoted by $y_{ij} \in \mathbb{R}$, $j \in [1, k_i]$; $\{y_{ij}\}$ are independant and are obtained at time $\{t_{ij}\}_{j \in [1, k_i]}$ where $t_{i,1} < \dots < t_{i,k}$

1.A A population model for longitudinal data

i) Average progression : $d(t) := p_0 + v_0 (t - t_0)$

Where $p_0 \sim \mathcal{N}(\bar{p}_0, \bar{\sigma}_{p_0}^2)$; $t_0 \sim \mathcal{N}(\bar{t}_0, \bar{\sigma}_{t_0}^2)$; $v_0 \sim \mathcal{N}(\bar{v}_0, \sigma_{v_0}^2)$ and σ_{p_0}, σ_{t_0}, σ_{v_0} are fixed variances, p_0 is also fixed.

ii) Individual :

$$d_i(t) = d(\alpha_i (t - t_0 - \tau_i) + t_0)$$

For any $j \in [\![1, k_i]\!]$:
$$\begin{cases} y_{ij} = d_i(t_{i,j}) + \varepsilon_{i,j} \text{ where } \varepsilon_{ij} \overset{iid}{\sim} \mathcal{N}(0, \sigma^2) \\ \alpha_i = e^{(\xi_i)} \text{ where } \xi_i \overset{i.i.d}{\sim} \mathcal{N}(0, \sigma_\xi^2) \\ \tau_i \overset{i.i.d}{\sim} \mathcal{N}(0, \sigma_\tau^2) \end{cases}$$

$\theta = (\bar{t}_0, \bar{v}_0, \sigma_\xi, \sigma_\tau, \sigma)$

1. log-likelihood of the previous model for the observations $\{y_{ij}\}_{i,j}$

$$q(y, z, \theta) = q(y|z, \theta)\, q(z|\theta)\, q(\theta)$$

$$\Rightarrow \log q(y, z, \theta) = \log q(y|z, \theta) + q(z|\theta) + \log q(\theta)$$

• Let's find $\log q(y|z, \theta)$:

$$q(y|z, \theta) = \prod_{i=1}^N \prod_{j=1}^N q(y_{ij}|z, \theta) \quad \text{where} \quad y_{ij} \sim \mathcal{N}(d(t_{ij}), \sigma^2)$$

$$\log q(y|z, \theta) = \sum_{i=1}^N \sum_{j=1}^N \log q(y_{ij}|z, \theta)$$

$$\boxed{\log q(y|z, \theta) = \sum_{i=1}^N \sum_{j=1}^N \left\{ -\frac{1}{2}\log(\sigma^2) - \frac{1}{2\sigma^2}(y_{ij} - d_i(t_{ij}))^2 \overbrace{- \frac{1}{2} \log(2\pi)}^{-C_1} \right\}}$$

• Let's find $\log q(z|\theta)$:

$$\log q(z|\theta) = \log q(z_{pop}|\theta) + \sum_{i=1}^N \log q(z_i|\theta)$$

- $z_{pop} = (t_0, v_0)$ where $t_0 \sim \mathcal{N}(\bar{t}_0, \sigma_{t_0}^2)$ and $v_0 \sim \mathcal{N}(\bar{v}_0, \sigma_{v_0}^2)$

$$\boxed{\log q(z_{pop}|\theta) = -\frac{1}{2\sigma_{t_0}^2}(t_0 - \bar{t}_0)^2 - \frac{1}{2\sigma_{v_0}^2}(v_0 - \bar{v}_0)^2 + C_2}$$

- $z_i = (\alpha_i, \tau_i)$

$$\boxed{\log q(z_i|\theta) = \sum_{i=1}^N \left(-\log(\sigma_\xi \sigma_\tau) - \frac{1}{2\sigma_\xi^2}\xi_i^2 - \frac{1}{2\sigma_\tau^2}\tau_i^2\right) + C_3}$$

• let's find $\log q(\theta)$

$\theta = (\bar{t}_0, \bar{v}_0, \sigma_\xi, \sigma_\tau, \sigma)$ where $\bar{t}_0 \sim \mathcal{N}(\bar{\bar{t}}_0, s_{t_0}^2)$

$\bar{v}_0 \sim \mathcal{N}(\bar{\bar{v}}_0, s_{v_0}^2)$

$\sigma_\xi \sim W^{-1}(v_\xi, m_\xi)$; $\sigma_\tau^2 \sim W^{-1}(v_\tau, m_\tau)$; $\sigma^2 \sim W(v, m)$

$W^{-1}(v, m)$ is the inverse Gamma-Distribution

$$f_{W^{-1}}(\sigma^2) = \frac{1}{\Gamma(\frac{m}{2})} \frac{1}{\sigma^2} \left(\frac{v}{\sigma\sqrt{2}}\right)^m e^{\left(-\frac{v^2}{2\sigma^2}\right)}$$

$$\log q(\theta) = -\frac{1}{2s_{t_0}^2}(\bar{t}_0 - \bar{\bar{t}}_0)^2 - \frac{1}{2v_{v_0}^2}(\bar{v}_0 - \bar{\bar{v}}_0)^2 + m_\xi \log\left(\frac{v_\xi}{\sigma_\xi}\right) - \log(\sigma_\xi^2) - \frac{1}{2\sigma_\xi^2} v_\xi^2$$
$$+ m_\tau \log\left(\frac{v_\tau}{\sigma_\tau}\right) - \log(\sigma_\tau^2) - \frac{v_\tau^2}{2\sigma_\tau^2} + m \log\left(\frac{v}{\sigma}\right) - \log(\sigma^2) - \frac{v^2}{2\sigma^2} + C$$

We then have:

$$\log(q, z, \theta) = \sum_{i=1}^N \sum_{j=1}^N \left[-\frac{1}{2}\log(\sigma^2) - \frac{1}{2\sigma^2}(y_{ij} - d_i(t_{ij}))^2 - \frac{1}{2}\log(2\pi)\right]$$
$$-\frac{1}{2\sigma_{t_0}^2}(t_0 - \bar{t}_0)^2 - \frac{1}{2\sigma_{v_0}^2}(v_0 - \bar{v}_0)^2$$
$$+\sum_{i=1}^N \left(-\log(\sigma_\xi \sigma_\tau) - \frac{1}{2\sigma_\xi^2}\xi_i^2 - \frac{1}{2\sigma_\tau^2}\tau_i^2\right)$$
$$-\frac{1}{2s_{t_0}^2}(\bar{t}_0 - \bar{\bar{t}}_0)^2 - \frac{1}{2v_{v_0}^2}(\bar{v}_0 - \bar{\bar{v}}_0)^2$$
$$+ m_\xi \log\left(\frac{v_\xi}{\sigma_\xi}\right) - \log(\sigma_\xi^2) - \frac{1}{2\sigma_\xi^2} v_\xi^2$$
$$+ m_\tau \log\left(\frac{v_\tau}{\sigma_\tau}\right) - \log(\sigma_\tau^2) - \frac{v_\tau^2}{2\sigma_\tau^2}$$
$$+ m \log\left(\frac{v}{\sigma}\right) - \log(\sigma^2) - \frac{v^2}{2\sigma^2} + C$$

Where C is a constant

- let's show that this model belongs to the curved exponential family

We have to show that $q(y, z, \theta) = \exp[-\phi(\theta) + \langle S(y,z), \Phi(\theta)\rangle]$

$\Rightarrow \log q(y, z, \theta) = -\phi(\theta) + \langle S(y,z), \Psi(\theta)\rangle$

By identification, we have:

$$\phi(\theta) = \frac{\bar{t}_0^2}{2\sigma_{t_0}^2} + \frac{\bar{t}_0^2}{2s_{t_0}^2} - \bar{t}_0 \frac{\bar{\bar{t}}_0}{s_{t_0}^2} + \frac{\bar{v}_0}{2\sigma_{v_0}^2} + \frac{\bar{v}_0}{2\sigma_{v_0}^2} - \bar{v}_0 \frac{\bar{\bar{v}}_0}{s_{v_0}^2}$$
$$+ \left(\frac{N}{2} + \frac{m_\xi}{2} + 1\right)\log\sigma_\xi^2 + \left(\frac{N}{2} + \frac{m_\tau}{2} + 1\right)\log\sigma_\tau^2 + \frac{1}{2}\sum_{i=1}^N l_i + \frac{m}{2} + 1)\log(\sigma^2)$$

$$S_1(y,z) = \frac{1}{KN}\sum_{i=1}^N \sum_{j=1}^K (y_{ij} - d(t_{ij}))^2 \qquad \Psi_1(\theta) = -\frac{KN}{\sigma^2} \qquad \theta = \begin{bmatrix} \sigma^2 \\ \sigma_\xi^2 \\ \sigma_\tau^2 \\ \bar{t}_0 \\ \bar{v}_0 \end{bmatrix}$$

$$S_2(y,z) = \frac{1}{N}\sum_{i=1}^N \xi_i^2 \qquad \Psi_2(\theta) = -\frac{N}{2\sigma_\xi^2}$$

$$S(y,z) = \frac{1}{N}\sum_{i=1}^N \tau_i^2 \qquad \Psi_3(\theta) = -\frac{N}{2\sigma_\tau^2}$$

$$S_4(y,z) = t_0 \qquad \Psi_4(\theta) = \frac{\bar{t}_0}{\sigma_{t_0}^2}$$

$$S_5(y,z) = v_0 \qquad \Psi_5(\theta) = \frac{\bar{v}_0}{\sigma_{v_0}^2}$$

We have shown that the model belongs to the curved exponential family.

Exercise 2: Multiplicative Hasting Metropolis

f: density of some distribution Π_f supported on $]-1;1[$

(i) $B \sim \mathcal{B}(1/2)$; (ii) $Y = \mathbb{1}_{\{B=1\}}\, \varepsilon X + \mathbb{1}_{\{B=0\}} \frac{X}{\varepsilon}$

1. Give a current state x, determine the proposal kernel $q(x,dy)$ of the MCMC step described:

let's $h: \mathbb{R} \to \mathbb{R}$ a continuous, bounded measurable function

$$\mathbb{E}[h(Y)] = \mathbb{E}[h(Y) \mid B=1]\, \mathbb{P}(B=1) + \mathbb{E}[h(Y) \mid B=0]\, \mathbb{P}(B=0)$$

$$= \frac{1}{2}\mathbb{E}[h(\varepsilon X)] + \frac{1}{2}\mathbb{E}[h(X/\varepsilon)]$$

$$= \frac{1}{2}\int_{-1}^{1} h(\varepsilon X) f(\varepsilon)\, d\varepsilon + \frac{1}{2}\int_{-1}^{1} h(X/\varepsilon) f(\varepsilon)\, d\varepsilon$$

We can set $v = X\varepsilon$ $\quad]-|X|, |X|[\to]-1;1[$ (first integral)

and $v = X/\varepsilon$ $\quad]-\infty; -|X|[\to]0; -\frac{X}{|X|}[$ (second integral)

$$\bullet\ \mathbb{E}[h(V)] = \int_{-|X|}^{|X|} g(v) f(v/x) \frac{1}{|X|}\, dv + \int_{-\infty}^{-|X|} g(v) f(X/v) |X|/v^2\, dv + \int_{|X|}^{+\infty} g(v) f(X/v) \frac{|X|}{v^2}\, dv$$

$$\Rightarrow \mathbb{E}[h(Y)] = \frac{1}{2}\left(\int_{-\infty}^{-|X|} g(v) f(X/v) |X|/v^2\, dv + \int_{-|X|}^{|X|} g(v) f(v/x) \frac{1}{|X|}\, dv + \int_{|X|}^{+\infty} g(v) f(X/v) \frac{|X|}{v^2}\, dv \right)$$

By identification, we have:

$$q(X,Y) = \frac{1}{2} \frac{f(Y/y)}{|X|} \mathbb{1}_{\{]-|X|,|X|[\}} + \frac{1}{2} f(X/Y) \frac{|X|}{Y^2} \mathbb{1}_{\{]-\infty,-|X|[\}} + \frac{1}{2} f(X/Y) \frac{|X|}{Y^2} \mathbb{1}_{\{]|X|,\infty[\}}$$

$$\boxed{q(X,Y) = \frac{1}{2} \mathbb{1}_{\{|Y|<|X|\}} \frac{f(Y/X)}{|X|} + \frac{1}{2} \mathbb{1}_{\{|Y|>|X|\}} f\left(\frac{X}{Y}\right) \frac{|X|}{Y^2}}$$

2. Compute the acceptation ratio $\alpha(x,y)$ so that the chain has given distribution π as invariant distribution

$$\alpha(X,Y) = \min\left(1, \frac{\pi(Y)\, q(Y,X)}{\pi(X)\, q(X,Y)}\right)$$

$$= \min\left(1, \frac{\pi(Y)}{\pi(X)} \frac{\mathbb{1}_{\{|X|>|Y|\}} f(Y|X)|Y|/2X^2 + \mathbb{1}_{\{|X|<|Y|\}} f(X/Y) 1/(2|Y|)}{\mathbb{1}_{\{|Y|>|X|\}} f(X/Y) |X|/(2Y^2) + \mathbb{1}_{\{|Y|<|X|\}} f(Y/X) 1/(2|X|)}\right)$$

$$= \min\left(1, \frac{\pi(Y)}{\pi(X)} \cdot \frac{\frac{|Y|}{|X|}\left(\mathbb{1}_{\{|X|>|Y|\}} f(Y/X)/\{2|X|\} + \mathbb{1}_{\{|X|<|Y|\}} f(X/Y) \frac{|X|}{2|Y|^2}\right)}{\mathbb{1}_{\{|Y|<|X|\}} f(Y/X)\, 1/(2|X|) + \mathbb{1}_{\{|Y|>|X|\}} f(X/Y)\, |X|/(2Y^2)}\right)$$

$$\alpha(X,Y) = \min\left(1, \frac{|Y|}{|X|} \frac{\pi(Y)}{\pi(X)}\right)$$