1 **Circle at most one answer per question.** 10 **points for each correct answer and -5 points for each incorrect answer (blank answer is 0 points). Don't guess!**

(a) $P(n)$ is a predicate ($n \in \mathbb{N}$). $P(1), P(2), P(3)$ are true, and $P(n) \to P(n+4)$ is true for $n \geq 1$. For which n can we be **sure** $P(n)$ is true?

$\boxed{A}$ All $n \geq 1$ except multiples of 2.

$\boxed{B}$ All $n \geq 1$ except multiples of 4.

$\boxed{C}$ All $n \geq 1$

$\boxed{D}$ Only $n = 1, 2, 3$.

(b) Of the following five sets, list *all* that are countable ($\mathcal{A}$ is countable if $\mathbb{N} \xrightarrow{\text{surj}} \mathcal{A}$):
(I) Prime numbers; (II) Rational numbers; (III) Integers; (IV) Even numbers; (V) Infinite binary strings.

$\boxed{A}$ I and III.

$\boxed{B}$ I and II and III and IV.

$\boxed{C}$ I and III and V.

$\boxed{D}$ II and III and IV.

(c) A class with 25 students needs to choose a representative committee which is a subset of 5 students. How many different committees can be formed?

$\boxed{A}$ 25^5.

$\boxed{B}$ $\dfrac{25!}{20! \times 5!}$.

$\boxed{C}$ $\dfrac{25!}{5!}$.

$\boxed{D}$ $25 \times 24 \times 23 \times 22 \times 21 = \dfrac{25!}{20!}$.

(d) A friendship network has 7 people and each person has at least 1 friend. 6 of the people have *exactly two friends*. How many friends can the 7th person have? Give all possibilities.

$\boxed{A}$ The seventh person could have either 2 or 4 friends.

$\boxed{B}$ The seventh person could have either 2 or 4 or 6 friends.

$\boxed{C}$ The seventh person could have either 1 or 2 or 3 friends.

$\boxed{D}$ The seventh person could have any number of friends that is greater than 1.

(e) Compute the summation $(0+1)+(1+2)+(2+4)+(3+8)+\cdots+(10+2^{10}) = \sum_{i=0}^{10}(i+2^i)$

$\boxed{A}$ 2048.

$\boxed{B}$ 2102.

$\boxed{C}$ 1078.

$\boxed{D}$ 2200.

(f) You have a known fact that $0 = 0$ and all the standard operations of algebra you learned in high-school math. Which of the following is a valid proof that $7 = 7$:

$$\begin{array}{lrcll} & & \text{I} & & \\ 1. & 7 & = & 7 & \\ 2. & 7-7 & = & 7-7 & \\ 3. & 0 & = & 0 & \checkmark \\ \hline \to & 7 & = & 7 & \end{array} \qquad \begin{array}{lrcll} & & \text{II} & & \\ 1. & 7 & \neq & 7 & \\ 2. & 7-7 & \neq & 7-7 & \\ 3. & 0 & \neq & 0 & \textbf{!FISHY} \\ \hline \to & 7 & = & 7 & \end{array} \qquad \begin{array}{lrcll} & & \text{III} & & \\ 1. & 0 & = & 0 & \\ 2. & 0+7 & = & 0+7 & \\ 3. & 7 & = & 7 & \checkmark \\ \hline \to & 7 & = & 7 & \end{array}$$

$\boxed{A}$ I & II & III. $\boxed{B}$ II & III. $\boxed{C}$ I & II $\boxed{D}$ I & III.

(g) Let $f(n) = \sum_{i=1}^{n} i$ and $g(n) = 2^{3\log_2 n}$. What is the big-Oh relationship between f and g?

$\boxed{A}$ $f(n) = O(g(n))$ and $g(n) = O(f(n))$.

$\boxed{B}$ $f(n) = O(g(n))$ and $g(n) \neq O(f(n))$.

$\boxed{C}$ $f(n) \neq O(g(n))$ and $g(n) = O(f(n))$.

$\boxed{D}$ $f(n) \neq O(g(n))$ and $g(n) \neq O(f(n))$.

(h) You independently generate the ten bits of a binary sequence $b_1 b_2 \cdots b_{10}$ with $\mathbb{P}[b_i = 0] = \frac{1}{2}$. Compute the probability that the sequence is sorted from low to high. For example 0000111111 is sorted.

$\boxed{A}$ $\dfrac{10}{1024}$

$\boxed{B}$ $\dfrac{11}{1024}$

$\boxed{C}$ $\dfrac{20}{1024}$

$\boxed{D}$ $\dfrac{12}{1024}$

(i) x_1, x_2, x_3 are non-negative integers. Compute the number of different solutions to $x_1 + x_2 + x_3 = 100$. (For example two different solutions are $1 + 2 + 97 = 100$ and $97 + 1 + 2 = 100$.)

$\boxed{A}$ 10302

$\boxed{B}$ 5151

$\boxed{C}$ 4949

$\boxed{D}$ 5050

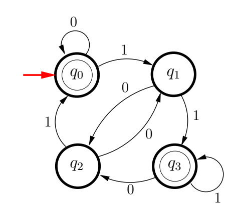
(j) For the automaton on the right, which input string is accepted?
(Strings are processed from left to right.)

$\boxed{A}$ 010101

$\boxed{B}$ 0101011

$\boxed{C}$ 01010110

$\boxed{D}$ 010101100

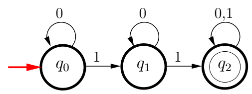


2 Proofs

1. Prove that for all integers $n \geq 1$: $\quad n2^n \leq 3^n$

2. Prove that *every* odd natural number is the difference of two square numbers.

3 Finite Automaton with a Random Input String



The automaton to the right processes a random binary string $b_1 b_2 \ldots b_n$ of length n generated as follows: you independently generate each bit b_i with $\mathbb{P}[b_i = 1] = p$ and $\mathbb{P}[b_i = 0] = 1 - p$. Show that the probability that the string is accepted is

$$\mathbb{P}[\text{ random input string is accepted }] = 1 - (1-p)^n - np(1-p)^{n-1}.$$

[Hints: (i) Figure out a simple property of a string for it to be accepted. (ii) Binomial distribution.]

4 Probability and Expectation

(a) You independently roll 3 fair dice D_1, D_2, D_3 and let $S = D_1 + D_2 + D_3$ be the sum. Compute:

(i) $\mathbb{P}[S = 8]$

(ii) $\mathbb{P}[S = 8 \mid D_1 = 1]$

(iii) Compute the expectation and variance of S.

(b) You toss a fair coin independently until you get two heads *in a row*. Let X be the number of tosses. Compute $\mathbb{E}[X]$ using the law of total expectation:

(i) Consider the 3 cases T, HT, HH for how the tosses may start and show that

$$\mathbb{E}[X] = \tfrac{1}{2}(1 + \mathbb{E}[X]) + \tfrac{1}{4}(2 + \mathbb{E}[X]) + \tfrac{1}{2}.$$

(ii) Use (i) to show that $\mathbb{E}[X] = 6$.

5 Context Free Grammars

This problem is about the language $\mathcal{L}$ generated by the CFG:

$$\begin{aligned} S &\rightarrow 1T \mid 0T \\ T &\rightarrow 1T1 \mid 0T0 \mid \epsilon \end{aligned}$$

(a) Is the string 1010010 in $\mathcal{L}$? If yes then give a derivation or parse tree; if no then explain why.

(b) Prove that the length of every string in $\mathcal{L}$ is odd.

6 Turing Machine

(a) What is the difference between a Turing-recognizable language and a Turing-decidable language?

(b) Consider the arithmetic task of squaring, which corresponds to the language $\mathcal{L} = \{0^n \# 0^{n^2} | n \geq 1\}$.

(i) Circle the simplest model of computing that you think solves the problem $\mathcal{L}$:

Finite Automaton   Context Free Grammar   Turing Machine

(ii) Give your machine from (i) that solves $\mathcal{L}$ (for a TM, a high level description will do).

1 Circle at most one answer per question. 10 points for each correct answer.

(1) The **negation** of "All Malik's friends are big and strong" is

A None of Malik's friends are big and strong.

B Malik has a friend who is either small or weak (or both).

C All Malik's friends are small and weak.

D All Malik's friends are either small or weak (or both).

E Malik has no friends who are small or weak.

(2) What is the *most accurate* order relation between $3^{\log_2 n}$ and n^2?

A $3^{\log_2 n} \in o(n^2)$.

B $3^{\log_2 n} \in O(n^2)$.

C $3^{\log_2 n} \in \Theta(n^2)$.

D $3^{\log_2 n} \in \Omega(n^2)$.

E $3^{\log_2 n} \in \omega(n^2)$.

(3) Compute the summation $\sum_{i=1}^{20} (-1)^i i^2$

A 190.

B 200.

C 210.

D 220.

E 230.

(4) Let $f(n) = \sum_{i=1}^{n} i$ and $g(n) = 4^{\log_2 n}$. What is the *most accurate* order relationship between f and g?

A $f \in o(g)$.

B $f \in O(g)$.

C $f \in \Theta(g)$.

D $f \in \Omega(g)$.

E $f \in \omega(g)$.

(5) Let $f(n)$ be a function satisfying the recurrence $f(0) = 0$; $f(n) = f(n-1) + \sqrt{n}$. Which order relationship describes f.

A $f \in \Theta(n)$.

B $f \in \Theta(n \log n)$.

C $f \in \Theta(n\sqrt{n})$.

D $f \in \Theta(n^2)$.

E $f \in \Theta(n^3)$.

(6) A class with 10 students needs to choose a president, vice-president and secretary (a student *cannot* fill multiple roles). In how many ways can this be done?

A 1000.

B 720.

C 120.

D 10!

E $\binom{10}{3}$.

(7) A fraternity orders 5 pizzas (eg. 2 with sausage and 3 with meatballs & onion). There are 5 toppings. A pizza can have 0,1 or 2 toppings. How many ways are there for the fraternity to make its order?

A 16.

B 16^5.

C $\binom{16}{5}$.

D $\binom{20}{15}$.

E $16 \times 15 \times 14 \times 13 \times 12$.

(8) A friendship network has 6 people ⒶⒷⒸⒹⒺⒻ. If you add up the number of friends of each person, you get a total of 26. How many *different* social network graphs could correspond to this friendship network. (Two graphs are different if they don't have exactly the same edges.)

A 0.

B 95.

C 105.

D 115.

E 125.

(9) You are thinking of a graph with 5 nodes ⒶⒷⒸⒹⒺ. *Approximately* how many such graphs are there?

A 100.

B 500.

C 1000.

D 5000.

E 10000.

(10) X and Y are random variables (not necessarily independent). Which of the following is an expression for $Var(X+Y)$ (variance of the sum)?

A $Var(X) + Var(Y)$.

B $\mathbb{E}[(X+Y)^2]$.

C $\mathbb{E}[X^2] + \mathbb{E}[Y^2] - \mathbb{E}[X]^2 - \mathbb{E}[Y]^2$.

D $Var(X) + Var(Y) + 2\,\mathbb{E}[XY] - 2\,\mathbb{E}[X]\,\mathbb{E}[Y]$.

E $Var(X) + Var(Y) - 2Var(XY)$.

(11) You independently generate two random ten bit binary sequences and compute a new sequence using the BITWISE-OR of the two random sequences (treating 0 as FALSE and 1 as TRUE). Let X be the number of 1s in the result. What is $\mathbb{E}[X]$. (for example, 0001110010 BITWISE-OR 1000111000 = 1001111010.)

A 2.5

B 3.5

C 5

D 6.5

E 7.5

(12) About 1 in a 1000 people have Coeliac disease. The outcome of a test for Coeliac is random: the test makes a mistake on 1 in 10 people who have it (90% accuracy if you have Coeliac); the test makes a mistake on 1 in 100 people who do not have it (99% accuracy if you do not have Coeliac). You got tested, and the result was positive. *Approximately* what are the chances that you have Coeliac?

A 0.1%

B 10%

C 40%

D 80%

E 90%

(13) Which set is *not countable*?

A {1,3,5,7}.

B The prime numbers {2,3,5,7,...}.

C All possible angles between 0 and 360.

D All even numbers which are not a sum of two primes.

E All possible pairs of integers, $\mathbb{Z}^2$.

(14) A random binary string $b_1b_2 \ldots b_{10}$ of length 10 is the input to the automaton.
What is the probability that the string is accepted?

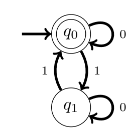
A 0.25

B 0.4

C 0.5

D 0.6

E 0.75



(15) Which string below is *not* in the language of the CFG: $S \longrightarrow \varepsilon \mid 0S|S0|11S$

A ε

B 1111

C 11011

D 0011000

E 001010

2 Positive Integer Partitions

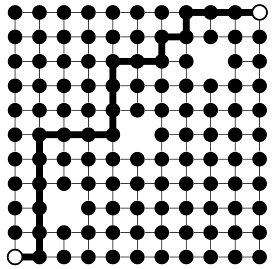
A positive partition of n is a *sequence* of *positive* integers that add up to n. For example, $(6, 4)$, $(4, 6)$ and $(2, 4, 2, 2)$ are different partitions of 10. How many positive partitions of n are there? Prove your answer.

3 Proofs

(a) *Prove* that $n^2 \leq 3^n$ for integer $n \geq 0$.

(b) *Prove* that $n^3 \notin O(n^2)$. You must *prove* that there is no *constant* C for which $n^3 \leq Cn^2$ for all $n \geq 1$.

4 Counting Paths on Graphs with Holes



A grid is missing nodes at $(2,2)$, $(5,5)$ and $(8,8)$. A *shortest* path from the bottom left node (0,0) to the top right node (10,10) is shown.

How many <u>different</u> shortest paths go from (0,0) to (10,10)? (Two paths are different if they do not have exactly the same edges).

You may leave your answer in the form of a combination of binomial coefficients – you do not need to compute a numerical answer.

5 Turing Machine and Exponentiation

(a) <u>*Prove*</u>: the problem (language) $\mathcal{L} = \{0^n \# 1^{2^n} \mid n \geq 1\}$ <u>*cannot*</u> be solved (accepted) by a finite automaton.

(b) Give a high-level description of a Turing Machine that solves $\mathcal{L} = \{0^n \# 1^{2^n} \mid n \geq 1\}$.

1 Circle at most one answer per question. 10 points for each correct answer.

(1) The **negation** of "Every student is a friend of some other student" is

A Some student has a friend who is a student.

B Some student is a friend of all students.

C Some student is not a friend of some other student.

D Some student is not a friend of all other students.

E Some student has no friends.

(2) Estimate $2^1 \times 2^2 \times 2^3 \times \cdots \times 2^{20} = \prod_{i=1}^{20} 2^i$.

A 1.65×10^{61}

B 1.65×10^{63}

C 1.65×10^{65}

D 1.65×10^{67}

E 1.65×10^{69}

(3) What is the *most accurate* order relation between 2^n and e^n?

A $2^n \in o(e^n)$.

B $2^n \in O(e^n)$.

C $2^n \in \Theta(e^n)$.

D $2^n \in \Omega(e^n)$.

E $2^n \in \omega(e^n)$.

(4) $f(n)$ satisfies the recurrence $f(0) = 1$; $f(n) = nf(n-1)$. Which order relationship describes f.

A $f \in \Theta(2^n)$.

B $f \in O(2^n)$.

C $f \in o(2^n)$.

D $f \in \Theta(n^n)$.

E $f \in o(n^n)$.

(5) What is the greatest common divisor of 756 and 840?

A 12.

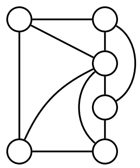
B 28.

C 63.

D 84.

E 189.

(6) What is the minimum number of colors needed to color the graph on the right?



A 2.

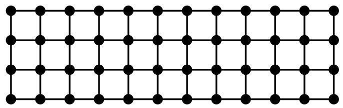
B 3.

C 4.

D 5.

E 6.

(7) On the right is the 4×12 grid graph. What is the average degree of a node?



A 3.

B $3\frac{1}{4}$.

C $3\frac{1}{3}$.

D $3\frac{1}{2}$.

E $3\frac{2}{3}$.

(8) Shirts come in 6 colors. 4 students are in a row. You must assign shirts to the students, and two students standing next to each other cannot get the same color shirt. In how many ways can you do this?

A $\binom{9}{3}$.

B $6 \times 5 \times 4 \times 3$.

C $\binom{6}{4}$.

D 6×5^3.

E 6^4.

(9) Pokemons have 4-digit serial numbers, e.g. 0255. A pokemon is defective if any digit repeats (e.g. 0255, 5250, 5255 are defective). *Approximately* what fraction of the possible serial numbers are defective?

$\boxed{\text{A}}$ 0.

$\boxed{\text{B}}$ 0.25.

$\boxed{\text{C}}$ 0.5.

$\boxed{\text{D}}$ 0.75.

$\boxed{\text{E}}$ 1.

(10) A senate committee of 10 senators must pick a president. 3 candidates will be proposed from the 10 senators, and everyone votes. In how many ways can the 3 candidates be chosen.

$\boxed{\text{A}}$ 1000.

$\boxed{\text{B}}$ 720.

$\boxed{\text{C}}$ 120.

$\boxed{\text{D}}$ 10!

$\boxed{\text{E}}$ $\dfrac{10!}{3!}$.

(11) Three integers z_1, z_2, z_3 satisfy $0 \leq z_1 \leq z_2 \leq z_3 \leq 6$ (the sequence is non-decreasing and bounded between 0 and 6). How many such sequences are there?

$\boxed{\text{A}}$ 28.

$\boxed{\text{B}}$ 42.

$\boxed{\text{C}}$ 84.

$\boxed{\text{D}}$ 165.

$\boxed{\text{E}}$ 168.

(12) You are thinking of a graph with 4 nodes Ⓐ Ⓑ Ⓒ Ⓓ. How many such graphs are there?

$\boxed{\text{A}}$ 24.

$\boxed{\text{B}}$ 64.

$\boxed{\text{C}}$ 81.

$\boxed{\text{D}}$ 256.

$\boxed{\text{E}}$ 4096.

(13) $\mathbf{X}, \mathbf{Y}$ are random variables (not necessarily independent) and $\mathbf{Z} = a\mathbf{X} + b\mathbf{Y}$. What is $\mathbb{E}[\mathbf{Z}]$?

$\boxed{\text{A}}$ $a\,\mathbb{E}\,[\mathbf{X}] + b\,\mathbb{E}\,[\mathbf{Y}]$

$\boxed{\text{B}}$ $a^2\,\mathbb{E}\,[\mathbf{X}] + b^2\,\mathbb{E}\,[\mathbf{Y}]$

$\boxed{\text{C}}$ $(a+b)(\mathbb{E}[\mathbf{X}] + \mathbb{E}[\mathbf{Y}])$

$\boxed{\text{D}}$ $a(\mathbb{E}[\mathbf{X}] + \mathbb{E}[\mathbf{Y}]) + b(\mathbb{E}[\mathbf{X}] + \mathbb{E}[\mathbf{Y}])$

$\boxed{\text{E}}$ None of the above are true in general.

(14) This test has 20 multiple choice questions, each with 5 possible choices. If you answer questions randomly, what is the expected number of multiple questions you get correct?

$\boxed{\text{A}}$ 3

$\boxed{\text{B}}$ 4

$\boxed{\text{C}}$ 5

$\boxed{\text{D}}$ 6

$\boxed{\text{E}}$ 10

(15) About 1 in a 1000 people have Coeliac disease. The test for Coeliac randomly makes a mistake 5% of the time (95% accuracy). You tested positive. *Approximately* what are the chances you have Coeliac?

$\boxed{\text{A}}$ 0.2%

$\boxed{\text{B}}$ 2%

$\boxed{\text{C}}$ 20%

$\boxed{\text{D}}$ 50%

$\boxed{\text{E}}$ 95%

(16) A random binary string $b_1 b_2 \ldots b_{10}$ of 10 bits is the input to the automaton. What is the probability that the string is accepted?

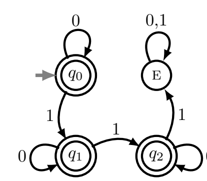
$\boxed{\text{A}}$ $\frac{2}{1024}$

$\boxed{\text{B}}$ $\frac{45}{1024}$

$\boxed{\text{C}}$ $\frac{56}{1024}$

$\boxed{\text{D}}$ $\frac{90}{1024}$

$\boxed{\text{E}}$ $\frac{512}{1024}$



(17) What is a computing problem?

$\boxed{\text{A}}$ A Person.

$\boxed{\text{B}}$ An automaton (machine which transitions between states as it reads the input).

$\boxed{\text{C}}$ An automaton with stack memory.

$\boxed{\text{D}}$ An automaton with random access memory.

$\boxed{\text{E}}$ A set containing finite binary strings.

(18) The computing problem $\mathcal{L} = \{$strings with an even number of 1s$\}$ can be solved by:
(I) DFA. (II) CFG. (III) Turing Machine.

$\boxed{\text{A}}$ I,II,III

$\boxed{\text{B}}$ I,III

$\boxed{\text{C}}$ II,III

$\boxed{\text{D}}$ III only

$\boxed{\text{E}}$ None of these models of computing

(19) The computing problem $\mathcal{L} = \{$strings corresponding to programs which HALT$\}$ can be solved by:
(I) DFA. (II) CFG. (III) Turing Machine.

$\boxed{\text{A}}$ I,II,III

$\boxed{\text{B}}$ I,III

$\boxed{\text{C}}$ II,III

$\boxed{\text{D}}$ III only

$\boxed{\text{E}}$ None of these models of computing

(20) A DFA has *two* states a start state q_0 and a second state q_1. The DFA is described by a list of its accept states and a list of its transition instructions. The order in which you list the accept states and the transition instructions does not matter. We draw a DFA as a graph with nodes q_0, q_1 and add a directed arrow for each transition instruction (the accepting states have double circles).

How many different DFA's are there with two states? (*Different* DFA's *can* have the same YES-set)

$\boxed{\text{A}}$ 4.

$\boxed{\text{B}}$ 8.

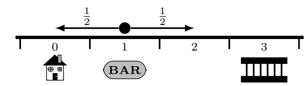
$\boxed{\text{C}}$ 16.

$\boxed{\text{D}}$ 32.

$\boxed{\text{E}}$ 64.

2 Random Walk

A drunk leaves the bar (at position 1), and takes independent steps: left (L) with probability $\frac{1}{2}$ or right (R) with probability $\frac{1}{2}$. The drunk stops when he reaches home (at 0) or the jail (at 3). Compute the *expected* number of steps the drunk makes.



3 Induction

(a)
$$\begin{aligned} G(1) &= 1; \\ G(n) &= G(n-1)\left(1-\frac{1}{n}\right) \quad \text{for } n>1; \end{aligned}$$

Prove that $G(n) = \dfrac{1}{n}$ for integer $n \geq 1$.

(b) The nth Harmonic number is $H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$. Prove that $H_1 + H_2 + \ldots + H_n = (n+1)H_n - n$.

4 Turing Machine

Give a high-level description of a Turing Machine that solves the problem $\mathcal{L} = \{0^n\texttt{\#}1^{n^2} \mid n \geq 0\}$ (squaring). (You may find it useful to illustrate how your TM works on 00#1111.)

5 [Hard] Unsolvable Problems

Prove: There is an undecidable computing problem which is a subset of $\{1\}^*$.

1 Circle at most one answer per question. 10 points for each correct answer.

(1) Every card has a letter and a number. **Rule:** If a card has a P on it, then the other side *must* be a 5.

$\boxed{\mathbf{S}}$ $\boxed{\mathbf{5}}$ $\boxed{\mathbf{P}}$ $\boxed{\mathbf{3}}$

Which of the above cards *must* be turned over to verify the rule has not been broken.

A $\boxed{\mathbf{S}}$ $\boxed{\mathbf{5}}$

B $\boxed{\mathbf{5}}$ $\boxed{\mathbf{P}}$

C $\boxed{\mathbf{S}}$ $\boxed{\mathbf{3}}$

D $\boxed{\mathbf{P}}$ $\boxed{\mathbf{3}}$

E None of the above.

(2) Which set relationship does not hold in general.

A $\overline{A \cap B} = \overline{A} \cup \overline{B}$.

B $\overline{A \cup B} = \overline{A} \cap \overline{B}$.

C $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

D $(A \cup B) \cap \overline{A} = B \cap \overline{A}$.

E They all hold in general.

(3) $T_0 = 2$ and $T_n = T_{n-1}^2$ for $n > 0$. Estimate T_{20}.

A $10^{3{,}156{,}500}$

B $10^{1{,}156{,}500}$

C $10^{315{,}650}$

D $10^{156{,}500}$

E $10^{31{,}565}$

(4) $T_0 = 2$ and $T_n = T_{n-1}^2$ for $n > 0$, as in problem (3). Which order relationship is accurate?

A $T_n \in O(n)$.

B $T_n \in O(2^n)$.

C $T_n \in O(n!)$.

D $T_n \in O(2^{n!})$.

E None of the above.

(5) What is the last digit of $3^{1000} \times 5^{2000} + 7^{3000} \times 9^{4000}$?

$\boxed{\text{A}}$ 1.

$\boxed{\text{B}}$ 2.

$\boxed{\text{C}}$ 3.

$\boxed{\text{D}}$ 4.

$\boxed{\text{E}}$ None of the above.

(6) Let $d = \gcd(m, n)$, where $m, n > 0$. Bezout's identity gives $d = mx + ny$ where $x, y \in \mathbb{Z}$. Which of the statements A, B, C or D are false?

$\boxed{\text{A}}$ It is always possible to choose $x > 0$.

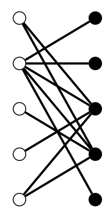
$\boxed{\text{B}}$ It is always possible to choose $x < 0$.

$\boxed{\text{C}}$ It is possible to find another $x, y \in \mathbb{Z}$ for which $0 < mx + ny < d$.

$\boxed{\text{D}}$ It is always possible to find $a, b \in \mathbb{Z}$ for which $ax + by = 1$.

$\boxed{\text{E}}$ All the statements A, B, C and D are true.

(7) The left nodes are tasks and the right nodes are resources. A resource can perform at most one task. What is the maximum number of tasks that can be performed?



$\boxed{\text{A}}$ 1.

$\boxed{\text{B}}$ 2.

$\boxed{\text{C}}$ 3.

$\boxed{\text{D}}$ 4.

$\boxed{\text{E}}$ 5.

(8) A queen covers a square if that square is on the same row, column or diagonal as the queen. What is the minimum number of queens required to cover all squares on a 5×5 chessboard?

$\boxed{\text{A}}$ 1.

$\boxed{\text{B}}$ 2.

$\boxed{\text{C}}$ 3.

$\boxed{\text{D}}$ 4.

$\boxed{\text{E}}$ 5.

(9) A friendship network has 100 people (vertices) and 2000 edges (friendships). You pick a person at random. What is the expected number of friends this person has?

$\boxed{\text{A}}$ 10.

$\boxed{\text{B}}$ 20.

$\boxed{\text{C}}$ 30.

$\boxed{\text{D}}$ 40.

$\boxed{\text{E}}$ None of the above, or not enough information to say for sure.

(10) To get into a certain US-college, all students submit at least one of SAT or ACT. 80% of students submit SAT; 40% of students submit ACT. How many students submit both SAT and ACT?

$\boxed{\text{A}}$ 10%.

$\boxed{\text{B}}$ 20%.

$\boxed{\text{C}}$ 30%.

$\boxed{\text{D}}$ 40%.

$\boxed{\text{E}}$ None of the above, or not enough information to say for sure.

(11) How many 4 digit strings (digits are 0,1,...,9) from 0000 to 9999 have digits which sum to 8. For example 0071, 0233 and 2033 are different digit-strings with digit-sum 8.

$\boxed{\text{A}}$ $\binom{8}{4} = 70$.

$\boxed{\text{B}}$ $\binom{11}{3} = 165$.

$\boxed{\text{C}}$ $10 \times 9 \times 8 \times 7 = 5040$.

$\boxed{\text{D}}$ $10^4 = 10,000$.

$\boxed{\text{E}}$ None of the above.

(12) How many different friendship networks are possible with the 5 people, Ⓐ Ⓑ Ⓒ Ⓓ Ⓔ? (Two networks are different if they have different edge-sets.)

A Approximately 10.

B Approximately 100.

C Approximately 1000.

D Approximately 10,000.

E Approximately 100,000.

(13) A friendship network has 5 people, Ⓐ Ⓑ Ⓒ Ⓓ Ⓔ. Each pair of people independently flips a fair coin and forms a friendship-edge if the flip is H. What is the probability that the network has exactly 5 edges?

A Approximately 1%.

B Approximately 2%.

C Approximately 10%.

D Approximately 25%.

E Approximately 50%.

(14) A friendship network has 5 people, Ⓐ Ⓑ Ⓒ Ⓓ Ⓔ. Each pair of people independently flips a coin and forms a friendship if they get H. What is the expected number of edges in the friendship network?

A 2.

B 3.

C 4.

D 5.

E None of the above.

(15) A tennis club has 20 members who are paired up in twos for the first round of a tournament. In the first round, we only care about who plays whom. How many ways are there of forming the first round matches? *[Hint: With 4 members, there are 3 ways to form the first round matches.]*

A $20!$.

B $\binom{20}{2}^{10}$.

C $\binom{20}{2}\times\binom{18}{2}\times\binom{16}{2}\times\binom{14}{2}\times\binom{12}{2}\times\binom{10}{2}\times\binom{8}{2}\times\binom{6}{2}\times\binom{4}{2}\times\binom{2}{2}$.

D $20!/(2^{10}\times 10!)$

E None of the above.

(16) $\mathbf{X}$ is a random variable and $\mathbf{Z} = a\mathbf{X} + b\mathbf{X}^2$. What is $\mathbb{E}[\mathbf{Z}]$?

A $\mathbb{E}[\mathbf{Z}] = a\,\mathbb{E}[\mathbf{X}] + b\,\mathbb{E}[\mathbf{X}]^2$

B $\mathbb{E}[\mathbf{Z}] = a\,\mathbb{E}[\mathbf{X}] + b^2\,\mathbb{E}[\mathbf{X}]^2$

C $\mathbb{E}[\mathbf{Z}] = a\,\mathbb{E}[\mathbf{X}] + b\,\mathbb{E}[\mathbf{X}^2]$

D $\mathbb{E}[\mathbf{Z}] = a\,\mathbb{E}[\mathbf{X}] + b^2\,\mathbb{E}[\mathbf{X}^2]$

E None of the above are true in general.

(17) $\mathbf{X}, \mathbf{Y}$ are *independent* random variables and $\mathbf{Z} = \mathbf{XY}$. What is $\sigma^2(\mathbf{Z})$, the variance of the product? *[Hint: Tinker with simple random variables. Make a conclusion and justify it.]*

A $\sigma^2(\mathbf{Z}) = \sigma^2(\mathbf{X})\sigma^2(\mathbf{Y})$

B $\sigma^2(\mathbf{Z}) = \sigma^2(\mathbf{X})\,\mathbb{E}[\mathbf{Y}^2] + \sigma^2(\mathbf{Y})\,\mathbb{E}[\mathbf{X}^2]$

C $\sigma^2(\mathbf{Z}) = \sigma^2(\mathbf{X})\,\mathbb{E}[\mathbf{Y}]^2 + \sigma^2(\mathbf{Y})\,\mathbb{E}[\mathbf{X}]^2$

D $\sigma^2(\mathbf{Z}) = \sigma^2(\mathbf{X})\,\mathbb{E}[\mathbf{Y}^2] + \sigma^2(\mathbf{Y})\,\mathbb{E}[\mathbf{X}]^2$

E None of the above are true in general.

(18) About 1 in a 100 people have Coeliac disease. The test for Coeliac has 90% accuracy, randomly making a mistake only 10% of the time. You tested positive. What are the chances you have Coeliac?

$\boxed{A}$ 1/100.

$\boxed{B}$ 1/12

$\boxed{C}$ 1/8

$\boxed{D}$ 1/4

$\boxed{E}$ 9/10

(19) The computing problem $\mathcal{L} = \{0^{\bullet n}\, 1^{\bullet (n+m)}\, 0^{\bullet m} \mid m, n \geq 0\}$ can be solved by:

(I) DFA. (II) CFG. (III) Turing Machine.

$\boxed{A}$ I,II,III

$\boxed{B}$ I,III

$\boxed{C}$ II,III

$\boxed{D}$ III only

$\boxed{E}$ None of these models of computing

(20) Which of these problems can be solved by a computer (Turing Machine)?

$\boxed{A}$ Determine if some other program halts or loops forever – ULTIMATEDEGUGGER

$\boxed{B}$ Determine YES or NO if some other program says YES on its input and halts.

$\boxed{C}$ Given $n \in \mathbb{N}$, compute $f(n)$, where $f(n) = 1$ if the nth Turing Machine halts and 0 otherwise.

$\boxed{D}$ Given m-bit and n-bit binary sequences $b_1 \cdots b_m$ and $c_1 \cdots c_n$ with $m < n$, is it possible to add $n - m$ bits into various positions of the first sequence so that the two sequences match exactly?

$\boxed{E}$ None of these problems can be solved.

2 Independent Sets and Vertex Covers in a Graph. (Tinker, tinker,. . .)

A graph G has vertices $V = \{v_1, \ldots, v_n\}$ and edges $E = \{e_1, \ldots, e_m\}$. Let $S \subseteq V$ be a subset of the vertices.

S is a **vertex cover** if every edge in E has at least one endpoint in S.

S is an **inpdependent set** if no pair of vertices in S is connected by an edge.

Prove: The subset S is a vertex cover *if and only if* $\overline{S}$ (the vertices not in S) is an independent set.

3 Conditional Probability and Expected Value.

A box has 1 fair coin and 1 two-headed coin. You picked a random coin, flipped it 2 times and both flips were H. You now keep flipping the *same* coin you picked until you flip *two heads in a row*. Let $\mathbf{X}$ be the number of additional flips you make. Compute $\mathbb{E}[\mathbf{X}]$, the expected value of $\mathbf{X}$.

4 Sums and Induction. (Tinker, tinker,...)

Obtain a formula that does not use a sum for $S(n) = \sum_{i=1}^{2n}(-1)^i i^2$. Prove your formula by *induction*.

5 Transducer Turing Machine for Unary to Binary.

Give a high-level description of a transducer Turing Machine to solve unary to binary conversion. The input is $0^{\bullet n}$ (if not reject). The Turing Machine should halt with the tape showing $0^{\bullet n}\#w$, where w is the binary representation of n. (E.g. for input 00000, the the tape should be 00000#101 when the machine halts.)

1 Circle at most one answer per question. 10 points for each correct answer.

(1) Is this claim true or false. $\forall n \in \mathbb{Z} : n^2 \geq 0$.

$\boxed{\text{A}}$ True.

$\boxed{\text{B}}$ False.

$\boxed{\text{C}}$ You can't say because it depends on n.

$\boxed{\text{D}}$ You can't assign true or false to quantified statements.

$\boxed{\text{E}}$ It is not a proper statement to which you can assign true or false.

(2) If it rains on a day, it must rain the next day. Today it did not rain. What can you conclude?

$\boxed{\text{A}}$ It won't rain tomorrow.

$\boxed{\text{B}}$ It won't rain on any future day.

$\boxed{\text{C}}$ It rained yesterday.

$\boxed{\text{D}}$ It did not rain yesterday but it could have rained on some day prior to yesterday.

$\boxed{\text{E}}$ It did not rain yesterday and it did not rain on any day prior to yesterday.

(3) To prove $P(n)$ by induction, which is *not* a valid induction step to prove $P(n) \to P(n+1)$.

$\boxed{\text{A}}$ *Assume* that $P(n)$ is true and *prove* that $P(n+1)$ is true.

$\boxed{\text{B}}$ *Assume* two things, that $P(n)$ is true and that $P(n+1)$ is false. Now derive a contradiction.

$\boxed{\text{C}}$ *Assume* that $P(n)$ is false and *prove* that $P(n+1)$ is false.

$\boxed{\text{D}}$ *Assume* that $P(n+1)$ is false and *prove* that $P(n)$ is false.

$\boxed{\text{E}}$ All of the above are valid induction steps.

(4) What is the approximate value of the sum $\displaystyle\sum_{i=0}^{20}(2^i+i)(2^i-i)$.

$\boxed{\text{A}}$ 1.5×10^{11}.

$\boxed{\text{B}}$ 4.0×10^{11}.

$\boxed{\text{C}}$ 1.5×10^{12}.

$\boxed{\text{D}}$ 4.0×10^{12}.

$\boxed{\text{E}}$ 1.5×10^{13}.

(5) $T_1 = 1$ and $T_n = T_{n-1} + n^2$ for $n > 1$. Which order relationship is accurate?

$\boxed{\text{A}}$ $T_n \in \Theta(n)$.

$\boxed{\text{B}}$ $T_n \in \Theta(n^2)$.

$\boxed{\text{C}}$ $T_n \in \Theta(n^3)$.

$\boxed{\text{D}}$ $T_n \in \Theta(2^n)$.

$\boxed{\text{E}}$ None of the above.

(6) What is the remainder when 2^{2019} is divided by 5?

$\boxed{\text{A}}$ 0.

$\boxed{\text{B}}$ 1.

$\boxed{\text{C}}$ 2.

$\boxed{\text{D}}$ 3.

$\boxed{\text{E}}$ 4.

(7) Define the set $A = \{3x + 7y \mid x \text{ and } y \text{ are in } \mathbb{Z}\}$. Which numbers are *not* in A?

$\boxed{\text{A}}$ -11.

$\boxed{\text{B}}$ 11.

$\boxed{\text{C}}$ 37.

$\boxed{\text{D}}$ 142.

$\boxed{\text{E}}$ They are all in A.

(8) Ayfos is in a social network with 14 others, so 15 people in all with Ayfos. There are 25 friendship links in this network. Everyone but Ayfos has 3 friends. How many friends does Ayfos have?

$\boxed{\text{A}}$ 6.

$\boxed{\text{B}}$ 7.

$\boxed{\text{C}}$ 8.

$\boxed{\text{D}}$ 9.

$\boxed{\text{E}}$ Can't be determined or such a social network cannot exist.

(9) In the previous problem regarding Ayfos' social network, you pick a person randomly. What is the expected number of friends that person has.

$\boxed{\text{A}}$ $3\frac{1}{3}$.

$\boxed{\text{B}}$ $3\frac{1}{2}$.

$\boxed{\text{C}}$ $3\frac{3}{4}$.

$\boxed{\text{D}}$ 4.

$\boxed{\text{E}}$ None of the above, or not enough information to say for sure.

(10) From 1000 students, 900 are CS and 200 are MATH. How many are CS-MATH duals?

$\boxed{\text{A}}$ 50.

$\boxed{\text{B}}$ 100.

$\boxed{\text{C}}$ 150.

$\boxed{\text{D}}$ 200.

$\boxed{\text{E}}$ None of the above, or not enough information to say for sure.

(11) Digits are 0,1,..., 9. How many of the three digit strings 000 to 999 have a digit-sum 10? (For example, 307 and 811 have digit sum 10, but 846 and 213 do not.)

$\boxed{\text{A}}$ 60.

$\boxed{\text{B}}$ 63.

$\boxed{\text{C}}$ 66.

$\boxed{\text{D}}$ 69.

$\boxed{\text{E}}$ None of the above.

(12) A and B are sets. $|A| = 5$ and $|B| = 3$. How many functions are there from A to B?

$\boxed{\text{A}}$ 3^5.

$\boxed{\text{B}}$ 5^3.

$\boxed{\text{C}}$ $5!$.

$\boxed{\text{D}}$ $\binom{5}{3}$.

$\boxed{\text{E}}$ None of the above.

(13) A and B are sets. $|A| = 5$ and $|B| = 3$. How many injections (1-to-1) are there from A to B?

$\boxed{\text{A}}$ 0.

$\boxed{\text{B}}$ 100.

$\boxed{\text{C}}$ 150.

$\boxed{\text{D}}$ 200.

$\boxed{\text{E}}$ None of the above.

(14) A and B are sets. $|A| = 5$ and $|B| = 3$. How many surjections (onto) are there from A to B?

$\boxed{\text{A}}$ 0.

$\boxed{\text{B}}$ 100.

$\boxed{\text{C}}$ 150.

$\boxed{\text{D}}$ 200.

$\boxed{\text{E}}$ None of the above.

(15) You roll a die 4 times. What is the probability to get (exactly) 2 sixes?

$\boxed{\text{A}}$ $6/6^4$.

$\boxed{\text{B}}$ $12/6^4$.

$\boxed{\text{C}}$ $36/6^4$.

$\boxed{\text{D}}$ $150/6^4$.

$\boxed{\text{E}}$ None of the above.

(16) Al and Jo each independently pick 4 restaurants randomly from 10 restaurants $r_1, \ldots, r_{10}$. They must eat at a restaurant that both picked. Compute the probability they can eat at (exactly) 2 restaurants.

A 2/7

B 3/7

C 4/7

D 5/7

E None of the above

(17) Compute the expected number of restaurants Al and Jo from the previous problem can eat at.

A 1.2.

B 1.4

C 1.6.

D 1.8

E None of the above

(18) Which computing problem *cannot* be solved by a DFA?

A Strings with an even number of 1s.

B Strings which have more 1s than 0s.

C Strings whose number of 1s is a multiple of 3.

D Strings whose number of 1s is not a multiple of 3.

E Each problem is solvable using a DFA

(19) Which string cannot be generated by the CFG $S \to \varepsilon|0S|1S$?

A $11111111110000000000 = 1^{\bullet 10}0^{\bullet 10}$.

B $10101010101010101010 = (10)^{\bullet 10}$.

C $00000000000000000000 = 0^{\bullet 20}$.

D $00110011001100110011 = (0011)^{\bullet 5}$.

E They can all be generated.

(20) Which answer is a valid conclusion about the decidability of the language $\mathcal{L}_B$?

A $\mathcal{L}_A$ is decidable. A decider for $\mathcal{L}_B$ can be converted to a decider for $\mathcal{L}_A$. So, $\mathcal{L}_B$ is decidable.

B $\mathcal{L}_A$ is decidable. A decider for $\mathcal{L}_A$ can be converted to a decider for $\mathcal{L}_B$. So, $\mathcal{L}_B$ is decidable.

C $\mathcal{L}_A$ is undecidable. A decider for $\mathcal{L}_A$ can be converted to a decider for $\mathcal{L}_B$. So, $\mathcal{L}_B$ is undecidable.

D $\mathcal{L}_A$ is undecidable. A decider for $\mathcal{L}_B$ can be converted to a decider for $\mathcal{L}_A$. So, $\mathcal{L}_B$ is decidable.

E None of the above is valid.

2 Determine the Type of Proof and Prove

Prove that for $n \in \mathbb{N}$, $\sqrt{n(n+1)} \le n + \frac{1}{2}$.

3 Induction and Sums. Tinker, Tinker, Tinker.

For $n \in \mathbb{N}$, obtain a formula for the sum $S(n) = \sum_{i=1}^{2n} (-1)^i i$ and prove your formula by induction.

4 Expected Waiting Time to 3 Heads In A Row

You flip a fair coin until you get 3 heads *in a row*. Compute the expected number of flips you make.

5 CFGs and Induction. (Tinker, tinker,...)

For the CFG $S \rightarrow 0|0S1$, *prove* that every string that can be generated has odd length.

6 Turing Machine for Squaring.

Give a high level pseudo-code description of a Turing Machine that solves the problem $\mathcal{L} = \{0^{\bullet n}1^{\bullet n \times n} | n \geq 1\}$. (You do not need to give machine level details but your pseudo-code should demonstrate understanding of how the Turing Machine moves back and forth to solve the problem. Tinker.)

1 **Circle at most one answer per question. 10 points for each correct answer.**

(1) "For a constant $c > 0$, $1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} > c\sqrt{n}$, where n is any natural number." Which claim is this?

$\boxed{\text{A}}$ $\exists c > 0 : (\exists n \in \mathbb{N} : 1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} > c\sqrt{n})$.

$\boxed{\text{B}}$ $\exists c > 0 : (\forall n \in \mathbb{N} : 1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} > c\sqrt{n})$.

$\boxed{\text{C}}$ $\exists n \in \mathbb{N} : (\forall c > 0 : 1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} > c\sqrt{n})$.

$\boxed{\text{D}}$ $\forall n \in \mathbb{N} : (\exists c > 0 : 1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} > c\sqrt{n})$.

$\boxed{\text{E}}$ None of the above.

(2) You will pick a constant $C > 0$ such that no matter which $n \in \mathbb{N}$ I pick, $\sum_{i=1}^{n} i \leq Cn$. Which is true?

$\boxed{\text{A}}$ You can pick a C satisfying $C \leq 10$.

$\boxed{\text{B}}$ You can pick a C satisfying $10 < C < 100$.

$\boxed{\text{C}}$ You can pick a C satisfying $100 < C < 1000$.

$\boxed{\text{D}}$ You can pick a C satisfying $1000 < C$.

$\boxed{\text{E}}$ There is no constant $C > 0$ that you can pick.

(3) $T_1 = 2$ and $T_n = T_{n-1} + 2n$ for $n > 1$. What is T_{100}?

$\boxed{\text{A}}$ 5050.

$\boxed{\text{B}}$ 10100.

$\boxed{\text{C}}$ 20200.

$\boxed{\text{D}}$ 40400.

$\boxed{\text{E}}$ None of the above.

(4) $T_1 = 1$ and $T_n = n \times T_{n-1}$ for $n > 1$. Which is true?

$\boxed{\text{A}}$ $T(n) \in O(n^2)$.

$\boxed{\text{B}}$ $T(n) \in o(2^n)$.

$\boxed{\text{C}}$ $T(n) \in \Theta(2^n)$.

$\boxed{\text{D}}$ $T(n) \in \omega(2^n)$.

$\boxed{\text{E}}$ None of the above.

(5) You divide 2^{2016} candies evenly among 11 kids. How many candies are left over?

$\boxed{\text{A}}$ 0.

$\boxed{\text{B}}$ 3.

$\boxed{\text{C}}$ 6.

$\boxed{\text{D}}$ 9.

$\boxed{\text{E}}$ None of the above.

(6) Estimate the sum $S = \sum_{i=1}^{\infty} \frac{1}{i} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$.

$\boxed{\text{A}}$ $0 < S \leq 2$.

$\boxed{\text{B}}$ $2 < S \leq 2000$.

$\boxed{\text{C}}$ $2000 < S \leq 20000$.

$\boxed{\text{D}}$ $20000 < S \leq 200000$.

$\boxed{\text{E}}$ None of the above.

(7) How many of the numbers $100, 101, 102, \ldots, 999$ do not contain the digit 2?

$\boxed{\text{A}}$ 100.

$\boxed{\text{B}}$ 504.

$\boxed{\text{C}}$ 648.

$\boxed{\text{D}}$ 729.

$\boxed{\text{E}}$ None of the above.

(8) Let S be the sum of the reciprocals of all natural numbers not containing the digit 2. Estimate S.

$$S = 1 + \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{5} + \tfrac{1}{6} + \tfrac{1}{7} + \tfrac{1}{8} + \tfrac{1}{9} + \tfrac{1}{10} + \tfrac{1}{11} + \tfrac{1}{13} + \tfrac{1}{14} + \tfrac{1}{15} + \tfrac{1}{16} + \tfrac{1}{17} + \tfrac{1}{18} + \tfrac{1}{19} + \tfrac{1}{30} + \tfrac{1}{31} + \tfrac{1}{33} + \tfrac{1}{34} + \tfrac{1}{35} + \cdots$$

$\boxed{\text{A}}$ $0 < S \leq 2$.

$\boxed{\text{B}}$ $2 < S \leq 2000$.

$\boxed{\text{C}}$ $2000 < S \leq 20000$.

$\boxed{\text{D}}$ $20000 < S \leq 200000$.

$\boxed{\text{E}}$ None of the above.

(9) Shirts come in 3 colors R, G or B. In how many ways can you distribute shirts to 7 students?

$\boxed{\text{A}}$ $\binom{7}{3}$.

$\boxed{\text{B}}$ 7^3.

$\boxed{\text{C}}$ 3^7.

$\boxed{\text{D}}$ $7!/3!$.

$\boxed{\text{E}}$ None of the above.

(10) Repeat problem 9 if at least two shirts of each color are distributed to students (7 shirts in total).

$\boxed{\text{A}}$ 570

$\boxed{\text{B}}$ 600.

$\boxed{\text{C}}$ 630.

$\boxed{\text{D}}$ 660.

$\boxed{\text{E}}$ None of the above.

(11) Every vertex in a graph G has degree 1. Which is true?

$\boxed{\text{A}}$ The graph G must be disconnected.

$\boxed{\text{B}}$ The graph G could have 5 vertices.

$\boxed{\text{C}}$ The graph G must have a cycle.

$\boxed{\text{D}}$ The graph G is not possible.

$\boxed{\text{E}}$ None of the above

(12) You rolled a pair of dice. What are the chances you rolled exactly one 5?

$\boxed{\text{A}}$ 9/36.

$\boxed{\text{B}}$ 10/36.

$\boxed{\text{C}}$ 11/36.

$\boxed{\text{D}}$ 12/36.

$\boxed{\text{E}}$ None of the above.

(13) You rolled a pair of dice. What are the chances you rolled exactly one 5 if the sum is even?

$\boxed{\text{A}}$ 4/10.

$\boxed{\text{B}}$ 5/10.

$\boxed{\text{C}}$ 4/11.

$\boxed{\text{D}}$ 5/11.

$\boxed{\text{E}}$ None of the above.

(14) Which of the following random variables $\mathbf{X}$ is not a binomial random variable.

$\boxed{\text{A}}$ Randomly throw 100 darts at a dart board. $\mathbf{X}$ is the number of darts hitting the bulls-eye.

$\boxed{\text{B}}$ Randomly answer 100 5-choice multiple choice questions. $\mathbf{X}$ is the number of questions correct.

$\boxed{\text{C}}$ Randomly answer 100 5-choice multiple choice questions. $\mathbf{X}$ is the number of questions wrong.

$\boxed{\text{D}}$ 1000 students randomly line up, 500 are boys. $\mathbf{X}$ is the number of boys in the first 100 students.

$\boxed{\text{E}}$ They are all binomial random variables.

(15) A social network (graph) is a *tree* with 20 people. The edges are friendships. Each person randomly picks red or blue. Friends compare to see if they match. What is the expected number of matches.

$\boxed{\text{A}}$ 4.75.

$\boxed{\text{B}}$ 5.

$\boxed{\text{C}}$ 9.5

$\boxed{\text{D}}$ 10.

$\boxed{\text{E}}$ None of the above or not enough information.

(16) On BlueToe, your first child is equally likely to be a boy or girl. From then on, the sex of a child is the same as the previous child with probability 2/3 and different with probability 1/3. What is the expected number of kids to get a girl?

$\boxed{\text{A}}$ 1.5.

$\boxed{\text{B}}$ 2.

$\boxed{\text{C}}$ 2.5.

$\boxed{\text{D}}$ 3.

$\boxed{\text{E}}$ None of the above.

(17) On BlueToe, as in problem 16, what is the expected number of kids to two girls?

$\boxed{\text{A}}$ 3.25.

$\boxed{\text{B}}$ 4.

$\boxed{\text{C}}$ 4.5.

$\boxed{\text{D}}$ 5.25.

$\boxed{\text{E}}$ None of the above.

(18) Estimate the number of DFA you can draw with 4 states, q_0, q_1, q_2, q_3. Tinker!

$\boxed{\text{A}}$ About a hundred.

$\boxed{\text{B}}$ About a thousand.

$\boxed{\text{C}}$ About a million.

$\boxed{\text{D}}$ About a billion.

$\boxed{\text{E}}$ About a trillion.

(19) Which string can be generated by the CFG $S \to 0|1|SSS$?

$\boxed{\text{A}}$ 1111.

$\boxed{\text{B}}$ 0000.

$\boxed{\text{C}}$ 000111.

$\boxed{\text{D}}$ 111000.

$\boxed{\text{E}}$ None of the above.

(20) If $\mathcal{L}_A$ is decidable, then $\mathcal{L}_B$ is decidable. We know that $\mathcal{L}_B$ is undecidable. Therefore:

$\boxed{\text{A}}$ $\mathcal{L}_A$ must be finite.

$\boxed{\text{B}}$ $\mathcal{L}_A$ must be infinite.

$\boxed{\text{C}}$ $|\mathcal{L}_A| > |\mathcal{L}_B|$.

$\boxed{\text{D}}$ $|\mathcal{L}_B| < |\mathcal{L}_A|$.

$\boxed{\text{E}}$ None of the above.

2 Determine the Type of Proof and Prove

Prove that there is a constant $c > 0$ for which, no matter which $n \in N$ you pick,

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} > c\sqrt{n}.$$

3 Product of 5 Consecutive Numbers.

Prove that the product of any 5 consecutive natural numbers is divisible by 5! (e.g. $5! | 3 \times 4 \times 5 \times 6 \times 7$).

4 Expected Waiting Time to All Colors of Starburst.

Starburst is sold in 2-packs, and there are 3 colors of starburst. What is the expected number of 2-packs you will buy if your goal is to get all colors?

5 DFA or no DFA

Give a DFA for $\mathcal{L} = \{0^{\bullet n^2} | n \geq 1\} = \{0, 0000, 000000000, \ldots\}$, or prove that $\mathcal{L}$ can't be solved with DFA.

6 Transducer Turing Machine for Reversal.

Give a high level pseudo-code description of a transducer Turing Machine for reversal. The input on the tape is any binary string w. When the Turing Machine halts, the reversal of w should have replaced w. E.g:

Start

*	1	0	1	0	0	1	1	␣

End

*	1	1	0	0	1	0	1	␣

(Don't give machine level details, but you should make it clear how the Turing Machine moves back and forth. Tinker.)