2

$$N(x;\mu,\sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}}$$

$$p(x|\mu_1,\mu_2,\sigma_1^2,\sigma_2^2) = \frac{1}{2}\frac{1}{\sqrt{2\pi\sigma_1^2}} e^{-\frac{1}{2}\frac{(x-\mu_1)^2}{\sigma_1^2}} + \frac{1}{2}\frac{1}{\sqrt{2\pi\sigma_2^2}} e^{-\frac{1}{2}\frac{(x-\mu_2)^2}{\sigma_2^2}}$$

A) General PDs have the following properties:

fa a PDF $p(\vec{x})$, for all $\vec{x} \in \Omega_{\vec{x}}$ applies:

① $p(\vec{x}) \geq 0$, and ② $\int_{\vec{x} \in \Omega_{\vec{x}}} p(\vec{x})\, d\vec{x} = 1$

① The first condition is fulfilled because this new PDF is composed of Gaussian functions $N(x;\mu,\sigma^2)$ which already comply with this principle, and these both being added cannot allow to make a point $p(\vec{x}) < 0$, as we are adding 2 poite functions together.

②
$$\int_{\vec{x} \in \Omega_{\vec{x}}} P(\vec{x}|\mu_1,\mu_2,\sigma_1^2,\sigma_2^2)\cdot dx = \frac{1}{2}\underbrace{\int_{\vec{x} \in \Omega_{\vec{x}}} N(\vec{x};\mu_1,\sigma_1^2)\, dx}_{1} + \dots$$

$$\dots + \frac{1}{2}\underbrace{\int_{\vec{x} \in \Omega_{\vec{x}}} N(\vec{x};\mu_2,\sigma_2^2)\, dx}_{1} = \boxed{\frac{1}{2}\cdot(1) + \frac{1}{2}(1) = 1}$$

B) To find a Maximum likelihood solution:

$$\mathcal{L}(x^{(1)}, \dots x^{(N)}); \theta) = \sum_{n=1}^{N} \log \left(p(x^{(n)} | \mu_1, \mu_2, \sigma_1^2, \sigma_2^2) \right)$$

$$= \sum_{n=1}^{N} \log \left(\frac{1}{2} \frac{1}{\sqrt{2\pi\sigma_1^2}} e^{-\frac{1}{2}\frac{(x-\mu_1)^2}{\sigma_1^2}} + \frac{1}{2} \frac{1}{\sqrt{2\pi\sigma_2^2}} e^{-\frac{1}{2}\frac{(x-\mu_2)^2}{\sigma_2^2}} \right)$$

The sum within the logarith, means that we cannot just simply solve the equation analytically.

We could use instead ~~use~~ a gradient ascend method,

or try to do further analytical steps with the EM algorithm.