Computationnal statistics Homework 1

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Exercise 1 : Box-Muller and Marsaglia-Bray algorithm

let R a random variable $\sim$ Rayleigh(1) $\sim f_R(r) = re^{(-\frac{r^2}{2})}\mathbb{1}_{\mathbb{R}^+}(r)$, $\forall r \in \mathbb{R}$

1. Let X and Y such that : $X = r\cos(\theta)$ and $Y = r\sin(\theta)$

Prove that both X and Y $\sim \mathcal{N}(0,1)$ and are independant

We want to show that : $\mathbb{E}[h(X,Y)] = \int_{\mathbb{R}^2} h(x,y) f_X(x) f_Y(y)\, dx\, dy$ $\forall h$ continuous and bounded.

$$\mathbb{E}[h(R\cos\theta, R\sin\theta)] = \int_{\mathbb{R}^2} h(r\cos\theta, r\sin\theta) f_{R,\Theta}(r,\theta)\, dr\, d\theta$$

Now we want to change variables from (r,θ) to (x,y) using the transformation

$\varphi(r,\theta) \longmapsto (x,y)$ with φ a diffeomorphism : $\varphi : \mathbb{R}^+ \times [0,2\pi] \longrightarrow \mathbb{R}^2$

$\mathbb{R}^2 \longmapsto \mathbb{R}^2$ $(r,\theta) \longrightarrow r\cos\theta, r\sin\theta$

$$\Rightarrow \mathbb{E}[h(X,Y)] = \mathbb{E}[h(\varphi(R,\Theta))]$$
$$= \int_{\mathbb{R}^+}\int_0^{2\pi} h(\varphi(r,\theta)) f_R(r) f_\Theta(\theta)\, d\theta\, dr$$

$$|J_{\varphi(r,\theta)}| = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r$$

$dr\, d\theta$

$$\Rightarrow \mathbb{E}[h(X,Y)] = \iint_{\mathbb{R}^2} h(x,y) \frac{1}{2\pi} e^{-r^2/2}\, dx\, dy$$

$\boxed{\text{We know that } x^2+y^2=r^2}$

$$= \iint_{\mathbb{R}^2} h(x,y)\, e^{-\left(\frac{x^2+y^2}{2}\right)} \cdot \frac{1}{2\pi}\, dx\, dy = \iint_{\mathbb{R}^2} h(x,y) \underbrace{\frac{1}{\sqrt{2\pi}} e^{-x^2/2}}_{f_X(x)} \underbrace{\frac{1}{\sqrt{2\pi}} e^{-y^2/2}}_{f_Y(y)}\, dx\, dy$$

$$= \mathbb{E}[h(N_1, N_2)]$$

With $N_1, N_2 \sim \mathcal{N}(0,1)$ that are independant. We conclude that $X, Y \sim \mathcal{N}(0,1)$, $X \perp Y$

2) Write an algorithm for sampling 2 independent gaussian distributions $\mathcal{N}(0,1)$

Cumulative distribution function (C.D.F) : $F_R(r) = \int_{-\infty}^{r} f_R(r) = P(R \leq r)$

Let's $U([0,1])$, then $F_R^{-1}(u) \sim R$

Méthode de la transformée inverse

$F_X^{-1}(u)$, $u \sim U([0,1])$, then $F_X^{-1}(u) \sim X$

$$F_R(r) = -\int_0^r -re^{-\frac{r^2}{2}} dr = \left[-e^{-r^2/2}\right] = 1 - e^{-r^2/2}$$

$$y = F_R(x) \longrightarrow y = 1 - e^{-x^2/2} \Rightarrow e^{-x^2/2} = 1 - y \quad \text{with } 0 < y < 1$$
$$\Rightarrow -\frac{x^2}{2} = \ln(1-y) \qquad y_i \sim U([0,1])$$
$$\Rightarrow x = \sqrt{-2\ln(1-y)}$$
$$\Rightarrow F_R^{-1}(u) = \sqrt{-2\ln(1-u)}$$

Algorithm :

- Draw $U_1 \sim U([0,2\pi])$
  $U_2 \sim U([0,2\pi])$
- Compute $R = \sqrt{-2\ln(1-U_1)}$ where $R = \sqrt{-2\ln(U_1)}$ because $1-U_1 \sim U([0,1])$
- Set $\begin{cases} X = R\cos U_1 \sim \mathcal{N}(0,1) \\ Y = R\sin U_2 \sim \mathcal{N}(0,1) \end{cases}$, $X, Y \sim \mathcal{N}(0,1)$ and $X \perp Y$

3. Consider the given Marsaglia-Bray algorithm

a- Distribution of (V_1, V_2) at the end of "while loop"

At the end of the while loop, we have $V_1^2 + V_2^2 \leq 1$. So V_1 and V_2 are on the disk with center being the origin and with radius = 1.

$V_1 = 2U_1 - 1$ and $V_2 = 2U_2 - 1$; U_1 and $U_2 \sim U([0,1])$ and $U_1 \perp U_2 \Rightarrow V_1 \perp\!\!\!\perp V_2$

Finally we conclude that $(V_1, V_2) \sim U(D(0,1))$

b. $T_1 = \dfrac{V_1}{\sqrt{V_1^2+V_2^2}}$ $T_2 = \dfrac{V_2}{\sqrt{V_1^2+V_2^2}}$ and $V = V_1^2 + V_2^2$

Show that (T_1, T_2) and V are independant, $V \sim \mathcal{U}([0,1])$ and (T_1, T_2) has the same distribution as $(\cos(\Theta), \sin(\Theta))$ with $\Theta \sim \mathcal{U}([0, 2\pi])$

• $T_1 \perp\!\!\!\perp V$

let's $h : \mathbb{R}^2 \to \mathbb{R}^2$ a measurable, bounded function

$$\mathbb{E}[h(T_1, V)] = \mathbb{E}\left[h\left(\frac{V_1}{\sqrt{V_1^2+V_2^2}}, (V_1+V_2^2)\right)\right] = \iint h\left(\frac{v_1}{\sqrt{v_1^2+v_2^2}}, (v_1^2+v_2^2)\right) f_{V_1,V_2}(v_1, v_2)\, dv_1 dv_2$$

$$E = \{(v_1, v_2) \mid v_1^2 + v_2^2 \leq 1\}$$

$$f_{v_1,v_2}(v_1, v_2) = \mathbb{1}_{\{v_1^2+v_2^2 \leq 1\}}(v_1, v_2) \cdot \frac{1}{|E|} \quad \mathbb{1}_{\{v_1, v_2 \in [-1,1]^2\}} - (v_1, v_2) \qquad |E| = \pi 1^2 = \pi$$

Thus $\boxed{f_{v_1,v_2}(v_1, v_2) = \mathbb{1}_{v_1^2+v_2^2}(v_1, v_2) \cdot \frac{1}{\pi}}$

$$\mathbb{E}[h(T_1, V)] = \frac{1}{\pi}\iint_{V_1^2+V_2^2 \leq 1} h\left(\frac{V_1}{\sqrt{V_1^2+V_2^2}}, (V_1^2+V_2^2)\right) dV_1\, dV_2$$

$$= \frac{2}{\pi}\iint_{\{V_1^2+V_2^2 \leq 1\} \cap \{V_2 > 0\}} h\left(\frac{V_1}{\sqrt{V_1^2+V_2^2}}, (V_1^2+V_2^2)\right) dV_1\, dV_2$$

let's $\phi : \mathcal{D}(0,1) \cap \{V_2 > 0\} \longrightarrow [-1, 1] \times [0, 1]$

$(v_1, v_2) \longrightarrow \left(\frac{V_1}{\sqrt{V_1^2+V_2^2}}, V_1^2+V_2^2\right)$

ϕ is a diffeomorphism : $\varphi : \mathcal{D}(0,1) \cap \{V_2 > 0\} \to [-1,1] \times [0,1]$

$$|J_\phi(V_1, V_2)| = \begin{vmatrix} \frac{V_2^2}{(V_1^2+V_2^2)^{3/2}} & \frac{-V_1 V_2}{(V_1^2+V_2^2)^{3/2}} \\ 2V_1 & 2V_2 \end{vmatrix}$$

$$= \frac{2V_2^3 + 2V_1^2 V_2}{(V_1^2+V_2^2)^{3/2}} = \frac{2V_2(V_2^2+V_1^2)}{(V_1^2+V_2^2)^{3/2}} = \frac{2V_2}{(V_1^2+V_2^2)^{1/2}}$$

$$= 2\sqrt{\left(1 - \frac{V_1^2}{V_1^2+V_2^2}\right)} > 0$$

We have:

$$\mathbb{E}[h(T_1, V)] = \frac{1}{\pi}\int_{-1}^{1}\int_0^1 h(t, v) \frac{1}{\sqrt{1-t^2}}\, dt\, dv$$

We can conclude that $f_{T_1, V}(t, v) = \frac{1}{\pi\sqrt{1-t^2}} \mathbb{1}_{[-1,1]}(t) \times \mathbb{1}_{[0,1]}(v) \Rightarrow f_{V,T_1}(t, v) = f_{T_1}(t) \times f_V(v)$

$$\Rightarrow \boxed{T_1 \perp\!\!\!\perp V}$$

• Independance of T_2 and V:

$$\mathbb{E}[h(T_2, V)] = \frac{2}{\pi}\iint_{\{V_1^2+V_2^2 \leq 1\} \cap \{V_1 > 0\}} h\left(\frac{v_2}{\sqrt{V_1^2+V_2^2}}, (V_1^2+v_2^2)\right) dv_1\, dv_2$$

let's $\phi : \mathcal{D}(0,1) \cap \{V_1 > 0\} \to [-1, 1] \times [0, 1]$ a diffeomorphism

$(v_1, v_2) \to \frac{v_2}{\sqrt{v_1^2+v_2^2}}, v_1^2 + v_2^2$

The Jacobian is : $|J_\phi(v_1, v_2)| = \begin{vmatrix} \frac{-v_1 v_2}{(v_1^2+v_2^2)^{3/2}} & \frac{v_1^2}{(v_1^2+v_2^2)^{3/2}} \\ 2v_1 & 2v_2 \end{vmatrix}$

$$|J_{\varphi(v_1,v_2)}| = 2\sqrt{1-\frac{v_2^2}{v_1^2+v_2^2}} > 0$$

We have

$$\mathbb{E}[h(T_2,V)] = \frac{1}{\pi}\int_{-1}^{1}\int_0^1 h(t_2,v)\,\frac{1}{\sqrt{1-t^2}}\,dt\,dv$$

Hence: $\boxed{T_2 \perp\!\!\!\perp V}$ and $\Rightarrow$ $\boxed{T_1, T_2 \perp\!\!\!\perp V}$

. let's show that (T_1, T_2) has the same distribution as: $(\cos(\Theta), \sin(\Theta))$

let h a bounded measurable function, we have

$$\mathbb{E}[h(\cos\Theta)] = \frac{1}{2\pi}\int_0^{2\pi} h(\cos(\theta))\,d\theta = \frac{1}{2\pi}\underbrace{\int_0^{\pi} h(\cos(\theta))\,d\theta}_{(1)} + \underbrace{\int_{\pi}^{2\pi} h(\cos(\theta))\,d\theta}_{(2)}$$

$$(1): \int_0^{\pi} h(\cos(\theta))\,d\theta = -\int_1^{-1} h(t)\sin(\theta)\,dt = \int_{-1}^{1} h(t)\frac{1}{\sqrt{1-t^2}}\,dt$$

with a change of variable $t = \cos(\theta)$ and $dt = -\sin(\theta)dt$

$t^2 = \cos^2(\theta) \Rightarrow t^2 = 1-\sin^2(\theta) \Rightarrow \sin(\theta) = \sqrt{1-t^2}$

$w = (1, 2)$
$x - i = (1, 2$

$$(2): \int_{\pi}^{2\pi} h(\cos(\theta))\,d\theta = \int_0^{\pi} h\cos(u+\pi)\,du = \int_0^{\pi} h(-\cos(\theta))\,du$$

$(1, 2) \times (1, 1 - (1, 2)(2, 2)$

with a change of variable $u = \theta - \pi$, $du = d\theta$

let's $t = -\cos(u)$ and $dt = \sin(u)du = \sqrt{1-t^2}\,du$

$$= \int_{-1}^{1} h(t)\frac{1}{\sqrt{1-t^2}}\,dt$$

Hence: $\mathbb{E}[h(\cos(\Theta))] = \frac{1}{2\pi}\left[2\int_{-1}^{1} h(t)\frac{1}{\sqrt{1-t^2}}\,dt\right]$

$$= \int_{-1}^{1} h(t)\frac{1}{\pi\sqrt{1-t^2}}\,dt$$

We have here $f_{T_1}(t) = \frac{1}{\pi\sqrt{1-t^2}}\mathbb{1}_{[-1,1]}$, thus we conclude that $T_1 \sim \cos(\Theta)$ and $V \sim \mathcal{U}([0,1])$

By the same procedure we can get $T_2 \sim \sin(\Theta)$.

Finally $(T_1, T_2) \sim (\cos(\Theta), \sin(\Theta))$

c) Distribution of the output (X, Y)

Dans l'algorithme:

$S = \sqrt{-2\log r} = F_R^{-1}(1-v)$ with $V \sim \mathcal{U}([0,1])$

Thus $1-v \sim \mathcal{U}([0,1]) \Rightarrow S \sim \text{Rayleigh}(1)$

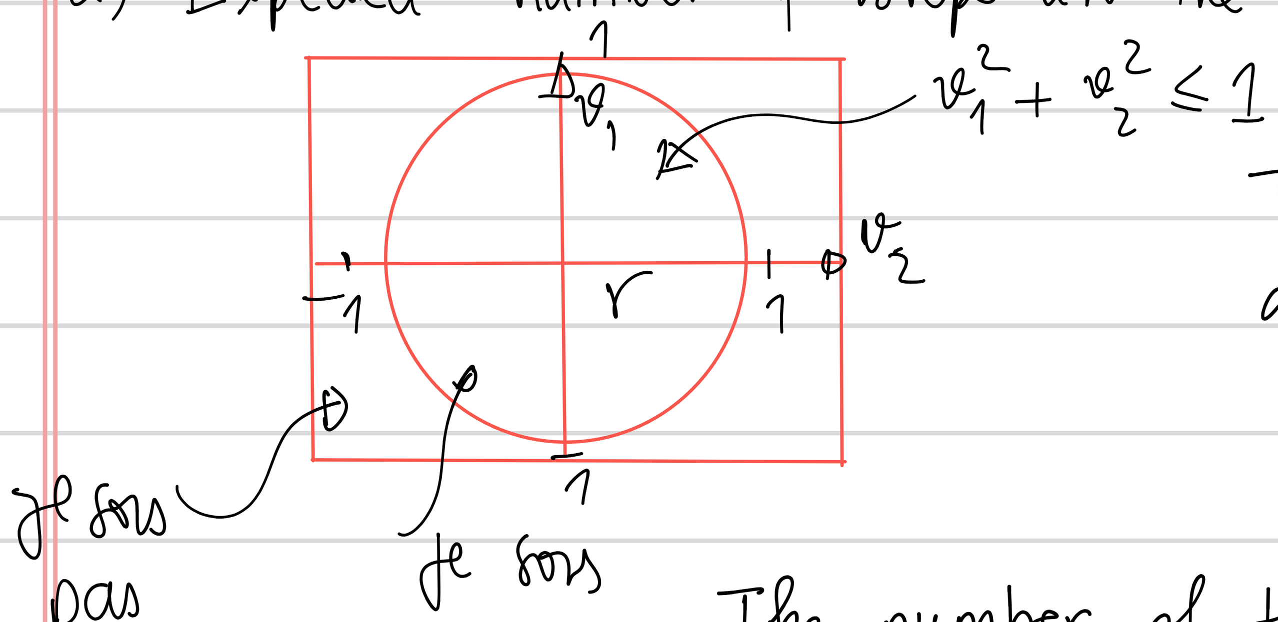
We can also remark that:

$X = ST_1$ and $Y = ST_2$ with $(T_1, T_2) \sim (\cos(\Theta), \sin(\Theta))$, $\Theta \sim \mathcal{U}([0,1])$

$V \perp\!\!\!\perp (T_1, T_2)$ thus $S \perp\!\!\!\perp \Theta$

From question (1), we know that $X \perp\!\!\!\perp Y$, we conclude that $X, Y \overset{i.i.d}{\sim} \mathcal{N}(0,1)$

d) Expected number of steps in the while loop



The probability that we are in the circle given that we are in $[-1,1]^2$ is

$$P = \frac{|E|}{|C|} = \frac{\pi \cancel{r^2}}{4 \cancel{r^2}} = \frac{\pi}{4}$$

The number of times we will go through the loop follows a geometric law

$$P(S = n) = \underbrace{\left(1 - \frac{\pi}{4}\right)^{n-1}}_{\text{stay in the loop}} \underbrace{\frac{\pi}{4}}_{\text{get out the while loop}}$$

It is thus the expected value of a geometric law with parameter $p = \frac{\pi}{4}$

$\Rightarrow N \sim \text{Geometric}\left(\frac{\pi}{4}\right)$

$$\text{Number} = \mathbb{E}[N] = \frac{1}{p} = \frac{4}{\pi} \approx 2$$

The number of times we will go through the while loop is : 2

Exercise 2: Invariant distribution

We define a Markov chain $(X_n)_{n\geq 0}$ in $[0,1]$, ---

- If $X_n = \frac{1}{m}$ ($\forall m > 0$), we let:

$$\begin{cases} X_{n+1} = \frac{1}{m+1} & \text{with probability } 1 - X_n^2 \\ X_{n+1} \sim \mathcal{U}([0,1]) & \text{with probability } X_n^2 \end{cases}$$

- If not, $X_{n+1} \sim \mathcal{U}([0,1])$

1. Prove that the transition kernel of the chain $(X_n)_{n\geq 0}$ is given by:

$$P(x, A) = \begin{cases} x^2 \int_{A \cap [0,1]} dt + (1-x^2)\,\delta_{\{\frac{1}{m+1}\}}(A) & \text{if } x = \frac{1}{m} \\ \int_{A \cap [0,1]} dt & \text{otherwise} \end{cases}$$

where δ_a is the Dirac measure at a

let's $Q = \{\frac{1}{m}, m \in \mathbb{N}^*\}$, transition kernel = distribution $(X_{n+1} | X_n)$

$\mathbb{E}[h(X_{n+1}|X_n)] = \int h(y)\, \underbrace{P(x, dy)}_{\text{density of } X_{n+1}|X_n}$ $\forall h$ bounded and continuous

let's h a continuous and bounded function, we have: $\mathbb{E}[h(X_{n+1})|X_n = x_n] = \int h(y)\, P(x, y)$

. If $x \in Q$ $x \notin Q$

$$\mathbb{E}[h(X_{n+1}|X_n = x] = \int_{\mathbb{R}} h(y)\, \underbrace{P(x, dy)}_{f_{X_{n+1}|X_n}(y)dy} = \int_{\mathbb{R}} h(y)\, \underbrace{\mathbb{1}_{[0,1]}(y)\, dy}_{P(X, dy) \text{ does not depend on } X}$$

$$\Longrightarrow P(x, A) = \int_A P(x, dy) = \int_{A \cap [0,1]} dy$$

. If $x \in Q$, B_m is a random Bernoulli variable, with parameter $\frac{1}{m^n}$ $[\equiv B_m \sim \mathcal{B}(\frac{1}{m^n})]$

$$\mathbb{E}[h(X_{n+1}|X_n = x] = \mathbb{E}[h(X_{n+1})|X_n = x, B_m = 0]\, \mathbb{P}(B_m = 0) + \mathbb{E}[h(X_{n+1}|X_n = x, B_m = 1]\, \mathbb{P}(B_m = 1)$$

$$= h\left(\frac{1}{m+1}\right)\left(1 - \frac{1}{m^2}\right) + \frac{1}{m}\int_{[0,1]} h(y)\, dy$$

$$\Longrightarrow P(x, A) = (1 - x^2)\,\delta_{\frac{1}{m+1}}(A) + x^2 \int_{[0,1]} h(y)\, dy \quad \text{with } x = \frac{1}{m}$$

We have:
$$P(x, A) = \begin{cases} x^2 \int_{A \cap [0,1]} dt + (1-x^2)\,\delta_{\{\frac{1}{m+1}\}}(A) & \text{if } x = \frac{1}{m} \\ \int_{A \cap [0,1]} dt & \text{otherwise} \end{cases}$$

2. Prove that uniform distribution on $[0,1]$ is invariant for P. In the following, this invariant distribution will be denoted by π

A distribution π is invariant for a Markov chain with transition kernel P if for every measurable set A, we have:
$$\pi(A) = \int_{[0,1]} P(x, A)\, \pi(dx) \qquad [\pi(A) = \pi P(A)]$$

We want to show that the uniform distribution on $[0,1]$ is invariant for P.

$$\pi P(A) = \int_{\mathbb{R}} P(x, A)\, \pi(dx) = \underbrace{\int_Q P(x, A)\, \pi(dx)}_{(1)} + \underbrace{\int_{\bar{Q}} P(x, A)\, \pi(dx)}_{(2)}$$

(1) $= 0$, because it's over a countable set of points [the points $x = \frac{1}{m}$, for $m = 1, 2, \ldots$]

(2) $= \int_{A \cap [0,1]} dt\, \pi(dx)$

$$\Rightarrow \pi P(A) = \int_{\bar{Q}} \int_{A \cap [0,1]} dt\, \pi(dx) = \int_{A \cap [0,1]} \int_{\bar{Q}} \pi(dx)$$

$$\Rightarrow \pi P(A) = \int_{\bar{\mathbb{Q}}} \int_{A\cap[0,1]} dt\, \pi(dx) = \underbrace{\int_{A\cap[0,1]}}_{\pi(A)} \underbrace{\int_{\bar{\mathbb{Q}}} \pi(dx)}_{1}$$

$$= \pi(A)$$

3. Let $x \notin \{\frac{1}{m}, m \in \mathbb{N}^*\}$. Compute the value $Pf(x) = \mathbb{E}[f(X_1)|X_0 = x]$ for a bounded measurable function f.

x_1 is uniformly distributed in $[0,1]$

$$Pf(x) = \mathbb{E}[f(x_1)|X_0 = x] = \int_0^1 f(u)\,P(x,du) = \int_0^1 f(u)\,du$$

Deduce $P^n f(x)$ $\forall\, n \geq 1$:

$$P^1 f(x) = Pf(x) = \pi(f)$$

$$P^2 f(x) = \int_0^1 \int_0^1 f(u)\,du\,dv = \int_0^1 f(u)\,du$$

let's assume $P^n f(x) = \int_0^1 f(u)\,d(u)$ $\forall\, n \geq 1$

We need to show that the statement is true for $n+1$, i.e

$$P^{n+1} f(x) = \int_0^1 f(u)\,du$$

$$P^{n+1} f(x) = P(P^n(f(x))) = \int_{\mathbb{R}} P^n f(v)\,P(x,dv) = \int_0^1 P^n f(v)\,dv$$

$$= \int_0^1 \int_0^1 f(u)\,du\,dv = \int_0^1 f(u)\,du$$

Thus we conclude that $P^n f(x) = \int_0^1 f(u)\,du$ $\forall\, n \geq 1$

4. let $x = \frac{1}{m}$ with $m \geq 2$

a- Let $n \in \mathbb{N}$. Compute $P^n(x, \frac{1}{n+m})$ in terms of m and n

. If $n = 1$:

$$P(x, \frac{1}{1+m}) = P(\frac{1}{m}, \frac{1}{m+1})$$

$$= P(X_{n+1} = \frac{1}{m+1} | X_n = \frac{1}{m})$$

$$P(x, \frac{1}{n+m}) = 1 - X_n^2 = 1 - (\frac{1}{m})^2$$

For $n = 2$:

$$P(x, \frac{1}{m+2}) = P(P(\frac{1}{m}, \frac{1}{m+2})) = \int P(y, \frac{1}{m+2})\,P(x,dy)$$

$$= \int P(y, \frac{1}{m+2})\left[x^2 \int_{dy\cap[0,1]} du + (1-x^2)\,\delta_{1/m+1}\,dy\right]$$

$$= \int P(y, \frac{1}{m+2})(1-x^2)\,\delta_{1/m+1}\,dy = (1-x^2)\int P(y, \frac{1}{m+2})\,\delta_{1/m+1}\,dy$$

$$= (1-(\frac{1}{m})^2)\int P(y, \frac{1}{m+2})\,\delta_{1/m+1}\,dy = (1-\frac{1}{m^2})\,P(\frac{1}{m+1}, \frac{1}{m+2})$$

$$= (1-(\frac{1}{m})^2)(1-(\frac{1}{m+1})^2) = \prod_{i=0}^{1}(1-(\frac{1}{m+i})^2)$$

$$P^2\left(x, \frac{1}{m+2}\right) = \prod_{i=0}^{1} \left(1 - \left(\frac{1}{m+i}\right)^2\right)$$

Hérédité:

Let's assume for any k we have : $P^k\left(x, \frac{1}{m+k}\right) = \prod_{i=0}^{k-1} \left(1 - \left(\frac{1}{m+i}\right)^2\right)$

Let's try to show that we have for $k+1$: $P^{k+1}\left(x, \frac{1}{m+k+1}\right) = \prod_{i=0}^{k} \left(1 - \left(\frac{1}{m+i}\right)^2\right)$

$$P^{k+1}\left(x, \frac{1}{m+k+1}\right) = P\left(P^k\left(x, \frac{1}{m+k+1}\right)\right)$$

$$= \int P^k\left(y, \frac{1}{m+k+1}\right) P(x, dy) = \int P^k\left(y, \frac{1}{m+k+1}\right)\left[x^2 \int_{dy \cap [0,1]} du + (1-x^2)\delta_{1/m+1}\, dy\right.$$

$$= \left(1 - \left(\frac{1}{m}\right)^2\right) P^k\left(\frac{1}{m+1}, \frac{1}{m+k+1}\right)$$

Taking $m' = m+1$, we have

$$P^{k+1}\left(x, \frac{1}{m+k+1}\right) = \left(1 - \left(\frac{1}{m}\right)^2\right) P^k\left(\frac{1}{m}, \frac{1}{m'+k}\right)$$

Using our hypothesis on $P^k\left(x, \frac{1}{m+k}\right)$, we have

$$P^{k+1}\left(x, \frac{1}{m+k+1}\right) = \left(1 - \left(\frac{1}{m}\right)^2\right) \prod_{i=0}^{k-1} \left(1 - \left(\frac{1}{m'+i}\right)^2\right)$$

$$= \left(1 - \left(\frac{1}{m}\right)^2\right) \prod_{i=0}^{k-1} \left(1 - \left(\frac{1}{m+i+1}\right)^2\right)$$

$$= \left(1 - \left(\frac{1}{m}\right)^2\right) \prod_{i=0}^{k} \left(1 - \left(\frac{1}{m+i}\right)^2\right)$$

$$P^{k+1}\left(x, \frac{1}{m+k+1}\right) = \prod_{i=0}^{k} \left(1 - \left(\frac{1}{m+i}\right)^2\right)$$

We have shown that $P^n\left(x, \frac{1}{m+n}\right) = \prod_{i=0}^{n-1} \left(1 - \left(\frac{1}{m+i}\right)^2\right)$

b) Do we have $\lim_{n \to +\infty} P^n(x, A) = \pi(A)$ when $A = \bigcup_{q \in \mathbb{N}} \left\{\frac{1}{m+1+q}\right\}$

$$\pi(A) = \sum_{q \in \mathbb{N}} \pi\left(\left\{\frac{1}{m+1+q}\right\}\right) = 0$$

$\forall n \in \mathbb{N}_+^*$, we have:

$$P^n(x, A) = \sum_{q \in \mathbb{N}} P^n\left(\frac{1}{m}, \frac{1}{m+q+1}\right) = P^n\left(\frac{1}{m}, \frac{1}{m+n}\right)$$

$$= \prod_{i=0}^{n-1} \left(1 - \left(\frac{1}{m+i}\right)^2\right) = \prod_{i=0}^{n-1} \left(\frac{(m+i)^2 - 1}{(m+i)^2}\right)$$

$$= \prod_{i=0}^{n-1} \left(\frac{(m+i+1)(m+i-1)}{(m+i)^2}\right) = \frac{\prod_{i=0}^{n-1} (m+i-1) \cdot \prod_{i=0}^{n-1} (m+i+1)}{\prod_{i=0}^{n-1} (m+i)^2}$$

$$= \frac{\prod_{i=-1}^{n-2} (m+i) \times \prod_{i=1}^{n} (m+i)}{\prod_{i=0}^{n-1} (m+i)^2} = \frac{(m-1)(m+n)}{m(m+n-1)} = \frac{m-1}{m}\left(1 + \frac{1}{m+n-1}\right)$$

$$\geq \frac{1}{2} > \pi(A) = 0$$

We conclude that $\lim_{n \to +\infty} P^n(x, A) \neq \pi(A)$