Mathematics for Political Science

Day 2 – Algebra

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Overview

- Algebra manipulates objects using operations (from yesterday)
- Apply operations to equations to determine what value(s) a variable (parameter) must take on to make a mathematical expression true (that is, to make the expression hold with equality or inequality)
- "Solving for unknown values"

Algebra in Political Science

- Algebra is a fundamental basis for more advanced mathematical manipulation
  - Use to derive statistical estimators, and to understand their properties and the assumptions necessary to apply them.
  - Use to evaluate the optimal choices of strategic actors

Agenda

(1) Basic Algebra

(2) Multivariate Algebra

(3) Vector Algebra*

(4) Matrix Algebra*

Solving for x: One Unknown

- Bring elements with that variable to one side of the expression and elements without it to the other
  - Add/subtract the same term from both sides
  - Multiply/divide all terms by the same (non-zero) number
- Factor out the variable to isolate it
- Divide to move all other terms to the other side to solve
- Check your answer*

$$\begin{aligned} ax + b &= c \\ ax &= c - b \\ x &= \frac{c-b}{a} \end{aligned}$$

$$\begin{aligned} dx + e &= fx + g \\ dx - fx &= g - e \\ x(d - f) &= g - e \\ x &= \frac{g-e}{d-f} \end{aligned}$$

Solving for x

A numerical example:

$$\begin{aligned} 4x - 3 &= 2x + 8 \\ 4x &= 2x + 11 \\ 2x &= 11 \\ x &= 5.5 \end{aligned}$$

With another variable:

$$\begin{aligned} 4xy + 3x + 2 &= 8y + 7x + 1 \\ 4xy + 3x &= 8y + 7x - 1 \\ 4xy - 4x &= 8y - 1 \\ x(4y - 4) &= 8y - 1 \\ x &= \frac{8y - 1}{4y - 4} \end{aligned}$$

Solving for a Range of x

With inequality expressions, a *range* of values may make the expression true rather than a specific value.

$$\begin{aligned} 3x - 1 &> 2x + 3 \\ 3x &> 2x + 4 \\ x &> 4 \end{aligned}$$

$$\begin{aligned} 5x + 1 &\leqslant 15x + 3 \\ 5x &\leqslant 15x + 2 \\ -10x &\leqslant 2 \\ x &\geqslant -.2 \end{aligned}$$

Be careful of the direction of the inequality!

- Addition or subtraction has no effect, regardless of what is added or subtracted
- Multiplication or division by negative values flips the direction of the inequality if the number is negative

Solving for x

Solve the following equations for x:

- $10x + 8 = 3x + 30$
- $3(x + 4) - (2 - x) = 5x + 2$
- $3x + 18 \leqslant 12 - 6x$

Dealing with x^2

With higher order terms, it may not be possible to isolate the variable

$$\begin{aligned} x^2 + 14 &= 7x + 2 \\ x^2 - 7x &= -12 \\ x(x-7) &= -12 \end{aligned}$$

Two solutions:

- Factorization
- Quadratic Formula

Both begin by manipulating the expression to make one side zero:

$$\begin{aligned} x^2 + 14 &= 7x + 2 \\ x^2 - 7x &= -12 \\ x^2 - 7x + 12 &= 0 \end{aligned}$$

Factorization

To factor, look for two simpler polynomials that multiply to produce your given expression and set equal to zero.

Factoring

$$(x-r)(x+s) = x^2 + bx + c = 0$$

Finding factors can be more art than science:

- Consider pairs of numbers r and s such that $r * s = c$
- Determine whether for any such pair:
  - $r + s = b$
  - $r - s = b$
  - $-r + s = b$
  - $-r - s = b$
- Adjust the positives and negatives to make it work
- Check carefully

Factorization

Consider $0 = x^2 - 7x + 12$:

- note that $4 * 3 = 12$ and $6 * 2 = 12$
- note that $-4 - 3 = -7$
- try $(x - 3) * (x - 4)$
- check:

$$\begin{aligned} 0 &= (x-3)(x-4) \\ 0 &= x^2 - 3x - 4x + 12 \\ 0 &= x^2 - 7x + 12 \\ x &= 4 \text{ or } x = 3 \end{aligned}$$

Consider $0 = x^2 - 4$:

- note that $4 * 1 = 4$ and $2 * 2 = 4$
- note that $2 - 2 = 0$
- try $(x + 2) * (x - 2)$
- check:

$$\begin{aligned} 0 &= (x+2)(x-2) \\ 0 &= x^2 + 2x - 2x - 4 \\ 0 &= x^2 + 0x - 4 \\ 0 &= x^2 - 4 \\ x &= -2 \text{ or } x = 2 \end{aligned}$$

Quadratic Formula

Quadratic Formula

$$ax^2 + bx + c = 0 \quad \Rightarrow \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

A less intuitive but more general method, the quadratic formula will always find any real numbers x that solve the equation (if there are any). It's particularly useful when the expression does not factor easily.

Quadratic Formula

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Consider $0 = x^2 - 7x + 12$:

- $a = 1, b = -7, c = 12$

$$0 = x^2 - 7x + 12$$
$$x = \frac{-(-7) \pm \sqrt{(-7)^2 - 4(1)(12)}}{2(1)}$$
$$x = \frac{7 \pm \sqrt{1}}{2}$$
$$x = \frac{8}{2} \text{ or } x = \frac{6}{2}$$
$$x = 4 \text{ or } x = 3$$

Consider $0 = 3x^2 + 8x + 3$:

- $a = 3, b = 8, c = 3$

$$0 = 3x^2 + 8x + 3$$
$$x = \frac{-(8) \pm \sqrt{(8)^2 - 4(3)(3)}}{2(3)}$$
$$x = \frac{-8 \pm \sqrt{28}}{6}$$
$$x \approx \frac{-2.71}{6} \text{ or } x \approx \frac{-13.29}{6}$$
$$x \approx -0.45 \text{ or } x \approx -2.22$$

Solving for x, dealing with x^2

Solve for x in the equations below using either factorization or the quadratic formula. At home, show that the other method could produce the same answer.

- $x^2 - 4x - 5 = 0$
- $5x^2 - 20x - 2 = 4 - 11x^2$
- $0 = 4x^2 + 9x + 1$

Solving for x and y and z and ...

The same principle of isolating variables applies, but more information is needed.

A single equation such as $x + y = 5$ does not provide enough information to solve because there are infinitely many pairs of values for x and y that would make that true (e.g. $(4, 1)$, $(2.5, 2.5)$, $(-2, 7)$, etc).

In general, you need at least as many equations as you have unknown parameters.

1 equation	1 unknown	:-)
1 equation	2 unknowns	:-(
2 equations	1 unknown	:-D
2 equations	2 unknowns	:-)
2 equations	3 unknowns	:-(

Solving for x and y and z and ...

There are two main approaches to solving these systems of equations using basic algebra:

- Direct substitution
  - isolate one variable in terms of another, substitute this expression into the next equation to solve for that variable, use that solution to solve for the initial variable
  - fairly intuitive, simple for easy problems, quite messy for complicated problems
- Elimination
  - line up equations such that like terms are stacked, multiply one or more of them so the coefficients on variables cancel each other out, add/subtract one equation from another to simplify
  - less intuitive, less flexible, elegantly handles complicated problems

Solving for x and y: Direct Substitution

With $ax + by = c$ and $dx + ey = f$, isolate x in terms of y in equation 1:

$$ax + by = c$$
$$ax = c - by$$
$$x = \frac{c - by}{a}$$

Plug this into equation 2, and solve for y:

$$dx + ey = f$$
$$d(\frac{c - by}{a}) + ey = f$$
$$d(c - by) + aey = af$$
$$dc - dby + aey = af$$
$$aey - dby = af - dc$$
$$y(ae - db) = af - dc$$
$$y = \frac{af - dc}{ae - db}$$

Use the result to solve for x, using either of the equations.

Solving for x and y: Direct Substitution

This seems complicated with parameterized equations, but is often quite routine arithmetic with actual numbers. Consider $2x + 3y = 22$ and $5x + y = 16$.

$$\begin{aligned} 5x + y &= 16 \\ y &= 16 - 5x \end{aligned}$$

$$\begin{aligned} 2x + 3y &= 22 \\ 2x + 3(16 - 5x) &= 22 \\ 2x + 48 - 15x &= 22 \\ 48 - 13x &= 22 \\ 26 &= 13x \\ x &= 2 \end{aligned}$$

$$\begin{aligned} 5x + y &= 16 \\ y &= 16 - 5(2) \\ y &= 6 \end{aligned}$$

Solving for x and y: Elimination

Consider $fx + gy = h$ and $x + ry = s$. First, multiply one or both equations so the coefficients "match" on one of the variables:

$$\begin{aligned} x + ry &= s \\ f(x + ry) &= f(s) \\ fx + fry &= fs \end{aligned}$$

Now align the two equations and add/subtract*** all like terms to eliminate that variable:

$$\begin{aligned} fx + gy &= h \\ fx + fry &= fs \\ (fx - fx) + (gy - fry) &= (h - fs) \\ y(g - fr) &= (h - fs) \\ y &= \frac{h - fs}{g - fr} \end{aligned}$$

Use the result to solve for x, using either of the equations.

Solving for x and y: Elimination

Consider again $2x + 3y = 22$ and $5x + y = 16$.

$$\begin{aligned} 5x + y &= 16 \\ 3(5x + y) &= 3(16) \\ 15x + 3y &= 48 \end{aligned}$$

$$\begin{aligned} 15x + 3y &= 48 \\ 2x + 3y &= 22 \\ 13x &= 26 \\ x &= 2 \end{aligned}$$

$$\begin{aligned} 5x + y &= 16 \\ y &= 6 \end{aligned}$$

Solving for x and y: Elimination

Now consider $4x + 5y = 7$ and $3x + 4y = 6$.

$$\begin{aligned} 4x + 5y &= 7 \\ 3(4x + 5y) &= 3(7) \\ 12x + 15y &= 21 \end{aligned}$$

$$\begin{aligned} 3x + 4y &= 6 \\ 4(3x + 4y) &= 4(6) \\ 12x + 16y &= 24 \end{aligned}$$

$$\begin{aligned} 12x + 16y &= 24 \\ 12x + 15y &= 21 \\ y &= 3 \end{aligned}$$

$$\begin{aligned} 4x + 5y &= 7 \\ 4x + 15 &= 7 \\ 4x &= -8 \\ x &= -2 \end{aligned}$$

Solving for x and y

Solve the following systems of equations for x and y (use direct substitution for one and elimination for the other):

- $3x - 6y = -6$
  $2x + 2y = 14$

- $4x + 2y = 10$
  $3x + 3y = 18$

Solving for x and y and z: Elimination

The elimination approach tends to be easier when dealing with increasingly complicated systems of equations.

Consider:

$$\begin{aligned} 2x + 4y + z &= 21 \\ 5x - y + 3z &= 10 \\ -3x + 2y - 4z &= -7 \end{aligned}$$

Solving for x and y and z: Elimination

First, get two equations in terms of x and y by combining the first and second, and first and third, equations to eliminate z:

$$
\begin{aligned}
2x + 4y + z &= 21 \qquad \leftarrow \text{first equation} \\
3(2x + 4y + z) &= 3(21) \\
6x + 12y + 3z &= 63 \\
5x - y + 3z &= 10 \qquad \leftarrow \text{second equation} \\
x + 13y &= 53
\end{aligned}
$$

$$
\begin{aligned}
2x + 4y + z &= 21 \qquad \leftarrow \text{first equation} \\
4(2x + 4y + z) &= 4(21) \\
8x + 16y + 4z &= 84 \\
-3x + 2y - 4z &= -7 \qquad \leftarrow \text{third equation} \\
5x + 18y &= 77
\end{aligned}
$$

Solving for x and y and z: Elimination

This leaves a system of two equations with two unknowns:

$$\begin{aligned} x + 13y &= 53 \\ 5x + 18y &= 77 \end{aligned}$$

Solve these two equations for one variable:

$$\begin{aligned} x + 13y &= 53 \\ 5(x + 13y) &= 5(53) \\ 5x + 65y &= 265 \\ 5x + 18y &= 77 \\ 47y &= 188 \\ y &= 4 \end{aligned}$$

Using this you can solve for the other variables ($x = 1, z = 3$).

Solving for x and y and z

Solve the system of equations for p, q, and r:

- $p + 2q + 4r = -7$
  $3p - 7q + r = 12$
  $2p + q + 2r = 4$

Vectors

A *vector* is an ordered series of values

$$\mathbf{x} = [x_1, x_2, x_3, x_4, x_5]$$

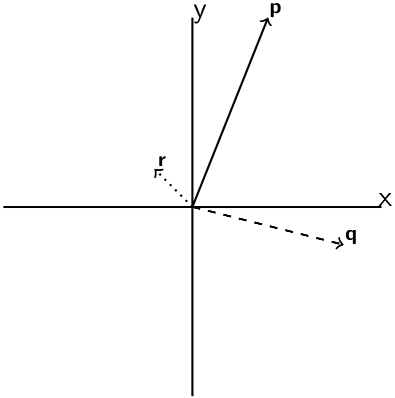
x is a row vector with 5 elements

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$$

y is a column vector with 4 elements

Vectors in Space

Vectors can be thought of as lines from the origin in k-dimensional space (where k is the number of vector elements) going to a point with the coordinates of the elements of the vector.



$$\mathbf{p} = [2, 5]$$
$$\mathbf{q} = [4, -1]$$
$$\mathbf{r} = [-1, 1]$$

Vector Transpose

The *transpose* of a vector switches it from a row vector to a column vector, or vice-versa.

The transpose is denoted with a superscript $^{\mathsf{T}}$ or simply a prime symbol $'$.

$$\mathbf{v} = [v_1, v_2, v_3]$$

$$\mathbf{v}' = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

Vector Addition and Subtraction

To add (or subtract) vectors, simply add (or subtract) the corresponding elements.

$$[a, b, c, d] + [e, f, g, h] = [a + e, b + f, c + g, d + h]$$

$$[3, 4, 5] + [0, 1, 2] = [3, 5, 7]$$
$$[3, 4, 5] - [0, 1, 2] = [3, 3, 3]$$

***Note that vectors must have the same size (*conformable*) to be added or subtracted!

Vector (Scalar) Multiplication

To multiply (or divide) a vector by a scalar, simply multiply (or divide) each element of the vector by the scalar.

$$c * [r, s, t, u] = [cr, cs, ct, cu]$$

$$2 * [4, 2, 7] = [8, 4, 14]$$
$$[3, 1, 8]/2 = [1.5, 0.5, 4]$$

You can think of this as "scaling" the vector up/down or as distributing.

Vector Multiplication: Dot Product

The simplest way to think about multiplying vectors is the *dot product* (or *inner product*).

Each element of each vector is multiplied by the corresponding element of the other vector, and the results are summed to produce a scalar.

$$[a, b, c] \cdot [e, f, g] = (ae + bf + cg)$$

$$[x_1, x_2, x_3] \cdot [y_1, y_2, y_3] = (x_1y_1 + x_2y_2 + x_3y_3) = \sum_1^3 x_i y_i$$

$$\begin{aligned}[1, 8, 3] \cdot [3, 0, 4] &= (1 * 3 + 8 * 0 + 3 * 4) \\ &= 15\end{aligned}$$

There is also a different (more complicated) conception of vector multiplication, the *cross product*.

Vector Norm

The *norm* of a vector (denoted $\|\mathbf{v}\|$) is a measure of its length (calculated using a formula derived from the Pythagorean Theorem).

$$\mathbf{v} = [v_1, v_2, v_3, v_4]$$

$$\begin{aligned} \|\mathbf{v}\| &= (v_1^2 + v_2^2 + v_3^2 + v_4^2)^{\frac{1}{2}} \\ &= \sqrt{v_1^2 + v_2^2 + v_3^2 + v_4^2} \end{aligned}$$

$$\begin{aligned} \mathbf{g} &= [3, 2, 4] \\ \|\mathbf{g}\| &= (3^2 + 2^2 + 4^2)^{\frac{1}{2}} \\ &= 29^{\frac{1}{2}} \\ &= \sqrt{29} \end{aligned}$$

Vectors

Given vectors $\mathbf{x} = [1, 2, 0, 4]$ and $\mathbf{y} = [5, 3, 2, 3]$, find:

- $\mathbf{x}^T$
- $\|\mathbf{y}\|$
- $\mathbf{x} + \mathbf{y}$
- $\mathbf{x} \cdot \mathbf{y}$

The Matrix

A matrix is a rectangular array of values.

Matrices have dimensions denoted RxC, where R is the number of rows (written and subscripted first) and C the number of columns.

$$\mathbf{X} = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \end{bmatrix}$$

X is a 2 x 3 matrix.

Alternatively, matrices can be viewed as stacked vectors.

Special Matrices

Zero Matrix (Z)

- $A + Z = A$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Identity Matrix (I)

- $AI = IA = A$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Diagonal Matrix (D)

$$\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$$

Symmetric Matrix (S)

- $S = S'$

$$\begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix}$$

Matrix Transpose

The *transpose* of a matrix switches its rows and columns.

This is denoted with a superscript ^{T} or simply a prime symbol $'$.

$$\mathbf{M} = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$$

$$\mathbf{M}' = \mathbf{M}^T = \begin{bmatrix} a & d \\ b & e \\ c & f \end{bmatrix}$$

This switches the dimensions (here, from 2x3 to 3x2).

Matrix Addition and Subtraction

To add (or subtract) matrices, simply add (or subtract) the corresponding elements.

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}}_{\mathbf{A}} + \underbrace{\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}}_{\mathbf{B}} = \underbrace{\begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}}_{\mathbf{C}}$$

Such that, e.g. $a_{21} + b_{21} = c_{21}$.

Note that matrices can only be added to (or subtracted from) others with the same number of rows and columns.

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} - \begin{bmatrix} l & m & n \\ o & p & q \end{bmatrix} = \begin{bmatrix} a-l & b-m & c-n \\ d-o & e-p & f-q \end{bmatrix}$$

Matrix Addition and Subtraction

Consider:

$$\begin{bmatrix} 5 & 0 \\ 2 & 8 \end{bmatrix} + \begin{bmatrix} 7 & 1 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 5+7 & 0+1 \\ 2+4 & 8+3 \end{bmatrix} = \begin{bmatrix} 12 & 1 \\ 6 & 11 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 4 \\ 9 & 2 \\ 5 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 7 \\ 3 & 1 \\ 8 & 2 \end{bmatrix} = \begin{bmatrix} 4-0 & 4-7 \\ 9-3 & 2-1 \\ 5-8 & 0-2 \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ 6 & 1 \\ -3 & -2 \end{bmatrix}$$

Matrix (Scalar) Multiplication

To multiply (or divide) a matrix by a scalar, simply multiply (or divide) each element of the matrix by the scalar.

$$c * \begin{bmatrix} p & q \\ r & s \\ t & u \end{bmatrix} = \begin{bmatrix} cp & cq \\ cr & cs \\ ct & cu \end{bmatrix}$$

$$4 * \begin{bmatrix} 0 & 1 & 3 \\ 2 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 4 & 12 \\ 8 & 8 & 0 \end{bmatrix}$$

Matrix Multiplication

To multiply a matrix by another matrix, each element of the result is found by taking the corresponding row of the first matrix, turning it sideways, multiplying each element by the corresponding column in the second matrix, and summing the result.

$$\underset{\mathbf{A}}{\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}} \underset{\mathbf{B}}{\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix}} = \underset{\mathbf{C}}{\begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}}$$

Such that, e.g. $a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} = c_{21}$.

The product of a 1x2 matrix and a 2x1 matrix is a 1x1 matrix - a scalar.

$$\begin{bmatrix} a & b \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = ac + bd$$

Matrix Multiplication

Consider:

$$\begin{bmatrix} 2 & 1 & 5 \\ 4 & 3 & 0 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 1 & 2 \\ 3 & 3 \end{bmatrix} =$$

$$\begin{bmatrix} 2*2+1*1+5*3 & 2*4+1*2+5*3 \\ 4*2+3*1+0*3 & 4*4+3*2+0*3 \end{bmatrix} =$$

$$\begin{bmatrix} 20 & 25 \\ 11 & 22 \end{bmatrix}$$

Matrix Multiplication

To multiply matrices, they must *conform*: the number of columns in the first must be equal to the number of rows in the second (the inner dimensions must be equal).

The result of matrix multiplication will have dimensions equal to the outer dimensions of the two matrices.

Unlike with scalars, order matters. Reversing the order may result in a different product, or may not even be possible depending on the dimensions of the matrices.

Matrices

$$\mathbf{A} = \begin{bmatrix} 4 & 1 \\ 0 & 5 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 3 & 2 \\ 7 & 2 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} 0 & 2 \\ 3 & 3 \\ 1 & 5 \end{bmatrix} \quad \mathbf{D} = \begin{bmatrix} 4 & 1 \end{bmatrix}$$

Given the matrices above, calculate:

- $\mathbf{A} + \mathbf{B}$
- $\mathbf{C}^{\mathsf{T}}$
- $\mathbf{DB}$
- $\mathbf{CD}^{\mathsf{T}}$

Matrix Inversion

The operation most closely analogous to division for matrices is inversion. The *inverse* of a matrix (denoted with the superscript $^{-1}$) is the matrix that, when multiplied by the original, produces the identity matrix:

$$\mathbf{X}\,\mathbf{X}^{-1} = \mathbf{I}$$

Not all matrices have an inverse. If the inverse does exist, it is unique.

To find the inverse of a 2x2 matrix, calculate the *determinant* (product of the main diagonal minus the product of the off diagonal) and adjust the elements as such:

$$\mathbf{X} = \begin{bmatrix} p & q \\ r & s \end{bmatrix} \quad \Longrightarrow \quad \mathbf{X}^{-1} = \begin{bmatrix} \frac{s}{ps-rq} & -\frac{q}{ps-rq} \\ -\frac{r}{ps-rq} & \frac{p}{ps-rq} \end{bmatrix}$$

Calculating inverses of larger (square) matrices is more complicated.

Matrix Inversion

$$\mathbf{X} = \begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix} \quad \Longrightarrow \quad \mathbf{X}^{-1} = \begin{bmatrix} \frac{1}{2} & 0 \\ -\frac{3}{2} & 1 \end{bmatrix}$$

Multiplying these matrices together yields:

$$\begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix} * \begin{bmatrix} \frac{1}{2} & 0 \\ -\frac{3}{2} & 1 \end{bmatrix} = \begin{bmatrix} 2 * \frac{1}{2} + 0 * -\frac{3}{2} & 2 * 0 + 0 * 1 \\ 3 * \frac{1}{2} + 1 * -\frac{3}{2} & 3 * 0 + 1 * 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Solving Systems of Equations Using Matrices

Consider a system of equations:

$$\begin{aligned} 3x + 1y + 0z &= 7 \\ 2x + 4y + 7z &= 8 \\ 1x + 8y + 5z &= 10 \end{aligned}$$

This can be represented as a product of matrices:

$$\begin{bmatrix} 3 & 1 & 0 \\ 2 & 4 & 7 \\ 1 & 8 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3x + 1y + 0z \\ 2x + 4y + 7z \\ 1x + 8y + 5z \end{bmatrix} = \begin{bmatrix} 7 \\ 8 \\ 10 \end{bmatrix}$$

Solving Systems of Equations Using Matrices

Call these matrices:

$$\mathbf{Q} = \begin{bmatrix} 3 & 1 & 0 \\ 2 & 4 & 7 \\ 1 & 8 & 5 \end{bmatrix}$$

$$\mathbf{V} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad \mathbf{A} = \begin{bmatrix} 7 \\ 8 \\ 10 \end{bmatrix}$$

$\mathbf{QV} = \mathbf{A}$ and $\mathbf{Q}^{-1}\mathbf{Q} = \mathbf{I}$ so:

$$\begin{aligned} \mathbf{QV} &= \mathbf{A} \\ \mathbf{Q}^{-1}\mathbf{QV} &= \mathbf{Q}^{-1}\mathbf{A} \\ \mathbf{IV} &= \mathbf{Q}^{-1}\mathbf{A} \\ \mathbf{V} &= \mathbf{Q}^{-1}\mathbf{A} \end{aligned}$$

To solve for the values of the variables, pre-multiply both sides by the inverse (analogous to dividing both sides by the constant in simple one-variable scalar algebra).

Solving Systems of Equations Using Matrices

Consider the equations:

$$2x + 0y = 10$$
$$3x + 1y = 12$$

In matrix algebra form, this is represented:

$$\begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 10 \\ 12 \end{bmatrix}$$

Recall:

$$\mathbf{X} = \begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix} \quad \Longrightarrow \quad \mathbf{X}^{-1} = \begin{bmatrix} \frac{1}{2} & 0 \\ -\frac{3}{2} & 1 \end{bmatrix}$$

Solving Systems of Equations Using Matrices

$$\begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 10 \\ 12 \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{2} & 0 \\ -\frac{3}{2} & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ -\frac{3}{2} & 1 \end{bmatrix} \begin{bmatrix} 10 \\ 12 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{2} * 10 + 0 * 12 \\ -\frac{3}{2} * 10 + 1 * 12 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \end{bmatrix}$$

These values correctly solve the system of equations:

$$2(5) + 0(-3) = 10$$
$$3(5) + 1(-3) = 12$$