Boubacar SOW - M2 MVA ①

Homework 1 - Convex optimization

Exercise 1: Wich of the following sets are convex?

1. A rectangle; i.e a set of the form
$$\{x \in \mathbb{R}^n \mid \alpha_i \leq x_i \leq \beta_i, i=1 \dots n\}$$

A rectangle is a convex set because it is a finite intersection of halfspaces.

Moreover, we can show that for any two points x, y s.t $\alpha_i \leq x_i \leq \beta_i$ and $\alpha_i \leq y_i \leq \beta_i$, $i=1 \dots n$, the line segment connecting x and y is entirely with the rectangle

We have
$$(\theta x + (1-\theta) y)_i = \theta x_i + (1-\theta) y_i$$
$$\theta x_i + (1-\theta) y_i \geq \theta \alpha_i + (1-\theta)\alpha_i = \alpha_i \qquad (1)$$
$$\theta x_i + (1-\theta) y_i \leq \beta \alpha_i + (1-\theta)\beta_i = \beta_i \qquad (2)$$

From (1) and (2), we write: $\alpha_i \leq \theta x_i + (1-\theta) y_i \leq \beta_i$

2. The hyperbolic set $\{x \in \mathbb{R}^2_+ \mid x_1 x_2 \geq 1\}$

The set is convex if any combination between two points in the set is in the set.

Let's $x = (x_1, x_2)$ and $y = (y_1, y_2)$ and $z = \theta x + (1-\theta) y$ a point in the segment $[x, y]$

We want to show that z is also in the set ②

This means that we want to show that
$$(\theta x_1 + (1-\theta) y_1)(\theta x_2 + (1-\theta) y_2) \geq 1$$
$$= \theta^2 x_1 x_2 + (1-\theta)^2 y_1 y_2 + \theta(1-\theta) x_1 y_2 + \theta(1-\theta) x_2 y_1$$
$$= \theta^2 x_1 x_2 + (1-\theta)^2 y_1 y_2 + \theta(1-\theta) \underbrace{(x_1 y_2 + x_2 y_1)}_{\geq 0 \text{ as } x \in \mathbb{R}^2_+}$$
$$\geq \theta^2 + (1-\theta)^2 + \theta - \theta^2$$
$$\geq \theta^2 + 1 - 2\theta + \cancel{\theta^2} + \theta - \cancel{\theta^2}$$
$$\geq \theta^2 + 1 - \theta$$
$$\geq 1$$

We can conclude that the hyperbolic set is convex.

3). The set of point closer to a given point than a given set, i.e $\{x \mid \|x - x_0\|_2 \leq \|x - y\|_2 \ \forall y \in S\}$

$$\|x - x_0\|_2 \leq \|x - y\|_2 \Leftrightarrow (x - x_0)^T (x - x_0) \leq (x-y)^T (x-y)$$
$$\Leftrightarrow x^T x - x^T x_0 - x_0^T x + x_0^T x_0 \leq x^T x - x^T y - y^T x + y^T y$$
$$\Leftrightarrow -2 x_0^T x + x_0^T x_0 \leq -2 y^T x + y^T y$$
$$\Rightarrow 2 (y - x_0)^T x \leq y^T y - x_0^T x_0$$

This is the equation of a half space expressed as $Ax \leq b$ with $A = 2\begin{bmatrix} y_1 - x_0 \\ y_2 - x_0 \\ \dots \\ y_n - x_0 \end{bmatrix}$ and $b = \begin{bmatrix} y_1^T y - x_0^T x_0 \\ y_2^T y - x_0^T x_0 \\ \dots \\ y_n^T y - x_0^T x_0 \end{bmatrix}$

The set is then an intersection of half spaces, we can conclude that the set is convex

$$\bigcap_{y \in S} \{x \mid \|x - x_0\|_2 \leq \|x - y\|_2\}$$

$$\bigcap_{y \in S} \{x \mid Ax \leq b\}$$

4. The set of points closer to one set than another, i.e

$$\{x \mid dist(x, S) \leq dist(x, T)\}$$

Let's have $A = \{-1; 1\}$ and $B = \{0\}$

The set of points closer to A are

$$\{x \mid dist(x, A) \leq dist(x, B)\} = \{x \in \mathbb{R} \mid x \leq -\frac{1}{2} \text{ or } x \geq \frac{1}{2}\}$$

This is clearly not convex as if you take two points in the set $x = -1$ and $y = 1$ for example, the line segment connecting these two points is not entierly in the set. For instance the point 0 lies between 0 and 1 but does not satisfy $x \leq -\frac{1}{2}$ or $x \geq \frac{1}{2}$.

5. The set $\{x \mid x + S_2 \subseteq S_1\}$ where $S_1, S_2 \subseteq \mathbb{R}^n$ and S_1 convex

let's take any two points x, y in the set

$\equiv$ $x + S_2 \subseteq S_1$

$y + S_2 \subseteq S_1$

let's $z = \theta x + (1-\theta) y$ where $\theta \in [0, 1]$

let's show that $z + S_2 \subseteq S_1$.

$\forall s \in S_2$, we have $z + s = \theta x + (1-\theta) y + s$

$$= \theta(x + s) + (1-\theta)(y + s)$$

$x + s \in S_1$ and $y + s \in S_1 \Rightarrow z + s \in S_1 \quad \forall s \in S_2$

$$\Rightarrow z + S_2 \subseteq S_2$$

and z is in the set.

We conclude that the set is convex since for any $\theta \in [0,1]$ any two points in the set, the points on the line segment are also on the set.

Exercise 2: For each of the following functions, determine whether it is convex or not.

1. $f(x_1, x_2) = x_1 x_2$ on $\mathbb{R}^2_{++}$

f is convex iff it's domain is convex and $\nabla^2 f(x_1, x_2) \geq 0$

. $\mathbb{R}^2_{++}$ is convex

. $\frac{\partial f}{\partial x_1} = x_2$; $\frac{\partial f}{\partial x_2} = x_1$; $\frac{\partial^2 f}{\partial x_1} = 0$; $\frac{\partial^2 f}{\partial x_2} = 0$

$\frac{\partial^2 f}{\partial x_1 \partial x_2} = 1$; $\frac{\partial^2 f}{\partial x_2 \partial x_1} = 1$

$$\Rightarrow \nabla^2 f(x_1, x_2) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$\det(\nabla^2 f - \lambda Id) = 0 \Rightarrow \det\begin{pmatrix} -\lambda & 1 \\ 1 & -\lambda \end{pmatrix} = 0$ (5)

$\Rightarrow \lambda^2 - 1 = 0 \Rightarrow \lambda = 1$ or $\lambda = -1$

$\Rightarrow \nabla^2 f(x_1, x_2) \not\succeq 0$ and $\nabla^2 f(x_1,x_2) \not\preceq 0$

We conclude that the function is not convex nor concave.

Optional: quasiconvexity and quasiconcavity.

$f(x_1,x_2)$ is quasiconvex if it's sublevel sets $S_\alpha = \{x_1, x_2 \in \mathbb{R}^2_{++} \mid f(x_1,x_2) \leq \alpha\}$ are convex for all α.
$x_1 x_2 \leq \alpha$

let's $x = (x_1, x_2)$ and $y = (y_1, y_2)$, we need to show that $z = \theta x + (1-\theta) y$, $\forall \theta \in [0,1]$ is in S_α

$z_1 z_2 = [\theta x_1 + (1-\theta) y_1][\theta x_2 + (1-\theta) y_2]$

$= \theta^2 \underbrace{x_1 x_2}_{\geq \alpha} + (1-\theta)^2 \underbrace{y_1 y_2}_{\geq \alpha} + \underbrace{\theta(1-\theta)}_{\geq 0} (\underbrace{x_1 y_2}_{\geq 0} + \underbrace{y_1 x_2}_{\geq 0})$

$z_1 z_2 \geq \theta^2 \alpha + (1-\theta)^2 \alpha$

$z_1 z_2 \geq \alpha$

Since $z_1 z_2 \geq \alpha$, the function is not quasiconvex

Since $-z_1 z_2 \leq \alpha$, the function is quasiconcave.

$f(x_1, x_2)$ is not convex, nor concave; it is quasiconcave and not quasiconvex.

2) $f(x_1,x_2) = \dfrac{1}{x_1 x_2}$ on $\mathbb{R}^2_{++}$ (6)

$\dfrac{\partial f(x_1,x_2)}{\partial x_1} = \dfrac{-1}{x_1^2 x_2}$; $\dfrac{\partial^2 f(x_1,x_2)}{\partial^2 x_1} = \dfrac{2}{x_1^3 x_2}$; $\dfrac{\partial f(x_1,x_2)}{\partial x_1 \partial x_2} = \dfrac{+1}{x_1^2 x_2^2}$

$\dfrac{\partial f(x_1,x_2)}{\partial x_2} = \dfrac{-1}{x_1 x_2^2}$; $\dfrac{\partial^2 f(x_1,x_2)}{\partial^2 x_2} = \dfrac{2}{x_1 x_2^3}$; $\dfrac{\partial f(x_1,x_2)}{\partial x_2 \partial x_1} = \dfrac{+1}{x_1^2 x_2^2}$

$$\nabla^2 f(x_1,x_2) = \begin{bmatrix} \dfrac{2}{x_1^3 x_2}\,a & \dfrac{1}{x_1^2 x_2^2}\,b \\ \dfrac{1}{x_1^2 x_2^2}\,b & \dfrac{2}{x_1 x_2^3}\,c \end{bmatrix}$$

$\det(\nabla^2 f(x_1,x_2)) = \dfrac{4}{x_1^4 x_2^4} - \dfrac{1}{x_1^4 x_2^4} = \dfrac{3}{x_1^4 x_2^4} > 0$

$Tr(\nabla^2 f(x_1,x_2)) = \dfrac{2}{x_1^3 x_2} + \dfrac{2}{x_1 x_2^3} > 0$

Since $\nabla^2 f(x,y)$ is 2×2 matrix and both determinant and trace are positive, we can deduce that the matrix $\nabla^2 f(x,y)$ is positive definite.

The domain $\mathbb{R}^2_{++}$ is convex. We finally conclude that $f(x_1,x_2)$ is a convex function

Optional: • quasiconvexity

Since the fonction is convex, it is thus quasiconvex.

• quasiconcavity:

$f(x_1,x_2)$ is quasiconcave, iff it's superlevel sets $S_\alpha = \{(x_1,x_2) \in \mathbb{R}^2_{++} \mid f(x_1,x_2) \geq \alpha\}$ is convex

$S_\alpha = \{(x_1,x_2) \in \mathbb{R}^2_{++} \mid \frac{1}{x_1 x_2} \geq \alpha\}$ (7)

$S_\alpha = \{ \ldots \mid x_1 x_2 \leq \frac{1}{\alpha}\}$

$S_\alpha = \{(x_1,x_2) \in \mathbb{R}^2_{++} \mid x_1 x_2 \leq \alpha'\}$

From the previous exercise, we conclude that S_α is not convex because a point z that is a convex combination between two points in the set is not in the set

$f(x,x_2)$ is convex and quasiconvex. It is not quasiconcave nor concave.

) $f(x_1,x_2) = \frac{x_1}{x_2}$ on $\mathbb{R}^2_{++}$

$\frac{\partial f}{\partial x_1} = \frac{1}{x_2}$; $\frac{\partial f(x,x_2)}{\partial x_2} = \frac{-x_1}{x_2^2}$;

$\frac{\partial f(x,x_2)}{\partial x_1^2} = 0$; $\frac{\partial f(x_1,x_2)}{\partial^2 x_2} = \frac{2x_2 x_1}{x_2^4}$

$$\nabla^2 f(x,y) = \begin{pmatrix} 0 & -\frac{1}{x_2^2} \\ -\frac{1}{x_2^2} & \frac{2x_1}{x_2^3} \end{pmatrix}$$

et $\nabla^2 f(x_1,x_2) = 0 - \frac{1}{x^4} = -\frac{1}{x^4} < 0$

This matrix can not be positive semi-definite, since the eigen values are of different signs. Then (8) $\nabla f^2(x,y) \not\geq 0$ and $\nabla f^2(x,y) \not\leq 0$. The function not concave nor convex.

Optional: Quasiconvexity and quasiconcavity

$f(x,y)$ is quasiconvex if dom f is convex and the sublevel sets $S_\alpha = \{(x_1,x_2) \in \text{dom } f \mid f(x_1,x_2) \leq \alpha\}$ is convex

$= \{ \ldots \mid \frac{x_1}{x_2} \leq \alpha$

$= \{ \ldots \mid x_1 \leq \alpha x_2\}$

This is a half space in $\mathbb{R}^2_{++}$, wich is convex. Since $\mathbb{R}^2_{++}$ is also convex, we conclude that $f(x_1,x_2)$ is quasiconvex.

- $f(x_1,x_2)$ is quasiconcave if $S_\alpha = \{(x_1,x_2) \in \mathbb{R}^2_{++} \mid f(x_1,x_2) \ldots$ is convex

$S_\alpha = \{(x_1,x_2) \in \mathbb{R}^2_{++} \mid \frac{x_1}{x_2} \geq \alpha\}$

$= \{(x_1,x_2) \in \mathbb{R}^2_{++} \mid x_1 \geq \alpha x_2\}$

Again it is a half space. Since $\mathbb{R}^2_{++}$ is convex, S_α being convex, we conclude that $f(x_1,x_2)$ is quasiconcave

$f(x_1,x_2)$ is not convex nor concave, it is quasiconvex

4. $f(x_1, x_2) = x_1^{\alpha} x_2^{1-\alpha}$, where $0 \leq \alpha \leq 1$, on $\mathbb{R}^2_{++}$ 9

$\operatorname{dom} f = \mathbb{R}^2_{++}$ is convex.

$$\frac{\partial f}{\partial x_1} = \alpha x_1^{\alpha-1} x_2^{1-\alpha} \; ; \; \frac{\partial f}{\partial x_2} = (1-\alpha) x_1^{\alpha} x_2^{-\alpha}$$

$$\nabla^2 f(x_1, x_2) = \begin{pmatrix} \alpha(\alpha-1) x_1^{\alpha-2} x_2^{1-\alpha} & \alpha(1-\alpha) x_1^{\alpha-1} x_2^{-\alpha} \\ \alpha(1-\alpha) x_1^{\alpha-1} x_2^{-\alpha} & -\alpha(1-\alpha) x_1^{\alpha} x_2^{-\alpha-1} \end{pmatrix}$$

$$\det \nabla^2 f(x_1, x_2) = \alpha^2(\alpha-1)^2 x_1^{2\alpha-2} x_2^{-2\alpha} - \alpha^2(\alpha-1)^2 x_1^{2\alpha-2} x_2^{-2\alpha} = 0$$

$$\operatorname{Tr}(\nabla^2 f(x_1, x_2)) = \underbrace{\alpha(\alpha-1)}_{\leq 0} [\underbrace{x_1^{\alpha} x_2^{-\alpha-1} + x_1^{\alpha-2} x_2^{1-\alpha}}_{\geq 0}] \leq 0$$

Given that the determinant is zero and the trace is negative, all eigen values are non positive $\Rightarrow$ The Hessian is negative semi-definite, $f(x_1, x_2)$ is concave.

- Optional: quasiconvexity and quasiconcavity

Since $f(x_1, x_2)$ is concave, $f(x_1, x_2)$ is necessary quasiconcave.

$f(x_1, x_2)$ is quasiconvex if $S_\beta = \{(x_1, x_2) \in \mathbb{R}^2_{++} \mid f(x_1, x_2) \leq \beta\}$ are ~~is~~ convex.

~~Let's~~ $\alpha = \frac{1}{2}$ and consider $S_\beta = \{(x_1, x_2) \in \mathbb{R}^2 \mid f(x)$

Let's $\alpha = \frac{1}{2}$ and consider $S_\beta = \{(x_1, x_2) \in \mathbb{R}^2 \mid f(x) \leq \beta\}$ 10

$S_\beta = \{x \in \mathbb{R}^2 \mid \sqrt{x_1 x_2} \leq \beta\}$ for $\beta > 0$

$S_\beta = \{x \in \mathbb{R}^2 \mid x_1 x_2 \leq \beta^2\}$

S_β is not convex, see exercise 2.L for more details

Exercise 4: Monotone negative cone as

$$K_{m+} \equiv \{x \in \mathbb{R}^n \mid x_1 \geq x_2 \geq \cdots \geq x_n \geq 0\}$$

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1. Show that K_{m+} is a proper cone

We need to verify the closedness, pointedness and the solidness of K_{m+}

* K_{m+} is closed:

$$K_{m+} = \{x \in \mathbb{R}^n \mid x_1 \geq x_2 \geq \cdots \geq x_n \geq 0\}$$

Let's $H_i = \{x_i \in \mathbb{R}^n \mid x_i \geq x_{i+1}\}$ Halfspaces

$H_0 = \{x \in \mathbb{R}^n \mid x_n \geq 0\}$ Halfspace

$$K_{m+} = H_0 \cap H_1 \cap \cdots \cap H_{n-1}$$

Since every half space is closed, K_{m+} the intersection of halfspaces is also closed

* pointedness:

K_{m+} is pointed $\Rightarrow \nexists\, x, y \in K$ s.t $x = -y$ and $x \neq 0$

Suppose that $\exists\, x, y$ s.t. $x = -y$ and $x \neq 0$

$\Rightarrow x_i = -y_i \quad \forall i$

Since x and $y \in K_{m+}$, then $x \geq 0$ and $y \geq 0$

If $x_i = -y_i \Rightarrow x_i = y_i = 0 \Rightarrow$ Contradiction.

Hence K_{m+} is pointed.

• solidness: K_{m+} is solid because it's interior is nonempty

We have $x = (n, n-1, \ldots, n)$ which statisfis the inequalities with strict inequality $\Rightarrow K_{m+} \neq \emptyset$

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$\Rightarrow K_{m+}$ is a proper cone.

2. Find the dual cone K^*_{m+}

$$K^*_{m+} = \{y \in \mathbb{R}^n \mid y^T x \geq 0, \forall x \in K_{m+}\}$$

$$y^T x = \sum_{i=1}^n x_i y_i = (x_1 - x_2) y_1 + (x_2 - x_3)(y_1 + y_2) + \cdots + (x_{n-1} - x_n)(y_1 + \cdots + y_{n-1}) + x_n(y_1 + \cdots + y_n) \geq 0$$

Since $x_i \geq x_{i+1}\ \forall i$ and $x_i \geq 0\ \forall i$, we have

$y^T x \geq 0$ iff:

$$\left.\begin{array}{l} y_1 \geq 0 \\ (y_1 + y_2) \geq 0 \\ y_1 + y_2 + y_3 \geq 0 \\ \cdots \\ (y_1 + y_2 + \cdots + y_n) \geq 0 \end{array}\right\} \Rightarrow y \in K_m^+$$

$$K^*_{m+} = \{y \mid \sum_{i=1}^k y_i \geq 0, k = 1, \ldots, n\}$$

Exercise 3: Show that the following functions are convex:

1) $f(X) = Tr(X^{-1})$ on $dom\, f = S^n_{++}$

. S^n_{++} is first a convex set.

We define $g(t) = Tr((X+tV)^{-1})$ to study how $f(X)$ change when we move in the direction of V from the matrix point X

We have
$$g(t) = Tr((X+tV)^{-1})$$
$$= Tr[(X^{1/2}(I+tX^{-1/2}VX^{-1/2})X^{-1/2})^{-1}]$$
$$= Tr[X^{-1/2}((I+tX^{-1/2}VX^{-1/2})X^{1/2})^{-1}]$$
$$= Tr[X^{-1}(I+tX^{-1/2}VX^{-1/2})^{-1}]$$

Eigen value decomposition on the matrix $X^{-1/2}VX^{-1/2} \Rightarrow Q\Lambda Q^T$, Q orthogonal matrix whose columns are the eigenvectors of $X^{-1/2}VX^{-1/2}$ and Λ a diagonal matrix whose diagonal elements are the corresponding eigenvalues $\Lambda_{ii} = \lambda_i$

$$\Rightarrow g(t) = Tr(X^{-1}(I+tQ\Lambda Q^T)^{-1})$$
$$= Tr(X^{-1}(QQ^T + tQ\Lambda Q^T)^{-1})$$
$$= Tr(X^{-1}(Q(Q^T + t\Lambda Q^T))^{-1})$$
$$= Tr(X^{-1}Q(I+t\Lambda)^{-1}Q^T)$$

Since the trace operator is invariant under cyclic permutations, we get
$$g(t) = Tr(Q^T Z^{-1} Q(I+t\Lambda)^{-1})$$

$$\Rightarrow g(t) = \sum_{i=1}^n \underbrace{(Q^T Z^{-1} Q)_{ii}}_{\geq 0} \underbrace{(1+t\lambda_i)^{-1}}_{\text{convex in } t}$$

(Positive weighted sum of convex functions) $\Rightarrow$ convex

$\Rightarrow$ We conclude that $f(X)$ is convex.

2). $f(X,y) = y^T X^{-1} y$ on $dom\, f = S^n_{++} \times \mathbb{R}^n$

. First, dom f is convex

. $f(x,y) = y^T X^{-1} y = \sup_{z \in \mathbb{R}^n} (y^T z - \frac{1}{2} z^T X z)$ — sum of convex functions

$g(x,y) = y^T z - \frac{1}{2} z^T X z$ is linear in y and z so it is convex. The supremum of a convex function is convex $\Rightarrow$ We conclude that $f(x,y)$ is convex in X and y.

3) $f(x) = \sum_{i=1}^n \sigma_i(X)$ on $dom\, f = S^n$, $\sigma_i(X)$ are singular values of $X \in \mathbb{R}^{n\times n}$

. First dom f is convex.

. The singular value decomposition X is given by $X = U\Sigma V^T$, where U and V are orthogonal matrices and Σ is a diagonal matrix with the singular values of X on its diagonal

let's Q a diagonal matrix an orthogonal matrix we have
$$|Tr(QX)| = |Tr(QU\Sigma V^T)|$$
$$= |Tr(V^T QU\Sigma)|$$
$$= \sum_{i=1}^n |C_{ii}|\, \sigma(X)$$

where $C_{ii} = (V^T QU)_{ii}$, $|C_{ii}| \leq 1$ because it's the entries of the product of orthogonal matrices [C_{ii} is the entries of the product $V^T U Q$ or dot product of the rows of V^T and the colums of UQ, since the dot product of two orthonormal vectors is either 0 (for different vectors) or 1 (for the same vector), the absolute for each entry in the resulting matrix is less than or equal to 1 $\Rightarrow |C_{ii}| \leq 1$]

$$\leq \sum_{i=1}^n \sigma_i(X)$$

$\Rightarrow f(x) = \sup_{Q\in O(n)} |Tr(QX)| = \sup_{Q\in O(n)} h(Q,X)$ where $h(Q,X) = |Tr(QX)|$

The function $h(Q,X)$ is linear in X for fixed Q. Since linear functions are both convex and concave $h(Q,X)$ is convex in X. The supremum of a family of convex functions is also convex

$\Rightarrow$ $f(x)$ is convex.

Exercise 5: Derive the conjugates of the following functions:

1. Max function $f(x) = \max_{i\in 1,\dots,n} x_i$ on $\mathbb{R}^n$

$$f^*(y) = \sup_{x\in\mathbb{R}^n} (\langle x,y\rangle - f(x))$$
$$= \sup_{x\in\mathbb{R}} y^Tx - \max_{1\le i\le n} x_i$$
$$\le \sup_{x\in\mathbb{R}} \max_{1\le i\le n} x_i \left(\sum_{i=1}^n y_i - 1\right)$$

Let's cover all the possible cases for y.

1. y has a negative component $y_k < 0$. In this case, there exists x s.t $x_i = 0$ for $i \neq k$
   If we choose $x_k = -t$ we have if we let $t \to +\infty$
   $$\Rightarrow y^Tx - \max x_i = -ty - 0 = -t\underbrace{y_k}_{\le 0} \xrightarrow{t\to\infty} \infty$$
2. y is non negative i.e $y \succeq 0$
   We have 3 cases dependin on $\sum_{i=1}^n y_i$
   - If $\sum y_i > 1$ (or $1^T y > 1$), there exist x st $x = t1$ (constant vector) and if $t \to +\infty$, we have
   $$x^Ty - \max x_i = t1^Ty - t = t(1^Ty - 1) \xrightarrow{t\to+\infty} +\infty$$
   - If $\sum y_i < 1$, $\exists\, x \mid x = -t1$, $t\to\infty$
   $$-t1^Ty + t = t(-1^Ty + 1) \xrightarrow{t\to+\infty} +\infty$$
   - If $\sum y_i = 1$, we have:
   $$y^T x_i - \max_{1\le i\le n} x_i \le 0$$
   $$f^*(y) \le \sup_{x\in\mathbb{R}} \max_{1\le i\le n} x_i \left(\sum_{i=1}^n y_i - 1\right) = \max x_i (1-1) = 0$$

Finally
$$f^*(y) = \begin{cases} 0 & \text{if } y \ge 0 \text{ and } y^T1 = 1 \\ +\infty & \text{Otherwise} \end{cases}$$

2. Sum of largest element:
$$f(x) = \sum_{i=1}^r x[i] \text{ on } \mathbb{R}^n$$
$$f^*(y) = \sup \langle y, x\rangle - \sum_{i=1}^r x[i]$$

Let's cover all the possible cases for y.

1. y has a negative component $y_k < 0$. In this case, there exists x s.t $x_i = 0$ for $i \neq k$
   If we choose $x_k = -t$ we have if we let $t \to +\infty$
   $$\_^*(y) = -t\underbrace{y_k}_{\le 0} \xrightarrow{t\to\infty} +\infty$$
2. Let's $y_k > 1$, $\exists\, x_k = t$, $x_i = 0\ \forall i \neq k$ s.t if $t \to +\infty$:
   $$y^Tx - f(x) = y_k t - t = t(y_k - 1) \xrightarrow{t\to\infty} +\infty$$
3. We thus necessarily that $0 \le y \le 1$
   - If $\sum_{i=1}^r y_i \neq r$, $\exists$ $x = t.1$ (constant vector) st:
   $$ty^T - rt = t(\sum y_i - r) \xrightarrow{t\to\infty} +\infty$$
   - If $\sum_{i=1}^r y = r$, we have
   $$g(x) = y^Tx - \sum_i^n x[i] = \sum_{i=1}^n y_i x_i - \sum_{i=1}^r x[i]$$
   $$= \sum_{i=1}^r (y_i - 1)x_{[i]} + \sum_{i=r+1}^n y_i x_i$$
   $$\le \sum_{i=1}^r (y_i - 1)x_r + \sum_{i=r+1}^n y_i x_i$$
   $$= \left(\sum_{i=1}^n y_i - r\right) x_r$$
   $$g(x) = 0$$

   Finally, we write: $f^*(y) = \begin{cases} 0 & \text{if } 0 \le y \le 1,\ 1^Ty = r \\ \infty & \text{Otherwise} \end{cases}$

3. Piece wise linear function on $\mathbb{R}$: $f(x) = \max_{i=1,\dots,m}(a_i x + b_i)$ on $\mathbb{R}$

Assumptions : • a_i are sorted in increasing order. ie $a_1 \leq \dots \leq a_m$

• None of the functions $a_i x + b_i$ is redundant, ie. $\forall k, \exists x$ s.t $f(x) = a_k x + b_k$.

$$f^*(y) = \sup_{x \in \operatorname{dom} f} \langle y, x \rangle - f(x)$$

$$= \sup_{x \in \operatorname{dom} f} y^T x - \max_{i=1,\dots,m}(a_i x + b_i)$$

If $y > \max x_i$ or $y < \min x_i$, the expression $y^T x - f(x)$ is unbounded. This is because for any x, there will be an i s.t $a_i x + b_i < y^T x$ or $a_i x + b_i > y^T x$ $\Big\}$ as $x \to \infty$, $y^T x - f(x) \to \infty$

$\Rightarrow$ The domain of f^* is $[\min(a_i), \max(a_i)] = [a_1, a_m]$

Since $f(x)$ is piecewise linear function, then it is by several line segments each corresponding to a different linear function $a_i x + b_i$. The breakpoints between the segments i and $i+1$ is the value of x where the maximum switches from one segment to the next, i.e. where

$$a_i x + b_i = a_{i+1} x + b_{i+1}$$

$$\Rightarrow a_i x - a_{i+1} x = b_{i+1} - b_i$$

$$\Rightarrow x^* = \frac{b_{i+1} - b_i}{a_i - a_{i+1}}$$

If $y \in [a_1, a_m] \Rightarrow a_i \leq y \leq a_{i+1} \Rightarrow \begin{cases} y - a_j \geq 0 & \text{for } j = 1, \dots, i \\ y - a_{j+1} \leq 0 & \text{for } j = i, \dots, m-1 \end{cases}$

$\Rightarrow$ The maximum is reached at the breakpoint between the two segments i and $i+1$

$$\Rightarrow f^*(y) = y x^* - a_i x^* - b_i$$

$$= (y - a_i) \frac{b_{i+1} - b_i}{a_i - a_{i+1}} - b_i$$

$$f^*(y) = -b_i - (b_{i+1} - b_i) \frac{(y - a_i)}{a_{i+1} - a}$$

$$\Rightarrow f^*(y) = \begin{cases} -b_i - (b_{i+1} - b_i) \dfrac{(y - a_i)}{a_{i+1} - a} & \text{if } y \in [a_1, a_m] \\ +\infty & \text{Otherwise} \end{cases}$$