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TP 4: Improve the Metropolis – Hastings algorithm

Exercise 1: Adaptive Metropolis - Hastings within Gibbs sampler

1.A We aim to sample on $\mathbb{R}^2$ the target distribution π given by

$$d\pi(x,y) \propto \exp\left(-\frac{x^2}{a^2} - y^2 - \frac{1}{4}\left(\frac{x^2}{a^2} - y^2\right)^2\right) dx\,dy$$

where $a>0$. We consider a Markov kernel P defined by:

$$P = \frac{1}{2}(P_1 + P_2)$$

where $P_i((x,y), dx', dy')$ for $i = 1, 2$ is the Markov transition kernel, ...

1. Implementation:
   See the notebook

2. Running the algorithm
   See the notebook

3. Improvement
   See the notebook

1.B. Adaptive Metropolis – Hastings with Gibbs sampler

let π be a density on an open set $\mathcal{U}$ on $\mathbb{R}^d$, $d \geq 2$

See the notebook

Exercise 2:

See the notebook

Exercise 3: Bayesian analysis of a one way random effects

Inverse Gamma distribution: $x \mapsto \frac{1}{x^{a+1}} \exp\left(-\frac{b}{x}\right) \mathbb{1}_{\mathbb{R}^+}(x)$

....

$$\pi_{prior}(\mu, \sigma^2, \tau^2) \propto \frac{1}{\sigma^{2(1+\alpha)}} \exp\left(-\frac{\beta}{\sigma^2}\right) \frac{1}{\tau^{2(1+\gamma)}} \exp\left(-\frac{\beta}{\tau^2}\right)$$

where α, β and γ are known hyper-parameters

1. Write the density of a posteriori distribution $(X, \mu, \sigma^2, \tau^2)$

We have
$$\mathbb{P}(X, \mu, \sigma^2, \tau^2 | Y) \propto P(Y, X, \mu, \sigma^2, \tau^2)$$
$$\propto \mathbb{P}(Y|X, \mu, \sigma^2, \tau^2)\, \mathbb{P}(X|\mu, \sigma^2)\, \mathbb{P}(\mu, \sigma^2, \tau^2)$$

$$\mathbb{P}(Y|X, \mu, \sigma^2, \tau^2) = \mathbb{P}(\varepsilon = Y_{i,j} - X_i | \mu, \sigma^2, \tau^2) \propto \left(\frac{1}{\sigma^2}\right)^{N/2} \exp\left(-\frac{\sum_i^N (X_i - \mu)^2}{2\sigma^2}\right)$$

We have:
$$\mathbb{P}(X, \mu, \sigma^2, \tau^2 | Y) \propto \left(\frac{1}{\tau^2}\right)^{NK/2} \exp\left(-\frac{\sum_i^N \sum_j^{k_i} (y_{i,j} - X_i)^2}{2\tau^2}\right) \sigma^{-N} \exp\left(-\frac{1}{2\sigma^2} \sum_{i=1}^N (x_i - \mu)^2\right)$$
$$\times\, \sigma^{-2(1+\alpha)} \exp\left(-\frac{\beta}{\sigma^2}\right) \frac{1}{\tau^{2(1+\gamma)}} \exp(-\beta/\tau^2)$$

$$\boxed{\mathbb{P}(X, \mu, \sigma^2, \tau^2 | Y) \propto \left(\frac{1}{\tau^2}\right)^{NK/2 + \gamma + 1} \left(\frac{1}{\sigma^2}\right)^{N/2 + \alpha + 1} \exp\left(-\frac{\sum_i^N \sum_j^{k_i} (y_{i,j} - X_i)}{2\tau^2} - \frac{\sum_i^N (X_i - \mu)^2}{2\sigma^2} - \frac{\beta}{\sigma^2} - \frac{\beta}{\tau^2}\right)}$$

2. Implement a Gibbs sampler which updates in turn $(\sigma^2, \tau^2, \mu, X)$ one at a time

We have to find the conditional probability for each parameter

- $\mathbb{P}(\sigma^2 | Y, X, \mu, \tau^2) \propto \left(\frac{1}{\sigma^2}\right)^{N/2+\alpha+1} \exp\left(-\frac{\sum_i^N (X_i-\mu)^2}{2\sigma^2} - \frac{\beta}{\sigma^2}\right)$

$$\propto \text{Inverse Gamma}\left(\frac{N}{2}+\alpha, \frac{1}{2}\sum_{i=1}^N (X_i-\mu)^2 + \beta\right)$$

- $\mathbb{P}(\mu | Y, X, \sigma^2, \tau^2) \propto \exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^N (X_i-\mu)^2\right)$

$$\propto \exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^N (X_i - 2\mu X_i + \mu^2)\right)$$

$$\propto \exp\left(-\frac{N}{2\sigma^2}\left(\left(\frac{\sum_i^N X_i}{N}\right)^2 - 2\mu\left(\sum^N X_i/N\right) + \mu^2\right)\right)$$

$$\propto \exp\left(-\frac{N}{2\sigma^2}\left(\frac{\sum_{i=1}^N X_i}{N} - \mu\right)^2\right)$$

$$\propto \mathcal{N}\left(\frac{\sum_i^N X_i}{N}, \frac{\sigma^2}{N}\right)$$

- $\mathbb{P}(\tau^2 | Y, X, \mu, \sigma^2) \propto \left(\frac{1}{\tau^2}\right)^{NK/2+\gamma+1} \exp\left(-\frac{\sum_{i=1}^N \sum_j^{k_i} (y_{i,j}-X_i)^2}{2\tau^2} - \frac{\beta}{\tau^2}\right)$

$$\propto \text{Inverse Gamma}\left(\gamma + \frac{NK}{2}, \frac{1}{2}\sum_{i=1}^N \sum_{j=1}^{K_i} (y_{i,j} - X_i)\right)$$

- $\mathbb{P}(X | Y, \mu, \sigma^2, \tau^2) \propto \exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^N (X_i-\mu)^2 - \frac{1}{2\tau^2}\sum_{i=1}^N\sum_{j=1}^{k_i} (X_{i,j} - X_i)^2\right)$

$$\propto \prod_{i=1}^N \exp\left(-\frac{(X_i^2 - 2\mu X + \mu^2)}{2\sigma^2} - \frac{1}{2\tau^2}\sum_{j=1}^{k_i} (Y_{i,j}^2 - 2Y_{i,j}X_i + X_i^2)\right)$$

$$\propto \prod_{i=1}^N \exp\left(-X_i^2\left(\frac{1}{2\sigma^2} + \frac{k_i}{2\tau^2}\right) + X_i\left(\frac{\mu}{\sigma^2} + \frac{\sum_j^{k_i} y_{i,j}}{\tau^2}\right) - \left(\frac{\mu^2}{2\sigma^2} + \frac{\sum_j^{k_i} y_{i,j}^2}{2\tau^2}\right)\right)$$

$$\propto \prod_{i=1}^N \exp\left(-\frac{1}{2\frac{\sigma^2\sigma^2}{\tau^2+k_i\tau}}\left(\frac{\tau^2\mu + \sigma^2\sum_j^{K_i} y_{i,j}}{\tau^2 + k_i\sigma^2} - X_i\right)^2\right)$$

$$X_i | Y_i, \mu, \sigma^2, \tau^2 \sim \mathcal{N}\left(\frac{\tau^2\mu + \sigma^2\sum_j^{K_i} y_{i,j}}{\tau^2 + k_i\sigma^2}, \frac{\sigma^2\tau^2}{k\sigma^2 + \tau^2}\right)$$

since the X_i are independant.

See the notebook for the implementation

3. Implement a Block-Gibbs sampler which updates σ^2, then τ^2 and then the block (X, μ)

We have to compute the conditional probability of X, μ given Y, σ^2, τ^2

$$\mathbb{P}(X, \mu | Y, \sigma^2, \tau^2) \propto \prod_{i=1}^N \exp\left(-\frac{\sum_j^{k_i} (y_{i,j} - X_i)^2}{2\tau^2} - \frac{1}{2\sigma^2}(X_i - \mu)^2\right)$$

$$\propto \prod_{i=1}^N \exp\left(-\frac{1}{2\tau^2}\left(\sum_{j=1}^{k_i} X_i^2 - 2Y_{i,j}X_i + Y_{i,j}^2\right) - \frac{1}{2\sigma^2} X_i^2 - 2X_i\mu + \mu^2\right)$$

$$\propto \prod_{i=1}^N \exp\left(-\frac{1}{2}\left(\frac{(\sigma^2 k_i + \tau^2)X_i^2}{\tau^2\sigma^2} - \frac{2\sum_{j=1}^{k_i} y_{i,j}X_i}{\tau^2} - \frac{2X_i\mu}{\sigma^2} + \frac{\mu^2}{\sigma^2}\right)\right)$$

We have $X, \mu | Y, \sigma^2, \tau^2 \sim \mathcal{N}(\mu, \Sigma)$

$$\Sigma = \begin{pmatrix} \frac{\sigma^2 k + \gamma^2}{\gamma^2 \sigma^2} & 0 & 0 & -1/\sigma^2 \\ 0 & \ddots & & \\ 0 & & \frac{\sigma^2 k + \gamma^2}{\gamma^2 \sigma^2} & -1/\sigma^2 \\ -\frac{1}{\sigma^2} & & -1/\sigma^2 & N/\sigma^2 \end{pmatrix}$$

$$\mu = \Sigma \times \left(\frac{\sum_{j=1}^{k} y_{1,j}}{\gamma^2} \; \cdots \; \frac{\sum_{j=1}^{k} y_{i,j}}{\gamma^2} \right)$$