(1) $$p(x|\lambda) = \exp(-\lambda)\frac{\lambda^x}{x!} \quad , \; x \in \mathbb{N}$$

→ Likelyhood function:

$$L(\lambda; x_1, \ldots, x_N) = \prod_{j=1}^{N} \exp(-\lambda)\frac{\lambda^{x_j}}{x_j!}$$

→ Log-Likelyhood function:

$$\mathcal{L}(\lambda; x_1, \ldots, x_N) = \ln\left[\prod_{j=1}^{N} \exp(-\lambda)\frac{\lambda^{x_j}}{x_j!}\right]$$

$$= \sum_{j=1}^{N} \ln\left(\exp(-\lambda)\frac{\lambda^{x_j}}{x_j!}\right)$$

$$= \sum_{j=1}^{N}\left[\ln(\exp(-\lambda)) - \ln(x_j!) + \ln(\lambda^{x_j})\right]$$

$$= \sum_{j=1}^{N}\left[-\lambda - \ln(x_j!) + x_j \cdot \ln(\lambda)\right]$$

$$= -N\lambda - \sum_{j=1}^{N}\ln(x_j!) + \ln(\lambda)\cdot\sum_{j=1}^{N} x_j$$

→ maximum likelyhood solution:

$$\hat{\lambda} = \underset{\lambda}{\operatorname{argmax}}\left(\mathcal{L}(\lambda; x_1, \ldots, x_N)\right)$$

$$\frac{\partial}{\partial\lambda}\mathcal{L}(\lambda; x_1, \ldots, x_N) = 0 = \frac{\partial}{\partial\lambda}\left(-N\lambda - \sum_{j=1}^{N}\ln(x_j!) + \ln(\lambda)\cdot\sum_{j=1}^{N} x_j\right)$$

$$0 = -N + \frac{1}{\lambda}\sum_{j=1}^{N} x_j \quad \Big| : N \Big| \cdot \lambda$$

$$0 = -\lambda + \frac{1}{N}\sum_{j=1}^{N} x_j$$

$$\Leftrightarrow \quad \hat{\lambda} = \frac{1}{N}\sum_{j=1}^{N} x_j = \text{sample mean}$$

Poisson distribution: $\lambda = E(X) = Var(X)$