Mathematics for Political Science

Day 4 – Calculus II

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Overview

Yesterday, we introduced calculus through the notion of limits and a graphical interpretation. We learned some useful rules for first-order differentiation that you will be expected to use in department methods coursework—especially 835 and 836.

Today, we'll continue and conclude our study of calculus in the following way...

Agenda

(1) Second Derivatives
(2) Partial Derivatives
(3) Integrals
(4) Derivative & Integral Applications

Second Derivatives

For some purposes, you may need to know the derivative of the derivative—how fast the rate of change is changing.

These are known as *second derivatives* (denoted $\frac{d^2[f(x)]}{dx^2}$).

Consider $f(x) = 5x^3 + 8x^2 + 2x + 4$:

$$\frac{d[f(x)]}{dx} = 15x^2 + 16x + 2$$
$$\frac{d^2[f(x)]}{dx^2} = 30x + 16$$

Higher order (third, fourth, etc) derivatives also exist, but are rarely relevant.

Second Derivatives

Find the first and second derivative of the expressions below:

- $f(x) = 16x^3 - 3x^2 + 6$

Multivariate Functions & Partial Derivatives

When a function takes multiple variables as inputs, it is only possible (and sometimes useful) to take the derivative with respect to one variable at a time, treating the others as constants.

These are known as *partial derivatives* (denoted ∂).

Consider $f(x, y, z) = 4x^2y^4 + 2z^3x + 8y^2z^4 + 8x + 7y + 3z + 2$:

$$\frac{\partial[f(x, y, z)]}{\partial x} = 8y^4x + 2z^3 + 8$$
$$\frac{\partial[f(x, y, z)]}{\partial y} = 16x^2y^3 + 16z^4y + 7$$
$$\frac{\partial[f(x, y, z)]}{\partial z} = 6xz^2 + 32y^2z^3 + 3$$

Partial Derivatives

Find the partial derivatives of the expression below with respect to each variable:

- $8p^2q + 4pq - 7pq^2 + 18$

Partial Higher-Order Derivatives

It *is* possible to combine second-order (and higher) derivatives with partial derivatives. For example:

Consider $f(x, y) = 3x^3y^2$:

$$\frac{\partial^2}{\partial x \partial y} f(x, y) = \frac{\partial}{\partial y}(9x^2y^2) = 18x^2y$$

Pay attention to the denominator to give you guidance about what operations to perform. Here, we are taking the second derivative of the entire function, but are differentiating once with respect to x and once with respect to y overall. If instead we were given $\frac{\partial^3}{\partial x^2 \partial y}$, we would differentiate 3 times overall, twice with respect to x and once with respect to y.

Partial Higher-Order Derivatives

Consider again $f(x, y) = 3x^3y^2$. Find:

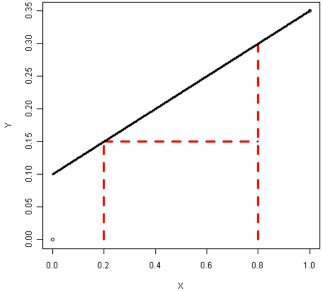
- $\frac{\partial^3}{\partial x^2 \partial y}$

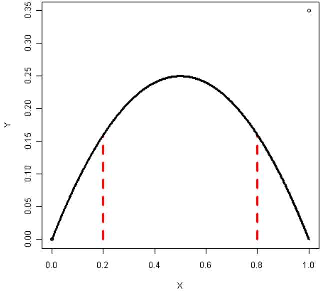
Integrals

The *integral* is the area of the region under a function and above the x-axis.

Either the total area as the function extends to infinity in either direction, or the area between two points.

Areas



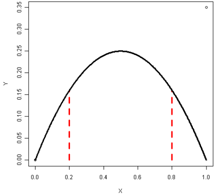


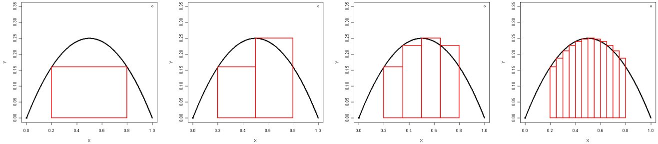
$$y = .25x + .1$$
$$\text{Area} = (.6)(.15) + \frac{1}{2}(.6)(.15)$$

$$y = x - x^2$$
$$\text{Area} = ???$$

Integral as the Limit of a Sum

Imagine dividing the region into intervals and drawing a rectangle to capture the area for each interval, with height equal to the value of the function at the left (or right) of the interval, then summing the area of those rectangles.





Approximation improves as the intervals become smaller.

Integrals

As you reduce the size of the interval to zero, the summed areas of the rectangles converges to the area under the curve—including more and more of the area inside and less and less of the area outside.

Integral

$$\int_a^b f(x)dx = \lim_{h \to 0} \sum_{i=1}^{H} f(x_i)h_i$$

Using this approach, we can find:

- ... the *exact* value of the area between those points
- ... a *general equation* for the area between any two points
- ... *the point* that has a given area to one side

However, it is mathematically difficult to solve these using this approach.

Antiderivatives

The *antiderivative* of a function $f(x)$ (denoted $F(x)$) is the function whose derivative returns the original function:

Antiderivative

$$\frac{d(F(x))}{dx} = f(x)$$

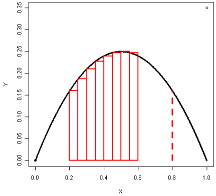
Essentially, this "unwinds" the derivative operation, or applies it backwards.

Fundamental Theorem of Calculus

Using this, can get to the fundamental theorem of calculus relating the derivative and the integral.

Fundamental Theorem of Calculus

$$\int_a^b f(x)dx = F(b) - F(a) = F(x)|_a^b$$



This is very powerful.

Indefinite Integrals

Suppose you had the same thing, but you just wanted the entire area under the function.

Indefinite Integrals

$$\int f(x)dx = F(x) + C$$

This uses exactly the same antiderivative... only differences are that you're using the whole thing so no subtraction, and that there's a constant C added (since that'd disappear when taking the derivative).

Fundamental Theorem of Calculus

This suggests an alternate (equivalent) statement of the Fundamental Theorem of Calculus.

Fundamental Theorem of Calculus

$$\int f(x)dx = F(x) + C$$

This is still very powerful.

Straightforward Integrals

The integral of a function relying on power components is antiderivative of the function.

Power Rule

$$\int ax^n dx = \frac{a}{n+1}x^{n+1} + C$$

Straightforward Integrals

$\int ax^n dx = \frac{a}{n+1}x^{n+1} + C$

Consider the example $f(x) = x - x^2$. The indefinite integral is:

$$\int (x - x^2)dx = \frac{1}{2}x^2 - \frac{1}{3}x^3 + C$$

Now consider the area of the curve specifically between .2 and .8:

$$\begin{aligned}
\int_{.2}^{.8} (x - x^2)dx &= \frac{1}{2}x^2 - \frac{1}{3}x^3|_{.2}^{.8} \\
&= [\frac{1}{2}(.8)^2 - \frac{1}{3}(.8)^3] - [\frac{1}{2}(.2)^2 - \frac{1}{3}(.2)^3] \\
&= [\frac{.64}{2} - \frac{.512}{3}] - [\frac{.04}{2} - \frac{.008}{3}] \\
&= .132
\end{aligned}$$

Straightforward Integrals

$\int ax^n dx = \frac{a}{n+1}x^{n+1} + C$

Let $f(x) = 9x^2 + 10x + 4$. The indefinite integral is:

$$\int (9x^2 + 10x + 4)dx = 3x^3 + 5x^2 + 4x + C$$

Now consider the area of the curve specifically between 2 and 5:

$$\begin{aligned}\int_2^5 (9x^2 + 10x + 4)dx &= 3x^3 + 5x^2 + 4x|_2^5 \\ &= [3(5)^3 + 5(5)^2 + 4(5)] - [3(2)^3 + 5(2)^2 + 4(2)] \\ &= [375 + 125 + 20] - [24 + 20 + 8] \\ &= 468\end{aligned}$$

Straightforward Integrals

Find the indefinite integral of the function below, and calculate the area under the curve between 0 and 1:

- $\int (2x^3 - 3x^2 + 7x + 4)dx$

Advanced Integrals

There are a number of techniques for computing the integrals of more complicated functions.

- Integration by Substitution
- Integration by Parts

These are beyond the scope of what we have time to cover here and, for the most part, beyond the scope of what you will need to do by hand in political science.

Optimization—Maximizing or Minimizing

Suppose we want to maximize the value of the output of some function...

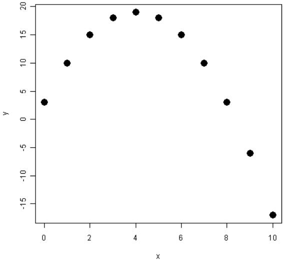
- A function that estimates the probability of any given set of coefficients (inputs) to produce the data we observed.
- A function that shows the utility some political actor would get out of choosing some policy (inputs) in terms of their chances of reelection, satisfaction with the policy, etc.

Or minimize the value of the output of some function...

- A function that calculates the sum of squared errors between data predicted by our coefficient estimates (inputs) and the actual data observed.

Optimization—Maximizing or Minimizing

If we have a simple function and discrete choices, it may be possible to check all options and compare them. Let $f(x) = -x^2 + 8x + 3$ and suppose we can choose any integer between 0 and 10:



x	f(x)
0	3
1	10
2	15
3	18
4	19
5	18
6	15
7	10
8	3
9	-6
10	-7

Maximum value:

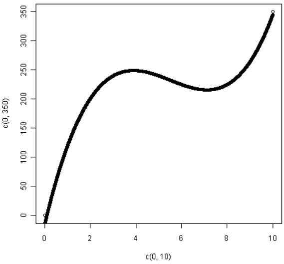
- $f(x) = 19$
- at $x = 4$

Minimum value:

- $f(x) = -7$
- at $x = 10$

Optimization - Maximizing or Minimizing

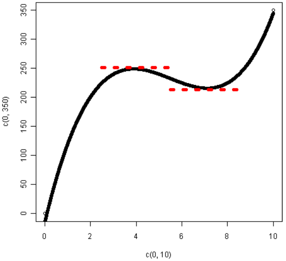
But suppose the function is more complicated, and the choice space is continuous. Let $f(x) = 2x^3 - 33x^2 + 168x - 15$ and suppose we can choose any real number on $[0, 10]$:



The absolute maximum occurs at $x = 10$, and the absolute minimum occurs at $x = 0$, but there also appear to be a local maximum and a local minimum.

Optimization - Maximizing or Minimizing

To determine the precise location of these maxima and minima, note that at these points, the slope of the line is flat. This means the derivative, which captures the slope of the tangent line at any given point, must be 0 at these points:



$$0 = \frac{d(f(x))}{dx} = \frac{d(2x^3 - 33x^2 + 168x - 15)}{dx} = 6x^2 - 66x + 168$$

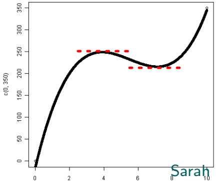
Optimization - Maximizing or Minimizing

Simplify and solve:

$$0 = 6x^2 - 66x + 168$$
$$0 = x^2 - 11x + 28$$

Factoring	Quadratic
$0 = x^2 - 11x + 28$	$x = \frac{11 \pm \sqrt{-11^2 - 4(1)(28)}}{2(1)}$
$0 = (x-7)(x-4)$	$x = \frac{11 \pm \sqrt{9}}{2}$
	$x = \frac{11 \pm 3}{2}$

$$x = 7 \text{ or } x = 4$$



$$f(4) = 2(4)^3 - 33(4)^2 + 168(4) - 15$$
$$= 249$$
$$f(7) = 2(7)^3 - 33(7)^2 + 168(7) - 15$$
$$= 216$$

Optimization - Maximizing or Minimizing

If the function is relatively simple to understand or graph, it may be possible to know whether you're dealing with a maximum or a minimum.

However, it may be necessary to check by taking the second derivative and evaluating it at the point where the first derivative equals 0:

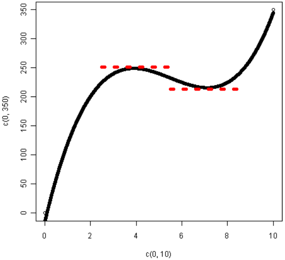
- Negative ($\frac{d^2[f(x)]}{dx^2} < 0$): local maximum
- Positive ($\frac{d^2[f(x)]}{dx^2} > 0$): local minimum
- Zero ($\frac{d^2[f(x)]}{dx^2} = 0$): saddle point - neither a minimum nor a maximum

Optimization - Maximizing or Minimizing

Checking the example:

$$\frac{d(f(x))}{dx} = 6x^2 - 66x + 168 = 0$$

$$\text{at } x = 4, x = 7$$



The second derivative is:

$$\frac{d^2(f(x))}{dx^2} = 12x - 66$$

$$\frac{d^2(f(x))}{dx^2}(\text{at } x = 4) = 12(4) - 66$$
$$= -18$$

$$\frac{d^2(f(x))}{dx^2}(\text{at } x = 7) = 12(7) - 66$$
$$= 18$$

This is negative, so $x = 4$ is a

This is positive, so $x = 7$ is a

Optimization - Maximizing or Minimizing

Find the local minimum and local maximum of the function below, and check mathematically which is the minimum and which is the maximum:

- $x^3 - x^2 + 1$

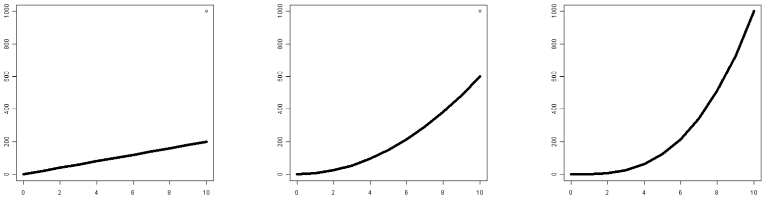
Accumulating Area

Suppose you want to compare different projections about the area accumulated under different growth functions.

- Aggregate profits from an economic investment under different estimates of the returns each year.
- National debt under different projections of how the deficit might rise over time.

Accumulating Area

Consider three different projections (functions):



$f(x) = 20x$ $g(x) = 6x^2$ $h(x) = x^3$

The linear $f(x)$ function initially increases most quickly, but is eventually overtaken by the exponential $g(x)$ and $h(x)$ functions.

Accumulating Area

Which function will have accumulated more area by the 2^{nd} year? the 5^{th}? the 10^{th}?

Can take the integral to calculate the area under each curve during any period of time:

	$f(x) = 20x$ $\int f(x) = 10x^2$	$g(x) = 6x^2$ $\int g(x) = 2x^3$	$h(x) = x^3$ $\int h(x) = \frac{1}{4}x^4$
2 years:	$10x^2\|_0^2$ 40	$2x^3\|_0^2$ 16	$\frac{1}{4}x^4\|_0^2$ 4
5 years:	$10x^2\|_0^5$ 250	$2x^3\|_0^5$ 250	$\frac{1}{4}x^4\|_0^5$ $156\ \frac{1}{4}$
10 years:	$10x^2\|_0^{10}$ 1000	$2x^3\|_0^{10}$ 2000	$\frac{1}{4}x^4\|_0^{10}$ 2500

Accumulating Area

Alternately, after what period of time will $g(x)$ and $h(x)$ have accumulated exactly the same amount of area?

Can take the integral to calculate the area under each curve during any period of time:

$$g(x) = 6x^2 \qquad h(x) = x^3$$
$$\int g(x) = 2x^3 \quad \int h(x) = \tfrac{1}{4}x^4$$

$$\begin{aligned} 2x^3 &= \frac{1}{4}x^4 \\ 8x^3 &= x^4 \\ 8 &= x \end{aligned}$$

After 8 years their area will be the same.

Accumulating Area

Solve:

- After what period of time will the first-order growth curve ($20x$) and the third-order growth curve (x^3) have accumulated the same area?