Probabilistic Unsupervised Learning
Exercise 1: Bayes Rule and Probability Densities

Submission deadline: Monday, Nov 7, 2022 at 10:00 a.m.
Exercises have to be solved individually or in groups of two people.
Send your solutions to `hamid.mousavi@uol.de`

For all the tasks, name the different probabilities
using the notations as shown in the lecture.

1. There is a small island somewhere in the Caribbean. Trading- and passenger-ships (200 per year) are travelling from the continent to the island and continue their route to other islands afterwards. Through the dangerous sea there are only 2 routes known. Also these 2 routes are not safe, there is a certain risk for a ship to sink. The danger of sinking is 10% if it takes the shorter route and 5% if it takes the longer route. Last year 20% of the ships decided to take the shorter route.
   Let us denote by "r" the random number "route of ship" and the possible values of "r" by "s" and "l" for "short" and "long", respectively. The probability of a ship to take the short route or the long route can then be denoted by the prior probabilities $p(r = s) = 0.2$ and $p(r = l) = 0.8$. Moreover, the risks to sink are given by conditional probabilities $p(a = 1|r = s) = 0.1$ and $p(a = 1|r = l) = 0.05$ where $a = 1$ means the ship sank and $a = 0$ means the ship did not sink.

   [2 *points*] Task A:
   How many ships do we expect to sink this year, assuming that the people behave similarly to last year?

   [4 *points*] Task B:
   We get a report that a ship sank but we have no coordinates. We notify the rescue-ship to find survivors. Now the captain of the rescue-ship asks you which of the 2 routes to check. What is the probability that the ship took the shorter route? So what will you answer to the rescue ship?

2. [4 *points*] X is a continuous random variable distributed according to the probability density function $p(x)$, $x \in \Re$. Suppose $p(x)$ is symmetric, i.e., there exists a value μ such that $p(\mu - x) = p(\mu + x)$, $\forall x \in \Re$. Show that the expectation value of X, $\langle X \rangle_p$, is equal to μ.

3. [4 *points*] Task A:
Write down a formula that allows to numerically solve the integrals:

$$\int_{-\infty}^{\infty} e^{-x^2} x^2 dx$$

$$\int_{-\infty}^{\infty} e^{-x^2} \cos(x) dx$$

Hint: Use a formula of the lecture (you may have to rewrite the integrals to be able to apply the formula).

[2 *bonus points*] Task B:
Draw 100 samples from a uniform distribution (interval [0, 1]) using any of the popular libraries or built in functions of matlab or python. Do the same for the normal distribution ($\sigma = 1$; $\mu = 0$).

[2 *bonus points*] Task C:
Numerically solve the integrals of Task A using samples drawn from a suitable distribution. Both integrals have an analytic solution (see e.g. Bronstein). How close do you get?

[2 *bonus point*] Task D:
Could you infer the analytical solution of the first integral using the formulas of Task C? If yes, how?