Kuramoto-Sivashinsky equation

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1 Introduction

The Kuramoto-Sivashinksy equation describes a falling liquid film. It is a nonlinear PDE for the velocity u at the surface of the liquid and in the following we will consider a one-dimensional system.

The equation shows two regimes for periodic boundary conditions, a stationary one and a turbulent one. There will be a special consideration of the numerical simulation for the turbulent regime.

2 Background

The PDE reads [2]

$$\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial u^2}{\partial x} + \frac{\partial^2 u}{\partial x^2} + \frac{\partial^4 u}{\partial x^4} = 0. \tag{1}$$

This is a reduced form, which can always be reached by scaling of t, u or x.

Any coefficient in front of the time-derivative can be reduced by scaling the time, without affecting any of the other coefficients, so we do not need to consider this term. This reduces our system to

$$\frac{\alpha}{2}\frac{\partial u^2}{\partial x} + \beta\frac{\partial^2 u}{\partial x^2} + \gamma\frac{\partial^4 u}{\partial x^4} = 0. \tag{2}$$

We will use the transformation $u = \beta^{3/2}/(\alpha\gamma^{1/2})v$.

$$\frac{1}{2}\frac{\beta^3}{\alpha\gamma}\frac{\partial v^2}{\partial x} + \frac{\beta^{5/2}}{\alpha\gamma^{1/2}}\frac{\partial^2 v}{\partial x^2} + \frac{\beta^{3/2}\gamma^{1/2}}{\alpha}\frac{\partial^4 v}{\partial x^4} = 0. \tag{3}$$

Next we scale x by $x = \frac{\gamma^{1/2}}{\beta^{1/2}}y$.

$$\frac{1}{2}\frac{\beta^{7/2}}{\alpha\gamma^{3/2}}\frac{\partial v^2}{\partial y} + \frac{\beta^{7/2}}{\alpha\gamma^{3/2}}\frac{\partial^2 v}{\partial y^2} + \frac{\beta^{7/2}}{\alpha\gamma^{3/2}}\frac{\partial^4 v}{\partial y^4} = 0. \tag{4}$$

There is the same coefficient in front of every term, by multiplying with the inverse we get a coefficient free equation.

This equation cannot be solved directly, even without the coefficients. A typical ansatz for such a PDE is to go to the Fourier-space, here we will use the Fourier-space of x. For u we

assume periodic boundary conditions to make the Fourier-space discrete, so $u(x,t) = u(x+L,t)$. Using this and $K = 2\pi/L$ transforms u to

$$C_n(t) = \frac{1}{L}\int_0^L dx\; u(x,t)e^{-inKx}, \tag{5}$$

$$u(x,t) = \sum_{n\in Z} C_n(t)e^{inKx}. \tag{6}$$

Because u is a real function we know that $C_{-n}(t) = C_n(t)^*$, where * denotes the complex conjugate.

Now we can plug (6) in (1) and use $1/2\partial_x u^2 = u\partial_x u$ to get

$$\sum_{n\in\mathbb{Z}} e^{inKx}\left[\frac{dC_n(t)}{dt} + C_n(t)\sum_{l\in\mathbb{Z}} C_l(t)ilKe^{ilKx} - n^2K^2C_n(t) + n^4K^4C_n(t)\right] = 0. \tag{7}$$

We can rewrite the $u\partial_x u$ further by first considering the sums

$$\sum_{n\in\mathbb{Z}} C_n(t)e^{inKx}\sum_{l\in\mathbb{Z}} C_l(t)ilKe^{ilKx} = \sum_{n\in\mathbb{Z}}\sum_{l\in\mathbb{Z}} C_n(t)C_l(t)ilKe^{i(l+n)Kx}. \tag{8}$$

The order of the sums can be switched, since they are bounded. Next we introduce a new variable $m = n + l$. This is a one-to-one-map for constant l, so we set $n = m - l$ and replace the sum over n by the sum over m,

$$\sum_{n\in\mathbb{Z}}\sum_{l\in\mathbb{Z}} C_n(t)C_l(t)ilKe^{i(l+n)Kx} = \sum_{l\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} e^{imKx}ilKC_l(t)C_{m-l}(t). \tag{9}$$

The sums are still bounded, so we can switch them again and reordering of the terms leads to

$$u\partial_x u = \sum_{m\in\mathbb{Z}} e^{imKx}\sum_{l\in\mathbb{Z}} ilKC_l(t)C_{m-l}(t). \tag{10}$$

This leads us to a system of coupled ODE with infinite dimensionality

$$\frac{dC_n(t)}{dt} + \sum_{m\in\mathbb{Z}} imKC_m(t)C_{n-m}(t) - n^2K^2C_n(t) + n^4K^4C_n(t) = 0. \tag{11}$$

It still cannot be solved analytically, but if we reduce the system to a finite number of modes we will be able to solve it numerically. To get a good estimate for the number of modes we use the linear approximation

$$\frac{dC_n(t)}{dt} = \left[n^2K^2 - n^4K^4\right]C_n(t). \tag{12}$$

C_n is a complex number, so we need to consider both parts, the real one and the imaginary one. But according to the linearization both will evolve according to the same ODE. Let us call c_n the real part of C_n, the calculation for the imaginary part will be exactly the same. If $c_n = 0$ or $n = 0$ we will always have $d_t c_n = 0$, nothing happens.

But if we assume $c_n > 0$, we need $d_t c_n > 0$ for it to grow. This leads us to $n^2K^2 - n^4K^4 > 0$. But this can only be fulfilled for $1/K > n$.

Next we assume $c_n < 0$, so we need $d_t c_n < 0$ for it to grow. This leads us again to $n^2K^2 - n^4K^4 > 0$.

Regardless of the sign of the real or imaginary part of C_n, we will always need $1/K > n$. Or in other words, the number of modes N that grow is given by

$$N = \lfloor 1/K \rfloor = \left\lfloor \frac{L}{2\pi} \right\rfloor. \tag{13}$$

We can see that most modes disappear over time. For $L < 2\pi$ we do not have any modes that grow according to the linear approximation.

To get a Galerkin approximation of the system, we use a number M of modes. M should be bigger than the linear approximation for it to work. We assume all modes with $|n| > M$ will tend to zero after a transient. Reordering (11) and using $C_{n>M}(t) = 0$ gives us

$$\frac{dC_n(t)}{dt} = \left[n^2K^2 - n^4K^4\right] C_n(t) - \sum_{|m|\leq M} imKC_m(t)C_{n-m}(t). \tag{14}$$

Since $C_{-n} = C_n^*$ it will be enough to evolve the positive modes. For $n > 0$ we can reduce the sum to

$$\frac{dC_n(t)}{dt} = \left[n^2K^2 - n^4K^4\right] C_n(t) - \sum_{m=-M+n}^{M} imKC_m(t)C_{n-m}(t). \tag{15}$$

This equation is the Galerkin approximation for the Kuramoto-Sivashinsky equation and can be solved numerically. It reduces the system of infinite dimensionality to a system with finite dimensionality by considering only the changing modes.

3 Simulation

In general there are three important parameters in simulating equation (15). The number of modes to consider M, the spatial periodicity L and the time step h. M and L were constantly varied and especially the way to choose M will be looked at later. The time step h has been chosen to 0.01.

The initial conditions were a random distribution of Fourier modes up to M and the integral over u has been chosen to zero, $C_0(t) = 0$, since it would only add a superimposed movement to the turbulence. The DFT (Discrete Fourier transform) was used to get the velocity field from the modes according to (6).

The ODE (15) is stiff, so we cannot use the Euler or Runge-Kutta algorithm. Instead we have to use the ETD2RK algorithm [1], which reduces the equation to Runge-Kutta and is of order $O(h^2)$. The system describes an equation of form

$$\frac{dC}{dt} = aC + F(C, t), \tag{16}$$

where C and F are vectors and a is a matrix. The ETD2RK algorithm then reads

$$\tilde{C}_n = C_n e^{ah} + F_n \frac{e^{ah} - 1}{a} \tag{17}$$

$$C_{n+1} = \tilde{C}_n + \left(F_n(\tilde{C}_n, t_n + h) - F_n\right) \frac{e^{ah} - 1 - ah}{a^2 h}, \tag{18}$$

where n is the index of time. If a is a diagonal matrix this algorithm reduces to an element-wise evolution of the vectors with the corresponding element of a. We have just such a case, which makes the implementation quite easy.

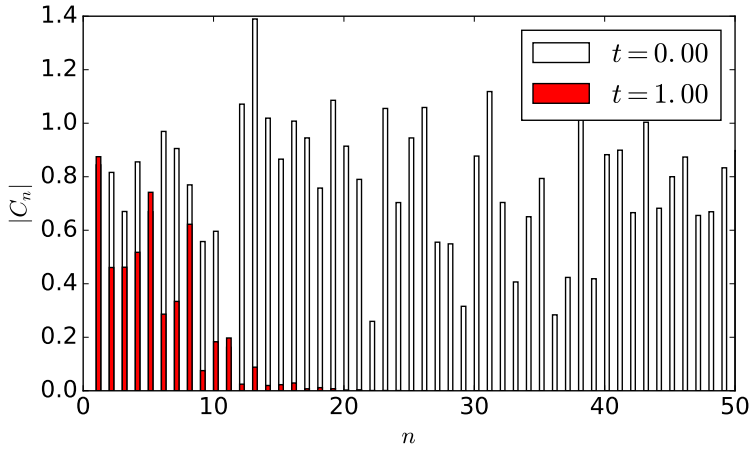


Figure 1: Evolution of the magnitude of modes for $L = 40$.

4 Results

4.1 M - Modes to consider

First we need to know, how many modes we should consider in the simulation. The evolution of the modes for $L = 40$ is plotted in figure 1. According to the linear approximation there will be $N = 6$ evolving modes. As we can see in the figure, there are in reality about 20 modes. If we choose $M = 4N$ we will be on the safe side. Even if there are more modes, they will decrease drastically over time. This value will be used for all later simulations.

4.2 Solutions for different L

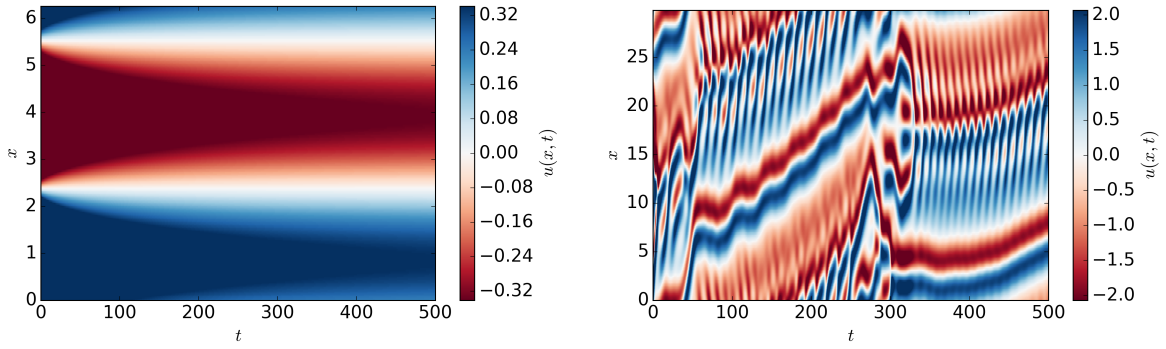
In the figure 2 we can see the evolution of the velocity for different L. In 2b we can see clear patterns, the system reaches the stationary state quite fast. In figure 2a the evolution is not finished, but it will also reach the stationary state after some time.

But if we look at 2c and 2d, we will see another regime. The field is no longer stationary, but turbulent. For $L = 30$ in 2c there are strong fluctuations and it is no longer periodic in time. In 2d there is also no periodicity. The system is in full turbulence.

4.2.1 Dependence on M

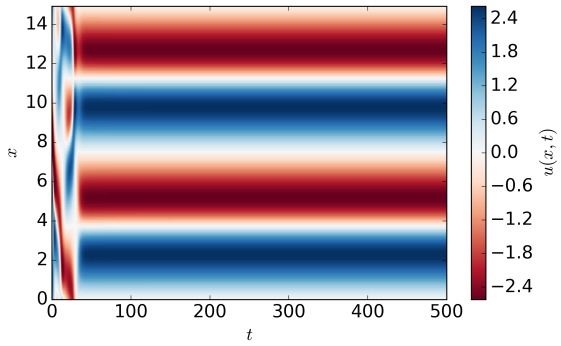
The solutions we have found depend on the number of modes M. An increase in M would not change much, since the higher modes die off quickly, but a decrease will have measurable consequences. An overview for $L = 30$ is given in figure 3. For $L = 30$ we have a linear approximation of $N = 4$.

Between figure 3a and 3b we see a loss of the fine detail. The structure retains it turbulent state, but for the bigger M the details are finer. Between 3b and 3c there is only a small change. And between 3c and 3d there is only one mode less and $M = 7 > N = 4$, yet we find that there are no turbulences.

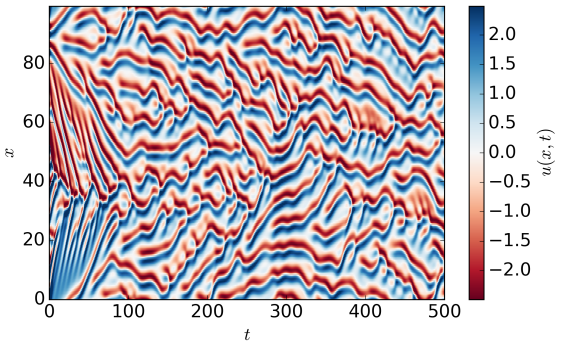


(a) $L = 6.29$

(c) $L = 30$

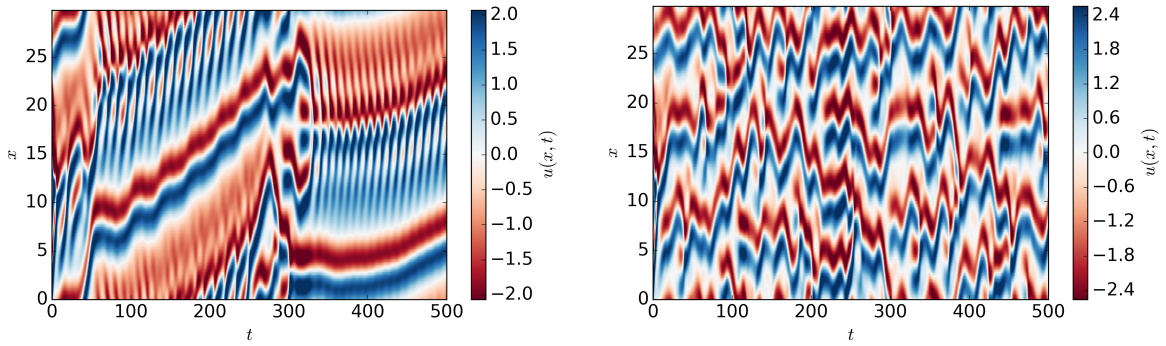


(b) $L = 15$

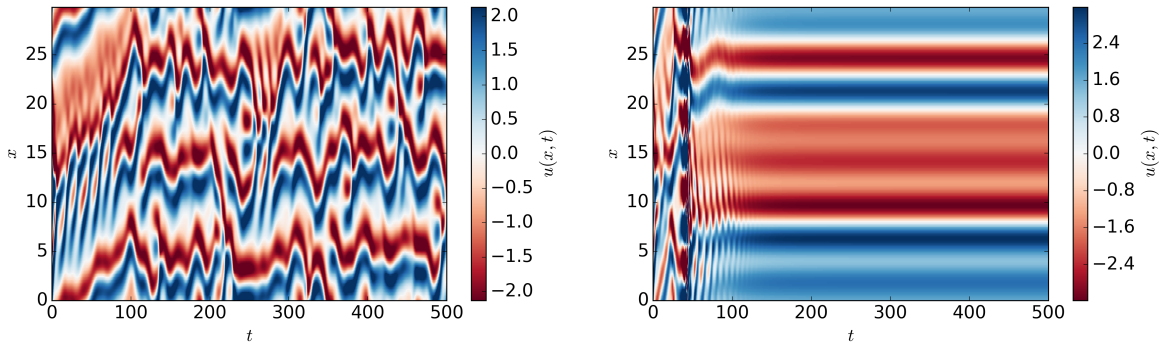


(d) $L = 100$

Figure 2: Evolution of the velocity field for different L.



(a) $M = 16$ (c) $M = 8$



(b) $M = 12$ (d) $M = 7$

Figure 3: Evolution of the velocity field for $L = 30$, but different a different amount of modes M.

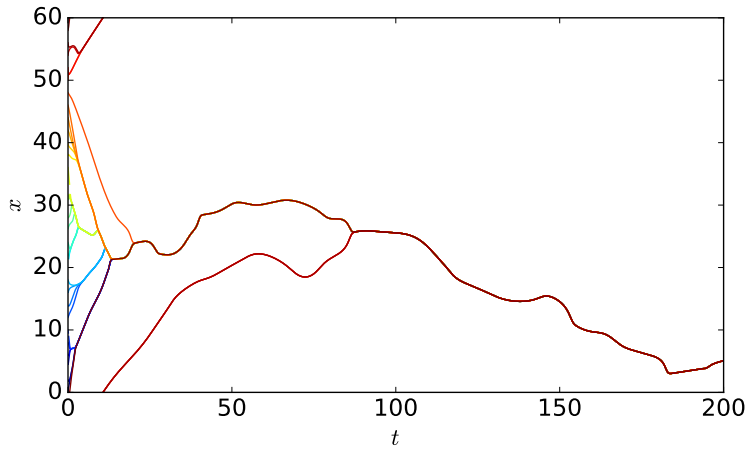


Figure 4: Advection of particles in the velocity field. The particles are initially spaced two units apart from each other.

4.3 Particle advection

Now we place particles on the fluid and look at their advection for $L = 60$, so we are in the turbulent regime. The particles are uniformly placed with a distance of two units. This is plotted in figure 4. The evolution of the position X is according to the equation

$$\frac{dX}{dt} = u(x,t). \tag{19}$$

The velocity field evolves using the ETD2RK algorithm, whereas the advection evolves using the Euler algorithm.

As we can see all the particles start at different positions, but after some time they all meet up. After about 100 time units they all have the same trajectory.

4.4 Conclusion

As we have seen, the Kuramoto-Sivashinsky equation describes two regimes, a stationary one and a turbulent one. The two regimes have been found numerically, where the turbulent regime starts somewhere above, but close by $L = 20$.

The accuracy of the simulation depends heavily on the number of Fourier modes, which are used for the evolution. Thanks to the Galerkin approximation there is a finite number of modes to consider, to describe the system accurately, but too strong a reduction lead to a loss of the turbulent regime.

References

[1] S.M. Cox and P.C. Matthews. Exponential time differencing for stiff systems. *Journal of Computational Physics*, 176(2):430 – 455, 2002.

[2] Christian Ruyer-Quil, Nicolas Kofman, Didier Chasseur, and Sophie Mergui. Dynamics of falling liquid films. *The European Physical Journal E*, 37(4):30, 2014.