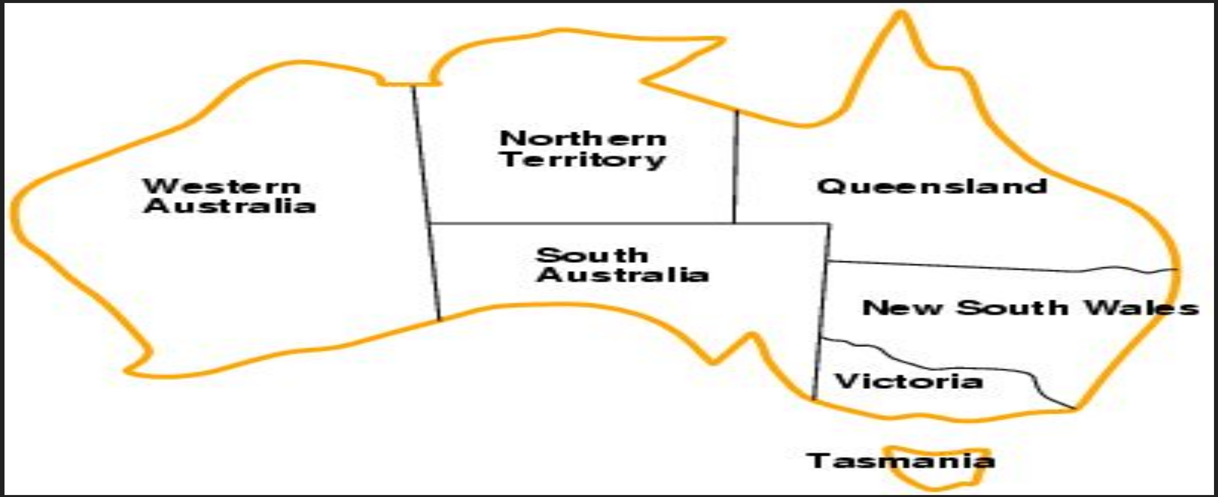
Constraint Satisfaction Problems

Constraint satisfaction problems (CSPs)

- CSP:
  - state is defined by variables X_i with values from domain D_i
  - goal test is a set of constraints specifying allowable combinations of values for subsets of variables

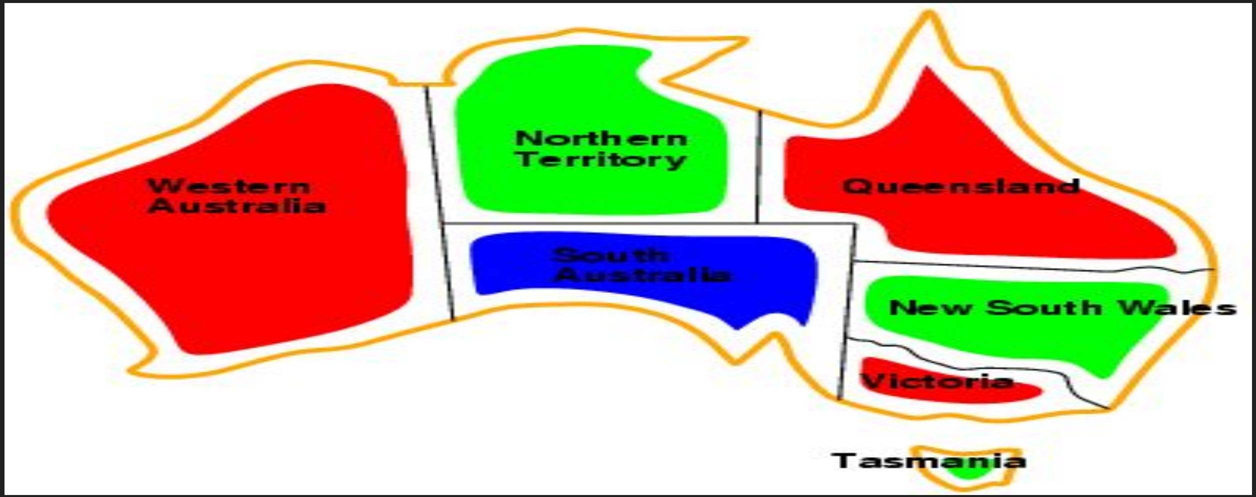
- Allows useful general-purpose algorithms with more power than standard search algorithms

Example: Map-Coloring



- Variables *WA, NT, Q, NSW, V, SA, T*
- Domains D_i = {red,green,blue}
- Constraints: adjacent regions must have different colors
- e.g., WA ≠ NT

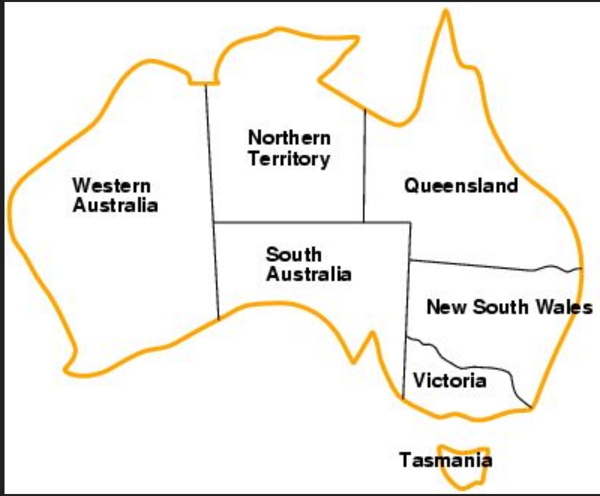
Example: Map-Coloring

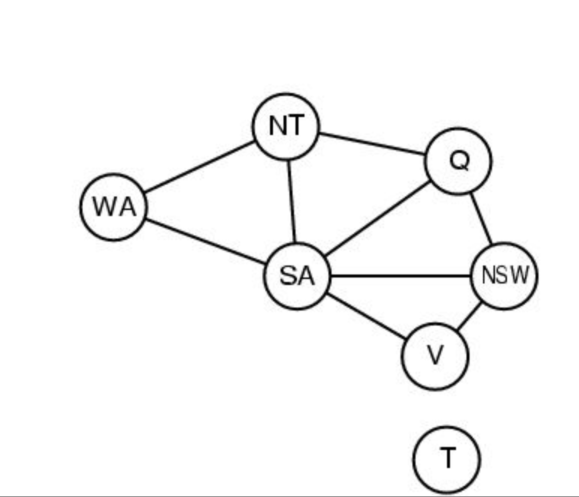


- Solutions are complete and consistent assignments, e.g., WA = red, NT = green,Q = red,NSW = green,V = red,SA = blue,T = green

Constraint graph

- Binary CSP: each constraint relates two variables
- Constraint graph: nodes are variables, arcs are constraints





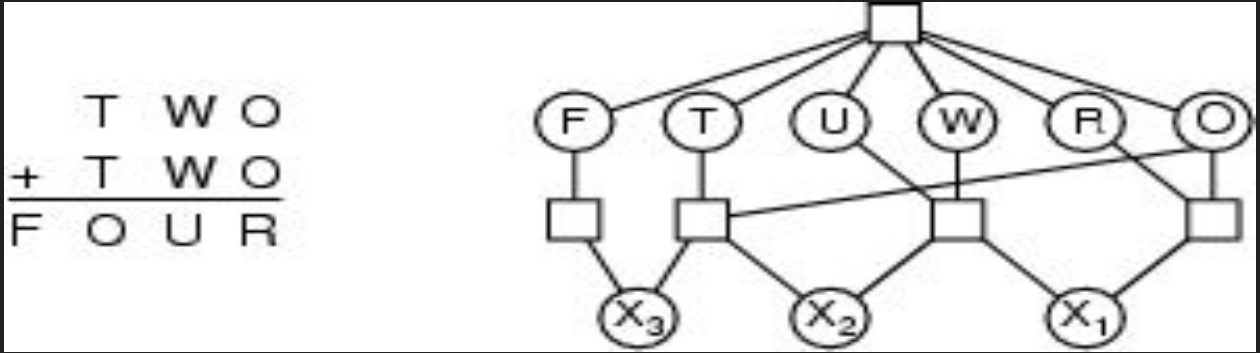
Varieties of CSPs

- Discrete variables
  - finite domains:
    - n variables, domain size d □ $O(d^n)$ complete assignments
    - e.g., 3-SAT (NP-complete)
  - countable domains:
    - integers, strings, etc.
    - e.g., job scheduling, variables are start/end days for each job:
      $StartJob_1 + 5 \le StartJob_3$
- Continuous variables
  - e.g., start/end times for Hubble Space Telescope observations
  - linear constraints solvable in polynomial time by linear programming

Varieties of constraints

- Unary constraints involve a single variable,
  - e.g., SA ≠ green
- Binary constraints involve pairs of variables,
  - e.g., SA ≠ WA
- Higher-order constraints involve 3 or more variables,
  - e.g., SA ≠ WA ≠ NT

Example: Cryptarithmetic

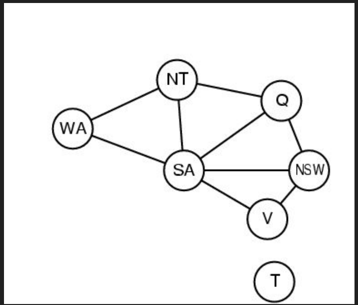


- Variables: *F T U W R O* $X_1\, X_2 X_3$
- Domains: {*0,1,2,3,4,5,6,7,8,9*} {0,1}
- Constraints: *Alldiff (F,T,U,W,R,O)*
  - $O + O = R + 10 \cdot X_1$
  - $X_1 + W + W = U + 10 \cdot X_2$
  - $X_2 + T + T = O + 10 \cdot X_3$
  - $X_3 = F,\ T \neq 0,\ F \neq 0$

Real-world CSPs

- Assignment problems
  - e.g., who teaches what class
- Timetabling problems
  - e.g., which class is offered when and where?
- Transportation scheduling
- Factory scheduling
- Notice that many real-world problems involve real-valued variables

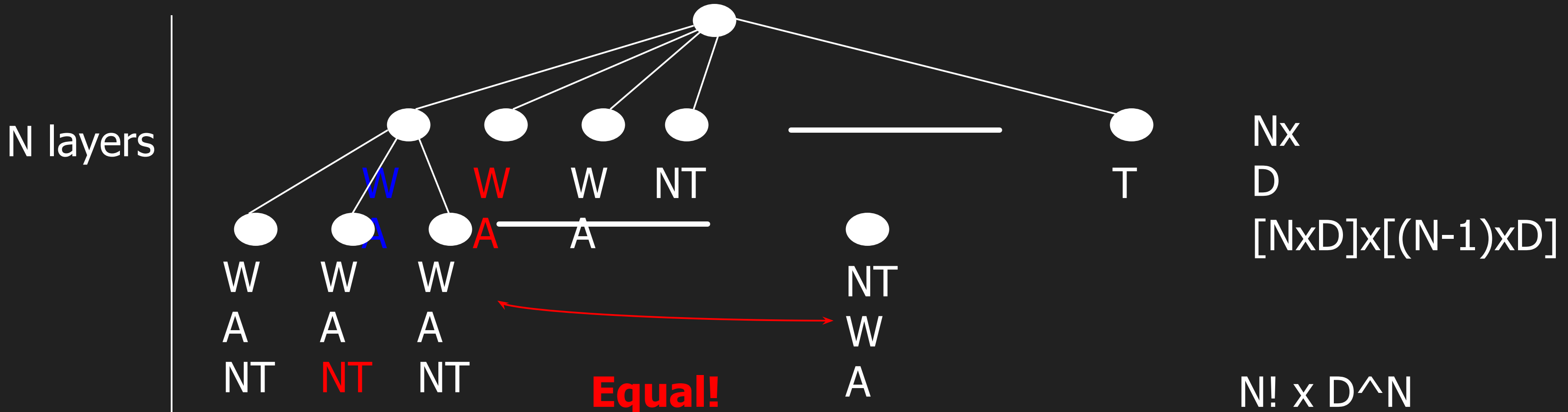
Standard search formulation



Let's try the standard search formulation.

We need:

- Initial state: none of the variables has a value (color)
- Successor state: one of the variables without a value will get some value.
- Goal: all variables have a value and none of the constraints is violated.



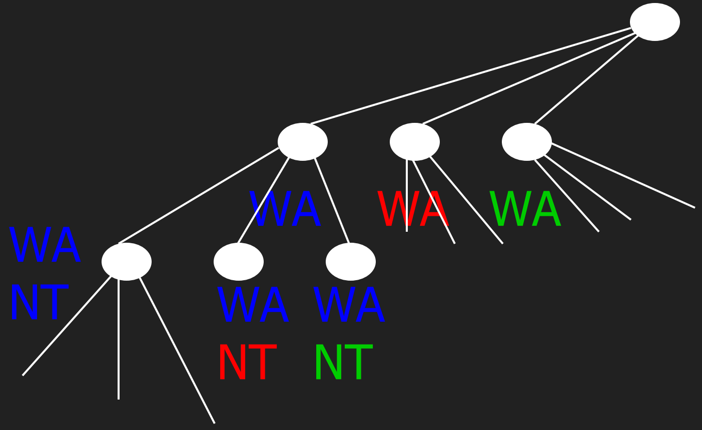
There are N! x D^N nodes in the tree but only D^N distinct states??

Backtracking (Depth-First) search

- Special property of CSPs: They are commutative:
  This means: the order in which we assign variables does not matter.

NT = W
W   A
A   NT

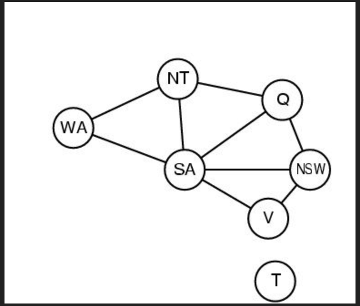
- Better search tree: First order variables, then assign them values one-by-one.



D

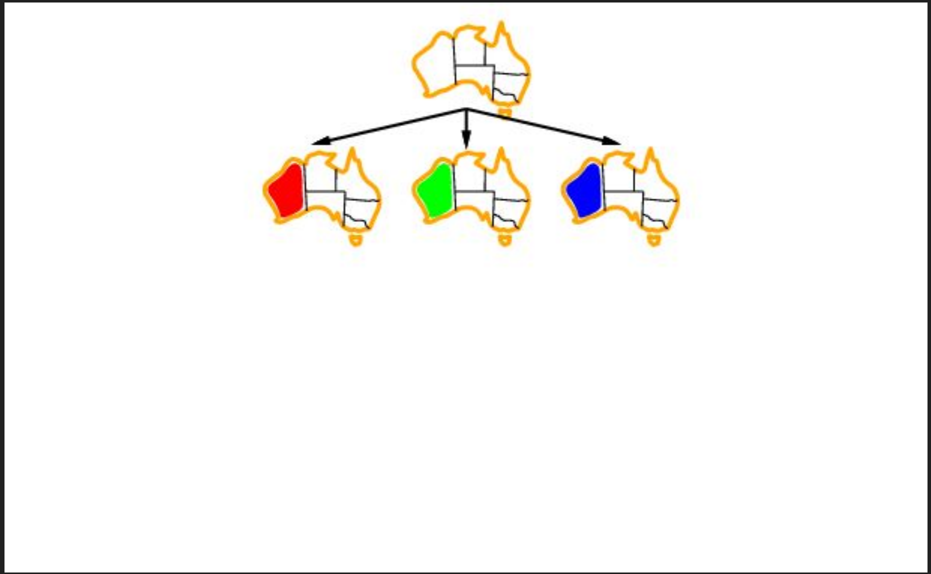
D^2

D^N

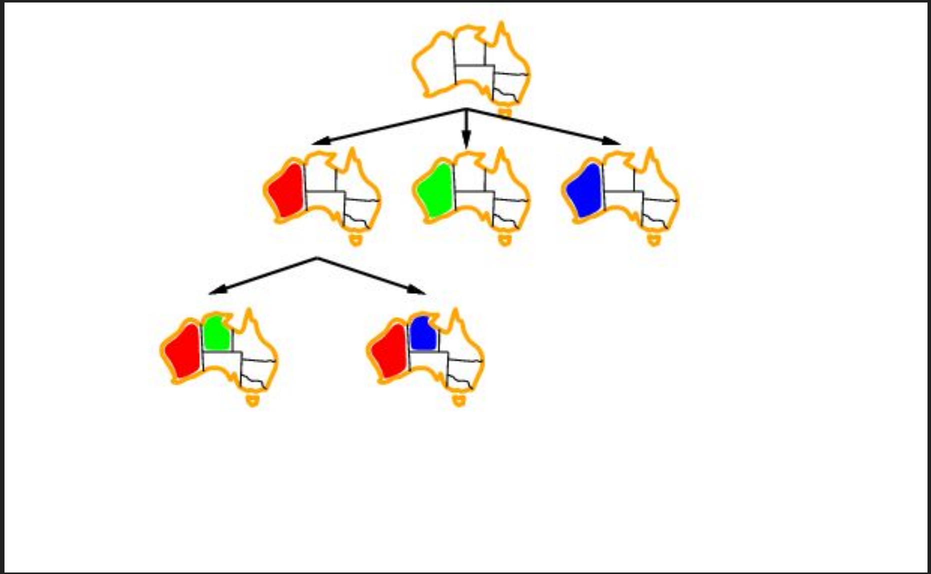


Backtracking example

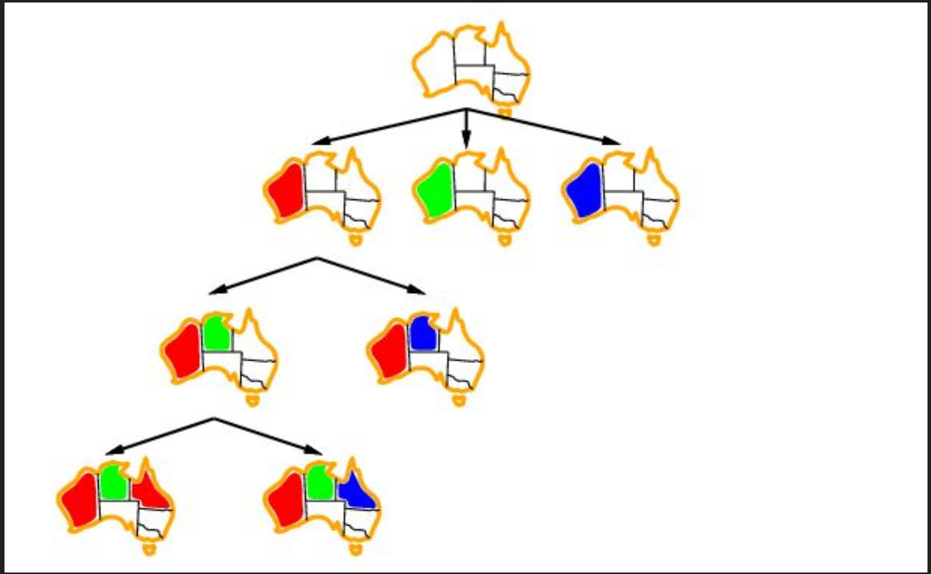
Backtracking example



Backtracking example



Backtracking example

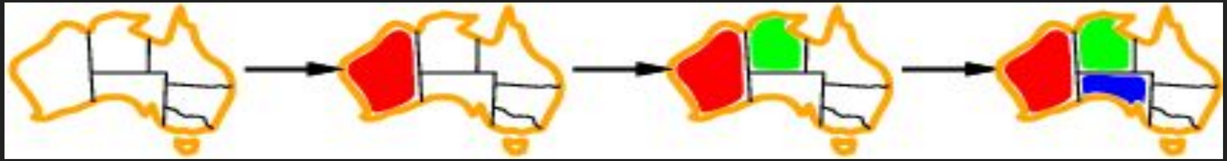


Improving backtracking efficiency

- General-purpose methods can give huge gains in speed:
  - Which variable should be assigned next?
  - In what order should its values be tried?
  - Can we detect inevitable failure early?
  - We'll discuss heuristics for all these questions in the following.

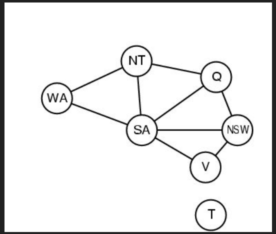
Which variable should be assigned next? □minimum remaining values heuristic

- Most constrained variable:
  choose the variable with the fewest legal values

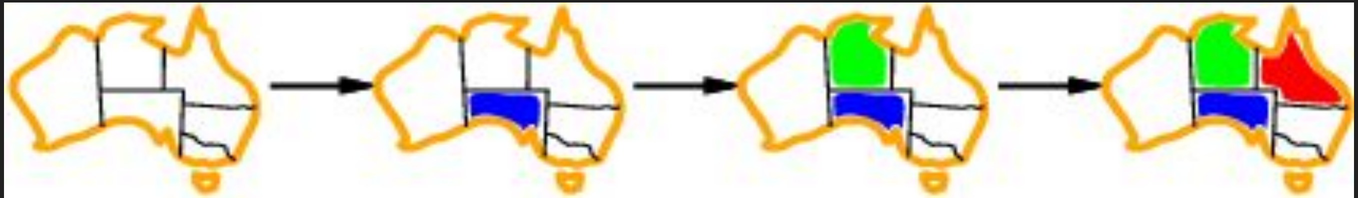


- a.k.a. minimum remaining values (MRV) heuristic
- Picks a variable which will cause failure as soon as possible, allowing the tree to be pruned.

Which variable should be assigned next? □ degree heuristic

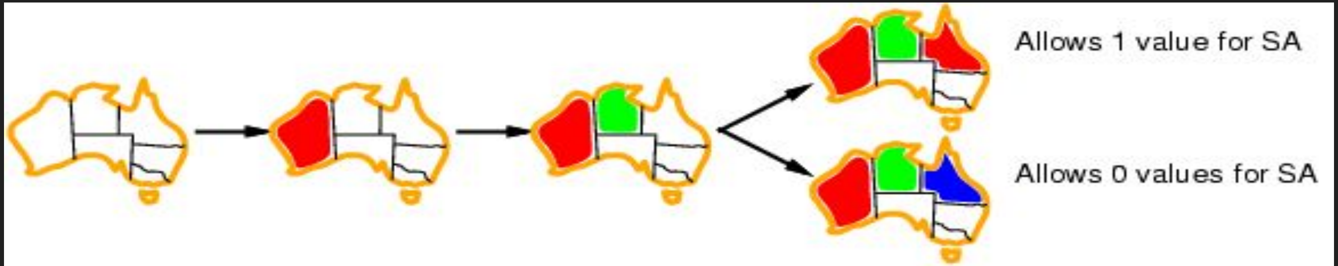


- Tie-breaker among most constrained variables
- Most constrain*ing* variable:
  - choose the variable with the most constraints on remaining variables (most edges in graph)



In what order should its values be tried? ☐ least constraining value heuristic

- *Given a variable*, choose the least constraining value:
  - the one that rules out the fewest values in the remaining variables



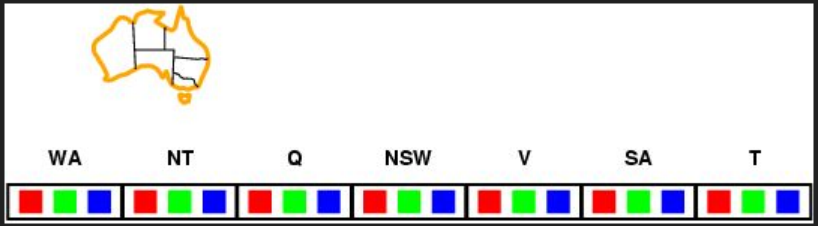
- Leaves maximal flexibility for a solution.
- Combining these heuristics makes 1000 queens feasible

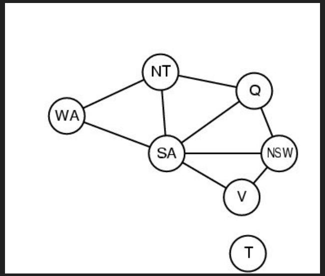
Rationale for MRV, DH, LCV

- In all cases we want to enter the most promising branch, but we also want to detect inevitable failure as soon as possible.
- MRV+DH: the variable that is most likely to cause failure in a branch is assigned first. E.g X1-X2-X3, values is 0,1, neighbors cannot be the same.
- LCV: tries to avoid failure by assigning values that leave maximal flexibility for the remaining variables.

Can we detect inevitable failure early? ☐ forward checking

- Idea:
  - Keep track of remaining legal values for unassigned variables that are connected to current variable.
  - Terminate search when any variable has no legal values





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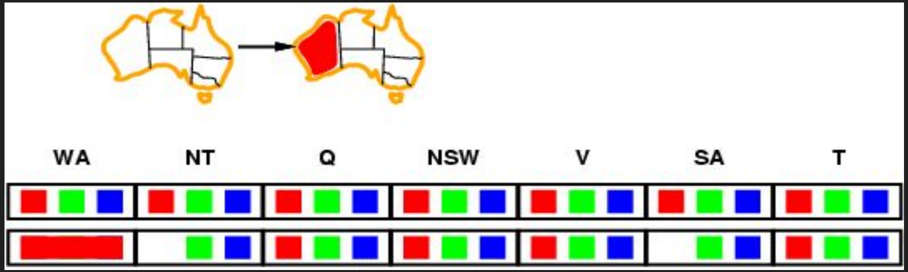

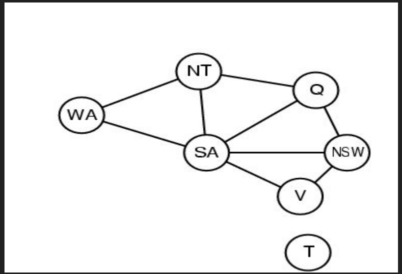

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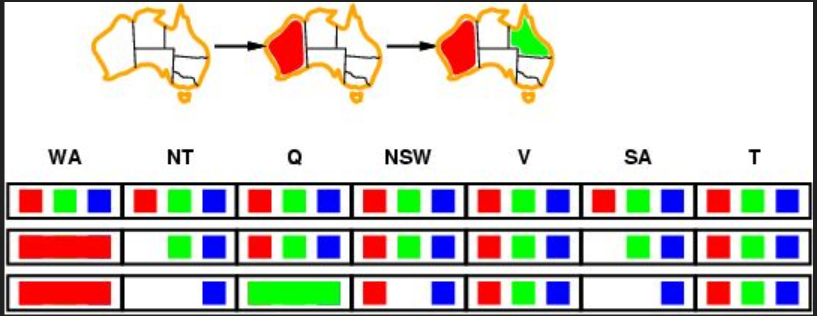

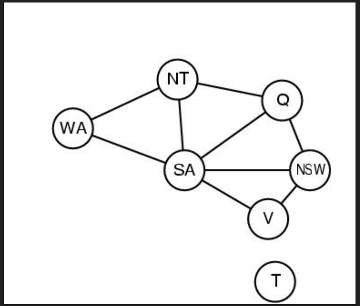

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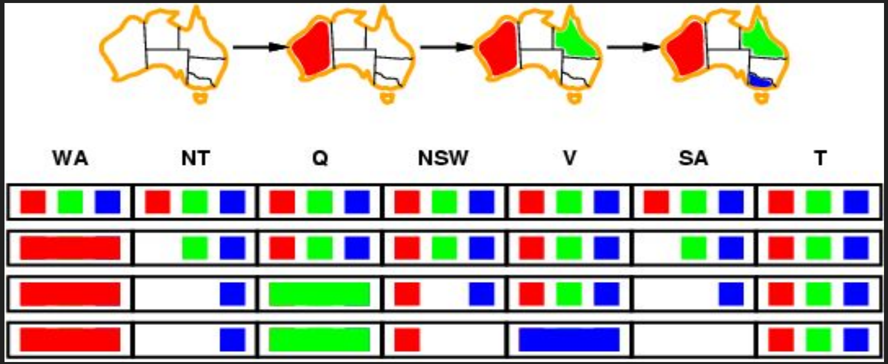
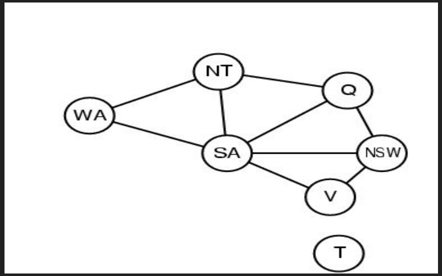
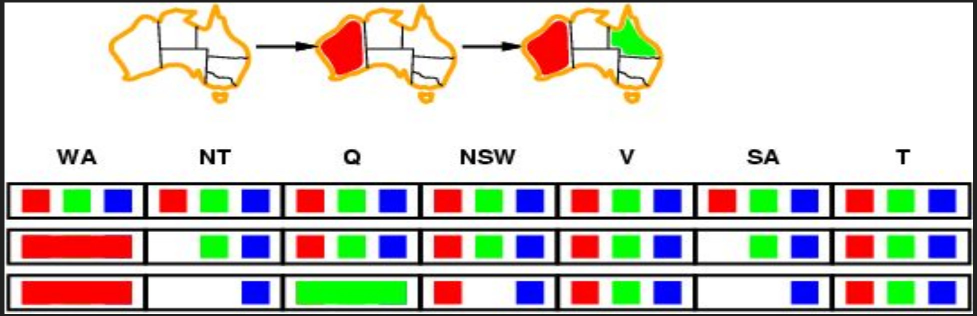
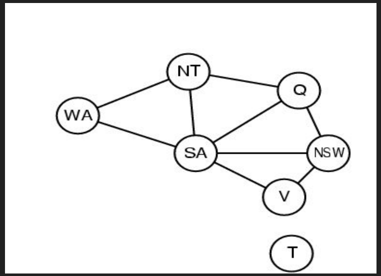




Constraint propagation

- Forward checking only looks at variables connected to current value in constraint graph.



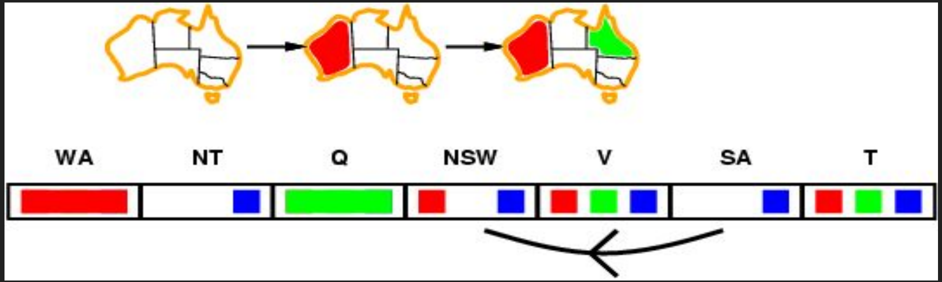


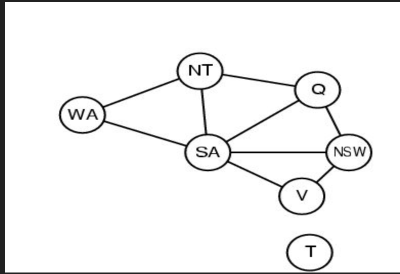
- NT and SA cannot both be blue!
- Constraint propagation repeatedly enforces constraints locally

Arc consistency

- Simplest form of propagation makes each arc consistent
- *X* □ *Y* is consistent iff

  for every value *x* of *X* there is some allowed *y*





consistent arc.

constraint propagation propagates arc consistency on the graph.

Arc consistency

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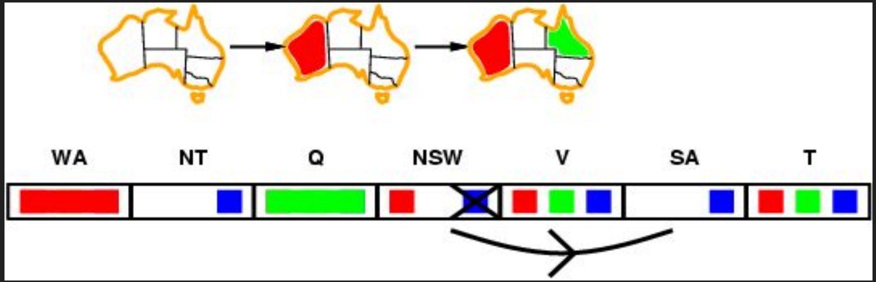

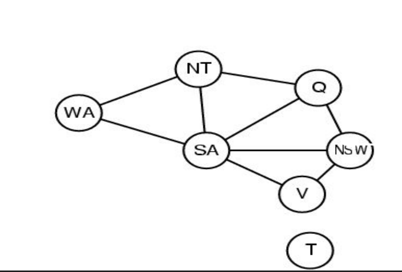

inconsistent arc.
remove blue from source□ consistent arc.

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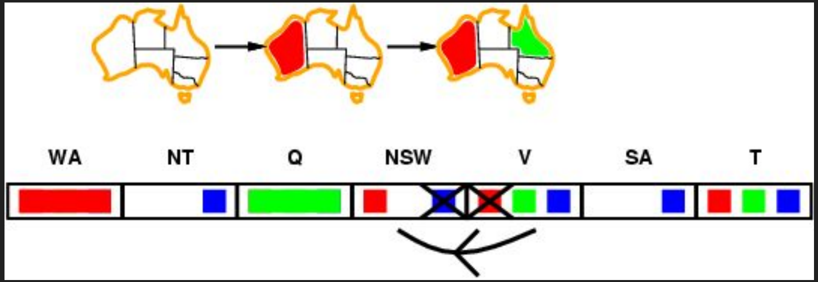
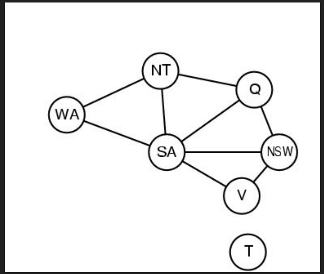




this arc just became inconsistent

- If *X* loses a value, neighbors of *X* need to be rechecked:

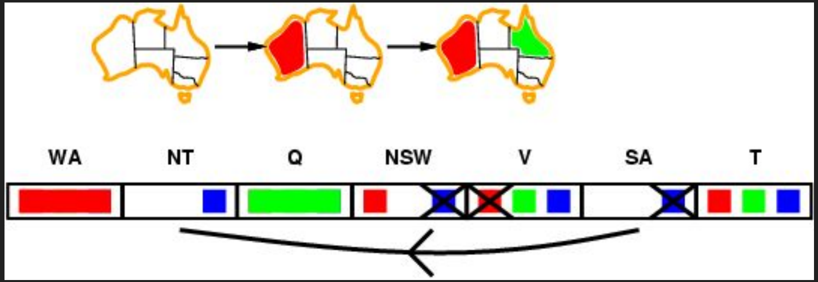
  i.e. incoming arcs can become inconsistent again

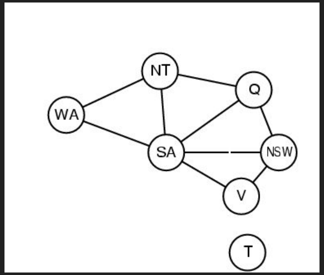
  (outgoing arcs will stay consistent).

Arc consistency

- Simplest form of propagation makes each arc consistent
- *X* □ *Y* is consistent iff

  for every value *x* of *X* there is some allowed *y*





- If *X* loses a value, neighbors of *X* need to be rechecked
- Arc consistency detects failure earlier than forward checking
- Can be run as a preprocessor or after each assignment
- Time complexity: $O(n^2d^3)$

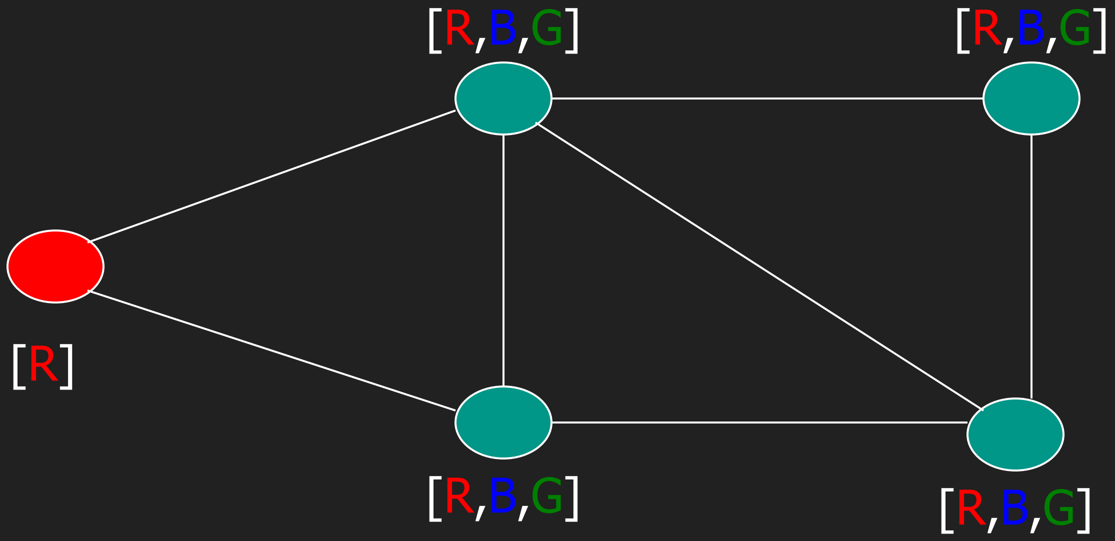
Arc Consistency

- This is a propagation algorithm. It’s like sending messages to neighbors on the graph! How do we schedule these messages?

- Every time a domain changes, all incoming messages need to be re-send. Repeat until convergence ☐ no message will change any domains.

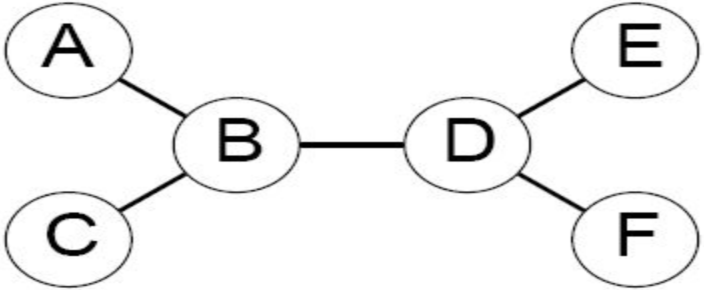
- Since we only remove values from domains when they can never be part of a solution, an empty domain means no solution possible at all ☐ back out of that branch.

Try it yourself



Use all heuristics including arc-propagation to solve this problem.

Tree-structured CSPs

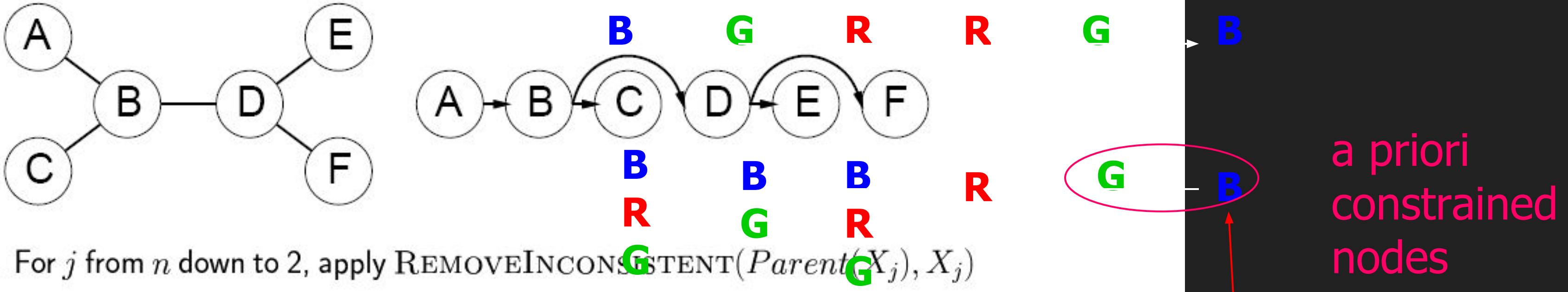


Theorem: if the constraint graph has no loops, the CSP can be solved in $O(n\,d^2)$ time

Compare to general CSPs, where worst-case time is $O(d^n)$

Algorithm for tree-structured CSPs

1. Choose a variable as root, order variables from root to leaves such that every node's parent precedes it in the ordering



2. For j from n down to 2, apply $\text{REMOVEINCONSISTENT}(Parent(X_j), X_j)$

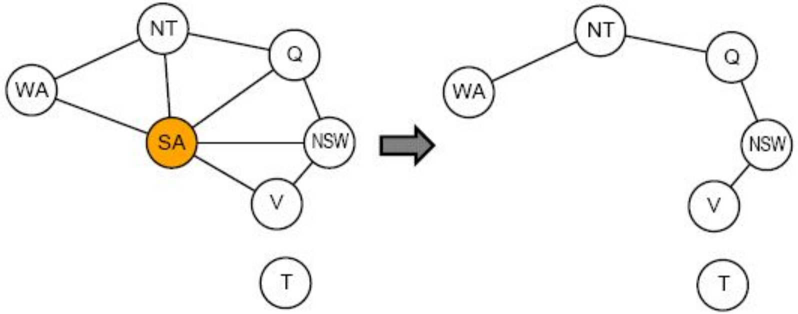
3. For j from 1 to n, assign X_j consistently with $Parent(X_j)$

Note: After the backward pass, there is guaranteed to be a legal choice for a child note for *any* of its leftover values.

This removes any inconsistent values from Parent(Xj), it applies arc-consistency moving backwards.

Nearly tree-structured CSPs

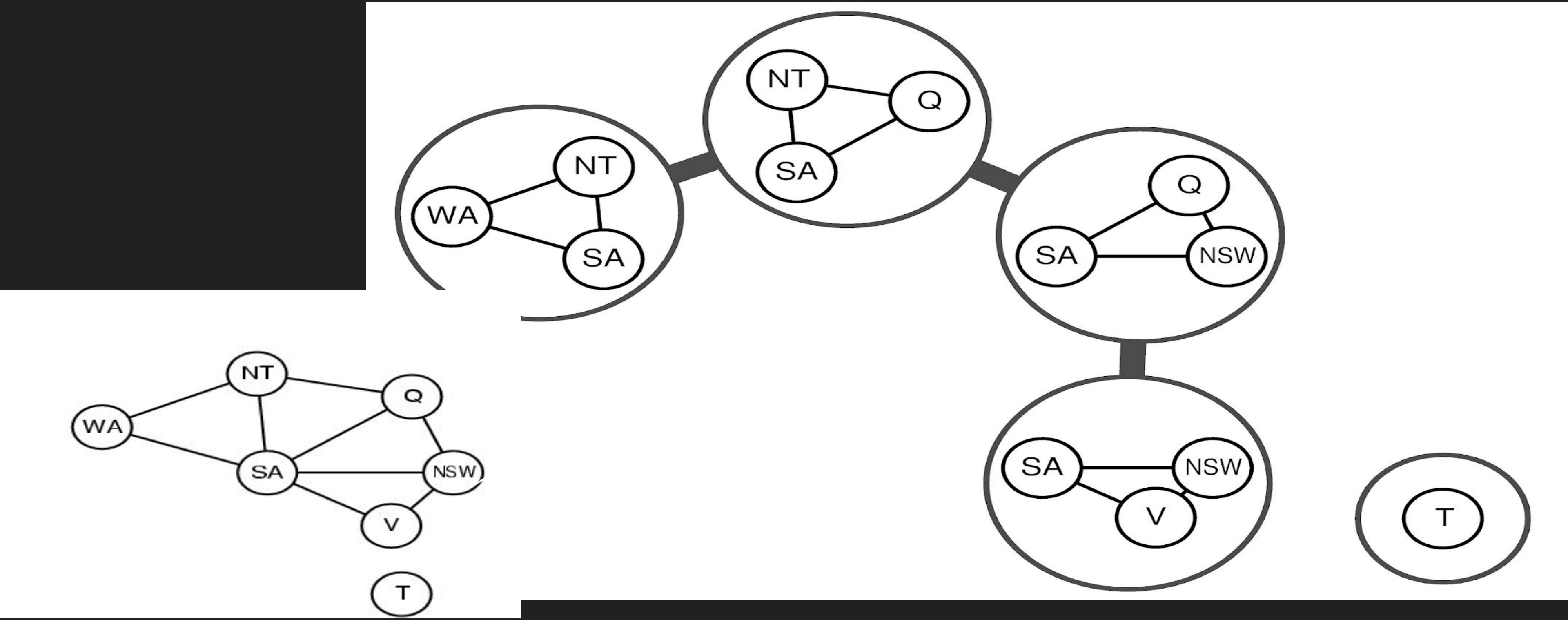
Conditioning: instantiate a variable, prune its neighbors' domains



Cutset conditioning: instantiate (in all ways) a set of variables such that the remaining constraint graph is a tree

Cutset size $c \Rightarrow$ runtime $O(d^c \cdot (n - c)d^2)$, very fast for small c

Junction Tree Decompositions

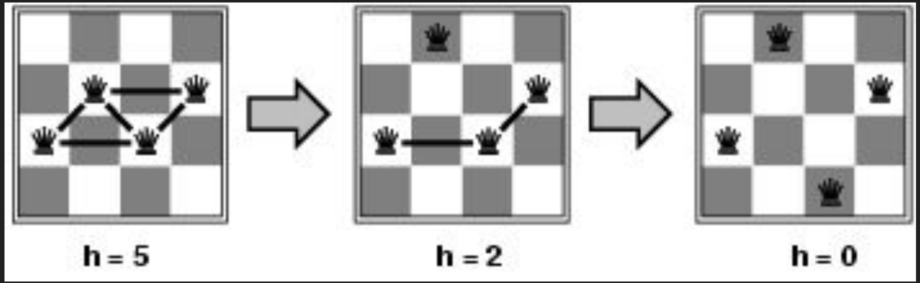


Local search for CSPs

- Note: The path to the solution is unimportant, so we can apply local search!
- To apply to CSPs:
  - allow states with unsatisfied constraints
  - operators reassign variable values
- Variable selection: randomly select any conflicted variable
- Value selection by min-conflicts heuristic:
  - choose value that violates the fewest constraints
  - i.e., hill-climb with $h(n)$ = total number of violated constraints

Example: 4-Queens

- States: 4 queens in 4 columns ($4^4 = 256$ states)
- Actions: move queen in column
- Goal test: no attacks
- Evaluation: $h(n)$ = number of attacks

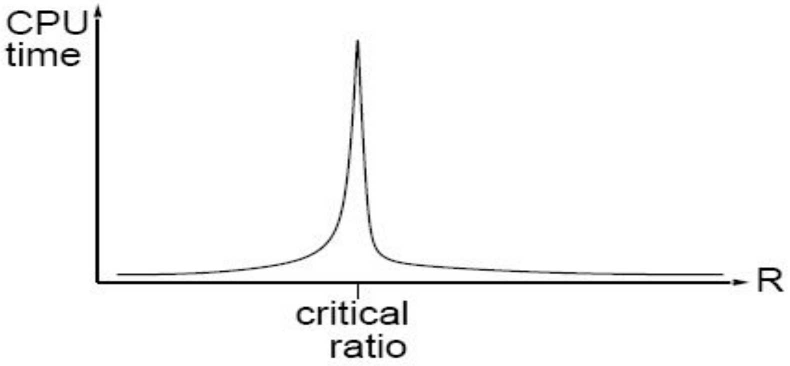


Performance of min-conflicts

Given random initial state, can solve n-queens in almost constant time for arbitrary n with high probability (e.g., n = 10,000,000)

The same appears to be true for any randomly-generated CSP
except in a narrow range of the ratio

$$R = \frac{\text{number of constraints}}{\text{number of variables}}$$



Summary

- CSPs are a special kind of problem:
  - states defined by values of a fixed set of variables
  - goal test defined by constraints on variable values
- Backtracking = depth-first search with one variable assigned per node
- Variable ordering and value selection heuristics help significantly
- Forward checking prevents assignments that guarantee later failure
- Constraint propagation (e.g., arc consistency) does additional work to constrain values and detect inconsistencies
- Iterative min-conflicts is usually effective in practice