FINAL: 180 Minutes

Last Name: ____________________

First Name: ____________________

RIN: ____________________

Section: ____________________

Answer **ALL** questions. You may use **two** single sided $8\frac{1}{2} \times 11$ crib sheets.
NO COLLABORATION or electronic devices. Any violations result in an F.
NO questions allowed during the test. Interpret and do the best you can.

GOOD LUCK!

1	2	3	4	5	6	Total
100	50	50	50	50	50	350

1 Circle at most one answer per question. 10 points for each correct answer and -5 points for each incorrect answer (blank answer is 0 points). Don't guess!

(a) $P(n)$ is a predicate ($n \in \mathbb{N}$). $P(1), P(2), P(3)$ are true, and $P(n) \to P(n+4)$ is true for $n \geq 1$. For which n can we be **sure** $P(n)$ is true?

A All $n \geq 1$ except multiples of 2.

B All $n \geq 1$ except multiples of 4.

C All $n \geq 1$

D Only $n = 1, 2, 3$.

(b) Of the following five sets, list *all* that are countable ($\mathcal{A}$ is countable if $\mathbb{N} \stackrel{\text{surj}}{\longrightarrow} \mathcal{A}$):
(I) Prime numbers; (II) Rational numbers; (III) Integers; (IV) Even numbers; (V) Infinite binary strings.

A I and III.

B I and II and III and IV.

C I and III and V.

D II and III and IV.

(c) A class with 25 students needs to choose a representative committee which is a subset of 5 students. How many different committees can be formed?

A 25^5.

B $\dfrac{25!}{20! \times 5!}$.

C $\dfrac{25!}{5!}$.

D $25 \times 24 \times 23 \times 22 \times 21 = \dfrac{25!}{20!}$.

(d) A friendship network has 7 people and each person has at least 1 friend. 6 of the people have *exactly two friends*. How many friends can the 7th person have? Give all possibilities.

A The seventh person could have either 2 or 4 friends.

B The seventh person could have either 2 or 4 or 6 friends.

C The seventh person could have either 1 or 2 or 3 friends.

D The seventh person could have any number of friends that is greater than 1.

(e) Compute the summation $(0+1) + (1+2) + (2+4) + (3+8) + \cdots + (10 + 2^{10}) = \sum_{i=0}^{10} (i + 2^i)$

A 2048.

B 2102.

C 1078.

D 2200.

(f) You have a known fact that $0 = 0$ and all the standard operations of algebra you learned in high-school math. Which of the following is a valid proof that $7 = 7$:

$$\begin{array}{c}\underline{\text{I}}\\ \begin{array}{lrcll} 1. & 7 & = & 7 & \\ 2. & 7-7 & = & 7-7 & \\ 3. & 0 & = & 0 & \checkmark \\ \hline \rightarrow & 7 & = & 7 & \end{array}\end{array} \qquad \begin{array}{c}\underline{\text{II}}\\ \begin{array}{lrcll} 1. & 7 & \neq & 7 & \\ 2. & 7-7 & \neq & 7-7 & \\ 3. & 0 & \neq & 0 & \textbf{!FISHY} \\ \hline \rightarrow & 7 & = & 7 & \end{array}\end{array} \qquad \begin{array}{c}\underline{\text{III}}\\ \begin{array}{lrcll} 1. & 0 & = & 0 & \\ 2. & 0+7 & = & 0+7 & \\ 3. & 7 & = & 7 & \checkmark \\ \hline \rightarrow & 7 & = & 7 & \end{array}\end{array}$$

[A] I & II & III. [B] II & III. [C] I & II [D] I & III.

(g) Let $f(n) = \sum_{i=1}^{n} i$ and $g(n) = 2^{3\log_2 n}$. What is the big-Oh relationship between f and g?

[A] $f(n) = O(g(n))$ and $g(n) = O(f(n))$.

[B] $f(n) = O(g(n))$ and $g(n) \neq O(f(n))$.

[C] $f(n) \neq O(g(n))$ and $g(n) = O(f(n))$.

[D] $f(n) \neq O(g(n))$ and $g(n) \neq O(f(n))$.

(h) You independently generate the ten bits of a binary sequence $b_1 b_2 \cdots b_{10}$ with $\mathbb{P}[b_i = 0] = \frac{1}{2}$. Compute the probability that the sequence is sorted from low to high. For example 0000111111 is sorted.

[A] $\dfrac{10}{1024}$

[B] $\dfrac{11}{1024}$

[C] $\dfrac{20}{1024}$

[D] $\dfrac{12}{1024}$

(i) x_1, x_2, x_3 are non-negative integers. Compute the number of different solutions to $x_1 + x_2 + x_3 = 100$. (For example two different solutions are $1 + 2 + 97 = 100$ and $97 + 1 + 2 = 100$.)

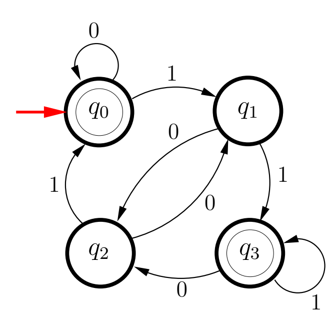
[A] 10302

[B] 5151

[C] 4949

[D] 5050

(j) For the automaton on the right, which input string is accepted? (Strings are processed from left to right.)



[A] 010101

[B] 0101011

[C] 01010110

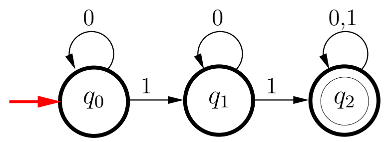
[D] 010101100

2 Proofs

1. Prove that for all integers $n \geq 1$: $\qquad n2^n \leq 3^n$

2. Prove that *every* <u>odd</u> natural number is the difference of two square numbers.

3 Finite Automaton with a Random Input String



The automaton to the right processes a random binary string $b_1 b_2 \dots b_n$ of length n generated as follows: you independently generate each bit b_i with $\mathbb{P}[b_i = 1] = p$ and $\mathbb{P}[b_i = 0] = 1 - p$. Show that the probability that the string is accepted is

$$\mathbb{P}[\text{ random input string is accepted }] = 1 - (1-p)^n - np(1-p)^{n-1}.$$

[Hints: (i) Figure out a simple property of a string for it to be accepted. (ii) Binomial distribution.]

4 Probability and Expectation

(a) You independently roll 3 fair dice D_1, D_2, D_3 and let $S = D_1 + D_2 + D_3$ be the sum. Compute:

(i) $\mathbb{P}[S = 8]$

(ii) $\mathbb{P}[S = 8 \mid D_1 = 1]$

(iii) Compute the expectation and variance of S.

(b) You toss a fair coin independently until you get two heads *in a row*. Let X be the number of tosses. Compute $\mathbb{E}[X]$ using the law of total expectation:

(i) Consider the 3 cases T, HT, HH for how the tosses may start and show that

$$\mathbb{E}[X] = \tfrac{1}{2}(1 + \mathbb{E}[X]) + \tfrac{1}{4}(2 + \mathbb{E}[X]) + \tfrac{1}{2}.$$

(ii) Use (i) to show that $\mathbb{E}[X] = 6$.

5 Context Free Grammars

This problem is about the language $\mathcal{L}$ generated by the CFG:

$$\begin{aligned} S &\rightarrow 1T \mid 0T \\ T &\rightarrow 1T1 \mid 0T0 \mid \epsilon \end{aligned}$$

(a) Is the string 1010010 in $\mathcal{L}$? If yes then give a derivation or parse tree; if no then explain why.

(b) Prove that the length of every string in $\mathcal{L}$ is odd.

6 Turing Machine

(a) What is the difference between a Turing-recognizable language and a Turing-decidable language?

(b) Consider the arithmetic task of squaring, which corresponds to the language $\mathcal{L} = \{0^n \# 0^{n^2} | n \geq 1\}$.

(i) Circle the simplest model of computing that you think solves the problem $\mathcal{L}$:

Finite Automaton   Context Free Grammar   Turing Machine

(ii) Give your machine from (i) that solves $\mathcal{L}$ (for a TM, a high level description will do).

SCRATCH

SCRATCH

FINAL: 180 Minutes

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NO COLLABORATION or electronic devices. Any violations result in an F.
NO questions allowed during the test. Interpret and do the best you can.

GOOD LUCK!

1	2	3	4	5	Total
150	50	50	50	50	350

1 Circle at most one answer per question. 10 points for each correct answer.

(1) The **negation** of "All Malik's friends are big and strong" is

$\boxed{\text{A}}$ None of Malik's friends are big and strong.

$\boxed{\text{B}}$ Malik has a friend who is either small or weak (or both).

$\boxed{\text{C}}$ All Malik's friends are small and weak.

$\boxed{\text{D}}$ All Malik's friends are either small or weak (or both).

$\boxed{\text{E}}$ Malik has no friends who are small or weak.

(2) What is the *most accurate* order relation between $3^{\log_2 n}$ and n^2?

$\boxed{\text{A}}$ $3^{\log_2 n} \in o(n^2)$.

$\boxed{\text{B}}$ $3^{\log_2 n} \in O(n^2)$.

$\boxed{\text{C}}$ $3^{\log_2 n} \in \Theta(n^2)$.

$\boxed{\text{D}}$ $3^{\log_2 n} \in \Omega(n^2)$.

$\boxed{\text{E}}$ $3^{\log_2 n} \in \omega(n^2)$.

(3) Compute the summation $\sum_{i=1}^{20}(-1)^i i^2$

$\boxed{\text{A}}$ 190.

$\boxed{\text{B}}$ 200.

$\boxed{\text{C}}$ 210.

$\boxed{\text{D}}$ 220.

$\boxed{\text{E}}$ 230.

(4) Let $f(n) = \sum_{i=1}^{n} i$ and $g(n) = 4^{\log_2 n}$. What is the *most accurate* order relationship between f and g?

$\boxed{\text{A}}$ $f \in o(g)$.

$\boxed{\text{B}}$ $f \in O(g)$.

$\boxed{\text{C}}$ $f \in \Theta(g)$.

$\boxed{\text{D}}$ $f \in \Omega(g)$.

$\boxed{\text{E}}$ $f \in \omega(g)$.

(5) Let $f(n)$ be a function satisfying the recurrence $f(0) = 0$; $f(n) = f(n-1) + \sqrt{n}$. Which order relationship describes f.

$\boxed{\text{A}}$ $f \in \Theta(n)$.

$\boxed{\text{B}}$ $f \in \Theta(n \log n)$.

$\boxed{\text{C}}$ $f \in \Theta(n\sqrt{n})$.

$\boxed{\text{D}}$ $f \in \Theta(n^2)$.

$\boxed{\text{E}}$ $f \in \Theta(n^3)$.

(6) A class with 10 students needs to choose a president, vice-president and secretary (a student *cannot* fill multiple roles). In how many ways can this be done?

A 1000.

B 720.

C 120.

D 10!

E $\binom{10}{3}$.

(7) A fraternity orders 5 pizzas (eg. 2 with sausage and 3 with meatballs & onion). There are 5 toppings. A pizza can have 0,1 or 2 toppings. How many ways are there for the fraternity to make its order?

A 16.

B 16^5.

C $\binom{16}{5}$.

D $\binom{20}{15}$.

E $16 \times 15 \times 14 \times 13 \times 12$.

(8) A friendship network has 6 people Ⓐ Ⓑ Ⓒ Ⓓ Ⓔ Ⓕ. If you add up the number of friends of each person, you get a total of 26. How many *different* social network graphs could correspond to this friendship network. (Two graphs are different if they don't have exactly the same edges.)

A 0.

B 95.

C 105.

D 115.

E 125.

(9) You are thinking of a graph with 5 nodes Ⓐ Ⓑ Ⓒ Ⓓ Ⓔ. *Approximately* how many such graphs are there?

A 100.

B 500.

C 1000.

D 5000.

E 10000.

(10) X and Y are random variables (not necessarily independent). Which of the following is an expression for $Var(X+Y)$ (variance of the sum)?

A $Var(X) + Var(Y)$.

B $\mathbb{E}[(X+Y)^2]$.

C $\mathbb{E}[X^2] + \mathbb{E}[Y^2] - \mathbb{E}[X]^2 - \mathbb{E}[Y]^2$.

D $Var(X) + Var(Y) + 2\,\mathbb{E}[XY] - 2\,\mathbb{E}[X]\,\mathbb{E}[Y]$.

E $Var(X) + Var(Y) - 2Var(XY)$.

(11) You independently generate two random ten bit binary sequences and compute a new sequence using the BITWISE-OR of the two random sequences (treating 0 as FALSE and 1 as TRUE). Let X be the number of 1s in the result. What is $\mathbb{E}[X]$. (for example, 0001110010 BITWISE-OR 1000111000 = 1001111010.)

A 2.5

B 3.5

C 5

D 6.5

E 7.5

(12) About 1 in a 1000 people have Coeliac disease. The outcome of a test for Coeliac is random: the test makes a mistake on 1 in 10 people who have it (90% accuracy if you have Coeliac); the test makes a mistake on 1 in 100 people who do not have it (99% accuracy if you do not have Coeliac). You got tested, and the result was positive. *Approximately* what are the chances that you have Coeliac?

A 0.1%

B 10%

C 40%

D 80%

E 90%

(13) Which set is *not countable*?

A {1,3,5,7}.

B The prime numbers {2,3,5,7,...}.

C All possible angles between 0 and 360.

D All even numbers which are not a sum of two primes.

E All possible pairs of integers, $\mathbb{Z}^2$.

(14) A random binary string $b_1b_2\ldots b_{10}$ of length 10 is the input to the automaton.
What is the probability that the string is accepted?

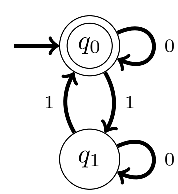
A 0.25

B 0.4

C 0.5

D 0.6

E 0.75



(15) Which string below is *not* in the language of the CFG: $S \longrightarrow \varepsilon \mid 0S|S0|11S$

A ε

B 1111

C 11011

D 0011000

E 001010

2 Positive Integer Partitions

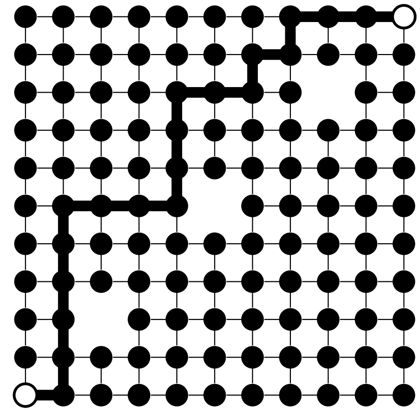
A positive partition of n is a *sequence* of *positive* integers that add up to n. For example, $(6, 4)$, $(4, 6)$ and $(2, 4, 2, 2)$ are different partitions of 10. How many positive partitions of n are there? Prove your answer.

3 Proofs

(a) *Prove* that $n^2 \leq 3^n$ for integer $n \geq 0$.

(b) *Prove* that $n^3 \notin O(n^2)$. You must *prove* that there is no *constant* C for which $n^3 \leq Cn^2$ for all $n \geq 1$.

4 Counting Paths on Graphs with Holes



A grid is missing nodes at $(2,2)$, $(5,5)$ and $(8,8)$. A *shortest* path from the bottom left node (0,0) to the top right node (10,10) is shown.

How many different shortest paths go from (0,0) to (10,10)? (Two paths are different if they do not have exactly the same edges).

You may leave your answer in the form of a combination of binomial coefficients – you do not need to compute a numerical answer.

5 Turing Machine and Exponentiation

(a) *Prove*: the problem (language) $\mathcal{L} = \{0^n\#1^{2^n} \mid n \geq 1\}$ *cannot* be solved (accepted) by a finite automaton.

(b) Give a high-level description of a Turing Machine that solves $\mathcal{L} = \{0^n\#1^{2^n} \mid n \geq 1\}$.

SCRATCH

SCRATCH

FINAL: 180 Minutes

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Answer **ALL** questions. You may use **two** double sided $8\frac{1}{2} \times 11$ crib sheets.
NO COLLABORATION or electronic devices. Any violations result in an F.
NO questions allowed during the test. Interpret and do the best you can.

GOOD LUCK!

1	2	3	4	5	Total
200	40	40	40	40	350

(10 bonus points)

1 Circle at most one answer per question. 10 points for each correct answer.

(1) The **negation** of “Every student is a friend of some other student” is

A Some student has a friend who is a student.

B Some student is a friend of all students.

C Some student is not a friend of some other student.

D Some student is not a friend of all other students.

E Some student has no friends.

(2) Estimate $2^1 \times 2^2 \times 2^3 \times \cdots \times 2^{20} = \prod_{i=1}^{20} 2^i$.

A 1.65×10^{61}

B 1.65×10^{63}

C 1.65×10^{65}

D 1.65×10^{67}

E 1.65×10^{69}

(3) What is the *most accurate* order relation between 2^n and e^n?

A $2^n \in o(e^n)$.

B $2^n \in O(e^n)$.

C $2^n \in \Theta(e^n)$.

D $2^n \in \Omega(e^n)$.

E $2^n \in \omega(e^n)$.

(4) $f(n)$ satisfies the recurrence $f(0) = 1$; $f(n) = nf(n-1)$. Which order relationship describes f.

A $f \in \Theta(2^n)$.

B $f \in O(2^n)$.

C $f \in o(2^n)$.

D $f \in \Theta(n^n)$.

E $f \in o(n^n)$.

(5) What is the greatest common divisor of 756 and 840?

$\boxed{\text{A}}$ 12.

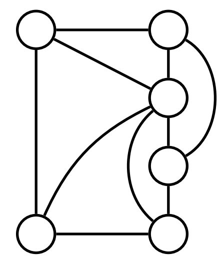
$\boxed{\text{B}}$ 28.

$\boxed{\text{C}}$ 63.

$\boxed{\text{D}}$ 84.

$\boxed{\text{E}}$ 189.

(6) What is the minimum number of colors needed to color the graph on the right?



$\boxed{\text{A}}$ 2.

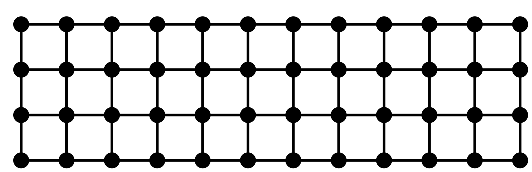
$\boxed{\text{B}}$ 3.

$\boxed{\text{C}}$ 4.

$\boxed{\text{D}}$ 5.

$\boxed{\text{E}}$ 6.

(7) On the right is the 4×12 grid graph. What is the average degree of a node?



$\boxed{\text{A}}$ 3.

$\boxed{\text{B}}$ $3\frac{1}{4}$.

$\boxed{\text{C}}$ $3\frac{1}{3}$.

$\boxed{\text{D}}$ $3\frac{1}{2}$.

$\boxed{\text{E}}$ $3\frac{2}{3}$.

(8) Shirts come in 6 colors. 4 students are in a row. You must assign shirts to the students, and two students standing next to each other cannot get the same color shirt. In how many ways can you do this?

$\boxed{\text{A}}$ $\binom{9}{3}$.

$\boxed{\text{B}}$ $6 \times 5 \times 4 \times 3$.

$\boxed{\text{C}}$ $\binom{6}{4}$.

$\boxed{\text{D}}$ 6×5^3.

$\boxed{\text{E}}$ 6^4.

(9) Pokemons have 4-digit serial numbers, e.g. 0255. A pokemon is defective if any digit repeats (e.g. 0255, 5250, 5255 are defective). *Approximately* what fraction of the possible serial numbers are defective?

A 0.

B 0.25.

C 0.5.

D 0.75.

E 1.

(10) A senate committee of 10 senators must pick a president. 3 candidates will be proposed from the 10 senators, and everyone votes. In how many ways can the 3 candidates be chosen.

A 1000.

B 720.

C 120.

D 10!

E $\frac{10!}{3!}$.

(11) Three integers z_1, z_2, z_3 satisfy $0 \leq z_1 \leq z_2 \leq z_3 \leq 6$ (the sequence is non-decreasing and bounded between 0 and 6). How many such sequences are there?

A 28.

B 42.

C 84.

D 165.

E 168.

(12) You are thinking of a graph with 4 nodes A B C D. How many such graphs are there?

A 24.

B 64.

C 81.

D 256.

E 4096.

(13) $\mathbf{X}, \mathbf{Y}$ are random variables (not necessarily independent) and $\mathbf{Z} = a\mathbf{X} + b\mathbf{Y}$. What is $\mathbb{E}[\mathbf{Z}]$?

A $a\,\mathbb{E}[\mathbf{X}] + b\,\mathbb{E}[\mathbf{Y}]$

B $a^2\,\mathbb{E}[\mathbf{X}] + b^2\,\mathbb{E}[\mathbf{Y}]$

C $(a+b)(\mathbb{E}[\mathbf{X}] + \mathbb{E}[\mathbf{Y}])$

D $a(\mathbb{E}[\mathbf{X}] + \mathbb{E}[\mathbf{Y}]) + b(\mathbb{E}[\mathbf{X}] + \mathbb{E}[\mathbf{Y}])$

E None of the above are true in general.

(14) This test has 20 multiple choice questions, each with 5 possible choices. If you answer questions randomly, what is the expected number of multiple questions you get correct?

A 3

B 4

C 5

D 6

E 10

(15) About 1 in a 1000 people have Coeliac disease. The test for Coeliac randomly makes a mistake 5% of the time (95% accuracy). You tested positive. *Approximately* what are the chances you have Coeliac?

A 0.2%

B 2%

C 20%

D 50%

E 95%

(16) A random binary string $b_1 b_2 \ldots b_{10}$ of 10 bits is the input to the automaton. What is the probability that the string is accepted?

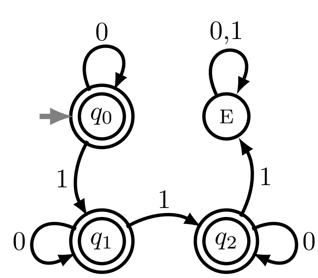
A $\frac{2}{1024}$

B $\frac{45}{1024}$

C $\frac{56}{1024}$

D $\frac{90}{1024}$

E $\frac{512}{1024}$



(17) What is a computing problem?

$\boxed{\text{A}}$ A Person.

$\boxed{\text{B}}$ An automaton (machine which transitions between states as it reads the input).

$\boxed{\text{C}}$ An automaton with stack memory.

$\boxed{\text{D}}$ An automaton with random access memory.

$\boxed{\text{E}}$ A set containing finite binary strings.

(18) The computing problem $\mathcal{L} = \{\text{strings with an even number of 1s}\}$ can be solved by:
(I) DFA. (II) CFG. (III) Turing Machine.

$\boxed{\text{A}}$ I,II,III

$\boxed{\text{B}}$ I,III

$\boxed{\text{C}}$ II,III

$\boxed{\text{D}}$ III only

$\boxed{\text{E}}$ None of these models of computing

(19) The computing problem $\mathcal{L} = \{\text{strings corresponding to programs which HALT}\}$ can be solved by:
(I) DFA. (II) CFG. (III) Turing Machine.

$\boxed{\text{A}}$ I,II,III

$\boxed{\text{B}}$ I,III

$\boxed{\text{C}}$ II,III

$\boxed{\text{D}}$ III only

$\boxed{\text{E}}$ None of these models of computing

(20) A DFA has *two* states a start state q_0 and a second state q_1. The DFA is described by a list of its accept states and a list of its transition instructions. The order in which you list the accept states and the transition instructions does not matter. We draw a DFA as a graph with nodes q_0, q_1 and add a directed arrow for each transition instruction (the accepting states have double circles).

<u>How many different DFA's are there with two states?</u> (*Different* DFA's *can* have the same YES-set)

$\boxed{\text{A}}$ 4.

$\boxed{\text{B}}$ 8.

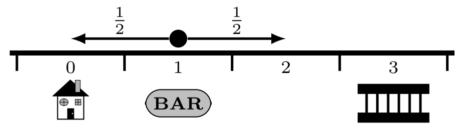
$\boxed{\text{C}}$ 16.

$\boxed{\text{D}}$ 32.

$\boxed{\text{E}}$ 64.

2 Random Walk

A drunk leaves the bar (at position 1), and takes independent steps: left (L) with probability $\frac{1}{2}$ or right (R) with probability $\frac{1}{2}$. The drunk stops when he reaches home (at 0) or the jail (at 3). Compute the *expected* number of steps the drunk makes.



3 Induction

(a) $$\begin{aligned} G(1) &= 1; \\ G(n) &= G(n-1)\left(1-\frac{1}{n}\right) \quad \text{for } n > 1; \end{aligned}$$ Prove that $G(n) = \dfrac{1}{n}$ for integer $n \geq 1$.

(b) The nth Harmonic number is $H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$. Prove that $H_1 + H_2 + \ldots + H_n = (n+1)H_n - n$.

4 Turing Machine

Give a high-level description of a Turing Machine that solves the problem $\mathcal{L} = \{0^n\#1^{n^2} \mid n \geq 0\}$ (squaring). (You may find it useful to illustrate how your TM works on 00#1111.)

5 [Hard] Unsolvable Problems

Prove: There is an undecidable computing problem which is a subset of $\{1\}^*$.

SCRATCH

SCRATCH

FINAL: 180 Minutes

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Answer **ALL** questions. You may use **two** double sided $8\frac{1}{2} \times 11$ crib sheets.

You **MUST** show work (even for multiple choice) to receive full credit.

NO COLLABORATION or electronic devices. Any violations result in an F.

NO questions allowed during the test. Interpret and do the best you can.

GOOD LUCK!

1	**2**	**3**	**4**	**5**	**Total**
200	**40**	**40**	**40**	**40**	**350**

(10 bonus points)

1 Circle at most one answer per question. 10 points for each correct answer.

(1) Every card has a letter and a number. **Rule:** If a card has a P on it, then the other side *must* be a 5.

$\boxed{\mathbf{S}}$ $\boxed{\mathbf{5}}$ $\boxed{\mathbf{P}}$ $\boxed{\mathbf{3}}$

Which of the above cards *must* be turned over to verify the rule has not been broken.

A $\boxed{\mathbf{S}}$ $\boxed{\mathbf{5}}$

B $\boxed{\mathbf{5}}$ $\boxed{\mathbf{P}}$

C $\boxed{\mathbf{S}}$ $\boxed{\mathbf{3}}$

D $\boxed{\mathbf{P}}$ $\boxed{\mathbf{3}}$

E None of the above.

(2) Which set relationship does not hold in general.

A $\overline{A \cap B} = \overline{A} \cup \overline{B}$.

B $\overline{A \cup B} = \overline{A} \cap \overline{B}$.

C $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

D $(A \cup B) \cap \overline{A} = B \cap \overline{A}$.

E They all hold in general.

(3) $T_0 = 2$ and $T_n = T_{n-1}^2$ for $n > 0$. Estimate T_{20}.

A $10^{3,156,500}$

B $10^{1,156,500}$

C $10^{315,650}$

D $10^{156,500}$

E $10^{31,565}$

(4) $T_0 = 2$ and $T_n = T_{n-1}^2$ for $n > 0$, as in problem (3). Which order relationship is accurate?

A $T_n \in O(n)$.

B $T_n \in O(2^n)$.

C $T_n \in O(n!)$.

D $T_n \in O(2^{n!})$.

E None of the above.

(5) What is the last digit of $3^{1000} \times 5^{2000} + 7^{3000} \times 9^{4000}$?

$\boxed{\text{A}}$ 1.

$\boxed{\text{B}}$ 2.

$\boxed{\text{C}}$ 3.

$\boxed{\text{D}}$ 4.

$\boxed{\text{E}}$ None of the above.

(6) Let $d = \gcd(m, n)$, where $m, n > 0$. Bezout's identity gives $d = mx + ny$ where $x, y \in \mathbb{Z}$. Which of the statements A, B, C or D are false?

$\boxed{\text{A}}$ It is always possible to choose $x > 0$.

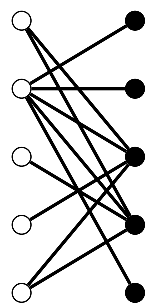
$\boxed{\text{B}}$ It is always possible to choose $x < 0$.

$\boxed{\text{C}}$ It is possible to find another $x, y \in \mathbb{Z}$ for which $0 < mx + ny < d$.

$\boxed{\text{D}}$ It is always possible to find $a, b \in \mathbb{Z}$ for which $ax + by = 1$.

$\boxed{\text{E}}$ All the statements A, B, C and D are true.

(7) The left nodes are tasks and the right nodes are resources. A resource can perform at most one task. What is the maximum number of tasks that can be performed?



$\boxed{\text{A}}$ 1.

$\boxed{\text{B}}$ 2.

$\boxed{\text{C}}$ 3.

$\boxed{\text{D}}$ 4.

$\boxed{\text{E}}$ 5.

(8) A queen covers a square if that square is on the same row, column or diagonal as the queen. What is the minimum number of queens required to cover all squares on a 5×5 chessboard?

$\boxed{\text{A}}$ 1.

$\boxed{\text{B}}$ 2.

$\boxed{\text{C}}$ 3.

$\boxed{\text{D}}$ 4.

$\boxed{\text{E}}$ 5.

(9) A friendship network has 100 people (vertices) and 2000 edges (friendships). You pick a person at random. What is the expected number of friends this person has?

$\boxed{\text{A}}$ 10.

$\boxed{\text{B}}$ 20.

$\boxed{\text{C}}$ 30.

$\boxed{\text{D}}$ 40.

$\boxed{\text{E}}$ None of the above, or not enough information to say for sure.

(10) To get into a certain US-college, all students submit at least one of SAT or ACT. 80% of students submit SAT; 40% of students submit ACT. How many students submit both SAT and ACT?

$\boxed{\text{A}}$ 10%.

$\boxed{\text{B}}$ 20%.

$\boxed{\text{C}}$ 30%.

$\boxed{\text{D}}$ 40%.

$\boxed{\text{E}}$ None of the above, or not enough information to say for sure.

(11) How many 4 digit strings (digits are 0,1,. . . ,9) from 0000 to 9999 have digits which sum to 8. For example 0071, 0233 and 2033 are different digit-strings with digit-sum 8.

$\boxed{\text{A}}$ $\binom{8}{4} = 70$.

$\boxed{\text{B}}$ $\binom{11}{3} = 165$.

$\boxed{\text{C}}$ $10 \times 9 \times 8 \times 7 = 5040$.

$\boxed{\text{D}}$ $10^4 = 10,000$.

$\boxed{\text{E}}$ None of the above.

(12) How many different friendship networks are possible with the 5 people, Ⓐ Ⓑ Ⓒ Ⓓ Ⓔ? (Two networks are different if they have different edge-sets.)

$\boxed{A}$ Approximately 10.

$\boxed{B}$ Approximately 100.

$\boxed{C}$ Approximately 1000.

$\boxed{D}$ Approximately 10,000.

$\boxed{E}$ Approximately 100,000.

(13) A friendship network has 5 people, Ⓐ Ⓑ Ⓒ Ⓓ Ⓔ. Each pair of people independently flips a fair coin and forms a friendship-edge if the flip is H. What is the probability that the network has exactly 5 edges?

$\boxed{A}$ Approximately 1%.

$\boxed{B}$ Approximately 2%.

$\boxed{C}$ Approximately 10%.

$\boxed{D}$ Approximately 25%.

$\boxed{E}$ Approximately 50%.

(14) A friendship network has 5 people, Ⓐ Ⓑ Ⓒ Ⓓ Ⓔ. Each pair of people independently flips a coin and forms a friendship if they get H. What is the expected number of edges in the friendship network?

$\boxed{A}$ 2.

$\boxed{B}$ 3.

$\boxed{C}$ 4.

$\boxed{D}$ 5.

$\boxed{E}$ None of the above.

(15) A tennis club has 20 members who are paired up in twos for the first round of a tournament. In the first round, we only care about who plays whom. How many ways are there of forming the first round matches? *[Hint: With 4 members, there are 3 ways to form the first round matches.]*

$\boxed{\text{A}}$ 20!.

$\boxed{\text{B}}$ $\binom{20}{2}^{10}$.

$\boxed{\text{C}}$ $\binom{20}{2} \times \binom{18}{2} \times \binom{16}{2} \times \binom{14}{2} \times \binom{12}{2} \times \binom{10}{2} \times \binom{8}{2} \times \binom{6}{2} \times \binom{4}{2} \times \binom{2}{2}$.

$\boxed{\text{D}}$ $20!/(2^{10} \times 10!)$

$\boxed{\text{E}}$ None of the above.

(16) $\mathbf{X}$ is a random variable and $\mathbf{Z} = a\mathbf{X} + b\mathbf{X}^2$. What is $\mathbb{E}[\mathbf{Z}]$?

$\boxed{\text{A}}$ $\mathbb{E}[\mathbf{Z}] = a\,\mathbb{E}[\mathbf{X}] + b\,\mathbb{E}[\mathbf{X}]^2$

$\boxed{\text{B}}$ $\mathbb{E}[\mathbf{Z}] = a\,\mathbb{E}[\mathbf{X}] + b^2\,\mathbb{E}[\mathbf{X}]^2$

$\boxed{\text{C}}$ $\mathbb{E}[\mathbf{Z}] = a\,\mathbb{E}[\mathbf{X}] + b\,\mathbb{E}[\mathbf{X}^2]$

$\boxed{\text{D}}$ $\mathbb{E}[\mathbf{Z}] = a\,\mathbb{E}[\mathbf{X}] + b^2\,\mathbb{E}[\mathbf{X}^2]$

$\boxed{\text{E}}$ None of the above are true in general.

(17) $\mathbf{X}, \mathbf{Y}$ are *independent* random variables and $\mathbf{Z} = \mathbf{XY}$. What is $\sigma^2(\mathbf{Z})$, the variance of the product? *[Hint: Tinker with simple random variables. Make a conclusion and justify it.]*

$\boxed{\text{A}}$ $\sigma^2(\mathbf{Z}) = \sigma^2(\mathbf{X})\sigma^2(\mathbf{Y})$

$\boxed{\text{B}}$ $\sigma^2(\mathbf{Z}) = \sigma^2(\mathbf{X})\,\mathbb{E}[\mathbf{Y}^2] + \sigma^2(\mathbf{Y})\,\mathbb{E}[\mathbf{X}^2]$

$\boxed{\text{C}}$ $\sigma^2(\mathbf{Z}) = \sigma^2(\mathbf{X})\,\mathbb{E}[\mathbf{Y}]^2 + \sigma^2(\mathbf{Y})\,\mathbb{E}[\mathbf{X}]^2$

$\boxed{\text{D}}$ $\sigma^2(\mathbf{Z}) = \sigma^2(\mathbf{X})\,\mathbb{E}[\mathbf{Y}^2] + \sigma^2(\mathbf{Y})\,\mathbb{E}[\mathbf{X}]^2$

$\boxed{\text{E}}$ None of the above are true in general.

(18) About 1 in a 100 people have Coeliac disease. The test for Coeliac has 90% accuracy, randomly making a mistake only 10% of the time. You tested positive. What are the chances you have Coeliac?

$\boxed{\text{A}}$ 1/100.

$\boxed{\text{B}}$ 1/12

$\boxed{\text{C}}$ 1/8

$\boxed{\text{D}}$ 1/4

$\boxed{\text{E}}$ 9/10

(19) The computing problem $\mathcal{L} = \{0^{\bullet n}\, 1^{\bullet (n+m)}\, 0^{\bullet m} \mid m, n \geq 0\}$ can be solved by:

(I) DFA. (II) CFG. (III) Turing Machine.

$\boxed{\text{A}}$ I,II,III

$\boxed{\text{B}}$ I,III

$\boxed{\text{C}}$ II,III

$\boxed{\text{D}}$ III only

$\boxed{\text{E}}$ None of these models of computing

(20) Which of these problems can be solved by a computer (Turing Machine)?

$\boxed{\text{A}}$ Determine if some other program halts or loops forever – ULTIMATEDEGUGGER

$\boxed{\text{B}}$ Determine YES or NO if some other program says YES on its input and halts.

$\boxed{\text{C}}$ Given $n \in \mathbb{N}$, compute $f(n)$, where $f(n) = 1$ if the nth Turing Machine halts and 0 otherwise.

$\boxed{\text{D}}$ Given m-bit and n-bit binary sequences $b_1 \cdots b_m$ and $c_1 \cdots c_n$ with $m < n$, is it possible to add $n - m$ bits into various positions of the first sequence so that the two sequences match exactly?

$\boxed{\text{E}}$ None of these problems can be solved.

2 Independent Sets and Vertex Covers in a Graph. (Tinker, tinker,...)

A graph G has vertices $V = \{v_1, \ldots, v_n\}$ and edges $E = \{e_1, \ldots, e_m\}$. Let $S \subseteq V$ be a subset of the vertices.

S is a **vertex cover** if every edge in E has at least one endpoint in S.

S is an **inpdependent set** if no pair of vertices in S is connected by an edge.

<u>*Prove:*</u> The subset S is a vertex cover *if and only if* $\overline{S}$ (the vertices not in S) is an independent set.

3 Conditional Probability and Expected Value.

A box has 1 fair coin and 1 two-headed coin. You picked a random coin, flipped it 2 times and both flips were H. You now keep flipping the *same* coin you picked until you flip *two heads in a row.* Let $\mathbf{X}$ be the number of additional flips you make. Compute $\mathbb{E}[\mathbf{X}]$, the expected value of $\mathbf{X}$.

4 Sums and Induction. (Tinker, tinker,...)

Obtain a formula that does not use a sum for $S(n) = \sum_{i=1}^{2n} (-1)^i i^2$. Prove your formula by *induction*.

5 Transducer Turing Machine for Unary to Binary.

Give a high-level description of a transducer Turing Machine to solve unary to binary conversion. The input is $0^{\bullet n}$ (if not reject). The Turing Machine should halt with the tape showing $0^{\bullet n}$#w, where w is the binary representation of n. (E.g. for input 00000, the the tape should be 00000#101 when the machine halts.)

SCRATCH

SCRATCH

FINAL: 180 Minutes

Last Name: ____________________

First Name: ____________________

RIN: ____________________

Section: ____________________

Answer **ALL** questions. You may use **two** double sided $8\frac{1}{2} \times 11$ crib sheets.
You **MUST** show **CORRECT** work (even for multiple choice) to receive full credit.
NO COLLABORATION or electronic devices. Any violations result in an F.
NO questions allowed during the test. Interpret and do the best you can.

GOOD LUCK!

1	2	3	4	5	6	Total
200	30	30	30	30	30	350

1 Circle at most one answer per question. 10 points for each correct answer.

(1) Is this claim true or false. $\forall n \in \mathbb{Z} : n^2 \geq 0$.

$\boxed{\text{A}}$ True.

$\boxed{\text{B}}$ False.

$\boxed{\text{C}}$ You can't say because it depends on n.

$\boxed{\text{D}}$ You can't assign true or false to quantified statements.

$\boxed{\text{E}}$ It is not a proper statement to which you can assign true or false.

(2) If it rains on a day, it must rain the next day. Today it did not rain. What can you conclude?

$\boxed{\text{A}}$ It won't rain tomorrow.

$\boxed{\text{B}}$ It won't rain on any future day.

$\boxed{\text{C}}$ It rained yesterday.

$\boxed{\text{D}}$ It did not rain yesterday but it could have rained on some day prior to yesterday.

$\boxed{\text{E}}$ It did not rain yesterday and it did not rain on any day prior to yesterday.

(3) To prove $P(n)$ by induction, which is *not* a valid induction step to prove $P(n) \rightarrow P(n+1)$.

$\boxed{\text{A}}$ *Assume* that $P(n)$ is true and *prove* that $P(n+1)$ is true.

$\boxed{\text{B}}$ *Assume* two things, that $P(n)$ is true and that $P(n+1)$ is false. Now derive a contradiction.

$\boxed{\text{C}}$ *Assume* that $P(n)$ is false and *prove* that $P(n+1)$ is false.

$\boxed{\text{D}}$ *Assume* that $P(n+1)$ is false and *prove* that $P(n)$ is false.

$\boxed{\text{E}}$ All of the above are valid induction steps.

(4) What is the approximate value of the sum $\displaystyle\sum_{i=0}^{20}(2^i + i)(2^i - i)$.

$\boxed{\text{A}}$ 1.5×10^{11}.

$\boxed{\text{B}}$ 4.0×10^{11}.

$\boxed{\text{C}}$ 1.5×10^{12}.

$\boxed{\text{D}}$ 4.0×10^{12}.

$\boxed{\text{E}}$ 1.5×10^{13}.

(5) $T_1 = 1$ and $T_n = T_{n-1} + n^2$ for $n > 1$. Which order relationship is accurate?

$\boxed{\text{A}}$ $T_n \in \Theta(n)$.

$\boxed{\text{B}}$ $T_n \in \Theta(n^2)$.

$\boxed{\text{C}}$ $T_n \in \Theta(n^3)$.

$\boxed{\text{D}}$ $T_n \in \Theta(2^n)$.

$\boxed{\text{E}}$ None of the above.

(6) What is the remainder when 2^{2019} is divided by 5?

$\boxed{\text{A}}$ 0.

$\boxed{\text{B}}$ 1.

$\boxed{\text{C}}$ 2.

$\boxed{\text{D}}$ 3.

$\boxed{\text{E}}$ 4.

(7) Define the set $A = \{3x + 7y \mid x \text{ and } y \text{ are in } \mathbb{Z}\}$. Which numbers are *not* in A?

$\boxed{\text{A}}$ -11.

$\boxed{\text{B}}$ 11.

$\boxed{\text{C}}$ 37.

$\boxed{\text{D}}$ 142.

$\boxed{\text{E}}$ They are all in A.

(8) Ayfos is in a social network with 14 others, so 15 people in all with Ayfos. There are 25 friendship links in this network. Everyone but Ayfos has 3 friends. How many friends does Ayfos have?

$\boxed{\text{A}}$ 6.

$\boxed{\text{B}}$ 7.

$\boxed{\text{C}}$ 8.

$\boxed{\text{D}}$ 9.

$\boxed{\text{E}}$ Can't be determined or such a social network cannot exist.

(9) In the previous problem regarding Ayfos' social network, you pick a person randomly. What is the expected number of friends that person has.

$\boxed{\text{A}}$ $3\frac{1}{3}$.

$\boxed{\text{B}}$ $3\frac{1}{2}$.

$\boxed{\text{C}}$ $3\frac{3}{4}$.

$\boxed{\text{D}}$ 4.

$\boxed{\text{E}}$ None of the above, or not enough information to say for sure.

(10) From 1000 students, 900 are CS and 200 are MATH. How many are CS-MATH duals?

$\boxed{\text{A}}$ 50.

$\boxed{\text{B}}$ 100.

$\boxed{\text{C}}$ 150.

$\boxed{\text{D}}$ 200.

$\boxed{\text{E}}$ None of the above, or not enough information to say for sure.

(11) Digits are 0,1,..., 9. How many of the three digit strings 000 to 999 have a digit-sum 10? (For example, 307 and 811 have digit sum 10, but 846 and 213 do not.)

A 60.

B 63.

C 66.

D 69.

E None of the above.

(12) A and B are sets. $|A| = 5$ and $|B| = 3$. How many functions are there from A to B?

A 3^5.

B 5^3.

C $5!$.

D $\binom{5}{3}$.

E None of the above.

(13) A and B are sets. $|A| = 5$ and $|B| = 3$. How many injections (1-to-1) are there from A to B?

A 0.

B 100.

C 150.

D 200.

E None of the above.

(14) A and B are sets. $|A| = 5$ and $|B| = 3$. How many surjections (onto) are there from A to B?

A 0.

B 100.

C 150.

D 200.

E None of the above.

(15) You roll a die 4 times. What is the probability to get (exactly) 2 sixes?

A $6/6^4$.

B $12/6^4$.

C $36/6^4$.

D $150/6^4$.

E None of the above.

(16) Al and Jo each independently pick 4 restaurants randomly from 10 restaurants $r_1, \ldots, r_{10}$. They must eat at a restaurant that both picked. Compute the probability they can eat at (exactly) 2 restaurants.

$\boxed{\text{A}}$ 2/7

$\boxed{\text{B}}$ 3/7

$\boxed{\text{C}}$ 4/7

$\boxed{\text{D}}$ 5/7

$\boxed{\text{E}}$ None of the above

(17) Compute the expected number of restaurants Al and Jo from the previous problem can eat at.

$\boxed{\text{A}}$ 1.2.

$\boxed{\text{B}}$ 1.4

$\boxed{\text{C}}$ 1.6.

$\boxed{\text{D}}$ 1.8

$\boxed{\text{E}}$ None of the above

(18) Which computing problem *cannot* be solved by a DFA?

$\boxed{\text{A}}$ Strings with an even number of 1s.

$\boxed{\text{B}}$ Strings which have more 1s than 0s.

$\boxed{\text{C}}$ Strings whose number of 1s is a multiple of 3.

$\boxed{\text{D}}$ Strings whose number of 1s is not a multiple of 3.

$\boxed{\text{E}}$ Each problem is solvable using a DFA

(19) Which string cannot be generated by the CFG $S \to \varepsilon|0S|1S$?

$\boxed{\text{A}}$ $11111111110000000000 = 1^{\bullet 10}0^{\bullet 10}$.

$\boxed{\text{B}}$ $10101010101010101010 = (10)^{\bullet 10}$.

$\boxed{\text{C}}$ $00000000000000000000 = 0^{\bullet 20}$.

$\boxed{\text{D}}$ $00110011001100110011 = (0011)^{\bullet 5}$.

$\boxed{\text{E}}$ They can all be generated.

(20) Which answer is a valid conclusion about the decidability of the language $\mathcal{L}_B$?

$\boxed{\text{A}}$ $\mathcal{L}_A$ is decidable. A decider for $\mathcal{L}_B$ can be converted to a decider for $\mathcal{L}_A$. So, $\mathcal{L}_B$ is decidable.

$\boxed{\text{B}}$ $\mathcal{L}_A$ is decidable. A decider for $\mathcal{L}_A$ can be converted to a decider for $\mathcal{L}_B$. So, $\mathcal{L}_B$ is decidable.

$\boxed{\text{C}}$ $\mathcal{L}_A$ is undecidable. A decider for $\mathcal{L}_A$ can be converted to a decider for $\mathcal{L}_B$. So, $\mathcal{L}_B$ is undecidable.

$\boxed{\text{D}}$ $\mathcal{L}_A$ is undecidable. A decider for $\mathcal{L}_B$ can be converted to a decider for $\mathcal{L}_A$. So, $\mathcal{L}_B$ is decidable.

$\boxed{\text{E}}$ None of the above is valid.

2 Determine the Type of Proof and Prove

Prove that for $n \in \mathbb{N}$, $\sqrt{n(n+1)} \leq n + \frac{1}{2}$.

3 Induction and Sums. Tinker, Tinker, Tinker.

For $n \in \mathbb{N}$, obtain a formula for the sum $S(n) = \sum_{i=1}^{2n} (-1)^i i$ and prove your formula by induction.

4 Expected Waiting Time to 3 Heads In A Row

You flip a fair coin until you get 3 heads *in a row*. Compute the expected number of flips you make.

5 CFGs and Induction. (Tinker, tinker,. . .)

For the CFG $S \to 0|0S1$, *prove* that every string that can be generated has odd length.

6 Turing Machine for Squaring.

Give a high level pseudo-code description of a Turing Machine that solves the problem $\mathcal{L} = \{0^{\bullet n}1^{\bullet n \times n} | n \geq 1\}$. (You do not need to give machine level details but your pseudo-code should demonstrate understanding of how the Turing Machine moves back and forth to solve the problem. Tinker.)

SCRATCH

SCRATCH

FINAL: 180 Minutes

Last Name: ____________________

First Name: ____________________

RIN: ____________________

Section: ____________________

Answer **ALL** questions. You may use **two** double sided $8\frac{1}{2} \times 11$ crib sheets.
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NO COLLABORATION or electronic devices. Any violations result in an F.
NO questions allowed during the test. Interpret and do the best you can.

GOOD LUCK!

1	2	3	4	5	6	Total
200	30	30	30	30	30	350

INSTRUCTIONS

1. This is a **closed book** test. No electronics, books, notes, internet, etc.
2. You can have two double sided $8\frac{1}{2} \times 11$ crib-sheets (handed in separately).
3. The test will become available in Submitty at 8am on the test date. Your PDF is due in Submitty by 8am the next day. You have 3 hours to do the exam and 3 additional hours to type your answers and submit a PDF.
4. By submitting the test you attest that the work is entirely your own and you obeyed the time limits of the exam.
5. Your submission *must* be typed PDF. The 3 hour test time for solving the problems must be continuous. The extra 3 hours is to type your answers and explanations: you may take breaks, but not change answers.
6. You *must* show your work for *every* answer immediately after the answer. The format for what you hand in is something like:

Problem 1
(1) A
Because x is even, therefore ...
(2) B
Because $\sqrt{2}$ is irrational, therefore ...
(4) D
By the law of total expectation, $\mathbb{E}[\mathbf{X}] = \cdots$
$\vdots$
(20) A
We proved in class that $\ell = n - 1$. Therefore ...
Problem 2
$\vdots$

   – Start each long-answer question on a new page.
   – Some problems may be "easy", so give a short explanation.
   – Some problems may require a detailed reasoning.
   – `3*3+1+3=13` is **not** an explanation. Everyone knows that `3*3+1+3=13`.
   Why this equation? Where do the numbers come from?

7. **If you don't show correct work, you won't get credit.**
8. Be especially careful in the multiple choice.
   – Correct answers get 10 points.
   – Wrong answers or correct answers without correct justification get 0.
9. Submit with plenty of time to spare. A late test won't be accepted.

1 Circle at most one answer per question. 10 points for each correct answer.

(1) "For a constant $c > 0$, $1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} > c\sqrt{n}$, where n is any natural number." Which claim is this?

$\boxed{\text{A}}$ $\exists c > 0 : (\exists n \in \mathbb{N} : 1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} > c\sqrt{n})$.

$\boxed{\text{B}}$ $\exists c > 0 : (\forall n \in \mathbb{N} : 1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} > c\sqrt{n})$.

$\boxed{\text{C}}$ $\exists n \in \mathbb{N} : (\forall c > 0 : 1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} > c\sqrt{n})$.

$\boxed{\text{D}}$ $\forall n \in \mathbb{N} : (\exists c > 0 : 1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} > c\sqrt{n})$.

$\boxed{\text{E}}$ None of the above.

(2) You will pick a constant $C > 0$ such that no matter which $n \in \mathbb{N}$ I pick, $\sum_{i=1}^{n} i \leq Cn$. Which is true?

$\boxed{\text{A}}$ You can pick a C satisfying $C \leq 10$.

$\boxed{\text{B}}$ You can pick a C satisfying $10 < C < 100$.

$\boxed{\text{C}}$ You can pick a C satisfying $100 < C < 1000$.

$\boxed{\text{D}}$ You can pick a C satisfying $1000 < C$.

$\boxed{\text{E}}$ There is no constant $C > 0$ that you can pick.

(3) $T_1 = 2$ and $T_n = T_{n-1} + 2n$ for $n > 1$. What is T_{100}?

$\boxed{\text{A}}$ 5050.

$\boxed{\text{B}}$ 10100.

$\boxed{\text{C}}$ 20200.

$\boxed{\text{D}}$ 40400.

$\boxed{\text{E}}$ None of the above.

(4) $T_1 = 1$ and $T_n = n \times T_{n-1}$ for $n > 1$. Which is true?

$\boxed{\text{A}}$ $T(n) \in O(n^2)$.

$\boxed{\text{B}}$ $T(n) \in o(2^n)$.

$\boxed{\text{C}}$ $T(n) \in \Theta(2^n)$.

$\boxed{\text{D}}$ $T(n) \in \omega(2^n)$.

$\boxed{\text{E}}$ None of the above.

(5) You divide 2^{2016} candies evenly among 11 kids. How many candies are left over?

$\boxed{\text{A}}$ 0.

$\boxed{\text{B}}$ 3.

$\boxed{\text{C}}$ 6.

$\boxed{\text{D}}$ 9.

$\boxed{\text{E}}$ None of the above.

(6) Estimate the sum $S = \sum_{i=1}^{\infty} \frac{1}{i} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$.

$\boxed{\text{A}}$ $0 < S \leq 2$.

$\boxed{\text{B}}$ $2 < S \leq 2000$.

$\boxed{\text{C}}$ $2000 < S \leq 20000$.

$\boxed{\text{D}}$ $20000 < S \leq 200000$.

$\boxed{\text{E}}$ None of the above.

(7) How many of the numbers $100, 101, 102, \ldots, 999$ do not contain the digit 2?

$\boxed{\text{A}}$ 100.

$\boxed{\text{B}}$ 504.

$\boxed{\text{C}}$ 648.

$\boxed{\text{D}}$ 729.

$\boxed{\text{E}}$ None of the above.

(8) Let S be the sum of the reciprocals of all natural numbers not containing the digit 2. Estimate S.

$\boxed{\text{A}}$ $0 < S \leq 2$.

$\boxed{\text{B}}$ $2 < S \leq 2000$.

$\boxed{\text{C}}$ $2000 < S \leq 20000$.

$\boxed{\text{D}}$ $20000 < S \leq 200000$.

$\boxed{\text{E}}$ None of the above.

$$\begin{aligned} S = \; & 1 + \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{5} + \tfrac{1}{6} + \tfrac{1}{7} + \tfrac{1}{8} + \tfrac{1}{9} + \tfrac{1}{10} + \tfrac{1}{11} + \tfrac{1}{13} + \tfrac{1}{14} + \\ & \tfrac{1}{15} + \tfrac{1}{16} + \tfrac{1}{17} + \tfrac{1}{18} + \tfrac{1}{19} + \tfrac{1}{30} + \tfrac{1}{31} + \tfrac{1}{33} + \tfrac{1}{34} + \tfrac{1}{35} + \cdots \end{aligned}$$

(9) Shirts come in 3 colors R, G or B. In how many ways can you distribute shirts to 7 students?

$\boxed{\text{A}}$ $\binom{7}{3}$.

$\boxed{\text{B}}$ 7^3.

$\boxed{\text{C}}$ 3^7.

$\boxed{\text{D}}$ $7!/3!$.

$\boxed{\text{E}}$ None of the above.

(10) Repeat problem 9 if at least two shirts of each color are distributed to students (7 shirts in total).

$\boxed{\text{A}}$ 570

$\boxed{\text{B}}$ 600.

$\boxed{\text{C}}$ 630.

$\boxed{\text{D}}$ 660.

$\boxed{\text{E}}$ None of the above.

(11) Every vertex in a graph G has degree 1. Which is true?

$\boxed{\text{A}}$ The graph G must be disconnected.

$\boxed{\text{B}}$ The graph G could have 5 vertices.

$\boxed{\text{C}}$ The graph G must have a cycle.

$\boxed{\text{D}}$ The graph G is not possible.

$\boxed{\text{E}}$ None of the above

(12) You rolled a pair of dice. What are the chances you rolled exactly one 5?

$\boxed{\text{A}}$ 9/36.

$\boxed{\text{B}}$ 10/36.

$\boxed{\text{C}}$ 11/36.

$\boxed{\text{D}}$ 12/36.

$\boxed{\text{E}}$ None of the above.

(13) You rolled a pair of dice. What are the chances you rolled exactly one 5 if the sum is even?

$\boxed{\text{A}}$ 4/10.

$\boxed{\text{B}}$ 5/10.

$\boxed{\text{C}}$ 4/11.

$\boxed{\text{D}}$ 5/11.

$\boxed{\text{E}}$ None of the above.

(14) Which of the following random variables $\mathbf{X}$ is not a binomial random variable.

$\boxed{\text{A}}$ Randomly throw 100 darts at a dart board. $\mathbf{X}$ is the number of darts hitting the bulls-eye.

$\boxed{\text{B}}$ Randomly answer 100 5-choice multiple choice questions. $\mathbf{X}$ is the number of questions correct.

$\boxed{\text{C}}$ Randomly answer 100 5-choice multiple choice questions. $\mathbf{X}$ is the number of questions wrong.

$\boxed{\text{D}}$ 1000 students randomly line up, 500 are boys. $\mathbf{X}$ is the number of boys in the first 100 students.

$\boxed{\text{E}}$ They are all binomial random variables.

(15) A social network (graph) is a *tree* with 20 people. The edges are friendships. Each person randomly picks red or blue. Friends compare to see if they match. What is the expected number of matches.

$\boxed{\text{A}}$ 4.75.

$\boxed{\text{B}}$ 5.

$\boxed{\text{C}}$ 9.5

$\boxed{\text{D}}$ 10.

$\boxed{\text{E}}$ None of the above or not enough information.

(16) On BlueToe, your first child is equally likely to be a boy or girl. From then on, the sex of a child is the same as the previous child with probability 2/3 and different with probability 1/3. What is the expected number of kids to get a girl?

A 1.5.

B 2.

C 2.5.

D 3.

E None of the above.

(17) On BlueToe, as in problem 16, what is the expected number of kids to two girls?

A 3.25.

B 4.

C 4.5.

D 5.25.

E None of the above.

(18) Estimate the number of DFA you can draw with 4 states, q_0, q_1, q_2, q_3. Tinker!

A About a hundred.

B About a thousand.

C About a million.

D About a billion.

E About a trillion.

(19) Which string can be generated by the CFG $S \to 0|1|SSS$?

A 1111.

B 0000.

C 000111.

D 111000.

E None of the above.

(20) If $\mathcal{L}_A$ is decidable, then $\mathcal{L}_B$ is decidable. We know that $\mathcal{L}_B$ is undecidable. Therefore:

A $\mathcal{L}_A$ must be finite.

B $\mathcal{L}_A$ must be infinite.

C $|\mathcal{L}_A| > |\mathcal{L}_B|$.

D $|\mathcal{L}_B| < |\mathcal{L}_A|$.

E None of the above.

2 Determine the Type of Proof and Prove

Prove that there is a constant $c > 0$ for which, no matter which $n \in N$ you pick,

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} > c\sqrt{n}.$$

3 Product of 5 Consecutive Numbers.

Prove that the product of any 5 consecutive natural numbers is divisible by $5!$ (e.g. $5!|3 \times 4 \times 5 \times 6 \times 7$).

4 Expected Waiting Time to All Colors of Starbust.

Starburst is sold in 2-packs, and there are 3 colors of starbust. What is the expected number of 2-packs you will buy if your goal is to get all colors?

5 DFA or no DFA

Give a DFA for $\mathcal{L} = \{0^{\bullet n^2} | n \geq 1\} = \{0, 0000, 000000000, \ldots\}$, or prove that $\mathcal{L}$ can't be solved with DFA.

6 Transducer Turing Machine for Reversal.

Give a high level pseudo-code description of a transducer Turing Machine for reversal. The input on the tape is any binary string w. When the Turing Machine halts, the reversal of w should have replaced w. E.g:

Start

*	1	0	1	0	0	1	1	␣

End

*	1	1	0	0	1	0	1	␣

(Don't give machine level details, but you should make it clear how the Turing Machine moves back and forth. Tinker.)

SCRATCH

SCRATCH